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On Summands Properties and Minkowski Subtraction

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Abstract. In this paper we generalise the Sallee theorem from [J. Geom. 29 (1987)(1), 1–11, Theorem 4.3] into non-symmetric sets and give its proof in the terms of Minkowski subtraction.

Keywords. Minkowski subtraction, summands of convex sets

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1. Preliminaries

Let $X = (X, \tau)$ be a Hausdorff topological vector space and let $\mathcal{B}(X)$ be the family of all closed bounded and convex subsets of X. Let $\mathcal{K}(X)$ be the family of all compact members of $\mathcal{B}(X)$. For a subset $A \subset X$ of a vector space X we denote by

$$\operatorname{conv} A = \left\{ x = \sum_{i=1}^{k} \alpha_i a_i : \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1,\ a_i \in A,\ k \in \mathbb{N} \right\}$$

the *convex hull* of A and by

$$\operatorname{aff} A = \left\{ x = \sum_{i=1}^{k} \alpha_i a_i : \alpha_i \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1,\ a_i \in A,\ k \in \mathbb{N} \right\}.$$

the *affine hull* of A. The *Minkowski sum* for A, $B \in \mathcal{K}(\mathrm{X})$ is defined by

$$A + B = \{a + b : a \in A,\, b \in B\}.$$

We also define for $\lambda \in \mathbb{R}$ the sets $\lambda A = \{\lambda a : a \in A\}$ and $A - B = A + (-1)B$. We say that a set $B \in \mathcal{K}(X)$ is *a summand of* $A \in \mathcal{K}(X)$ if there exists a set $C \in \mathcal{K}(X)$ such that $B + C = A$.

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We introduce an equivalence relation "$\sim$" on $\mathcal{K}^2(X) = \mathcal{K}(X)\times\mathcal{K}(X)$ by

$$(A, B) \sim (C, D) \iff A + D = B + C.$$

For a nonempty, compact, convex set $A \in \mathcal{K}(X)$ *the support function* $h(A, \cdot) : X^*\to\mathbb{R}$ is given by

$$h(A, x) = \max_{a\in A} \langle a, x\rangle ,$$

where $\langle\cdot, \cdot\rangle$ is the dual pairing between X and X^*, X^* is the dual space (see [4]). Let

$$A \dot{-} B = \{x \in X : x + B \subset A\}$$

be the *Minkowski subtraction* of A and B. The equality

$$A \dot{-} B = \bigcap_{b\in B}(A - b)$$

holds true (see [2]). For $A, B, C \in \mathcal{K}(X)$ the following conditions hold:

(i) $A + (B \dot{-} A) \subset B$.
(ii) If $A = B + C$, then $B = A \dot{-} C$.
(iii) If $B \subset C$, then $A \dot{-} C \subset A \dot{-} B$.
(iv) If $\alpha \in \mathbb{R}$, then $\alpha(A \dot{-} B) = \alpha A \dot{-} \alpha B$.
(v) If $\alpha, \beta \in \mathbb{R}, \alpha \geq \beta$, then $\alpha A \dot{-} \beta A = (\alpha - \beta)A$.
(vi) If $B + C \subset A$, then $B \subset A \dot{-} C$.

For more properties of Minkowski subtraction see [2].

If $\operatorname{aff} A \cap \operatorname{aff} B = \{p\}$, then we write $A\oplus B$ instead $A + B$ and we call it a *direct sum* of A and B. If $a_1, \ldots, a_{n+1}$ are $n + 1$ points affinelly independent, then the set $\operatorname{conv}\{a_1, \ldots, a_{n+1}\}$ we call the *n-simplex*. By the *cube* in the $\mathbb{R}^n$ we mean a cartesian product $[a_1, b_1] \times \cdots \times [a_n, b_n]$ of intervals, where $a_k \leq b_k$ for $k = 1, \ldots, n$. Let $f \in X^*$, $A \subset f^{-1}(0)$ and $B \subset f^{-1}(0) + x$ for some $x \in X$, then the set $\operatorname{conv}(A \cup B)$ we call a *frustum* with the bases A and B. We call the set $A \in \mathcal{K}(X)$ *centrally symmetric* if $A = -A$.

2. Introduction

In this paper we focus on the summands properties. We generalise the Sallee theorem from [5, Theorem 4.3] into non-symmetric sets and give its proof in the terms of Minkowski subtraction. We also introduce the operator D^n, prove some of its basic properties and show its connections with the operator Ω^n defined by Sallee for symmetric sets in $\mathbb{R}^n$ ([5, Theorems 3.1 and 3.2]). We use this connections to the modification of the proof of Theorem 4.3 given in [5]. By the way, we receive some new properties of Minkowski subtraction. Then, we give some examples of members of the family $\mathcal{F}$ for which the implication

$$C \dot{-} A = B \text{ and } C \dot{-} B = A \quad \Rightarrow \quad A + B = C$$

is not true. This problem was considered by Sallee in [5]. In the last chapter we investigate the following problem: for which sets $A \in \mathcal{K}(X)$ the set $A \dot{-} B$ is a summand of the set A. Although it is quite a complicated issue and we do not receive a full characterisation of that family, now we are able to define the family $\mathcal{C}$ whose each element has this property. That family is rather vast. This problem was solved in the space $\mathbb{R}^2$ (see [7]).

First let us define an operator

$$D^k : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X) \cup \{\emptyset\}, \quad k \in \mathbb{N}$$

$$D^k(B, A) = \begin{cases} B \dot{-} A & \text{for } k = 1 \\ D^1(B, D^{k-1}(B, A)) & \text{for } k > 1. \end{cases}$$

In [5] Sallee introduced the following operator:

$$\Omega^k : \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n), \quad k \in \mathbb{N}$$

$$\Omega^k(B, A) = \begin{cases} B & \text{for } k = 0 \\ \bigcap \{x + A : x \in \Omega^{k-1}(B, A)\} & \text{for } k > 0, \end{cases}$$

where A is a centrally symmetric set in $\mathbb{R}^n$.

3. The operators D^n and Ω^n

In this section we give some basic properties of the operations D^n and Ω^n.

Theorem 3.1. *Let $A, B \in \mathcal{K}(X)$. Then $D^3(A, B) = D^1(A, B)$.*

Proof. For $A, B, C, M, N \in \mathcal{K}(X)$ the following implication is true:

$$M \subset N \quad \Rightarrow \quad A \dot{-} N \subset A \dot{-} M. \tag{1}$$

Since $B + (A \dot{-} B) \subset A$ we obtain

$$B \subset A \dot{-} (A \dot{-} B). \tag{2}$$

Replacing B by $A \dot{-} B$ in the above inclusion we get $A \dot{-} B \subset A \dot{-} (A \dot{-} (A \dot{-} B))$ which is equivalent to the following expression $D^1(A, B) \subset D^3(A, B)$. On the other hand, putting $M = B$, $N = A \dot{-} (A \dot{-} B)$ and applying the equalities (2) and (1) we obtain $A \dot{-} (A \dot{-} (A \dot{-} B)) \subset A \dot{-} B$ which proved $D^3(A, B) \subset D^1(A, B)$. □

Theorem 3.2. *Let $A, B \in \mathcal{K}(X)$ and B be a centrally symmetric set. Then the following equation holds true:*

$$\Omega^n(A,B) = D^n(B,(-1)^n A) \quad \textit{for } n \in \mathbb{N}.$$

Proof. From the definition of Minkowski subtraction we have

$$\Omega^1(A,\, B) = \bigcap_{x\in A}(B-(-x)) = \bigcap_{y\in -A}(B-y) = B \dot{-} (-A) = D^1(B,\, -A).$$

We shall prove the equality $\Omega^1(A,B) = -(B \dot{-} A)$. From the definition of the subtraction we have $-\{y : y + A \subset B\} = -\{y : -y - A \subset -B\} = -\{y : -y - A \subset B\} = \{-y : -y - A \subset B\} = \{y : y - A \subset B\} = B \dot{-} (-A) = D^1(B,-A)$. Hence

$$\Omega^2(A,B) = \Omega^1(\Omega^1(A,B),B) = \Omega^1(D^1(B,-A),B) = D^1(B,-D^1(B,-A)). \quad (3)$$

It is easy to see that

$$D^1(B,\, -A) = B \dot{-} (-A) = -(B \dot{-} A) = -(B \dot{-} A) = -D^1(B,\, A). \quad (4)$$

Using the equalities (3) and (4) we obtain $\Omega^2(A,\, B) = D^1(B,(D^1(B,\, A))) = D^2(B,\, A)$. From the definition of Ω^n we have $\Omega^2(A,\, B) = \Omega^1(\Omega^1(A,\, B),\, B) = B \dot{-} [-(B \dot{-} (-A))] = B \dot{-} (B \dot{-} A)$.

In [5] was proved that $\Omega^3(A,\, B) = \Omega^1(A,\, B)$ for $A, B \subset \mathbb{R}^n$. Analogously it can be proved that $\Omega^3(A,\, B) = \Omega^1(A,\, B)$ for $A, B \subset X$. From the definition of Ω^n, Theorem 3.1 and the above we have $\Omega^3(A,\, B) = \Omega^1(\Omega^2(A,\, B),\, B) = \Omega^1(D^2(A,\, B),\, B) = D^1(B,-D^2(B,\, A)) = -D^1(B,\, D^2(B,\, A)) = -D^3(B,\, A) = D^3(B,\, -A) = -D^1(B,\, A) = D^1(B,\, -A) = \Omega^1(A,\, B)$.

Now, using the Theorem 3.1 we obtain that $\Omega^n(A,\, B) = D^n(B,\, (-1)^n A)$ for $n \in \mathbb{N}$. □

Corollary 3.3. *Let $A, B \in \mathcal{K}(X)$. Then*

$$D^n(B,\, A) = \begin{cases} B \dot{-} A & \textit{for odd } n \\ B \dot{-} (B \dot{-} A) & \textit{for even } n. \end{cases}$$

Moreover, if B is a centrally symmetric set, then

$$\Omega^n(A,\, B) = \begin{cases} B \dot{-} (-A) & \textit{for odd } n \\ B \dot{-} (B \dot{-} A) & \textit{for even } n. \end{cases}$$

Definition 3.4. Let X be a topological vector space and let $A, B, M \in \mathcal{K}(X)$. A pair (A, B) is called an *M-pair* if and only if $A + B = M$.

Definition 3.5. Let X be a topological vector space. A pair of sets $(A\,,B) \in \mathcal{K}^2(X)$ is called *the pair of sets of constant width relative to S* or *S-pair* if $A - B = \lambda$ for some $\lambda > 0$, where S is a centrally symmetric subset of X.

Notice, that S-pair and M-pair are different.

Theorem 3.6. *Let X be a locally convex topological vector space and let $A, B \in \mathcal{K}(X)$. If $(A,\, B)$ is an S-pair and there exists $(C,\, D)$ such that $(A,\, B) \sim (C,\, D)$ then $C - D$ is centrally symmetric.*

Proof. Using the quivalence

$$h(A,\, u) + h(B,\, -u) = \lambda h(S,\, u) \iff A - B = \lambda S$$

and the equivalence relation

$$(A,\, B) \sim (C,\, D) \iff A + D = B + C$$

we obtain

$$\begin{aligned} h(A,\, u) + h(D,\, u) &= h(B,\, u) + h(C,\, u) \\ h(B, -u) + h(C, -u) &= h(A, -u) + h(D, -u). \end{aligned}$$

Hence from the above equality

$$\lambda h(S,\, u) + h(D,\, u) + h(C, -u) = \lambda h(S, -u) + h(C,\, u) + h(D, -u).$$

Therefore $h(C,\, u) + h(D, -u) = h(D,\, u) + h(C, -u)$. It is easy to see that $h(D, -u) = \max\langle D, -u\rangle = \max\langle -D,\, u\rangle = h(-D,\, u)$, hence from the above equality we obtain $h(C,\, u) + h(-D,\, u) = h(D,\, u) + h(-C,\, u), h(C-\, D,\, u) = h(D-\, C,\, u)$. Therefore $C - D = D - C$. □

Definition 3.7. Let X be a normed space and $A \in \mathcal{K}(X)$. We call the set A *a set of constant width* if $A - A = \lambda B$,, where B is the unit ball.

Definition 3.8. Let X be a topological vector space and $A \in \mathcal{K}(X)$. We call the set A *a set of constant S-width* if $A - A = \lambda S$, where S is a centrally symmetric subset of X.

Definition 3.9. Let X be a normed space. A pair $(A,\, C) \in \mathcal{K}^2(X)$ is called *the pair of sets of constant width* if $A - C = \lambda B$, where B is a unit ball.

Theorem 3.10. *Let X be a normed space and let $A,\, B,\, M \in \mathcal{K}(X)$. Then the following statements are equivalent:*

(i) *A is a summand of M;*

(ii) *$(A, -D^1(M,\, A))$ is an M-pair.*

Proof. If A is a summand of M, then there exists $B \in \mathcal{K}(X)$ such that $A + B = M$. Hence $A + D^1(M, A) = M$, $B = M \dot{-} A = D^1(M, A)$. We have $-B = -D^1(M, A)$.

Conversely, assume that $(A, -D^1(M, A))$ is an M-pair. By the definition of an M-pair, A is a summand of M. □

For $A, B \in \mathcal{K}(X)$ we will use the notation $A \vee B = \operatorname{conv}(A \cup B)$. For two elements $a, b \in \mathcal{K}(X)$ the interval with end points a and b will be denoted by $[a, b] = a \vee b$.

Example 3.11. Let $K = a \vee b \vee c$ and $K' = -2K$, where $K \subset \mathbb{R}^2$ is the Reuleaux triangle (see [1],[6]). Then K and K' are the sets of constant width. Notice that $-2K \dot{-} K$ is not a set of constant width (see Figure 1).

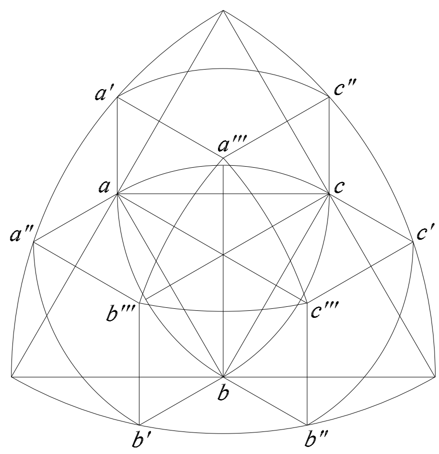

Figure 1: Reuleaux triangles

It is easy to observe that the set $-2K \dot{-} K$ is not a triangle. Its boundary is the union of three arcs with the vertices a''', b''', c'''. The curvature of every of those arcs is less than $\|a''' - b'''\|^{-1}$. So $-2K \dot{-} K$ is not a set of constant width.

Let $K = a \vee b \vee c \vee d \vee e$ be a pentagon of constant width and let $\alpha < -1$. Similary we can show that $\alpha K \dot{-} K$ is not a set of constant width (see Figure 2).

Corollary 3.12. *The Minkowski subtraction does not preserve the constant width of sets.*

Proposition 3.13. *The result of Minkowski subtraction of two centrally symmetric sets is centrally symmetric set.*

Proof. Let $A = -A$, $B = -B$. Then $-(A \dot{-} B) = (-1)A \dot{-} (-1)B = A \dot{-} B$. □

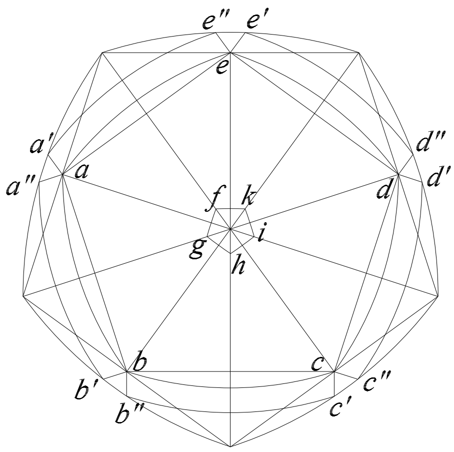

Figure 2: Reuleaux pentagons

If A, B, $S \in \mathcal{K}(X)$ and $A + B = S$, then $S \dot{-} A = B$ and $S \dot{-} B = A$. We define a class $\mathcal{S}$ of sets $S \in \mathcal{K}(X)$ such that $S \dot{-} A = B$ and $S \dot{-} B = A$ imply $A + B = S$. Moreover let us define $\mathcal{F} = \mathcal{K}(X) \setminus \mathcal{S}$. It is easy to observe that $S \in \mathcal{S}$ if and only if the equality $S \dot{-} (S \dot{-} A) = A$ implies that A is a summand of S.

Example 3.14. G. T. Sallee in [5] gives the following example (see Figure 3) of the member of the family $\mathcal{F}$.

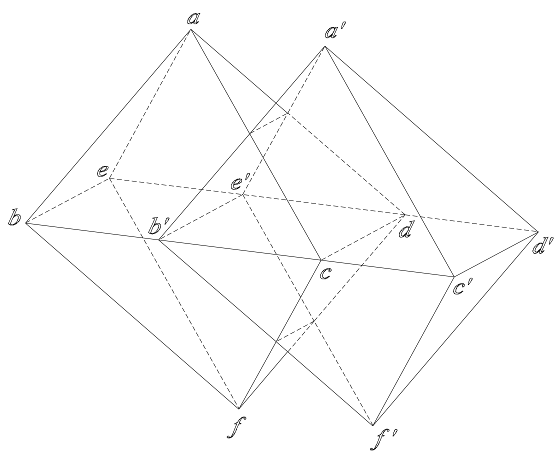

Figure 3: Octahedron

Let M be the octahedron, $B = b \vee b'$, A is the eight faceted set. Then

$A = M \dot{-} B$, $B = M \dot{-} A$, $A + B \neq M$:

$$\begin{aligned} A &= c \vee c' \\ M &= a \vee i \vee j \vee c' \vee d' \vee e' \vee f' \vee g' \vee b' \\ B &= h \vee c \vee c' \vee d' \vee e' \vee k' \vee k \vee e \vee d. \end{aligned}$$

Let the polytope A be the convex union of a hexagonal pyramid, a hexagonal prism and a wedge (see Figure 4). One of possible intersections $A \cap (A - x)$ is the convex union B of four-sided pyramid and four-sided prism, which is not a summand of A. From the construction of the above set we obtain a quite wide class of subsets of $\mathbb{R}^3$ which belongs to the family $\mathcal{F}$.

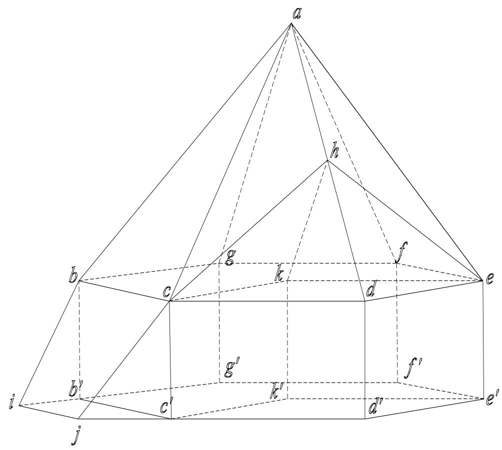

Figure 4: A polytope

The problem can be formulated as follows: to find all the sets S, for which the equality $S \dot{-} (S \dot{-} A) = A$ implies that A is a summand of S.

4. Some properties of Minkowski subtraction

In this section we give some properties of Minkowski subtraction and prove the Sallee theorem in its terms.

Theorem 4.1. *Let A, $B \in \mathcal{K}(X)$ and let L be a linear invertible transformation in X. Then $LA \dot{-} LB = L(A \dot{-} B)$.*

Proof. Let $y \in L(A \dot{-} B)$. Then there exists $x \in X$ such that $y = Lx$ and $x + B \subset A$. So $y + LB = Lx + LB \subset LA$. Hence $y \in LA \dot{-} LB$ and $L(A \dot{-} B) \subset LA \dot{-} LB$. If L is invertible then using the above inclusion we get $L^{-1}(LA \dot{-} LB) \subset A \dot{-} B$. Therefore $LA \dot{-} LB \subset L(A \dot{-} B)$. $\square$

Example 4.2. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(x,y) = (x, 0)$ and $A = [0,1] \times [0,1]$, $B = [0,1] \times [0,2]$, then $L(A \dot{-} B = \emptyset$ and $LA \dot{-} LB = \{(0,0)\}$. So the assumption of invertibility of L is essential.

Corollary 4.3. *Let L be a linear invertible transformation in X. If $L(A) = A$ and $L(B) = B$, then $L(A \dot{-} B) = A \dot{-} B$.*

Corollary 4.4. *If L is a linear invertible transformation in X, $L : X \to X$ and $S \in \mathcal{S}$, then $L(S) \in \mathcal{S}$.*

Proof. Assume that $L(S) \dot{-} (L(S) \dot{-} A) = A$. Then

$$L^{-1}(L(S) \dot{-} (L(S) \dot{-} A)) = L^{-1}(A).$$

Now by the linearity of L and L^{-1} we have $L^{-1}(L(S) \dot{-} L^{-1}(L(S) \dot{-} A)) = L^{-1}(A)$. Hence we obtain $S \dot{-} (L^{-1}(L(S)) \dot{-} L^{-1}(A)) = L^{-1}(A)$ and $S \dot{-} (S \dot{-} L^{-1}(A)) = L^{-1}(A)$. Putting $L^{-1}(A) = B$ we have $S \dot{-} (S \dot{-} B) = B$. Since $S \in \mathcal{S}$ so B is a summand of S. Therefore by the definition of the summand there exists a set $C \in \mathcal{K}(X)$ such that $B + C = S$. Putting $L^{-1}(A) + C = S$ and from properties of the operator L we obtain

$$LL^{-1}(A) + L(C) = L(S),$$

hence $A + L(C) = L(S)$. Then A is a summand of $L(S)$ and $L(S) \in \mathcal{S}$. □

Lemma 4.5. *Let $A, B, C \in \mathcal{K}(X)$ and* $\operatorname{aff} A = \operatorname{aff} B$. *Let moreover $C \subset W$, where W is a linear space, such that $W \cap \operatorname{aff} A = \{p\}$. Then*

$$(A \oplus C) \cap (B \oplus C) = (A \cap B) \oplus C$$
$$(A \oplus C) + B = (A + B) \oplus C.$$

Proof. We assume $V = \operatorname{aff} A = \operatorname{aff} B$ and W is a linear space with $W \cap V = \{p\}$. Let $U = V \oplus W$. Let us immerse isomorphicaly the space $V \oplus W$ into $V \times W \backsim V \oplus W \to V \times W$. We denote

$$\overset{\frown}{A} = \{(x,\, 0) \,:\, x \in A\} \subset V \times W$$
$$\overset{\frown}{B} = \{(y,\, 0) \,:\, y \in B\} \subset V \times W$$
$$\overset{\frown}{C} = \{(z,\, 0) \,:\, z \in C\} \subset V \times W.$$

We have $(\overset{\frown}{A} + \overset{\frown}{C}) \cap (\overset{\frown}{B} + \overset{\frown}{C}) = \{(x,z) \,:\, x \in A, z \in C \cap \{(y,z) \,:\, y \in B, z \in C\} = \{(u,z) \,:\, u \in A \cap B, z \in C\} = \overset{\frown}{A} \cap \overset{\frown}{B} + \overset{\frown}{C}$ since the isomorphic immersion is 1-1. So we have $(A \oplus C) \cap (B \oplus C) = A \cap B \oplus C$. To prove the second equality let us observe that $\operatorname{aff}(A + B) \subset \operatorname{aff} A + \operatorname{aff} B = 2 \operatorname{aff} A$. Hence $\operatorname{aff}(A + B) \cap \operatorname{aff} C \subset 2 \operatorname{aff} A \cap \operatorname{aff} W = 2\operatorname{aff} A \cap 2W = 2(\operatorname{aff} A \cap W) = 2\{p\}$. □

Corollary 4.6. *Let $A_t, A_{t_1}, A_{t_2}, C \subset X$ for $t, t_1, t_2 \in T$ and* $\operatorname{aff} A_{t_1} = \operatorname{aff} A_{t_2}$. *Moreover let $C \subset W$, where W is an affine subspace of X, such that $W \cap \operatorname{aff} A_t = \{p\}$. Then*

$$\bigcap_{t\in T}(A_t \oplus C) = \bigcap_{t\in T} A_t \oplus C.$$

Lemma 4.7. *Let $A \in \mathcal{K}(X)$ and let $x, y \in X$. Then* $\operatorname{aff}(A - x) = \operatorname{aff} A - x$. *Moreover, if $x, y \in \operatorname{aff} A$, then* $\operatorname{aff} A - x = \operatorname{aff} A - y$.

Proof. We prove that $\operatorname{aff}(A-x) = \operatorname{aff} A - x$. If $u \in \operatorname{aff}(A-x)$, then there exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, and $\alpha_1 + \dots + \alpha_n = 1$, $a_1, \dots, a_n \in A$ such that

$$u = \sum_{i=1}^{n} \alpha_i(a_i - x) = \sum_{i=1}^{n} \alpha_i a_i - x\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha_i a_i - x \in \operatorname{aff} \mathrm{A} - \mathrm{x}.$$

Moreover, let W be an affine space and let $p, q, w \in W$. Then $W - p = W - q$. Let $x \in W - p$ so $x = w - p + q - q \in W - q$, hence $W - p \subset W - q$. In a similar way, we obtain $W - q \subset W - p$. Hence $W - q = W - p$ what ends the proof. □

Lemma 4.8. *Let $\{A_s\}_{s\in S}, \{B_t\}_{t\in T}$ are two families of subsets of X such that*

(i) $\operatorname{aff} A_{s_1} = \operatorname{aff} A_{s_2}$, $s_1, s_2 \in S$

(ii) $\operatorname{aff} A_s \cap \operatorname{aff} B_t = \{p\}$, $s \in S, t \in T$.

Then

$$\bigcap_{s\in S, t\in T}(A_s \oplus B_t) = \bigcap_{s\in S} A_s \oplus \bigcap_{t\in T} B_t.$$

Proof. Let $t \in T$. Using the Corollary 4.6 for the B_t and A_s sets we have

$$\bigcap_{s\in S}(B_t + A_s) = \bigcap_{s\in S} A_s + B_t. \tag{5}$$

Since $\operatorname{aff}(\bigcap_{s\in S} A_s) \subset \bigcap_{s\in S} \operatorname{aff} A_s = \operatorname{aff} \mathrm{A_s}$, then using the previous collorary again for B_t and $C = \bigcap_{s\in S} A_s$ sets we have $\bigcap_{t\in T}(B_t + C) = \bigcap_{t\in T} B_t + C$. By (5) we have $\bigcap_{t\in T}\bigcap_{s\in S}(B_t + A_s) = \bigcap_{t\in T}(\bigcap_{s\in S} A_s + B_t) = \bigcap_{t\in T}(C + B_t) = \bigcap_{t\in T} B_t + C = \bigcap_{t\in T} B_t + \bigcap_{s\in S} A_s$. □

Lemma 4.9. *Let $S_1, S_2 \subset X$ be convex sets and* $\operatorname{aff} S_1 \cap \operatorname{aff} S_2 = \{p\}$. *Moreover let $A \subset \operatorname{aff} S_1 \oplus \operatorname{aff} S_2$ and let $S = S_1 \oplus S_2$. Then*

$$S \dot{-} A = (S_1 \dot{-} A_1) \oplus (S_2 \dot{-} A_2),$$

where $A_i = \pi(A, \operatorname{aff} S_i), i = 0, 1$, is a parallel projection onto the subspace $\operatorname{aff} S_i$.

Proof. Using the definition $S \dot{-} A = \bigcap_{a\in A}(S \dot{-} a)$ of Minkowski subtraction we have

$$S \dot{-} A = \bigcap_{x\in A_1, y\in A_2} (S_1 \oplus S_2 - x \oplus y) = \bigcap_{x\in A_1, y\in A_2} \big((S_1 - x) \oplus (S_2 - y)\big). \tag{6}$$

Putting $A_x = S_1 - x, x \in A_1$ and $B_y = S_2 - y, y \in A_2$ we have aff A_x = aff $(S_1 - x)$ and from the previous lemma, aff $(S_1 - x)$ = aff $S_1 - x$ = aff $S_1 - y$ = aff $(S_1 - y)$ = aff $A_y, x, y \in A_1$. Similary aff B_x = aff B_y for every $x, y \in A_2$. Moreover because $p \in \operatorname{aff} S_1 \cap \operatorname{aff} S_2$ so aff A_x = aff $(S_1 - x)$ = aff $S_1 - p$, $x \in A_1$, aff B_y = aff $S_2 - p$, $x \in A_2$. We have $\operatorname{aff} A_x \cap \operatorname{aff} B_y = (\operatorname{aff} S_1 - p) \cap (\operatorname{aff} S_2 - p) = \{0\}$. Now, using the Lemma 4.8 and equality (6) we get $\bigcap_{x\in A_1, y\in A_2} ((S_1 - x) \oplus (S_2 - y)) = \bigcap_{x\in A_1, y\in A_2} (A_x \oplus A_y) = \bigcap_{x\in A_1} A_x \oplus \bigcap_{y\in A_2} A_y = \bigcap_{x\in A_1} (S_1 - x) \oplus \bigcap_{y\in A_2} (S_2 - y) = (S_1 \dot{-} A_1) \oplus (S_2 \dot{-} A_2)$. □

Corollary 4.10. *Under the assumptions of the above lemma we have*

$$S \dot{-} (S \dot{-} A) = (S_1 \dot{-} (S_1 \dot{-} A_1)) \oplus (S_2 \dot{-} (S_2 \dot{-} A_2)).$$

Proof. We have $S - 2p = (S_1 - p) \oplus (S_2 - p) \subset \operatorname{aff}(S_1 - p) \oplus \operatorname{aff}(S_2 - p)$ and $S \dot{-} A = (S_1 \dot{-} A_1) \oplus (S_2 \dot{-} A_2)$, $S_1 \dot{-} A_1 \subset \operatorname{aff}(S_1 - p)$, $S_2 \dot{-} A_2 \subset \operatorname{aff}(S_2 - p)$, hence

$$S \dot{-} A = (S_1 \dot{-} A_1) \oplus (S_2 \dot{-} A_2) \subset \operatorname{aff}(S_1 - p) \oplus \operatorname{aff}(S_2 - p).$$

Using Lemma 4.9 we obtain $S \dot{-} (S \dot{-} A) - 2p = (S - 2p) \dot{-} (S \dot{-} A) = [(S_1 - p) \oplus (S_2 - p)] \dot{-} (S \dot{-} A) = [(S_1 - p) \dot{-} \pi(S \dot{-} A, \operatorname{aff}(S_1 - p))] \oplus [(S_2 - p) \dot{-} \pi(S \dot{-} A, \operatorname{aff}(S_2 - p))] = [(S_1 - p) \dot{-} (S_1 \dot{-} A_1)] \oplus [(S_1 - p) \dot{-} (S_1 \dot{-} A_1)] = [S_1 \dot{-} (S_1 \dot{-} A_1)] \oplus [S_1 \dot{-} (S_1 \dot{-} A_1)] - 2p$. Therefore

$$S \dot{-} (S \dot{-} A) = (S_1 \dot{-} (S_1 \dot{-} A_1)) \oplus (S_2 \dot{-} (S_2 \dot{-} A_2)).$$ □

Theorem 4.11. *If* $S_1, S_2 \in \mathcal{S}$ *and* $\operatorname{aff} S_1 \cap \operatorname{aff} S_2 = \{p\}$, *then* $S_1 \oplus S_2 \in \mathcal{S}$.

Proof. Let $A = (S_1 \oplus S_2) \dot{-} (S_1 \oplus S_2 \dot{-} A)$. Denote $B_1 = S_1 \dot{-} (S_1 \dot{-} A_1)$ and $B_2 = S_2 \dot{-} (S_2 \dot{-} A_2)$. From previous lemma we know that $A = B_1 \oplus B_2$, where $A_i = \pi(A, \operatorname{aff} S_i)$. Hence $B_1 = A_1$, $B_2 = A_2$. Because $S_1, S_2 \in \mathcal{S}$ so from condition $S_1 \dot{-} (S_1 \dot{-} A_1) = A_1$ there exists T_1 such that $S_1 = T_1 + B_1$. Similary $S_2 = T_2 + B_2$. So $S_1 \oplus S_2 = (B_1 + T_1) \oplus (B_2 + T_2) = (B_1 \oplus B_2) + (T_1 + T_2) = A + (T_1 + T_2)$. Hence A is a summand of the set $S_1 \oplus S_2$. Therefore $S_1 \oplus S_2 \in \mathcal{S}$. □

5. Some properties of a class $\mathcal{C}$ of sets

Now let us define a class $\mathcal{C} \subset \mathcal{K}(X)$ by the following condition: $A \in \mathcal{C}$ if and only if its intersection with any summand of A is still a summand of A.

Lemma 5.1. *Let* $A \in \mathcal{C}$. *Then the intersection of any finite number of translates of* A *is a summand of* A.

Proof. Let $n = 2$, hence $(A - x_1) \cap (A - x_2) = (A - x_1 + x_2) \cap A - x_2$. We denote

$$C = (A - x_1 + x_2) \cap A \quad \text{and} \quad D = C - x_2.$$

Since C is a summand of A so D is a summand of A. Let the lemma holds true for k translates, i.e., $(A - x_1) \cap (A - x_2) \cap \cdots \cap (A - x_k) \cap (A - x_{k+1}) = [(A - x_1) \cap (A - x_2) \cap \cdots \cap (A - x_k) + x_{k+1}] \cap A - x_{k+1}$. We denote

$$E = [(A - x_1) \cap (A - x_2) \cap \cdots \cap (A - x_k) + x_{k+1}] \cap A \quad \text{and} \quad F = E - x_{k+1}.$$

Hence the sets E and F are summands of A. □

Lemma 5.2. *Let X be a topological vector space and let A, $A_\lambda \in \mathcal{K}(X)$ for $\lambda \in \Lambda$. Suppose that the family $\{A_\lambda\}_{\lambda\in\Lambda}$ is a chain of summands of the set A, then the set $\bigcap_{\lambda\in\Lambda} A_\lambda$ is also a summand of the set A.*

Proof. It can be proved (see [4, p. 49, Lemma 4.11]) that, if C, $D_\lambda \in \mathcal{K}(X)$, and $\{D_\lambda\}_{\lambda\in\Lambda}$ is a chain, then

$$C + \bigcap_{\lambda\in\Lambda} D_\lambda = \bigcap_{\lambda\in\Lambda} (C + D_\lambda). \tag{7}$$

Let $A, A_\lambda \in \mathcal{K}(X)$ for $\lambda \in \Lambda$ and let the family $\{A_\lambda\}_{\lambda\in\Lambda}$ be a chain of summands of the set A. Then there exists $B_\lambda \in \mathcal{K}(X)$, such that $A = A_\lambda + B_\lambda$ for $\lambda \in \Lambda$. Since $B_\lambda = A \dot{-} A_\lambda \subset A \dot{-} (\bigcap_{\lambda\in\Lambda} A_\lambda)$, we deduce that the set $\overline{\bigcup_{\lambda\in\Lambda} B_\lambda}$ is compact, and since the family $\{B_\lambda\}_{\lambda\in\Lambda}$ is a chain it is convex. Using the equality (7) we obtain

$$A \subset \bigcap_{\lambda\in\Lambda} \left(A_\lambda + \overline{\bigcup_{\lambda\in\Lambda} B_\lambda} \right) = \bigcap_{\lambda\in\Lambda} A_\lambda + \overline{\bigcup_{\lambda\in\Lambda} B_\lambda}.$$

On the other hand

$$\bigcap_{\lambda\in\Lambda} A_\lambda + \overline{\bigcup_{\lambda\in\Lambda} B_\lambda} \subset \overline{\bigcap_{\lambda\in\Lambda} A_\lambda + \bigcup_{\lambda\in\Lambda} B_\lambda} \subset \overline{\bigcup_{\lambda\in\Lambda} (A_\lambda + B_\lambda)} = \overline{A} = A.$$

Therefore $A = \bigcap_{\lambda\in\Lambda} A_\lambda + \overline{\bigcup_{\lambda\in\Lambda} B_\lambda}$ and the set $\bigcap_{\lambda\in\Lambda} A_\lambda$ is a summand of the set A. □

Example 5.3. Let $X = c_0$ be the real Banach space of all sequences convergent to 0 with the supremum norm, $\|x\| = \sup_k |x_k|$. Let $A = \{x \in c_0 : \|x\| \le 1\}$ be the unit ball, $A_m = \{x \in A : x_1 = \cdots = x_m = 1\}$, $C_m = \{x \in A : x_k = 0 \text{ for } k > m\}$, $b_m = (-1, \ldots, -1, 0, \ldots)$, where the first m components equals -1. Let $B_m = b_m + C_m$ for $m \in \mathbb{N}$. Then A, A_m, $B_m \in \mathcal{B}(X)$, $A_m + B_m = A$ and $A_{m+1} \subset A_m$ for $m \in \mathbb{N}$. Hence the family $\{A_m\}_{m\in\mathbb{N}}$ is a chain of summands of the set A, but the set $\bigcap_{m\in N} A_m$ is empty and is obviously not a summand of A. This example shows that the assumption $A \in \mathcal{K}(X)$ in Lemma 5.2 is essential.

Lemma 5.4. *Let X be a normed vector space, $B \in \mathcal{B}(X)$ and let $\{b_n\} \subset B$ be a dense set in B. Then $\bigcap_{x \in B}(A - x) = \bigcap_{n=1}^{\infty}(A - b_n)$ for any $A \in \mathcal{B}(X)$.*

Proof. It is enough to show that $\bigcap_{n=1}^{\infty}(A - b_n) \subset \bigcap_{x \in B}(A - x)$. Let $z \in \bigcap_{n=1}^{\infty}(A - b_n)$ and let $x_0 \in B$. Then there exists a subsequence of (b_n), which we denote by (b_{n_k}) such that $b_{n_k} \to x_0$. We have $z = a_{n_k} - b_{n_k}$, $a_{n_k} \in A$. The closedness of A and the equality $\lim_{k \to \infty} a_{n_k} = z + x_0 \in A$ imply $z \in A - x_0$. □

Theorem 5.5. *Let A be a convex, compact subset of a normed, separable space X and let $A \in \mathcal{C}$. Then for any set $B \in \mathcal{K}(X)$ set $A \dot{-} B$ is a summand of A.*

Proof. From Lemma 5.4 we have $A \dot{-} B = \bigcap_{n=1}^{\infty}(A - b_n)$, where $\overline{\{b_n\}} = B$. Now from Lemmas 5.2 and 5.4 and equality $\bigcap_{n=1}^{\infty}(A - b_n) = \bigcap_{k=1}^{\infty}(\bigcap_{n=1}^{\infty}(A - b_n))$, we obtain that $A \dot{-} B$ is a summand of A. □

Proposition 5.6. *The cubes and simplexes belong to the family $\mathcal{C}$.*

Proof. Let $C, D \in \mathcal{K}(X)$ and $C + D = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Let $p_i : \mathbb{R}^n \to \mathbb{R}$ and $p_i(x) = x_i$. Then $p_k(C + D) = [a_k, b_k]$ and hence $p_k(C) + p_k(D) = [a_k, b_k]$. Let $C_1 = p_1(C) \times \cdots \times p_n(C)$ and $D_1 = p_1(D) \times \cdots \times p_n(D)$. Then $C \subset C_1, D \subset D_1$. We have

$$C + D_1 \subset C_1 + D_1 \subset [a_1, b_1] \times \cdots \times [a_n, b_n] = C + D ,$$

and by the cancellation law we obtain $D_1 \subset D$. Similary $C_1 \subset C$. Therefore $C = C_1, D = D_1$. We just proved that every summand of cube is still a cube. Hence the intersection of cube with any summand of cube is still a summand of cube. Therefore, cubes belongs to $\mathcal{C}$.

From the indecomposability of a simplex it follows that any summand of a simplex S is a simplex homothetic to S. Hence the intersection S with any summand of S is still a summand of S. Therefore, simplexes belongs to $\mathcal{C}$. □

There is still the open question how to characterise the class $\mathcal{C}$ for the space $\mathbb{R}^n$ $(n \geq 2)$.

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