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Existence Results for Strongly Nonlinear Elliptic Equations of Infinite Order

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Abstract. In this work, generalized Sobolev spaces and Sobolev spaces of infinite order are considered. Existence of solutions for strongly nonlinear equation of infinite order of the form $Au + g(x, u) = f$ is established. Here A is an elliptic operator from a functional space of Sobolev type to its dual and $g(x, s)$ is a lower order term satisfying a sign condition on s.

Keywords. Strongly nonlinear elliptic equations of infinite order, monotonicity condition, sign condition

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1. Introduction

This paper is devoted to the study of the following strongly nonlinear elliptic equation:

$$Au + g(x, u) = f, \quad x \in \Omega, \tag{1.1}$$

with boundary condition of Dirichlet type. Here Ω is a bounded domain in $\mathbb{R}^N$ and A is a nonlinear elliptic operator satisfying some growth and coerciveness conditions, the nonlinear term g has to fulfil a sign condition. If A is a Leray–Lions operator, let us mention that several studies have been devoted to the investigation of related problems and a lot of papers have appeared (cf. [2, 3, 5, 12]). In particular, Webb [12] has studied the isotropic case for the problem (1.1) and proved the existence of at least one solution u in the Sobolev space $W_0^{m,p}(\Omega)$ ($m \geq 1$, $1 < p < \infty$). Note that for our case, by establishing sufficient conditions, we obtain existence results for a general class of nonlinear elliptic equations, which includes as a special case problems involving Leray–Lions operators in the usual sense. We will separately consider a generalized

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problem and another one of infinite order to give an extension and complement the results stated in [5] and [12]. For problems of infinite order, let us point out that in this direction Dubinskij [7] proved, under hypothesis $(A_1 - A_4)$ (see Section 3) and certain monotonicity conditions, the existence of solutions for the Dirichlet problem associated with the equation $Au = f$ in some functional Sobolev spaces of infinite order. Our purpose is to prove the same result for strongly nonlinear equations of infinite order of the form (1.1), more precisely, we will assume more less restrictions on the operator A (no monotonicity condition) to show existence of solutions.

To treat problem (1.1), we will define in Section 2 a functional space of Sobolev type, the so-called generalized Sobolev space (or anisotropic Sobolev space), that is $W^{m,\vec{p}}(\Omega)$. We will define also the Sobolev space of infinite order denoted by $W^\infty(a_\alpha, p_\alpha)(\Omega)$. In Section 3, we assume that the operator A satisfy the monotonicity condition, this allows us the study of (1.1) in the generalized Sobolev space $W^{m,\vec{p}}(\Omega)$. And finally, we will consider the strongly nonlinear equation (1.1) with infinite order, where A is assumed to be a nonlinear elliptic operator of type

$$A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (A_\alpha(x, D^\gamma u)), \quad |\gamma| \leq |\alpha|,$$

without a monotonicity condition. The real functions $A_\alpha(x,\xi)$ are required to have polynomial growth in ξ. The term $g(x,u)$ is strongly nonlinear in that no such growth restriction is imposed, but it is supposed that g satisfies the sign condition $g(x,u)u \geq 0$. Here a_α, p_α are numbers and $\vec{p}$ is a vector of real numbers.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$. Further $a_\alpha \geq 0, p_\alpha > 1$ are real numbers for all multi-indices α, and $\|\cdot\|_{p_\alpha}$ is the usual norm in the Lebesgue space $L^{p_\alpha}(\Omega)$. For a positive integer m, we define the following vector of real numbers:

$$\vec{p} = \{p_\alpha, |\alpha| \leq m\},$$

and denote $\underline{p} = \min\{p_\alpha, |\alpha| \leq m\}$.

Now, let us consider the generalized functional Sobolev space

$$W^{m,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega), D^\alpha u \in L^{p_\alpha}(\Omega), |\alpha| \leq m\}$$

equipped with the norm

$$\|u\| = \sum_{|\alpha|=0}^{m} \|D^\alpha u\|_{p_\alpha}. \tag{2.1}$$

We define the space $W_0^{m,\vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,\vec{p}}(\Omega)$ with respect to the norm (2.1). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{m,\vec{p}}(\Omega)$. Both of $W^{m,\vec{p}}(\Omega)$ and $W_0^{m,\vec{p}}(\Omega)$ are reflexive, separable Banach spaces, $p_\alpha > 1$ for all $|\alpha| \leq m$ (the proof of this is an adaptation from Adams [1]). $W^{-m,\vec{p'}}(\Omega)$ designs its dual where $\vec{p'}$ is the conjugate of $\vec{p}$, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ for all $|\alpha| \leq m$.

The Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha, p_\alpha)(\Omega) = \left\{ u \in C^\infty(\Omega) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha(u)\|_{p_\alpha}^{p_\alpha} < \infty \right\}.$$

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.

Since we shall deal with the Dirichlet problem, we shall use the functional space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ defined by

$$W_0^\infty(a_\alpha, p_\alpha)(\Omega) = \left\{ u \in C_0^\infty(\Omega) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{p_\alpha}^{p_\alpha} < \infty \right\}.$$

We say that $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is a nontrivial space if it contains at least a nonzero function. The dual space of $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is defined as follows:

$$W^{-\infty}(a_\alpha, p'_\alpha)(\Omega) = \left\{ h : h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha h_\alpha,\ \rho'(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha\|_{p'_\alpha}^{p'_\alpha} < \infty \right\},$$

where $h_\alpha \in L^{p'_\alpha}(\Omega)$ and p'_α is the conjugate of p_α, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ (for more details about these spaces, see [6, 7, 8, 9]).

We need the anisotropic Sobolev embeddings result.

Lemma 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$.*

If $m \cdot \underline{p} < N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^q(\Omega)$ for all $q \in [\underline{p}, p^*[$ with $\frac{1}{p^*} = \frac{1}{\underline{p}} - \frac{m}{N}$.

If $m \cdot \underline{p} = N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^q(\Omega)$ for all $q \in [\underline{p}, +\infty[$.

If $m \cdot \underline{p} > N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ where $k = E(m - \frac{N}{\underline{p}})$.

Moreover, the embeddings are compacts.

The proof follows immediately from the corresponding embedding theorems in the isotropic case by using the fact that $W^{m,\vec{p}}(\Omega) \subset W^{m,\underline{p}}(\Omega)$.

3. Main results

In this section we formulate and prove the main result of the paper.

3.1. Strongly nonlinear equation of finite order. Let A be the nonlinear operator of order $2m$ defined as

$$A(u) = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma} u), \quad |\gamma| \leq |\alpha| ,$$

where $A_{\alpha} : \Omega \times \mathbb{R}^{\lambda_{\alpha}} \mapsto \mathbb{R}$ is a real function and λ_{α} is the number of multi-indices γ such that $|\gamma| \leq |\alpha|$. Consider the following strongly nonlinear problem with Dirichlet conditions:

$$Au + g(x, u) = f \quad \text{in } \Omega.$$

Here, the function $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is measurable and $f \in W^{-m,\vec{p}'}(\Omega)$. Note that to deal with the Dirichlet problem, we use the space $W_0^{m,\vec{p}}(\Omega)$.

In the following we apply the theory of pseudo-monotone operators.

Definition 3.1 ([4]). Let Y be a reflexive Banach space. A bounded mapping B from Y to Y^* is called *pseudo-monotone* if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in Y and $\limsup_{n \longrightarrow \infty} \langle Bu_n, u_n - v \rangle \leq 0$, one has

$$\liminf_{n \longrightarrow \infty} \langle Bu_n, u_n - v \rangle \geq \langle Bu, u - v \rangle \quad \text{for all } v \in Y.$$

We start by stating the following assumptions:

(A_0) $A : W_0^{m,\vec{p}}(\Omega) \mapsto W^{-m,\vec{p}'}(\Omega)$ is a bounded operator, pseudo-monotone and coercive, i.e.,

$$\lim_{\|u\|_{m,\vec{p}} \to +\infty} \frac{\langle Au, u \rangle}{\|u\|_{m,\vec{p}}} = +\infty,$$

$p_{\alpha} > 1$, for all $|\alpha| \leq m$.

(G_0) $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ satisfies the Carathéodory conditions, that is, it is measurable in x for each fixed $u \in \mathbb{R}$ and continuous in u for almost all $x \in \Omega$ such that

$$\sup_{|u|<s} |g(x, u)| \leq h_s(x),$$

for a.e. $x \in \Omega$, all $s > 0$ and some function $h_s \in L^1(\Omega)$. We assume also the "sign condition" $g(x, u)u \geq 0$, for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$.

Theorem 3.2. *Let $m \in \mathbb{N}^*$ such that $m\underline{p} > N$. Suppose* (A_0) *and* (G_0) *are satisfied. Then for all $f \in W^{-m,\vec{p}'}(\Omega)$, there exists $u \in W_0^{m,\vec{p}}(\Omega)$ such that*

$$\begin{cases} g(x,u) \in L^1(\Omega), \ g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\vec{p}}(\Omega). \end{cases}$$

Proof. Let $\varphi \in C_0^\infty(\Omega)$ such that $0 < \varphi(x) < 1$ and $\varphi(x) = 1$ for x close to 0. Set

$$g_k(x,u) = \varphi\Big(\frac{x}{k}\Big)P_k g(x,u) \qquad \text{with } \ P_k\xi = \begin{cases} \xi, & \text{if } |\xi| \le k \\ \frac{k\xi}{|\xi|}, & \text{if } |\xi| > k. \end{cases}$$

Thanks to this truncation and as in Webb [12], we prove that there exists a $u_k \in W_0^{m,\vec{p}}(\Omega)$, which is the solution of the problem

$$Au_k + g_k(x,u_k) = f,$$

or in its variational formulation,

$$\langle Au_k, v\rangle + \int_\Omega g_k(x,u_k)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\vec{p}}(\Omega).$$

Further we have $u_k \rightharpoonup u$ weakly in $W_0^{m,\vec{p}}(\Omega)$, $Au_k \rightharpoonup \chi$ weakly in $W^{-m,\vec{p}'}(\Omega)$, $g_k(x,u_k) \to g(x,u)$ in $L^1(\Omega)$ and $g(x,u)u \in L^1(\Omega)$. Consequently, we obtain

$$\langle \chi, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle \quad \forall v \in W_0^{m,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

In view of Lemma 2.1, the last equality holds true for $v = u$ since $W_0^{m,\vec{p}}(\Omega) \subset L^\infty(\Omega)$ with $m\underline{p} > N$. Hence

$$\langle \chi, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\vec{p}}(\Omega).$$

Now, we show that $\chi = Au$. Indeed, the Fatou lemma implies

$$\limsup_{k\to+\infty}\langle Au_k, u_k\rangle \le \langle f, u\rangle - \int_\Omega g(x,u)u\,dx = \langle \chi, u\rangle.$$

Hence we have $\limsup_{k\to+\infty}\langle Au_k, u_k\rangle \le \langle \chi, u\rangle$. Since A is pseudo-monotone, we get $\chi = Au$. □

3.2. Strongly nonlinear equation of infinite order. We denote by λ_α the number of multi-indices γ such that $|\gamma| \le |\alpha|$. Let A be an operator of infinite order defined by

$$A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \le |\alpha|,$$

with $A_\alpha : \Omega \times \mathbb{R}^{\lambda_\alpha} \mapsto \mathbb{R}$ is a real function. The function $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is measurable and $f \in W^{-\infty}(a_\alpha, p'_\alpha)(\Omega)$.

Let us now formulate the assumptions:

(A$_1$) $A_\alpha(x,\xi_\alpha)$ is a Carathéodory function for all $\alpha, |\gamma| \le |\alpha|$.

(A$_2$) For a.e. $x \in \Omega$, all $m \in \mathbb{N}^*$, all $\xi_\gamma, \eta_\alpha, |\gamma| \le |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\left|\sum_{|\alpha|=0}^{m} A_\alpha(x,\xi_\gamma)\eta_\alpha\right| \le c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha - 1}|\eta_\alpha|,$$

where $a_\alpha \ge 0, p_\alpha > 1$ are reals numbers for all multi-indices α, and for all bounded sequence $(p_\alpha)_\alpha$.

(A$_3$) There exist constants $c_1 > 0, c_2 \ge 0$ such that for all $m \in \mathbb{N}^*$, for all $\xi_\gamma, \xi_\alpha; |\gamma| \le |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x,\xi_\gamma)\cdot\xi_\alpha \ge c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha} - c_2.$$

(A$_4$) The space $W_0^\infty(a_\alpha,p_\alpha)(\Omega)$ is nontrivial.

(G$_1$) The function $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is of Carathéodory type such that, for all $\delta > 0$,

$$\sup_{|u|<\delta} |g(x,u)| \le h_\delta(x) \in L^1(\Omega).$$

(G$_2$) We assume the "sign condition" $g(x,u)u \ge 0$, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$.

Theorem 3.3. *Let us assume the conditions* (A$_1$) – (A$_4$), (G$_1$) *and* (G$_2$). *Then for all* $f \in W^{-\infty}(a_\alpha, p'_\alpha)(\Omega)$, *there exists* $u \in W_0^\infty(a_\alpha,p_\alpha)(\Omega)$ *such that*

$$\begin{cases} g(x,u) \in L^1(\Omega), \;\; g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \textit{for all } v \in W_0^\infty(a_\alpha,p_\alpha)(\Omega). \end{cases}$$

Proof. In order to get our result, we will deal with the following steps:

1. We prove the existence of approximate solutions u_m.
2. We establish the a priori estimates.
3. We prove that u_m converges to an element $u \in W_0^\infty(a_\alpha,p_\alpha)(\Omega)$ and we finally show that u is the solution of our problem.

`Step (1):` The approximate problem.
Define the operator of order $2m+2$ by

$$A_{2m+2}(u) = \sum_{|\alpha|=m+1} (-1)^{m+1} c_\alpha D^{2\alpha} u + \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \le m,$$

where c_α are constants small enough such that they fulfil the conditions of the following lemma introduced in [6].

Lemma 3.4 (cf. [6]). *For all nontrivial space* $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$, *there exists a nontrivial space* $W_0^\infty(c_\alpha, 2)(\Omega)$ *such that* $W_0^\infty(a_\alpha, p_\alpha)(\Omega) \subset W_0^\infty(c_\alpha, 2)(\Omega)$.

The operator A_{2m+2} is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition, this follows from the result of [11]. Moreover from assumptions (A_1), (A_2) and (A_3), we deduce that A_{2m+2} satisfies the growth, the coerciveness and the monotonicity conditions. Hence by Theorem 3.1, there exists an approximate solution u_m of the following problem:

$$(\mathrm{Pb_m}) \quad \begin{cases} g(x,u_m) \in L^1(\Omega), \; g(x,u_m)u_m \in L^1(\Omega) \\ \langle A_{2m+2}(u_m), v\rangle + \displaystyle\int_\Omega g(x,u_m)v\,dx = \langle f_m, v\rangle, \quad v \in W_0^{m+1,\vec{p}}(\Omega) \end{cases}$$

with $f_m = \sum_{|\alpha|=0}^m (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha$, $f_\alpha \in L^{p'_\alpha}(\Omega)$.

`Step (2):` A priori estimate.
Set $v = u_m$ and using (A_3), (G_2) and the Hölder inequality, we deduce the estimates

$$\sum_{|\alpha|=m+1} c_\alpha \|D^\alpha u_m\|_2^2 + \sum_{|\alpha|=0}^{m} a_\alpha \|D^\alpha u_m\|_{p_\alpha}^{p_\alpha} \le K \tag{3.1}$$

and

$$\int_\Omega g(x,u_m)u_m\,dx \le K \tag{3.2}$$

for some constant $K = K(f) > 0$. The estimate (3.1) is equivalent to

$$\sum_{|\alpha|=0}^{m+1} a_\alpha \|D^\alpha u_m\|_{p_\alpha}^{p_\alpha} \le K \tag{3.3}$$

with $a_\alpha = c_\alpha$ and $p_\alpha = 2$ for $|\alpha| = m+1$. Consequently, we have

$$\|u_m\|_{W^{m+1,\vec{p}}} \le K. \tag{3.4}$$

Then via a diagonalization process, there exists a subsequence still, denoted by u_m, which converges uniformly to an element $u \in C_0^\infty(\Omega)$, also for all derivatives there holds $D^\alpha u_m \to D^\alpha u$ (for more details we refer to [6]).

`Step (3):` Convergence of problem (Pb_m).
There exists a solution u_m of problem (Pb_m), $m = 1, 2, \ldots$. Then, by passing to the limit, we have

$$\lim_{m\to+\infty} \langle A_{2m+2}(u_m), v\rangle + \lim_{m\to+\infty} \int_\Omega g(x,u_m)v\,dx = \lim_{m\to+\infty} \langle f_m, v\rangle,$$

for $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. It is clear that $\lim_{m\to+\infty}\langle f_m, v\rangle = \langle f, v\rangle$ for all $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$.

Now, we shall prove that

$$\lim_{m\to+\infty}\langle A_{2m+2}(u_m),v\rangle = \langle Au,v\rangle, \quad \text{for all } v\in W_0^\infty(a_\alpha,p_\alpha)(\Omega).$$

In fact, let m_0 be a fix number sufficiently large ($m > m_0$) and let $v \in W_0^\infty(a_\alpha,p_\alpha)(\Omega)$. Set $\langle A(u) - A_{2m+2}(u_m), v\rangle = I_1 + I_2 - I_3$, where

$$I_1 = \sum_{|\alpha|=0}^{m_0} \langle A_\alpha(x, D^\gamma u) - A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle$$

$$I_2 = \sum_{|\alpha|=m_0+1}^{\infty} \langle A_\alpha(x, D^\gamma u), D^\alpha v\rangle$$

$$I_3 = -\sum_{|\alpha|=m_0+1}^{m} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle - \sum_{|\alpha|=m+1} c_\alpha\langle D^\alpha u, D^\alpha v\rangle,$$

or in another form,

$$I_3 = -\sum_{|\alpha|=m_0+1}^{m+1} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle$$

with $A_\alpha(x,\xi_\gamma) = c_\alpha\xi_\alpha$ and $c_\alpha \geq 0$ for $|\alpha| = m+1$.

The aim is to prove that I_1, I_2 and I_3 tend to 0. On the one hand, since $A_\alpha(x,\xi_\gamma)$ is of Carathéodory type, $I_1 \to 0$, and the term I_2 is the remainder of a convergent series, hence $I_2 \to 0$. On the other hand, for all $\varepsilon > 0$, there holds $k(\varepsilon) > 0$ (see [4, p. 56]) such that

$$\begin{aligned}
\left|\sum_{|\alpha|=m_0+1}^{m+1} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle\right| &\leq \sum_{|\alpha|=m_0+1}^{m+1} |\langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle| \\
&\leq c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \int_\Omega |D^\alpha u_m|^{p_\alpha-1}|D^\alpha v|\,dx \\
&\leq c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \|D^\alpha u_m\|_{p_\alpha}^{p_\alpha-1}\|D^\alpha v\|_{p_\alpha} \\
&\leq \varepsilon c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \|D^\alpha u_m\|_{p_\alpha}^{p_\alpha} + c_0 k(\varepsilon) \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \|D^\alpha v\|_{p_\alpha}^{p_\alpha} \\
&\leq \varepsilon c_0 K + c_0 k(\varepsilon) \sum_{|\alpha|=m_0+1}^{\infty} a_\alpha \|D^\alpha v\|_{p_\alpha}^{p_\alpha},
\end{aligned}$$

where K is the constant given in the estimate (3.1). Since the sequence (p_α) is bounded, this implies that $\sum_{|\alpha|=m_0+1}^\infty a_\alpha \|D^\alpha v\|_{p_\alpha}^{p_\alpha}$ is the remainder of a convergent series, therefore $I_3 \to 0$ holds. Hence $\langle A_{2m+2}(u_m), v\rangle \to \langle A(u), v\rangle$ as $m \to +\infty$ for all $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$.

Now we prove that

$$\lim_{m\to+\infty} \int_\Omega g(x, u_m)v\, dx = \int_\Omega g(x, u)v\, dx.$$

Indeed, we have $u_m \to u$ uniformly in Ω, hence $g(x, u_m) \to g(x, u)$ for a.e. $x \in \Omega$. In view of the Fatou lemma and (3.2), we obtain

$$\int_\Omega g(x, u)u\, dx \le \lim_{m\to+\infty} \int_\Omega g(x, u_m)u_m\, dx \le K,$$

this implies $g(x, u)u \in L^1(\Omega)$. On the other hand, for all $\delta > 0$ we have

$$|g(x, u_m)| \le \sup_{|t|\le\delta} |g(x, t)| + \delta^{-1}|g(x, u_m)u_m| \le h_\delta(x) + \delta^{-1}|g(x, u_m)u_m|.$$

If E is a measurable subset of Ω and $\varepsilon > 0$, we have

$$\int_E |g(x, u_m)|\, dx \le \int_E h_\delta(x)\, dx + \delta^{-1}K,$$

where K is the constant of (3.2) which is independent of m. For $|E|$ sufficiently small and $\delta = \frac{2K}{\varepsilon}$, we obtain $\int_E |g(x, u_m)|\, dx \le \varepsilon$. Using Vitali's theorem we get $g(x, u_m) \to g(x, u)$ in $L^1(\Omega)$. Hence it follows that $g(x, u) \in L^1(\Omega)$.

By passing to the limit, we obtain

$$\langle Au, v\rangle + \int_\Omega g(x, u)v\, dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega).$$

Finally, we conclude that

$$\begin{cases} g(x, u) \in L^1(\Omega),\ g(x, u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x, u)v\, dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega). \end{cases}$$

This completes the proof. □

4. Examples

1. Let Ω be a bounded open set in $\mathbb{R}^N, N \ge 2$, with Lipschitz boundary $\partial\Omega$. Let $m = 1$ and consider the Carathéodory functions

$$A_i(x, s, \xi) = |\xi_i|^{p_i-1}\mathrm{sgn}(\xi_i), \quad \text{for } i = 1, \dots, N.$$

It is easy to show that $A_i(x, s, \xi)$ are Carathéodory functions satisfying the condition (A_0).

2. The following example of an operator of infinite order is closely inspired from the one used in [7]. Let us consider the operator

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha \big(a_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u\big),$$

$a_\alpha \geq 0$ and $p_\alpha > 1$ are real numbers such that the space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is nontrivial (for example, if $a_\alpha = [(2\alpha)!]^{-p}$, $p > 1$ and $\dim \Omega = 1$), then the conditions (A_1), (A_2) and (A_3) are satisfied.

3. An explicit example of a function g that satisfies the conditions (G_0), (G_1) and (G_2) is $g(x,t) = t|t|^r h(x)$ with $r > 0$, where $h \in L^1(\Omega), h(x) \geq 0$ a.e.

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