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Homogenization of Scalar Problems for a Combined Structure with Singular or Thin Reinforcement

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Abstract. The homogenization of quadratic integral functionals for combined structures with singular or asymptotically singular reinforcement is studied in a model case in dimension $N = 2$. Generalizations to more general cases in dimension $N = 2$ or to some model cases in dimension $N > 2$ are discussed.
Such results are obtained in the frame of homogenization of problems depending on two parameters developed by V. V. Zhikov in [*Funct. Anal. Appl.* 33 (1999)(1)], [*Sb. Math.* 191 (2000)(7-8)], and [*Izv. Math.* 66 (2002)(2)]. In particular, an essential tool is the notion of two-scale convergence of sequences of functions belonging to Sobolev spaces with respect to variable measures.

Keywords. Combined structures, singular and thin reinforcement, two-scale convergence, Sobolev spaces with respect to variable measures

Mathematics Subject Classification (2000). Primary 35B27, secondary 35B40, 28A33

1. Introduction

We study the homogenization of a scalar problem in a composite medium with some physical properties being highly contrasting: e.g. mass density or heat conductivity. We shall refer to this medium as "combined structure" or, keeping in mind the first physical example, "reinforced structure". Reinforcement problems, in particular, are treated by many authors, and can roughly be divided into two different types, according as the reinforcing structure has spatial

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distribution or is concentrated near the boundary. We study the homogenization of quadratic integral functionals when the reinforcing structure has spatial distribution and is singular or asymptotically singular. Some problems of this kind are studied for example by [3].

For simplicity we consider a model case in dimension 2. We first describe the problem in the case the reinforcement is singular: then the unit cell is a square and the (periodic) reinforcing structure (see Figure 3) is given by two crossed wires (characterized by the "natural" measure $\mu = \frac{1}{2}dx + \frac{1}{2}\lambda$, where λ is the 1-dimensional normalized measure supported on the wires and having constant density). Then we consider a Lipschitz domain Ω and the minimization problem P_ε (see (14)) for the ε-periodic combined structure characterized by the measure μ_ε, obtained by periodizing and rescaling μ (see Section 3). Theorem 2 gives us the desired homogenization result. This result is (quite easily) obtained by general homogenization results with respect to measures (see [15]) and the direct verification of the connectedness (or ergodicity, see Definition 2) of the measure μ. In this case we analyze the classical formulation of a minimization problem of the same kind on the unit cell, paying attention to the equation that the solution has to satisfy on the singular reinforcement (the so-called Ventsel condition, cf. (24)).

We then describe the problem in the case the reinforcement is asymptotically singular (according to terminology of [17] we call "thin" a reinforcing structure of this type): then the unit cell is a square and the reinforcing structure (see Figure 4) is given by strips of width h (characterized by the "natural" measure $\mu^h = \frac{1}{2}dx + \frac{1}{2}\lambda^h$ where λ^h is the 2-dimensional normalized measure supported on the strips and having constant density; we obviously have $\mu^h \rightharpoonup \mu$ as $h \to 0$). Then we consider a Lipschitz domain Ω and the minimization problem P_ε^h (see (42)) for the ε-periodic combined structure characterized by the measure μ_ε^h obtained by periodizing and rescaling μ^h (see Section 4), depending on parameters ε and $h = h(\varepsilon) \to 0$, as $\varepsilon \to 0$. In this case we obtain the homogenization result by means of the measure approach elaborated in [14, 15, 16]. We also show that the same limit problem is obtained as parameters ε and h tend to zero separately, yielding commutative diagram in Figure 6.

The solutions of problems P_ε^h can be regarded as elements of a variable Sobolev space $H_0^1\left(\Omega, d\mu_\varepsilon^h\right)$ and the behavior of solutions as $\varepsilon \to 0$ is studied by means of Theorem 5, based on results of [16]. In order to apply such theorem, measures μ and μ^h on the unit cell must be linked by the so-called "approximation properties" (see Definition 4). These approximation properties are not trivial to check. We prove them showing that natural measure μ^h "almost" coincides with an appropriate smoothing measure (obtained from μ by convolution with a kernel proportional to the characteristic function of a square of width h). The fact that a measure μ and its approximations by smoothing measures are linked by approximation properties is well known and proved in [16] (see Section 6). In the case measure μ^h is a smoothing measure homogenization results have been also obtained in [5] (see Theorem 6.1).

We observe (Remark 2) that the same approach works for any dimension $N \geq 2$ when (periodic) reinforcing structure is given by the union of any number of plates of thickness h with the restriction that each plates will be taken parallel to some faces of the unit cell.

In the final section we sketch the proof of approximation properties for combined structures with reinforcing network of arbitrary form in dimension 2.

2. Definitions and preliminary results

Let $\square = [0,1)^2$ be the cell of periodicity and μ a $\square$-periodic Borel measure in $\mathbb{R}^2$ such that $\int_\square d\mu = 1$.

We define the Sobolev space $H^1_{per}(\square, d\mu)$ as the closure of the set of pairs $\{(u, \nabla u) : u \in C_0^\infty(\square)\}$ in the product norm of $L^2(\square, d\mu) \times L^2(\square, d\mu)^2$. The elements of this closure are pairs (u, v) in which v is called "gradient" of u and denoted by ∇u. In the following we will call the Sobolev space $H^1_{per}(\square, d\mu)$ as the set of first components of the above set, too; in this case for each function u, the gradient defined above is not unique (see Section 3 of [15]).

The set $\Gamma(u)$ of all gradients of a fixed function u in $H^1_{per}(\square, d\mu)$ has the structure $\nabla u + \Gamma(0)$, where ∇u is some gradient of u and $\Gamma(0)$ is the set of the gradients of zero. By definition, $g \in \Gamma(0)$ if there exists $\varphi_n \in C^\infty_{per}(\square)$ such that

$$\varphi_n \to 0 \quad \text{in } L^2(\square, d\mu) \quad \text{and} \quad \nabla\varphi_n \to g \quad \text{in } L^2(\square, d\mu)^2. \tag{1}$$

$\Gamma(0)$ is a subspace of the vector space $L^2(\square, d\mu)^2$.

A gradient ∇u can be represented as a sum of two orthogonal terms:

$$\nabla u = \nabla^t u + g, \qquad g \in \Gamma(0), \quad \nabla^t u \perp \Gamma(0). \tag{2}$$

The the first term $\nabla^t u$ is called tangential gradient of u and is minimal in the sense

$$\int_\square \left|\nabla^t u\right|^2 d\mu = \min_{\nabla u \in \Gamma(u)} \int_\square |\nabla u|^2 \, d\mu \, .$$

This definition of "minimal" gradient requires the knowledge of ∇u in the whole domain $\square$. But it can be characterized by pointwise properties because the space $\Gamma(0)$ admits a pointwise description, i.e., (see Theorem 9.3 of [15]) there exists a μ-measurable periodic subspace $T(y) \subset \mathbb{R}^2$ such that $\Gamma(0) = \left\{g \in L^2(\square, d\mu)^2 : g(y) \in T^\perp(y)\right\}$. Then it is possible to project pointwise the gradient ∇u to $T(y)$, and the gradient $\nabla^t u$ is determined by the tangentiality condition $\nabla^t u \in T(y)$ μ-a.e.. So the tangential gradient is a pointwise minimal gradient too: $|\nabla^t u(x)| \leq |\nabla u\,(x)|$ μ-a.e..

We define the space V_{pot} of *potential* vectors as the closure of the set $\left\{\nabla\varphi : \varphi \in C^{\infty}_{per}(\square)\right\}$ in $L^2_{per}(\square, d\mu)^2$.

A vector $b \in L^2(\square, d\mu)^2$ is said to be *solenoidal* if it is orthogonal to all potential vectors, that is $\int_\square b \cdot \nabla\varphi \, d\mu = 0$, for each $\varphi \in C^{\infty}_{per}(\square)$.

Definition 1. A $\square$-periodic measure μ is said to be *non-degenerate* if every non-zero constant vector is not potential.

Definition 2. A $\square$-periodic measure μ is *ergodic* or 2-*connected* if $u =$ constant μ-a.e. whenever there is a sequence $u_n \in C^{\infty}_{per}(\square)$ such that $u_n \to u$ in $L^2(\square, d\mu)$ and $\nabla u_n \to 0$ in $L^2(\Omega, d\mu)^2$.

We obviously have that the Lebesgue measure is ergodic. A sufficient condition for ergodicity is given by Poincaré inequality:

$$\int_\square \varphi^2 d\mu \le C\left[\left|\int_\square \varphi d\mu\right|^2 + \int_\square |\nabla\varphi|^2 d\mu\right], \quad \varphi \in C^{\infty}_{per}(\square). \tag{3}$$

Let us fix $\varepsilon > 0$. We now define the "rescaled" measure μ_ε by

$$\mu_\varepsilon(B) = \varepsilon^2 \mu\left(\varepsilon^{-1}B\right) \qquad \text{for every Borel set of } \mathbb{R}^2, \tag{4}$$

where $\varepsilon^{-1}B = \{\varepsilon^{-1}x : x \in B\}$.

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^2$. In a similar way to the Sobolev space of periodic functions $H^1_{per}(\square, d\mu)$, we can introduce the Sobolev space $H^1_0(\Omega, d\mu_\varepsilon)$ as the closure of the set of pairs $\{(u, \nabla u) : u \in C^\infty_0(\Omega)\}$ in the product norm of $L^2(\Omega, d\mu_\varepsilon) \times L^2(\Omega, d\mu_\varepsilon)^2$.

Definition 3. Let u_ε be a bounded sequence in $L^2(\Omega, d\mu_\varepsilon)$ ($\limsup_{\varepsilon\to 0} \int_\Omega u_\varepsilon^2 \, d\mu_\varepsilon < +\infty$). We say that u_ε *weakly converges to* u in $L^2(\Omega, d\mu_\varepsilon)$ and we write $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega, d\mu_\varepsilon)$, if

$$u \in L^2(\Omega, dx) \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_\Omega \varphi u_\varepsilon \, d\mu_\varepsilon = \int_\Omega \varphi u \, dx\,, \quad \forall \varphi \in C^\infty_0(\Omega).$$

We say that u_ε *strongly converges to* u in $L^2(\Omega, d\mu_\varepsilon)$, and we write $u_\varepsilon \to u$ in $L^2(\Omega, d\mu_\varepsilon)$, if

$$u \in L^2(\Omega, dx) \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon v_\varepsilon \, d\mu_\varepsilon = \int_\Omega uv \, dx \quad \text{for } v_\varepsilon \rightharpoonup v \text{ in } L^2(\Omega, d\mu_\varepsilon).$$

The following proposition holds (see Proposition 1.1 of [15]):

Proposition 1. *Let u_ε be a sequence in $L^2(\Omega, d\mu_\varepsilon)$. Then:*

i) *any sequence u_ε bounded in $L^2(\Omega, d\mu_\varepsilon)$ is compact with respect to weak convergence;*

ii) *strong convergence is implied by weak convergence* $u_\varepsilon \rightharpoonup u$ *in* $L^2(\Omega, d\mu_\varepsilon)$ *and by the relation*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon^2 \, d\mu_\varepsilon = \int_\Omega u^2 \, dx.$$

Let $A(y)$, $y \in \mathbb{R}^2$, be a μ-measurable, $\square$-periodic, symmetric matrix such that there exists $\nu > 0$:

$$\nu\xi^2 \le A(y)\xi \cdot \xi \le \nu^{-1}\xi^2, \quad \xi \in \mathbb{R}^2 \ \mu\text{-a.e. in } \square. \tag{5}$$

Let us consider now the following problem:

$$u_\varepsilon \in H_0^1(\Omega, d\mu_\varepsilon), \qquad -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) + u_\varepsilon = f_\varepsilon\,, \tag{6}$$

where $f_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)$. By definition, $u_\varepsilon \in H_0^1(\Omega, d\mu_\varepsilon)$ is solution of this equation if the following integral identity holds:

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon \cdot \nabla\varphi \, d\mu_\varepsilon + \int_\Omega u_\varepsilon \varphi \, d\mu_\varepsilon = \int_\Omega f_\varepsilon \varphi \, d\mu_\varepsilon, \quad \forall\varphi \in C_0^\infty(\Omega),$$

where ∇u_ε is some gradient of u_ε.

Let us consider the homogenized matrix $A^{\hom}$ defined by

$$A^{\hom}\xi \cdot \xi = \min_{v\in V_{pot}} \int_\square A(y)(\xi + v)\cdot(\xi + v)\, d\mu\,, \tag{7}$$

whose solution v satisfies the following Euler equation

$$\int_\square A(y)(\xi + v)\cdot \nabla w \, d\mu = 0, \quad \forall w \in C_{per}^\infty(\square). \tag{8}$$

Therefore $A^{\hom}\xi \cdot \xi = \int_\square A(y)(\xi + v)\cdot(\xi + v)\, d\mu = \int_\square A(y)(\xi + v)\, d\mu \cdot \xi$, and so

$$A^{\hom}\xi = \int_\square A(y)(\xi + v)\, d\mu. \tag{9}$$

We observe that if μ fails to be non-degenerate, by (7) $A^{\hom}$ has a non-trivial kernel.

We recall the following theorem proved in [15] (see Theorem 4.4).

Theorem 1. *Let* μ *be an ergodic measure. Let* u_ε *be a sequence of solutions of Problem* (6)*. If* $f_\varepsilon \rightharpoonup f$ *(strongly) in* $L^2(\Omega, d\mu_\varepsilon)$*, then* $u_\varepsilon \rightharpoonup u$ *(strongly) in* $L^2(\Omega, d\mu_\varepsilon)$*, where* u *is solution of the homogenized problem*

$$u \in H_0^1(\Omega), \qquad -\operatorname{div}\left(A^{\hom}\nabla u\right) + u = f.$$

3. Combined structure with singular reinforcement

3.1. Let us consider a combined structure with singular reinforcement: for example a structure that consists of a $\square$-periodic network F overlaid onto the plane and dividing it into tiles (see Figure 1).

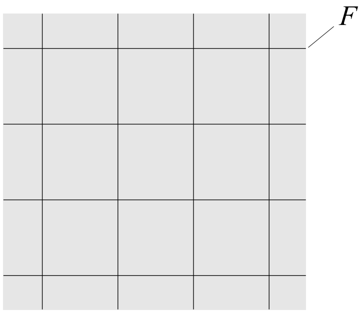

Figure 1: A combined structure with singular reinforcement.

This structure is characterized by the following $\square$-periodic normalized measure μ:

$$\mu = \frac{1}{2}dx + \frac{1}{2}\lambda, \tag{10}$$

where dx is the 2-dimensional Lebesgue measure on the tiles and λ is the measure proportional to 1-dimensional Lebesgue measure on the network F.

We shall call measures λ, μ *natural* related to this structure. By definition we have

$$\int_\square \varphi\, d\mu = \int_\square \varphi\, dx + \int_{\square\cap F} \varphi\, d\lambda\,, \quad \forall\varphi \in C^\infty_{per}(\square)\,, \qquad \text{and} \int_\square d\mu = 1.$$

We observe that the measure μ_ε and λ_ε obtained by μ and λ as in (4) are ε-periodic and $\mu_\varepsilon = \frac{1}{2}dx + \frac{1}{2}\lambda_\varepsilon$.

Proposition 2. *Let μ be the measure defined in* (10)*. If $u \in H^1_{per}(\square, d\mu)$, then*

i) *$u \in H^1_{per}(\square, dx)$ and therefore $u_{|F}$ (the trace of u on F) exists;*

ii) *$u_{|F} \in H^1_{per}(\square, d\lambda)$.*

Moreover μ is non-degenerate and ergodic.

Proof. The first part of proposition is easy to check.

If now E is the identity matrix and E^{hom} is given by relation (9) a simple calculation shows that $E^{\text{hom}} = \frac{3}{4}E$. This implies μ is non-degenerate.

We now prove the ergodicity of the combined measure μ. It is enough to prove that for the measure μ the Poincaré inequality holds. Assuming the inequality is not valid, we can find a sequence u_h such that

$$\int_\square u_h^2 d\mu = 1, \quad \int_\square |\nabla u_h|^2 d\mu \to 0, \quad \int_\square u_h d\mu = 0. \tag{11}$$

Then, up to a subsequence, $u_h \rightharpoonup u$ in $H^1_{per}(\Box, dx)$, $u_h \to u$ in $L^2(\Box, dx)$, $u_h \to u$ in $L^2(\Box, d\lambda)$, that implies by (11)

$$\int_\Box u^2 d\mu = 1, \quad \int_\Box |\nabla u|^2 d\mu = 0, \quad \int_\Box u d\mu = 0. \tag{12}$$

By Friedrichs inequality on the unit cell and on the network F, the second relation in (12) gives that u is (separately) constant on the cell and on the network F; by (i), (ii) and the last relation in (12) we have $u(y) = 0$ μ-a.e. that contradicts the first relation in (12). So the Poincaré inequality holds. $\Box$

Let $A(y)$, $y \in \mathbb{R}^2$, be a μ-measurable, $\Box$-periodic, symmetric matrix satisfying (5) and

$$A(y) = \begin{cases} \alpha(y) & \text{on tiles} \\ \beta(y) & \text{on network } F. \end{cases} \tag{13}$$

Let us consider the following problem:

$$(\mathrm{P}_\varepsilon): \qquad \min_{(u,\nabla u)\in H^1_0(\Omega, d\mu_\varepsilon)} \int_\Omega \left(A\left(\frac{x}{\varepsilon}\right) \nabla u \cdot \nabla u - 2G_\varepsilon u \right) d\mu_\varepsilon, \tag{14}$$

(about the non-uniqueness of ∇u, let us note that the set $\Gamma(0)$ defined by (1) actually is the set of vector fields that are non-zero only on the network $F_\varepsilon \cap \Omega$ and on each segment of F_ε they are orthogonal to the same segment), where

$$G_\varepsilon \in L^2(\Omega, d\mu_\varepsilon), \quad G_\varepsilon = \begin{cases} f_\varepsilon, & \text{on } \Omega \\ g_\varepsilon, & \text{on } \Omega \cap F_\varepsilon, \end{cases} \qquad F_\varepsilon = \varepsilon F.$$

The corresponding weak Euler equation has the form

$$u_\varepsilon \in H^1_0(\Omega, d\mu_\varepsilon), \quad \int_\Omega A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla \varphi \, d\mu_\varepsilon = \int_\Omega G_\varepsilon \varphi \, d\mu_\varepsilon \tag{15}$$

for every $\varphi \in C_0^\infty(\Omega)$, or more explicitly

$$\int_\Omega \alpha\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla \varphi \, dx + \int_{\Omega\cap F_\varepsilon} \beta\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla \varphi \, d\lambda_\varepsilon = \int_\Omega f_\varepsilon \varphi \, dx + \int_{\Omega\cap F_\varepsilon} g_\varepsilon \varphi \, d\lambda_\varepsilon$$

for every $\varphi \in C_0^\infty(\Omega)$. Sometimes we will write the problem (15) as

$$u_\varepsilon \in H^1_0(\Omega, d\mu_\varepsilon), \qquad -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \right) = G_\varepsilon. \tag{16}$$

By definition u_ε is a solution of (16) if the integral identity in (15) holds.

To prove existence of solutions to our problem we need Friedrichs inequality.

Proposition 3. *The following Friedrichs inequality holds for the measure μ_ε (uniformly with respect to ε):*

$$\int_\Omega \varphi^2 d\mu_\varepsilon \le c \int_\Omega |\nabla\varphi|^2 \, d\mu_\varepsilon, \quad \forall\varphi \in C_0^\infty(\Omega), \;\; c > 0.$$

Proof. We have the classical Friedrichs inequality for the Lebesgue measure on the domain Ω:

$$\int_\Omega \varphi^2 dx \le c \int_\Omega |\nabla\varphi|^2 \, dx, \quad \forall\varphi \in C_0^\infty(\Omega).$$

For any line l we have the one-dimensional Friedrics inequality

$$\int_{\Omega\cap l} \varphi^2 d\lambda_\varepsilon \le c \int_{\Omega\cap l} |\nabla\varphi|^2 \, d\lambda_\varepsilon, \quad \forall\varphi \in C_0^\infty(\Omega).$$

By summation we have the thesis. □

Now we state the main theorem of this section.

Theorem 2. *Let μ the measure defined in (10) and $G_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)$ such that*

$$G_\varepsilon \rightharpoonup G \quad \text{in } L^2(\Omega, d\mu_\varepsilon). \tag{17}$$

Then if u_ε is a sequence of solutions of problem (14), it results

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^2(\Omega, d\mu_\varepsilon), \tag{18}$$

where u is the solution of the homogenized problem

$$\left(\mathrm{P}^{\mathrm{hom}}\right): \qquad \min_{u\in H_0^1(\Omega)} \int_\Omega \left(A^{\mathrm{hom}}\nabla u \cdot \nabla u - 2Gu\right) dx, \tag{19}$$

where the matrix A^{hom} is defined by (7). If we have the strong convergence in (17), the same strong convergence holds in (18) and the convergence of energies takes place (i.e., minimum values of problem (P_ε) converge to the minimum value of the problem $(\mathrm{P}^{\mathrm{hom}})$).

Proof. Let us consider the equation. The left hand side defines a scalar product on $H_0^1(\Omega, d\mu_\varepsilon)$ (regarded as the pairs $(u, \nabla u)$) and the corresponding norm is equivalent to the original norm. Moreover we have the estimate

$$\int_\Omega G_\varepsilon \varphi \, d\mu_\varepsilon \le c\left(\int_\Omega \varphi^2 \, d\mu_\varepsilon\right)^{\frac{1}{2}} \le c_1 \left(\int_\Omega |\nabla\varphi|^2 \, d\mu_\varepsilon\right)^{\frac{1}{2}}.$$

So by Riesz' theorem on the representation of linear functionals in Hilbert spaces, we have that the problem admits a unique solution, i.e., a unique function u_ε in $H_0^1(\Omega, d\mu_\varepsilon)$ and a unique gradient ∇u_ε satisfying problem (15).

We have, for every $\varepsilon > 0$,

$$\int_\Omega \left(u_\varepsilon^2 + |\nabla u_\varepsilon|^2\right) d\mu_\varepsilon \le c \left(\int_\Omega G_\varepsilon^2 \, d\mu_\varepsilon\right)^{\frac{1}{2}}$$

and so u_ε is bounded in $H_0^1(\Omega, d\mu_\varepsilon)$, and then, up to a subsequence, $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega, d\mu_\varepsilon)$. Since u_ε is solution of problem (15), we have $-\operatorname{div}(A(\frac{x}{\varepsilon})\nabla u_\varepsilon) + u_\varepsilon = G_\varepsilon + u_\varepsilon$. Obviously, since $u_\varepsilon \rightharpoonup u$ and $G_\varepsilon \rightharpoonup G$ in $L^2(\Omega, d\mu_\varepsilon)$, then $G_\varepsilon + u_\varepsilon \rightharpoonup G + u$ in $L^2(\Omega, d\mu_\varepsilon)$. By Theorem 1 we have that u is a solution of the problem $u \in H_0^1(\Omega)$, $-\operatorname{div}(A^{\text{hom}}\nabla u) + u = G + u$ and so of the homogenized problem

$$u \in H_0^1(\Omega), \qquad -\operatorname{div}\left(A^{\text{hom}}\nabla u\right) = G. \tag{20}$$

Since u has to satisfy (20) then, for the whole sequence, $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega, d\mu_\varepsilon)$ and this proves the first part of theorem on the weak convergence of solutions. We will use an argument similar to Theorem 1 (see Theorem 4.4 of [15]).

Let us now prove the strong convergence $u_\varepsilon \to u$ in $L^2(\Omega, d\mu_\varepsilon)$ under the assumption that $G_\varepsilon \to G$ in $L^2(\Omega, d\mu_\varepsilon)$. Consider the problem

$$z_\varepsilon \in H_0^1(\Omega, d\mu_\varepsilon), \qquad -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla z_\varepsilon\right) = u_\varepsilon, \tag{21}$$

where u_ε is the solution of problem (16).

We know that $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega, d\mu_\varepsilon)$. Then, up to a subsequence, $z_\varepsilon \rightharpoonup z$ in $L^2(\Omega, d\mu_\varepsilon)$, where z is solution of the problem $z \in H_0^1(\Omega)$, $-\operatorname{div}\left(A^{\text{hom}}\nabla z\right) = u$. Let us take u_ε as test function in equation (21) and z_ε in equation (16); we have

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla z_\varepsilon \cdot \nabla u_\varepsilon \, d\mu_\varepsilon = \int_\Omega u_\varepsilon^2 \, d\mu_\varepsilon$$
$$\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon \cdot \nabla z_\varepsilon \, d\mu_\varepsilon = \int_\Omega G_\varepsilon z_\varepsilon \, d\mu_\varepsilon.$$

So $\int_\Omega u_\varepsilon^2 \, d\mu_\varepsilon = \int_\Omega G_\varepsilon z_\varepsilon \, d\mu_\varepsilon$ and since by hypothesis $G_\varepsilon \to G$ in $L^2(\Omega, d\mu_\varepsilon)$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon^2 \, d\mu_\varepsilon = \lim_{\varepsilon\to 0}\int_\Omega G_\varepsilon z_\varepsilon \, d\mu_\varepsilon = \int_\Omega Gz \, dx = \int_\Omega u^2 \, dx.$$

Since $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega, d\mu_\varepsilon)$ and the above convergence we have $u_\varepsilon \to u$ in $L^2(\Omega, d\mu_\varepsilon)$. Moreover, the convergence of the minimum values of the problems (P_ε) to the minimum value of the homogenized problem can readily be obtained by taking u_ε as test function in the Euler equation of the problem (P_ε) and passing to the limit as $\varepsilon \to 0$. □

3.2. Now we give an example of a variational problem on a combined structure with a singular reinforcement and obtain its classical formulation. For simplicity we consider the case $\Omega = \square$, $I = (0,1) \times \left\{\frac{1}{2}\right\}$ and $\mu = dx + \lambda$ where dx is the 2-dimensional Lebesgue measure and λ a measure proportional to 1-dimensional Lebesgue measure on I (see Figure 2).

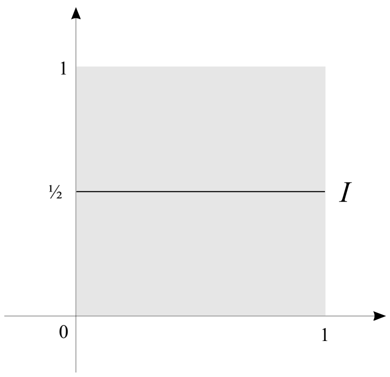

Figure 2: Singular reinforcement consists in one segment.

Let us consider the problem

$$\min_{u \in V_0} \int_{\square} \left(A(x) \left|\nabla u\right|^2 - 2Gu \right) d\mu,$$

where

$$A(x) = \begin{cases} a(x) & \text{on } \square \setminus I \\ b(x) & \text{on } I\,, \end{cases} \qquad G = \begin{cases} f & \text{on } \square \setminus I \\ g & \text{on } I\,, \end{cases}$$

$A(x)$ is a function such that $0 < \gamma_1 \leq A(x) \leq \gamma_2$ μ-a.e., $a(x), b(x) \in C^1_{per}(\square)$, $G \in L^2(\square, d\mu)$, $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D = \{x \in \partial\square : x_1 x_2 = 0\}$, $\Gamma_N = \partial\square \setminus \Gamma_D$ and $V_0 = \{u \in H^1(\square, d\mu) : u_{|\Gamma_D} = 0\}$. According to the decomposition (2) we have

$$\int_I b(x) \left|\nabla u\right|^2 \, d\lambda = \int_I b(x) \left(\left(\frac{\partial u}{\partial x_1} \right)^2 + \alpha^2 \right) d\lambda\,,$$

where $\alpha \in L^2(I, d\lambda)$ is arbitrary. So we get the equivalent variational formulation

$$\min_{u \in V_0} \left[\int_{\square} a(x) \left|\nabla u\right|^2 dx + \int_I b(x) \left(\frac{\partial u}{\partial x_1} \right)^2 d\lambda - 2 \int_{\square} f u \, dx - 2 \int_I g u \, d\lambda \right]$$

and the corresponding Euler equation

$$\int_{\square} a(x) \nabla u \cdot \nabla \varphi \, dx + \int_I b(x) \frac{\partial u}{\partial x_1} \cdot \frac{\partial \varphi}{\partial x_1} \, d\lambda = \int_{\square} f \varphi \, dx + \int_I g \varphi \, d\lambda, \quad \forall \varphi \in V_0, \quad (22)$$

$u \in V_0$. Taking $\varphi \in C^\infty(\mathbb{R}^2) \cap V_0$ such that $\varphi_{|I} = 0$, we see that

$$\begin{cases} -\operatorname{div}(a(x)\nabla u) = f \quad \text{in } \Omega \setminus I \\ u = 0 \ \text{ on } \Gamma_D, \quad \dfrac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_N. \end{cases} \qquad (23)$$

Now we obtain the equation for $u_{|I}$. Integrating by parts we get

$$\int_{\square} a(x)\nabla u\cdot\nabla\varphi\,dx=\int_{\square\setminus I} a(x)\nabla u\cdot\nabla\varphi\,dx=\int_{\square\setminus I} f\varphi\,dx-\int_I a\left[\frac{\partial u}{\partial x_2}\right]_I\varphi\,d\lambda.$$

By (22) we get $\int_I b\frac{\partial u}{\partial x_1}\frac{\partial\varphi}{\partial x_1}\,d\lambda=\int_I g\varphi\,d\lambda+\int_I a\left[\frac{\partial u}{\partial x_2}\right]_I\varphi\,d\lambda$, where the jump $\left[\frac{\partial u}{\partial x_2}\right]_I=\left(\frac{\partial u}{\partial x_2}\right)^+-\left(\frac{\partial u}{\partial x_2}\right)^-$ is the difference of values attained at segment I from upper and lower half-planes. It follows that

$$\begin{cases} -\dfrac{\partial}{\partial x_1}\left(b\,\dfrac{\partial u}{\partial x_1}\right)=g+a\left[\dfrac{\partial u}{\partial x_2}\right]_I \text{ on } I \\ u\left(0,\frac{1}{2}\right)=0, \quad \dfrac{\partial u}{\partial x_1}\left(1,\frac{1}{2}\right)=0. \end{cases} \tag{24}$$

Relations (24) and (23) can be considered as a coupled system. The first relation in (24) is called Ventsel condition.

In the previous example we can replace I with the union of two segments I and J in the domain $\square$ intersecting at the point P (see Figure 3).

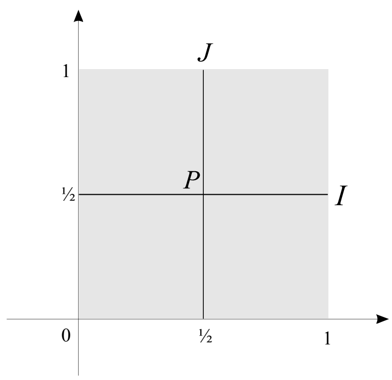

Figure 3: Singular reinforcement consists in two intersecting segments.

Then, together with boundary conditions of the type (24) on each segment, a natural condition in the point P arises, namely $\left(\left[\frac{\partial u}{\partial x_1}\right]_J+\left[\frac{\partial u}{\partial x_2}\right]_I\right)_{|P}=0$, where the jump $\left[\frac{\partial u}{\partial x_1}\right]_J$ is the difference of values attained at the vertical segment from right and left half-planes.

Remark 1. In the general case when $A(y)$ (see (13)) is a 2×2 symmetric and definite positive matrix of functions in $L^\infty(\Omega,d\mu)$, the problem (14) can be rewritten in terms of tangential gradient as follows:

$$\min_{u\in H_0^1(\Omega,d\mu_\varepsilon)}\int_\Omega\left(\widehat{A}\left(\frac{x}{\varepsilon}\right)\nabla^t u\cdot\nabla^t u-2G_\varepsilon u\right)\,d\mu_\varepsilon\,,$$

where $\widehat{A}(y)$ is the so-called *relaxed matrix* (see Section 9 in [15]) defined by $\widehat{A}(y)\xi \cdot \xi = \min_{\eta \in T^{\perp}(y)} A(y)(\xi+\eta)\cdot(\xi+\eta)$. In our case

$$T(y) = \begin{cases} \mathbb{R}^2 & \text{on tiles} \\ \mathcal{L}(e_1) \text{ or } \mathcal{L}(e_2) & \text{on the network,} \end{cases}$$

where (e_1, e_2) is the canonical base in $\mathbb{R}^2$, $\mathcal{L}(e_1)$ and $\mathcal{L}(e_2)$ are the linear space generated respectively by the vectors e_1 and e_2. Therefore

$$\widehat{A}(y) = \begin{cases} \quad \alpha(y) & & \text{on tiles} \\ \begin{pmatrix} \widehat{\beta}(y) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & \widehat{\beta}(y) \end{pmatrix} & & \text{on network } F, \end{cases} \tag{25}$$

with

$$\widehat{\beta}(y) = \begin{cases} \beta_{11} - \frac{\beta_{12}^2}{\beta_{22}} & \text{on horizontal lines} \\ \beta_{22} - \frac{\beta_{12}^2}{\beta_{11}} & \text{on vertical lines.} \end{cases}$$

4. An example of a variational problem for a combined structure with thin reinforcement

Let us consider a sequence of $\square$-periodic normalized measures μ^h such that $\mu^h \rightharpoonup \mu$ as $h \to 0$, i.e.,

$$\lim_{h\to 0} \int_{\square} \varphi d\mu^h = \int_{\square} \varphi d\mu, \quad \forall \varphi \in C^{\infty}_{per}(\square).$$

Let v^h a bounded sequence in $L^2(\square, d\mu^h)$, that is $\limsup_{h\to 0} \int_{\square} |v^h|^2 \varphi \, d\mu^h < +\infty$. We say that v^h weakly converges to v in $L^2(\square, d\mu^h)$, and we write $v^h \rightharpoonup v$ in $L^2(\square, d\mu^h)$, if $v \in L^2(\square, d\mu)$ and

$$\lim_{h\to 0} \int_{\square} v^h \varphi \, d\mu^h = \int_{\square} v \, \varphi \, d\mu, \quad \forall \varphi \in C^{\infty}_{per}(\square).$$

We say that v^h strongly converges to v in $L^2(\square, d\mu^h)$, and we write $v^h \to v$ in $L^2(\square, d\mu^h)$, if $v \in L^2(\square, d\mu)$ and

$$\lim_{h\to 0} \int_{\square} v^h z^h \, d\mu^h = \int_{\square} v \, z \, d\mu, \quad \text{whenever } v^h \rightharpoonup v \text{ in } L^2\left(\square, d\mu^h\right).$$

The compactness principle is valid for this type of convergences, i.e., each bounded sequence has a weakly convergent subsequence. The simplest combined structure with thin reinforcement can be obtained from the combined structure described in Section 3.2.

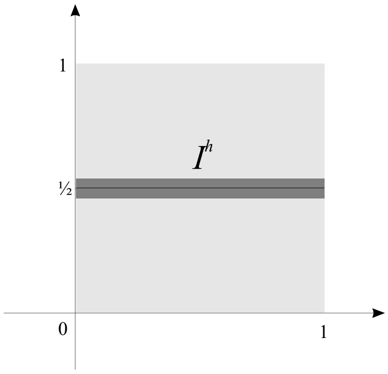

Figure 4: Simple combined structure with thin reinforcement.

Let $\square = (0,1)^2$, $I^h = (0,1) \times \left(\frac{1-h}{2}, \frac{1+h}{2}\right)$, $I = (0,1) \times \left\{\frac{1}{2}\right\}$ (see Figure 4) and $\mu^h = dx|_\square + \lambda^h$, with $\lambda^h = \frac{1}{h} dx|_{I_h}$. We observe that $\mu^h \rightharpoonup \mu = dx + \lambda$, with $\lambda = dx_1|_I$. Let us consider the problem

$$u^h \in V, \quad -\mathrm{div}\left(A^h(x)\nabla u^h\right) = G^h, \tag{26}$$

with

$$G^h = \begin{cases} f^h, & \text{on } \square \setminus I^h \\ g^h, & \text{on } I^h\,, \end{cases} \qquad \text{and} \qquad A^h(x) = \begin{cases} a(x), & \text{on } \square \setminus I^h \\ b(x), & \text{on } I^h\,, \end{cases}$$

where $G^h \in L^2(\square, d\mu^h)$, $G^h \rightharpoonup G$ in $L^2(\square, d\mu^h)$, $V = \{u \in H^1(\square, dx) : u_{|\Gamma_D} = 0\}$, $a(x)$ and $b(x)$ are the same as in Section 3.2.

By definition the solution u^h of the equation (26) satisfies the integral identity $u^h \in V, \int_\square A^h(x)\nabla u^h \cdot \nabla\varphi\, d\mu^h = \int_\square G^h \varphi\, d\mu^h$ for all $\varphi \in V$. In details, for all $\varphi \in V$,

$$\int_\square a(x)\nabla u^h \cdot \nabla\varphi\, dx + \int_{I^h} b(x)\nabla u^h \cdot \nabla\varphi\, d\lambda^h = \int_\square f^h\varphi\, dx + \int_{I^h} g^h\varphi\, d\lambda^h. \tag{27}$$

We want to show that $u^h \rightharpoonup u$ in $L^2\left(\square, d\mu^h\right)$ where u is the solution of the problem (22) considered in §3.2.

Firstly, by Friedrichs type inequality $\int_\square \varphi^2 d\mu^h \le c \int_\square |\nabla\varphi|^2 d\mu^h$, for all $\varphi \in V$, we obtain that u^h and ∇u^h are bounded in $L^2(\square, d\mu^h)$. In particular, u^h is bounded in $H^1(\square, dx) = H^1(\square)$ and so we can assume that

$$u^h \rightharpoonup u \;\text{ in } H^1(\square) \qquad \text{and} \qquad u \in V. \tag{28}$$

Then we have

$$\int_\square a(x)\nabla u^h \cdot \nabla\varphi\, dx \xrightarrow{h\to 0} \int_\square a(x)\nabla u \cdot \nabla\varphi\, dx\,. \tag{29}$$

Now consider the second term in (27). We state the following proposition involving a Sobolev space with variable measure $L^2(\square, d\lambda^h)$ (see [13]) which is similar to a Sobolev space with periodic variable measure $L^2(\square, d\mu^h)$ defined in Section 4.1.

Proposition 4. *If* $v^h \rightharpoonup v$ *in* $L^2(\square, d\lambda^h)$, *then*

$$\widetilde{v}^h \equiv \frac{1}{h}\int_{\frac{1-h}{2}}^{\frac{1+h}{2}} v^h\left(x_1, x_2\right) dx_2 \rightharpoonup v \quad \text{in } L^2(I, d\lambda)\,.$$

Let us now prove the following lemma.

Lemma 1. *Let us assume that* $u^h, \nabla u^h$ *are bounded in* $L^2(\square, d\lambda^h)$. *Then, up to a subsequence, we have*

i) $u^h \rightharpoonup u_1$ *in* $L^2(\square, d\lambda^h)$, $u_1 \in H^1(I, dx_1)$, $u_1(0) = 0$;
ii) $\nabla u^h \rightharpoonup \nabla u_1 = \left\{\frac{\partial u_1}{\partial x_1}, \alpha\right\}$ *in* $L^2(\square, d\lambda^h)$, $\alpha \in L^2(I, dx_1)$.

Proof. By hypothesis, we can assume that $u^h \rightharpoonup u_1$ in $L^2(\square, d\lambda^h)$, $\nabla u^h \rightharpoonup p$ in $L^2(\square, d\lambda^h)^2$. By Proposition 4, we have $\widetilde{u}^h \rightharpoonup u_1$ in $L^2(I, d\lambda)$ and

$$\frac{\partial}{\partial x_1}\widetilde{u}^h = \frac{1}{h}\int_{\frac{1-h}{2}}^{\frac{1+h}{2}} \frac{\partial}{\partial x_1} u^h dx_2 \rightharpoonup p_1 \quad \text{in } L^2\left(I, d\lambda\right). \qquad \square$$

Now we identify the function u_1 from Lemma 1 with $u|_I$, where u is the function given in (28).

Proposition 5. *If* $u^h \rightharpoonup u$ *in* $H^1(\square)$, *then* $u^h \to u|_I$ *in* $L^2(\square, d\lambda^h)$.

Proof. The thesis follows from the trace theorem

$$\int_\square u^2 d\lambda^h \le c\left(\delta\right)\int_\square u^2 dx + \delta \int_\square \left|\nabla u\right|^2 dx\,. \qquad \square$$

We observe that Lemma 1 and Proposition 5 also show the inclusion $u \in V_0$. From above the limit of the second term in (27) is equal to

$$\lim_{h\to 0}\int_{I^h} b(x)\nabla u^h \cdot \nabla\varphi\, d\lambda^h = \int_I b(x)\nabla u \cdot \nabla \varphi\, d\lambda\,, \tag{30}$$

where $\nabla u = \left\{\frac{\partial u}{\partial x_1}, \alpha\right\}$. So passing to the limit in (27) by (29) and (30) we get the integral identity

$$\int_\square a(x)\nabla u \cdot \nabla\varphi\, dx + \int_I b(x)\nabla u \cdot \nabla\varphi\, d\lambda = \int_\square G\varphi\, d\mu, \quad \forall \varphi \in V_0.$$

Taking here φ^h $\square$-periodic, smooth such that $\varphi^h \rightharpoonup 0$ in $L^2(\square, d\mu^h)$, $\nabla\varphi^h \rightharpoonup \{0, \beta\}$ in $L^2(\square, d\mu^h)^2$ (any gradient of zero), we get that $\alpha = 0$ and the last identity coincides with (22).

5. Homogenization for a combined structure with thin reinforcement

5.1. Let μ, μ^h normalized $\square$-periodic Borel measures such that $\mu^h \rightharpoonup \mu$. Given $a \in L^2(\square, d\mu)$, $b \in L^2(\square, d\mu)^2$ the relation $-\operatorname{div} b = a$ (in the sense of measure μ) means that the following identity holds:

$$\int_\square b \cdot \nabla\varphi \, d\mu = \int_\square a\varphi \, d\mu \,, \quad \forall \varphi \in C^\infty_{per}.$$

We first introduce the so-called approximation properties.

Definition 4. We say that *approximation properties* hold for measures μ, μ^h if for every $a \in L^2(\square, d\mu)$, $b \in L^2(\square, d\mu)^2$ such that $-\operatorname{div} b = a$ (in the sense of μ) there exist $a^h \in L^2(\square, d\mu^h)$, $b^h \in L^2(\square, d\mu^h)^2$ such that

i) $\operatorname{div} b^h = a^h$ (in sense of μ^h), $b^h \to b$ in $L^2(\square, d\mu^h)^2, a^h \to a$ in $L^2(\square, d\mu^h)$;
ii) if $a = 0$, then $a^h = 0$ in i) (strong approximability of solenoidal vectors).

Let us define the measure $\widetilde{\mu}^h$ from the initial measure μ by

$$\int_\square \varphi \, d\widetilde{\mu}^h = \int_\square (\varphi)_h \, d\mu \,, \quad \varphi \in C^\infty_{per}(\square) \,, \tag{31}$$

where $(\varphi)_h$ is the smoothing $(\varphi)_h(x) = h^{-2} \int_{\mathbb{R}^2} \varphi(x-y) w\,(h^{-1}y) \, dy$ in which w is suitable non negative smooth function such that $\int_{\mathbb{R}^2} w(x) \, dx = 1$. The measure $\widetilde{\mu}^h$ is called the *smoothing measure* and has the density

$$\widetilde{\rho}^h(x) = h^{-2} \int_{\mathbb{R}^2} w\left(h^{-1}(x-y)\right) d\mu(y) \,.$$

Now we recall the following result proved in Theorem 16.2 in [16]:

Theorem 3. *Let μ be an arbitrary periodic Borel measure and let $\widetilde{\mu}^h$, $h > 0$, be the smoothing measure defined in* (31)*. Then the approximation properties hold for μ, $\widetilde{\mu}^h$.*

We define the scaling measure μ^h_ε, $h > 0$, obtained by μ^h as in (4). It is easy to see that if $h = h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ it results $\mu^h_\varepsilon \rightharpoonup dx$. In this setting, if u^h_ε is a bounded sequence in $L^2(\Omega, d\mu^h_\varepsilon)$ we can define the convergence in $L^2(\Omega, d\mu^h_\varepsilon)$ in a similar way as in Definition 3; analogous properties to the ones expressed in Proposition 1 will hold.

We shall use also the concept of two-scale convergence with respect to variable 1-periodic measure μ^h, $\mu^h \rightharpoonup \mu$, $h = h(\varepsilon) \to 0$ introduced in Section 11 in [16].

Definition 5. Assume that v_ε^h is bounded in $L^2(\Omega, d\mu_\varepsilon^h)$ ($\limsup_{\varepsilon\to 0} \int_\Omega |v_\varepsilon^h|^2 d\mu_\varepsilon^h < +\infty$). We say v_ε^h is 2-*scale convergent* to v and we write $v_\varepsilon^h \overset{2}{\rightharpoonup} v(x,y)$, if $v(x,y) \in L^2(\Omega \times \Box, dx \times d\mu)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon^h(x)\varphi(x) b\left(\frac{x}{\varepsilon}\right) d\mu_\varepsilon^h = \int_\Omega \int_\Box v(x,y)\varphi(x) b(y)\, dx\, d\mu$$

for every $\varphi \in C_0^\infty(\Omega), b \in C_{per}^\infty(\Box)$.

The following properties (see Section 11 in [16]) will be used later.

Proposition 6. *Let v_ε^h a sequence in $L^2(\Omega, d\mu_\varepsilon^h)$. Then:*

i) *if v_ε^h is bounded in $L^2(\Omega, d\mu_\varepsilon^h)$, it is compact with respect to weak 2-scale convergence;*

ii) *if $v_\varepsilon^h \overset{2}{\rightharpoonup} v(x,y)$, then $v_\varepsilon^h \rightharpoonup \int_\Box v(x,y) d\mu$ in $L^2(\Omega, d\mu_\varepsilon^h)$;*

iii) *if $v_\varepsilon^h \overset{2}{\rightharpoonup} v(x,y)$, $b^h(y) \to b(y)$ in $L^2(\Box, d\mu^h)$, then*

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon^h(x)\varphi(x) b^h\left(\frac{x}{\varepsilon}\right) d\mu_\varepsilon^h = \int_\Omega \int_\Box v(x,y)\varphi(x) b(y)\, dx\, d\mu, \quad \forall \varphi \in C_0^\infty(\Omega).$$

For sake of completeness we now prove a slight generalization of Theorem 16.7 in [16].

Theorem 4. *Let μ^h, μ be $\Box$-periodic Borel measures such that $\mu^h \rightharpoonup \mu$, μ is ergodic and non-degenerate, μ^h, μ are connected through approximation properties. Let A^h be Borelian symmetric $\Box$-periodic matrices satisfying*

i) *$\nu \xi^2 \le A^h(y)\xi \cdot \xi \le \nu^{-1}\xi^2$, $\xi \in \mathbb{R}^2$, μ^h -a.e. in $\Box$, $\nu > 0$ independent on h;*

ii) *$A^h \to A$ in $L^2(\Box, d\mu^h)^{2\times 2}$.*

Let u_ε^h be a sequence of solutions of the problem

$$u_\varepsilon^h \in H_0^1(\Omega, d\mu_\varepsilon^h), \qquad -\operatorname{div}\left(A^h\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^h\right) + u_\varepsilon^h = G_\varepsilon^h. \tag{32}$$

If $\varepsilon \to 0$, $h = h(\varepsilon) \to 0$ and $G_\varepsilon^h \rightharpoonup G$ (strongly) in $L^2(\Omega, d\mu_\varepsilon^h)$, then $u_\varepsilon^h \rightharpoonup u$ (strongly) in $L^2(\Omega, d\mu_\varepsilon^h)$, where u is solution of the homogenized problem

$$u \in H_0^1(\Omega), \qquad -\operatorname{div}\left(A^{\mathrm{hom}}\nabla u\right) + u = G. \tag{33}$$

Moreover the convergence of the energies holds.

Proof. The variational formulation of (32) is

$$\int_\Omega A^h\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^h \cdot \nabla\varphi\, d\mu_\varepsilon^h + \int_\Omega u_\varepsilon^h\, \varphi\, d\mu_\varepsilon^h = \int_\Omega G_\varepsilon^h\, \varphi\, d\mu_\varepsilon^h, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{34}$$

The last equality implies that u_ε^h and ∇u_ε^h are bounded in $L^2(\Omega, d\mu_\varepsilon^h)$. Since μ is non-degenerate and approximation properties hold for μ^h and μ, then (see Section 16 in [16]) there exist $u_0(x) \in L^2(\Omega)$ and $v(x,y) \in L^2(\Omega, V_{pot})$ such that

$$u_\varepsilon^h \stackrel{2}{\rightharpoonup} u_0(x), \qquad \nabla u_\varepsilon^h \stackrel{2}{\rightharpoonup} \nabla u_0(x) + v(x,y). \tag{35}$$

Let us take $\varphi(x) = \varepsilon\psi(x)w(\frac{x}{\varepsilon})$ as test function in (34), where $\psi \in C_0^\infty(\Omega)$ and $w \in C_{per}^\infty(\square)$. We get

$$\begin{aligned}\int_\Omega A^h\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^h \cdot \nabla w\left(\frac{x}{\varepsilon}\right)\psi\, d\mu_\varepsilon^h + \varepsilon\int_\Omega A^h\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^h \cdot \nabla\psi w\left(\frac{x}{\varepsilon}\right) d\mu_\varepsilon^h \\ + \varepsilon\int_\Omega u_\varepsilon^h\, \varphi\, d\mu_\varepsilon^h = \varepsilon\int_\Omega G_\varepsilon^h\, \psi w\left(\frac{x}{\varepsilon}\right) d\mu_\varepsilon^h.\end{aligned} \tag{36}$$

Passing to the limit as $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\int_\Omega A^h\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^h \cdot \nabla w\left(\frac{x}{\varepsilon}\right)\psi\, d\mu_\varepsilon^h = 0 \tag{37}$$

since any other term in (36) tends to zero.

Since $\nabla u_\varepsilon^h \stackrel{2}{\rightharpoonup} \nabla u_0(x) + v(x,y)$ and $A^h \to A$ in $L^2(\square, d\mu^h)^{2\times 2}$, by (iii) and (ii) of Proposition 6, we have

$$A^h\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^h \stackrel{2}{\rightharpoonup} A(y)(\nabla u_0(x) + v(x,y)) \tag{38}$$

$$A^h\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^h \rightharpoonup \int_\square A(y)(\nabla u_0(x) + v(x,y))\, d\mu, \quad \text{in } L^2(\Omega, d\mu_\varepsilon^h)^2. \tag{39}$$

So by (38), equality (37) yields

$$\int_\Omega\int_\square A(y)\,(\nabla u_0(x) + v(x,y))\,\psi(x)\cdot\nabla w(y)\, d\mu\, dx = 0.$$

By the arbitrariness of $\psi \in C_0^\infty(\Omega)$ we get

$$\int_\square A(y)\,(\nabla u_0\,(x) + v(x,y))\cdot\nabla w\,(y)\; d\mu = 0, \quad \forall w \in C_{per}^\infty\,(\square)\,.$$

Then $v(x,\cdot)$ is a solution of periodic problem (8) with $\xi = \nabla u_0$. Moreover (9) implies that

$$\int_\square A(y)(\nabla u_0(x) + v(x,y))\, d\mu = A^{\text{hom}}\nabla u_0.$$

By (39) we obtain the weak convergence

$$A^h\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^h \rightharpoonup A^{\text{hom}}\nabla u_0\,, \quad \text{in } L^2(\Omega, d\mu_\varepsilon^h)^2. \tag{40}$$

We observe that by (35), we have $u_\varepsilon^h \rightharpoonup u_0$ in $L^2(\Omega, d\mu_\varepsilon^h)$; moreover by (40) and since $G_\varepsilon^h \rightharpoonup G$ in $L^2(\Omega, d\mu_\varepsilon^h$, passing to the limit in (34) we obtain

$$\int_\Omega A^{\mathrm{hom}} \nabla u_0 \cdot \nabla \varphi \, dx + \int_\Omega u_0 \, \varphi \, dx = \int_\Omega G \, \varphi \, dx \,, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Assume now that $G_\varepsilon^h \to G$ in $L^2(\Omega, d\mu_\varepsilon^h)$. Then the strong convergence $u_\varepsilon^h \to u_0$ in $L^2(\Omega, d\mu_\varepsilon^h)$ can be obtained by considering auxiliary problems like in (32) with u_ε^h in place of G_ε^h and arguing in a similar way to the last part of Theorem 2.

The convergence of the minimum values of the problems (32) to the minimum value of the homogenized problem (33) can be obtained by taking u_ε as test function in the Euler equation of the problem (32) and passing to the limit as $\varepsilon \to 0$. □

5.2. Now we pass to describe the combined structure with a □-periodic thin reinforcement of thickness h. Let us consider a □-periodic combined structure with singular reinforcement given by a network F like in (10); the corresponding □-periodic combined structure with thin reinforcement F^h will be obtained replacing the network F with the set $F^h = \left\{x \in \square : \mathrm{dist}(x, F) \leq \frac{h}{2}\right\}$ (see Figure 5).

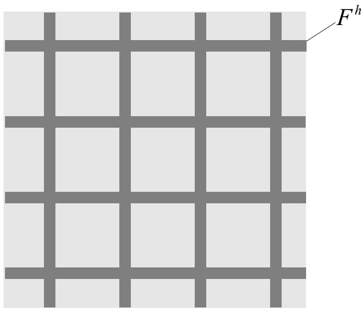

Figure 5: Combined structure with thin reinforcement of thickness h.

It is characterized by the □-periodic normalized measure μ^h on $\mathbb{R}^2$ defined as

$$\mu^h = \frac{1}{2} dx + \frac{1}{2} \lambda^h, \tag{41}$$

where dx is the 2-dimensional Lebesgue measure and λ^h is the □-periodic normalized measure proportional to 2-dimensional Lebesgue measure concentrated on F^h such that $\int_\square d\lambda^h = 1$. We shall call measures λ^h, μ^h *natural* relative to structure F^h. We have, as $h \to 0$, $\mu^h \rightharpoonup \mu = \frac{1}{2} dx + \frac{1}{2} \lambda$ where λ is the 1-dimensional Lebesgue measure supported on the singular network F we considered earlier (see (10)).

Let us now consider a homogenization problem depending on two parameters. Let μ_ε^h be the measure obtained by μ^h as in (4). It is easy to see that, as $h = h(\varepsilon) \to 0$, it results $\mu_\varepsilon^h \rightharpoonup dx$. For simplicity let

$$A^h(y) = \begin{cases} \alpha(y) & \text{outside } F^h \\ \beta(y) & \text{on } F^h, \end{cases} \quad \text{and} \quad A(y) = \begin{cases} \alpha(y) & \text{outside } F \\ \beta(y)_{|F} & \text{on } F, \end{cases}$$

where $\alpha(y)$ and $\beta(y)$ are symmetric, $\square$-periodic matrices satisfying (5), $\beta \in H^1_{per}(\square, d\mu)^{2\times 2}$, and consider the problem

$$(\mathrm{P}_\varepsilon^h): \qquad \min_{v \in H_0^1(\Omega)} \int_\Omega \left(A^h\left(\frac{x}{\varepsilon}\right) \nabla v \cdot \nabla v - 2G_\varepsilon^h\, v \right) d\mu\,_\varepsilon^h \tag{42}$$

where $G_\varepsilon^h \in L^2(\Omega)$ and $G_\varepsilon^h \rightharpoonup G$ in $L^2(\Omega, d\mu_\varepsilon^h)$. The Euler equation is

$$u_\varepsilon^h \in H_0^1(\Omega, d\mu_\varepsilon^h), \quad \int_\Omega A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon^h \cdot \nabla \varphi \, d\mu_\varepsilon^h = \int_\Omega G_\varepsilon^h \varphi \, d\mu_\varepsilon^h, \quad \forall \varphi \in C_0^\infty(\Omega).$$

To prove the existence result for our problem we need Friedrichs inequality.

Proposition 7. *The following Friedrichs inequality holds for the measure μ_ε^h:*

$$\int_\Omega \varphi^2 d\mu_\varepsilon^h \le c \int_\Omega |\nabla \varphi|^2 \, d\mu_\varepsilon^h\,, \quad \forall \varphi \in C_0^\infty(\Omega),$$

where c is a constant independent on ε and h.

Proof. The proof is similar to the one given in previous section (see Proposition 3). We only observe that instead of lines we have the corresponding strips. $\square$

We have the estimate

$$\int_\Omega G_\varepsilon^h \, \varphi \, d\mu_\varepsilon^h \le c \left(\int_\Omega \varphi^2 d\mu_\varepsilon^h \right)^{\frac{1}{2}} \le c_1 \left(\int_\Omega |\nabla \varphi|^2 \, d\mu_\varepsilon^h \right)^{\frac{1}{2}}.$$

So solution exists and is unique by the Riesz representation theorem and we have

$$\int_\Omega \left(\left|u_\varepsilon^h\right| + \left|\nabla u_\varepsilon^h\right|^2 \right) d\mu_\varepsilon^h \le c \int_\Omega \left|G_\varepsilon^h\right|^2 d\mu_\varepsilon^h.$$

So u_ε^h is bounded in $H_0^1(\Omega, d\mu_\varepsilon^h)$ and then, up to a subsequence, $u_\varepsilon^h \rightharpoonup u$ in $L^2(\Omega, d\mu_\varepsilon^h)$.

Theorem 5. *Let* μ^h *be the measure defined in* (41) *and* $G^h_\varepsilon \in L^2(\Omega)$ *such that*

$$G^h_\varepsilon \rightharpoonup G \quad in\ L^2\left(\Omega, d\mu^h_\varepsilon\right). \tag{43}$$

Then if u^h_ε *is a sequence of solutions of problem* $\left(\mathrm{P}^h_\varepsilon\right)$*, it results*

$$u^h_\varepsilon \rightharpoonup u \quad in\ L^2(\Omega, d\mu^h_\varepsilon), \tag{44}$$

where u is solution of the homogenized problem

$$(\mathrm{P}^{\mathrm{hom}}): \qquad \min_{v\in H^1_0(\Omega)} \int_\Omega \left(A^{\mathrm{hom}}\nabla v\cdot\nabla v - 2Gv\right) dx.$$

If we have the strong convergence in (43)*, the same strong convergence holds in* (44) *and the convergence of the energies holds.*

Proof. The proof is similar to the one of the main theorem of the previous section using Theorem 4 instead of Theorem 1. It remains to verify that hypotheses of Theorem 4 are verified, i.e., to check that $A^h \to A$ in $L^2(\square, d\mu^h)^{2\times 2}$ and that the measures μ^h and μ are connected through approximation properties.

It is not difficult to check that $A^h \rightharpoonup A$ in $L^2(\square, d\mu^h)^{2\times 2}$; moreover, since $\beta^2 \in W^{1,1}(\square)$ and $(\beta^2)_{|F} = (\beta_{|F})^2$, by the trace theorem we obtain $A^h \to A$ in $L^2(\square, d\mu^h)^{2\times 2}$.

Let $\widetilde{\mu}^h$ the smoothing measure obtained from μ (see (31)). We will use the kernel $w(y) = \chi_Q$, $Q = (-\frac{1}{2}, \frac{1}{2})^2$. Therefore

$$\widetilde{\rho}^h(x) = h^{-2}\int_{x+Q^h} d\mu(y)\, dy, \qquad Q^h = \left(-\frac{h}{2}, \frac{h}{2}\right)^2. \tag{45}$$

Let us compare the measures $\widetilde{\mu}^h$ and μ^h. By definition, μ^h has the density

$$\rho^h = \begin{cases} \frac{1}{4h(1-h)}, & \text{on } F^h \\ 0, & \text{outside}. \end{cases} \tag{46}$$

By (45) we have

$$\widetilde{\rho}^h = \begin{cases} \frac{1}{4h}, & \text{on } F^h \setminus Q^h \\ \frac{1}{2h}, & \text{on } Q^h. \end{cases} \tag{47}$$

It is known (see Theorem 3) that for the measures $\widetilde{\mu}^h$ and μ the approximation properties hold. So for any $a \in L^2(\square, d\mu)$, $b \in L^2(\Omega)^2$ such that $-\operatorname{div} b = a$ (in the sense of μ) there exist $\widetilde{a}^h \in L^2(\square, d\widetilde{\mu}^h)$, $\widetilde{b}^h \in L^2(\square, d\widetilde{\mu}^h)^2$ such that

$$\int_\square \widetilde{b}^h\cdot\nabla\varphi\, d\widetilde{\mu}^h = \int_\square \widetilde{a}^h\varphi\, d\widetilde{\mu}^h$$

$$\widetilde{b}^h \to b \ \text{ in } L^2(\square, d\widetilde{\mu}^h)^2, \quad \widetilde{a}^h \to a \ \text{ in } L^2(\square, d\widetilde{\mu}^h).$$

If we take $b^h = \widetilde{b}^h \frac{\widetilde{\rho}^h}{\rho^h}$ and $a^h = \widetilde{a}^h \frac{\widetilde{\rho}^h}{\rho^h}$ we get $\int_\Box b^h \cdot \nabla\varphi \, d\mu^h = \int_\Box a^h \varphi \, d\mu^h$.

In order to check (i) of Definition 4, it remains to verify that $b^h \to b$ in $L^2(\Box, d\mu^h)^2$, $a^h \to a$ in $L^2(\Box, d\mu^h)$, that can be derived from the following propositions.

Point (ii) of Definition 4 will then follows as particular case of (i) taking into account that if $a = 0$, $\widetilde{a}^h$ can be taken equal to zero too. □

Proposition 8. *Let v^h a sequence in $L^2(\Box, d\mu^h)$. We have that $v^h \rightharpoonup v$ in $L^2(\Box, d\mu^h)$ if*

$$\lim_{h\to 0} \int_\Box v^h \varphi \, d\mu^h = \int_\Box v\varphi \, d\mu \,, \qquad v \in L^2(\Box, d\mu)$$

for any $\varphi \in C^\infty_{per}$ and $\varphi = 0$ near the nods.

Proof. The thesis easily follows since the set $\{\varphi \in C^\infty_{per}(\Box) : \varphi = 0$ near the nods$\}$ is dense in $L^2(\Box, d\mu)$. □

By Proposition 8 and the structure of ρ^h and $\widetilde{\rho}^h$ (see (46) and (47)) we get:

Proposition 9. *The convergences $v^h \rightharpoonup v$ in $L^2(\Box, d\mu^h)$ and $v^h \rightharpoonup v$ in $L^2(\Box, d\widetilde{\mu}^h)$ are equivalent.*

Proposition 10. *If $\widetilde{v}^h \to v$ in $L^2(\Box, d\widetilde{\mu}^h)$, then $v^h = \widetilde{v}^h \frac{\widetilde{\rho}^h}{\rho^h} \to v$ in $L^2(\Box, d\mu^h)$.*

Proof. We know that the strong convergence $\widetilde{v}^h \to v$ in $L^2(\Box, d\widetilde{\mu}^h)$ implies the weak convergence $\widetilde{v}^h \rightharpoonup v$ in $L^2(\Box, d\widetilde{\mu}^h)$. By Proposition 8, $v^h \rightharpoonup v$ in $L^2(\Box, d\mu^h)$. Let $\psi^h \rightharpoonup \psi$ in $L^2(\Box, d\mu^h)$. Then by Proposition 9, $\psi^h \rightharpoonup \psi$ in $L^2(\Box, d\widetilde{\mu}^h)$. By the definition of strong convergence in $L^2(\Box, d\widetilde{\mu}^h)$,

$$\int_\Box v^h \psi^h d\mu^h = \int_\Box \widetilde{v}^h \psi^h d\widetilde{\mu}^h \to \int_\Box v\psi \, d\mu$$

that implies the strong convergence $v^h \to v$ in $L^2(\Box, d\mu^h)$. □

Remark 2. An analogous problem to (42) can be formulated in dimension $N > 2$ for a thin combined structure characterized by a measure $\mu^h = \frac{1}{2}dx + \frac{1}{2}\lambda^h$, where in this case dx denotes the N-dimensional Lebesgue measure and λ^h a normalized measure (on the unit cell) uniformly distributed on the union of plates of thickness h, each one being parallel to some face of the unit cell. A theorem analogous to Theorem 5 will still be valid, since approximation properties can be obtained in a similar way taking as $\widetilde{\mu}^h$ the smoothing measure obtained by convolution with a kernel proportional to the characteristic function of an N-dimensional cube of side h centered at the origin and with the faces parallel to the faces of the cell.

5.3. Consider now the case in which h and ε are independent and the case when A^h and A are defined as in Sections 5.2 and 3.1, G^h as in Section 4.

If ε is fixed and $h \to 0$, then $u_\varepsilon^h \overset{L^2(\Omega,d\mu_\varepsilon^h)}{\longrightarrow} u_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)$, where u_ε is the solution of problem (P_ε) given in (14). This convergence can be proved in a similar way as the corresponding one for the example considered in §4.2. By Theorem 2 we get $u_\varepsilon \overset{L^2(\Omega,d\mu_\varepsilon)}{\longrightarrow} u \in L^2(\Omega)$, where u is solution of homogenized problem (19).

If $h = \text{const} > 0$, we get by Theorem 4.4 in [15] $u_\varepsilon^h \overset{L^2(\Omega,d\mu_\varepsilon^h)}{\longrightarrow} u^h \in L^2(\Omega)$, where u^h is a solution of the homogenization problem

$$\left(\mathrm{P}_h^{\mathrm{hom}}\right): \qquad u^h \in H_0^1(\Omega)\,, \quad -\operatorname{div}\left(A_h^{\mathrm{hom}} \nabla u\right) = G$$

and

$$A_h^{\mathrm{hom}} \xi \cdot \xi = \inf_{v \in C_{per}^\infty} \int_\square A(\xi + \nabla v) \cdot (\xi + \nabla v) d\mu^h.$$

Approximation properties imply (see Lemma 16.5 of [16]), $u^h \longrightarrow u$ in $L^2(\Omega)$. In other words the following diagram is commutative:

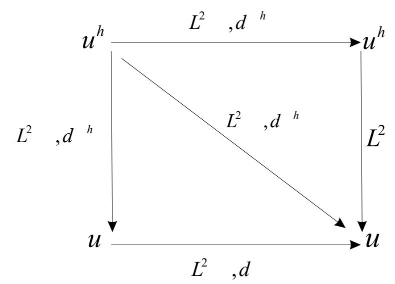

Figure 6: Commutative diagram.

6. Approximation properties for more general thin combined structures

Let us consider a singular (periodic) network F of more complicated geometry than the quadratic network considered before. Some examples are depicted in Figure 7, where fragments of F within the cell of periodicity are given.

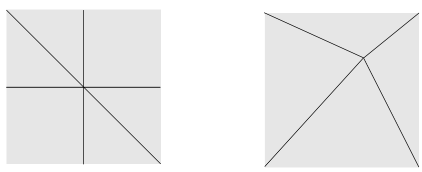

Figure 7: Structures with network F of more complicated geometry.

Corresponding thin network F^h is constructed (see [18]) by substituting every link I in structure F with a strip whose width equals to h and whose middle line is the segment I. Sometimes it is convenient also to add to this structure at each node O of network F a disc of radius $\frac{h}{2}$ with center O.

Let λ, λ^h be natural measures supported on F, F^h and μ, μ^h be relative combined measures defined by (10) and (41). For these measures μ, μ^h when reinforcing singular network F is of fairly general shape, the method used in the proof of approximation properties given in Section 4 can not be applied.

We now sketch the proof of approximation properties in this more general case. We need the following definition (cf. [13]). Let m, m^h be arbitrary Borel periodic normalized measures, $m^h \rightharpoonup m$.

Definition 6. We say that *passing to limit in $H^1(\square, dm^h)$ is possible* if whenever $u^h \rightharpoonup u$ in $L^2(\square, dm^h)$ and $\nabla u^h \rightharpoonup z$ in $L^2(\square, dm^h)^2$, we have $u \in H^1(\square, dm)$ and $z = \nabla u$.

In Section 4 we have proved the possibility of passing to limit in The non-periodic variable space $H^1(\square, d\mu^h)$, where μ^h is a natural measure corresponding to the combined structure consisting of a quadrate with strip.

Lemma 2.

(i) *Assume the uniform Poincaré inequality*

$$\int_\square |\varphi|^2 dm^h \leq C\left[\left|\int_\square \varphi dm^h\right|^2 + \int_\square |\nabla\varphi|^2 dm^h\right], \quad \forall\varphi \in C^\infty_{per}(\square), \tag{48}$$

holds, and passing to the limit in $H^1(\square, dm^h)$ is possible. Then approximation properties are valid for measures m, m^h.

(ii) *If approximation properties are valid for measures m, m^h, then passing to the limit in $H^1(\square, dm^h)$ is possible.*

Some relations between approximation properties and passing to limit in a variable Sobolev space were discovered in [16] while studying elasticity problems (see Section 16 in [16]). Some variant of assertion (i) of Lemma 2 is proved in [12]. Analogue of Lemma 2 for Sobolev spaces of elasticity theory is proved in [10], the scalar case is treated in the similar way. The proof of assertion (ii) is given with the help of dual definition of Sobolev spaces introduced in [17].

Lemma 3. *Let μ, μ^h be natural combined measures with reinforcing singular network F of general form. Then approximation properties are valid for μ and μ^h.*

We sketch the proof of this lemma, important for the homogenization principle in scalar problems on combined structures (see it in details in [10]].

Approximation properties for natural measures λ, λ^h corresponding to arbitrary plane networks are proved in [18]. Hence according to (ii) of Lemma 2 the passing to limit in $H^1(\square, d\lambda^h)$ is possible. From this we can deduce the similar fact for relative combined measures μ, μ^h.

Again we apply Lemma 2 (this time part (i) with respect to measures μ, μ^h) and derive the desirable properties for combined structures. It is necessary to mention that the uniform Poincaré inequality (48) really holds for the combined measure μ^h.

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