Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 26 (2007), 341–362

Metric and w^*-Differentiability of Pointwise Lipschitz Mappings

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Abstract. We study the metric and w^*-differentiability of pointwise Lipschitz mappings. First, we prove several theorems about metric and w^*-differentiability of pointwise Lipschitz mappings between $\mathbb{R}^n$ and a Banach space X (which extend results due to Ambrosio, Kirchheim and others), then apply these to functions satisfying the spherical Rado–Reichelderfer condition, and to absolutely continuous functions of several variables with values in a Banach space. We also establish the area formula for pointwise Lipschitz functions, and for (n, λ)-absolutely continuous functions with values in Banach spaces. In the second part of this paper, we prove two theorems concerning metric and w^*-differentiability of pointwise Lipschitz mappings $f : X \mapsto Y$ where X, Y are Banach spaces with X being separable (resp. X separable and $Y = G^*$ with G separable).

Keywords. Lipschitz mappings, pointwise Lipschitz mappings, metric differentiability, w^*-differentiability, absolutely continuous functions of several variables, area formula, Radon–Nikodým property, Aronszajn null sets

Mathematics Subject Classification (2000). 26B05, 26B30, 46G05, 58C20

1. Introduction

Let X, Y be Banach spaces, $L > 0$. We say that $f : A \subset X \mapsto Y$ is *L-Lipschitz* provided $\|f(x) - f(y)\| \leq L \|x - y\|$ for all $x, y \in A$. We say that $f : A \subset X \mapsto Y$ is *Lipschitz* if there exists an $L > 0$ such that f is L-Lipschitz. Let $\Omega \subset X$ be open. We say that $f : \Omega \mapsto Y$ is *Gâteaux differentiable at* $x \in \Omega$, provided the limit

$$D(f, x)(v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

exists for all $v \in X$, and $D(f, x)(\cdot)$ is a bounded linear operator. We say that $f : \Omega \mapsto Y$ is *Fréchet differentiable at* $x \in \Omega$, provided f is Gâteaux

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differentiable at x and

$$\lim_{h\to 0}\frac{f(x+h)-f(x)-D(f,x)(h)}{\|h\|}=0.$$

If f is Fréchet differentiable at x, we define $f'(x):=D(f,x)(\cdot)$. Fréchet differentiability obviously implies Gâteaux differentiability. If X is finite-dimensional, and f is Lipschitz, then the notions of Fréchet and Gâteaux differentiability coincide (see, e.g., [4, Proposition 4.3]).

The classical theorem of Rademacher [21], which says that every Lipschitz mapping $f:\mathbb{R}^n\mapsto\mathbb{R}^m$ is Fréchet differentiable almost everywhere, was generalized to Lipschitz mappings $f:X\mapsto Y$, where X is a separable Banach space, and Y has the RNP (*Radon–Nikodým property*; see e.g. [4] for a definition). The general theorem claims that such a mapping is Gâteaux differentiable outside of a "null" set. This was obtained by Aronszajn [3], Christensen [6], Mankiewicz [19], and Phelps [20]. Each of these authors introduced a different notion of null sets in a separable Banach space: Aronszajn null sets, Haar null sets, cube-null sets, and Gaussian null sets. In a remarkable paper [7], it was proved by Csörnyei that the first, third, and fourth definitions give the same notion. It is well known (see, e.g., [4]) that there exists a Haar null set (say in ℓ_2), which is not Aronszajn null.

We say that $f:X\mapsto Y$ is *pointwise Lipschitz at* x, provided

$$\operatorname{lip}(f,x)=\limsup_{y\to x}\frac{\|f(x)-f(y)\|}{\|x-y\|}<\infty. \tag{1.1}$$

If (1.1) holds for each $x\in X$, we say that f *is pointwise Lipschitz*.

Stepanoff's theorem (see [23, 24]) was an extension of Rademacher's theorem in a different direction: it says that a pointwise Lipschitz mapping $f:\mathbb{R}^n\mapsto\mathbb{R}^m$ is almost everywhere Fréchet differentiable. This was generalized by Bongiorno [5] to mappings $f:X\mapsto Y$, where X is a separable Banach space and Y has the RNP, using the Aronszajn null sets (see definition below). She proved that for every mapping $f:X\mapsto Y$, there exists an Aronszajn null set E so that f is Gâteaux differentiable at all points of $X\setminus E$ where f is pointwise Lipschitz.

Let X,Y be (real) Banach spaces. For $f:X\mapsto Y$ we shall denote

$$MD(f,x)(u)=\lim_{r\to 0}\frac{\|f(x+ru)-f(x)\|}{|r|}\quad\text{for } x,u\in X.$$

It was defined in [16]. If $MD(f,x)(u)$ exists for all $u\in X$, we say that f *is directionally metrically differentiable at* x. We will say that f *is metrically Gâteaux differentiable at* x provided f is directionally metrically differentiable at x, and $MD(f,x)(\cdot)$ is a continuous seminorm. We say that f *is metrically*

Fréchet differentiable[1] *at* x provided f is metrically Gâteaux differentiable at x, and

$$\|f(y) - f(x)\| - MD(f,x)(y-x) = o(\|y-x\|), \quad (y \to x). \tag{1.2}$$

We say that f *is metrically differentiable at* x, provided f is metrically Gâteaux differentiable, and

$$\|f(z)-f(y)\|-MD(f,x)(z-y) = o(\|z-x\|+\|y-x\|), \quad ((y,z) \to (x,x)). \tag{1.3}$$

Note that metric differentiability implies metric Fréchet differentiability, which in turn implies metric Gâteaux differentiability. It is easy to see that if X is finite-dimensional and f Lipschitz, then metric Gâteaux differentiability implies metric Fréchet differentiability. If f is Gâteaux differentiable at x, then it is metrically Gâteaux differentiable at x, and $MD(f,x)(\cdot) = \|D(f,x)(\cdot)\|$. Our goal is to extend the following theorem about metric differentiability due to Kirchheim [16] (see also [17]; for the case $n = 1$, see [1]; for the case of bi-Lipschitz maps, see [9]; for a different proof of a slightly different statement via w^*-differentiability, see the proof of Theorem 1.2 in [2]).

Theorem 1.1 ([16], Theorem 2). *Let* $f : \mathbb{R}^n \mapsto (X, \|\cdot\|)$ *be Lipschitz. Then, for almost each* $x \in \mathbb{R}^n$, *we have that* f *is metrically differentiable at* x.

As a tool, we will use the notion w^*-Gâteaux derivatives. It goes back to [13]. Let X, Y be separable Banach spaces, $f : X \mapsto Y^*$ be a mapping. For $v \in X$ we say that $wd(f,x)(v)$ *exists* provided

$$wd(f,x)(v) = w^* - \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

exists. We say that f *is* w^**-Gâteaux differentiable at* x provided $wd(f,x)(v)$ exists for all $v \in X$, and $wd(f,x)(\cdot)$ is a bounded linear map. We say that f *is* w^**-Fréchet differentiable at* x provided f is w^*-Gâteaux differentiable at x, and

$$w^* - \lim_{y \to x} \frac{f(y) - f(x) - wd(f,x)(y-x)}{\|y-x\|} = 0. \tag{1.4}$$

The w^*-derivatives were used in [2] to give a proof of a theorem about Fréchet metric differentiability related to [16, Theorem 2].

Theorem 1.2 ([2], Theorem 3.5). *Let* $Y = G^*$, *with* G *separable. Any Lipschitz* $f : \mathbb{R}^n \mapsto Y$ *is* w^**-Fréchet differentiable at* x, *metrically Fréchet differentiable at* x, *and fulfills*

$$MD(f,x)(v) = \|wd(f,x)(v)\| \quad \textit{for all } v \in \mathbb{R}^n,$$

for almost all $x \in \mathbb{R}^n$.

[1] Our definition of "metric Fréchet differentiability" corresponds to "metric differentiability" defined in [2]. However, "metric differentiability" as defined in [16] corresponds to our definition of "metric differentiability".

We will now describe the structure of the paper. In Section 2, we extend Theorems 1.1 and 1.2; first we prove that the same statements hold if we consider pointwise Lipschitz mappings. For details see Theorems 2.5, 2.6, and Corollary 2.7. As a corollary (see Corollary 2.8), we obtain the same results for functions satisfying the spherical Rado–Reichelderfer condition (see (2.13)). Malý [18] used a similar condition as a sufficient condition for absolute continuity (in the sense of [18]). Csörnyei [8] proved that the absolute continuity in the sense of [18] is actually equivalent to such a condition. Theorem 1 in [5] implies that functions satisfying the spherical condition of Rado and Reichelderfer with values in a Banach space with the Radon–Nikodým property are almost everywhere Fréchet differentiable; see Corollary 2.10. Another corollary of our approach is an area formula for pointwise Lipschitz mappings (see Theorem 2.11 and Corollary 2.12), which generalizes Kirchheim's results for Lipschitz mappings from [16]. In Section 3, we extend the definition of (n, λ)-absolutely continuous functions to functions with values in a Banach space (we also discuss the auxiliary notion of BV^n_λ functions as a tool). In the case $X = \mathbb{R}^m$, the $(n, 1)$-absolutely continuous functions (and BV^n_1 functions) were introduced by Malý [18], and (n, λ)-absolutely continuous functions (and BV^n_λ functions) were introduced by Hencl [14] (after a private communication with L. Zajíček). We obtain a version of Theorem 1.1 for such functions; see Remark 3.3. Theorem 1 in [5] can be used to obtain almost everywhere differentiability of (n, λ)-absolutely continuous functions with values in a space that has the Radon–Nikodým property; see Corollary 3.4. We also prove an area formula for Banach space-valued (n, λ)-absolutely continuous mappings. This is the content of Theorem 3.5. Our proof is a modification of the proof of [14, Theorem 3.4].

Section 4 contains auxiliary results for the proofs of Theorems 5.3 and 5.4, which are contained in Section 5; we also prove an infinite-dimensional version of [2, Theorem 3.5]; see Theorem 4.3. Prior to establishing Theorems 5.3 and 5.4, there are two examples showing why we can only prove a weaker statement, when considering mappings between Banach spaces. Theorem 5.3 is a generalization of Theorem 1.2 to pointwise Lipschitz mappings with infinite-dimensional domains. Finally, Theorem 5.4 is a generalization of Theorem 1.1 to mappings between Banach spaces, with the domain being separable. The main ingredients are the use of w^*-differentiability (as introduced by Ambrosio and Kirchheim in [2]) and an idea to use the differentiability of the distance function as a replacement for density considerations; in Bongiorno [5] this idea was used to prove the infinite-dimensional version of Stepanoff's theorem. We conclude the paper by two possible applications of metric derivatives.

2. Metric and w^*-differentiability of mappings with finite-dimensional domains

All Banach spaces are assumed to be real. Let X be a Banach space. By $\langle x^*, x\rangle$ we will denote the usual duality pairing between $x \in X$, and $x^* \in X^*$. By $B(x,r)$ we denote the closed ball with center x and radius $r > 0$. By $\mathcal{L}^n$ we will denote the n-dimensional Lebesgue measure on $\mathbb{R}^n$. We will often use the well-known fact that for every Lebesgue measurable set $A \subset \mathbb{R}^n$, almost all points of A are points of density, i.e., $\lim_{r\to 0+} \frac{\mathcal{L}^n(A\cap B(x,r))}{\mathcal{L}^n(B(x,r))} = 1$, for a.a. $x \in A$. By $\mathcal{H}^n(\cdot)$ we will denote the n-dimensional Hausdorff measure (as defined in [16]). Let X, Y be Banach spaces, $f : A \subset X \mapsto Y$. By $\operatorname{Osc}_B f$ we will denote $\operatorname{diam}(f(B)) = \sup_{x,y\in B} \|f(x) - f(y)\|$.

We will need the following lemma, which is an analogue of [12, Lemma 3.1.5] for "metric differentials".

Lemma 2.1. *Let X be a Banach space, $C \subset B \subset \mathbb{R}^n$, $a \in C$, $f, g : B \mapsto X$, $f(x) = g(x)$ for all $x \in C$, $0 < \eta$, $0 < M < \infty$,*

$$\|f(x) - f(z)\| \le M\|x - z\| \quad \text{for } z \in C \text{ and all } x \in B(z,\eta). \tag{2.1}$$

Let the function g be Lipschitz, and suppose that the set $\mathbb{R}^n \setminus C$ has $\mathcal{L}^n$-density 0 at a. Then

(i) *If g is metrically Fréchet differentiable at a, then f is directionally metrically differentiable, and $MD(f,a)(v) = MD(g,a)(v)$ for all $v \in \mathbb{R}^n$.*

(ii) *If g is metrically differentiable at a, then f is metrically differentiable at a.*

Proof. Without any loss of generality, we can (and do) assume that g is M-Lipschitz (by possibly enlarging M), $a = 0$, $f(a) = g(a) = 0$, and $M = 1$ (by rescaling). We shall prove (i) first. For a contradiction, suppose that

$$\big|\|f(t_m u)\| - MD(g,0)(t_m u)\big| > \varepsilon t_m, \tag{2.2}$$

for some $\varepsilon > 0$, $t_m \to 0+$, and $\|u\| = 1$. For a large enough m we obtain (by the density of C at 0, by (2.1), and by the fact that g is metrically Fréchet differentiable at a) that

$$\mathcal{L}^n(B(0,2t_m) \setminus C) < \left(\tfrac{\varepsilon}{16}\right)^n \mathcal{L}^n(B(0,2t_m)), \tag{2.3}$$

$B(0,2t_m) \subset B(0,\eta)$, and $\big|\|g(w)\| - MD(g,0)(w)\big| \le (\frac{\varepsilon}{8})\,\|w\|$, for all $\|w\| \le 2t_m$. Thus by (2.3) there exists $v \in C \cap B(0,2t_m)$ with $\|v - t_m u\| \le (\frac{\varepsilon}{8})t_m$. Now estimate

$$\begin{aligned}\big|\|f(t_m u)\| - MD(g,0)(t_m u)\big| &\le \|f(t_m u) - f(v)\| + \big|\|f(v)\| - MD(g,0)(v)\big| \\ &\quad + MD(g,0)(v - t_m u) \\ &\le \tfrac{\varepsilon}{8}\, t_m + \tfrac{\varepsilon}{8}\,\|v\| + \|v - t_m u\| \\ &\le \tfrac{\varepsilon}{8}\, t_m + \tfrac{\varepsilon}{4}\, t_m + \tfrac{\varepsilon}{8}\, t_m = \tfrac{\varepsilon}{2}\, t_m,\end{aligned}$$

and that is a contradiction with (2.2). Thus $MD(f,a)(u)$ exists and is equal to $MD(g,a)(u)$ for all $u \in \mathbb{R}^n$.

To prove (ii), by (i) we have that f is directionally metrically differentiable at a. Note that $MD(f,a)(\cdot)$ is a seminorm (as it is equal to $MD(g,a)(\cdot)$). For a contradiction, suppose that f fails the condition (1.3) at a. Then there are $\varepsilon > 0$, $u_m, v_m \in \mathbb{R}^n$ with $\|u_m\| > 0$, $\|v_m\| > 0$, $\lim_m u_m = \lim_m v_m = 0$, and

$$\big|\|f(u_m) - f(v_m)\| - MD(f,0)(u_m - v_m)\big| > \varepsilon(\|u_m\| + \|v_m\|). \tag{2.4}$$

For a large enough m we obtain (by the density of C at 0 and by (1.3) for g at 0)

$$\mathcal{L}^n(B(0,2\|u_m\|) \setminus C) < \left(\tfrac{\varepsilon}{32}\right)^n \mathcal{L}^n(B(0,2\|u_m\|)) \tag{2.5}$$

$$\mathcal{L}^n(B(0,2\|v_m\|) \setminus C) < \left(\tfrac{\varepsilon}{32}\right)^n \mathcal{L}^n(B(0,2\|v_m\|)), \tag{2.6}$$

$B(0, 2\max(\|u_m\|, \|v_m\|)) \subset B(0,\eta)$, and

$$\big|\|g(w) - g(z)\| - MD(g,0)(w - z)\big| \le \tfrac{\varepsilon}{8}\,(\|w\| + \|z\|),$$

for all $\max(\|w\|, \|z\|) \le 2\,\max(\|u_m\|, \|v_m\|)$. Thus by (2.5) and (2.6), there exist $u \in C \cap B(0, 2\|u_m\|)$ and $v \in C \cap B(0, 2\|v_m\|)$ with $\|u - u_m\| \le \frac{\varepsilon}{16}\|u_m\|$ and $\|v - v_m\| \le \frac{\varepsilon}{16}\|v_m\|$. Now estimate

$$\begin{aligned}
&\big|\|f(u_m) - f(v_m)\| - MD(f,0)(u_m - v_m)\big| \\
&\quad \le \|f(u_m) - f(u)\| + \|f(v_m) - f(v)\| + \big|\|f(u) - f(v)\| - MD(g,0)(u - v)\big| \\
&\qquad + MD(g,0)(u - u_m) + MD(g,0)(v - v_m) \\
&\quad \le \tfrac{\varepsilon}{16}\,\|u_m\| + \tfrac{\varepsilon}{16}\,\|v_m\| + \tfrac{\varepsilon}{8}\,\big(\|u\| + \|v\|\big) + \|u - u_m\| + \|v - v_m\| \\
&\quad \le \tfrac{\varepsilon}{8}\,\|u_m\| + \tfrac{\varepsilon}{8}\|v_m\| + \tfrac{\varepsilon}{4}\,\big(\|u_m\| + \|v_m\|\big) \\
&\quad \le \tfrac{\varepsilon}{2}\,\big(\|u_m\| + \|v_m\|\big),
\end{aligned}$$

and that is a contradiction with (2.4). Thus (1.3) holds for f at a, and f is metrically differentiable at a. □

Lemma 2.2. *Let Y be a separable Banach space, $C \subset B \subset \mathbb{R}^n$, $a \in C$, $f, g : B \mapsto Y^*$, $f(x) = g(x)$ for all $x \in C$, $0 < \eta$, $0 < M < \infty$,*

$$\|f(x) - f(z)\| \le M\|x - z\| \quad \textit{for } z \in C \textit{ and all } x \in B(z,\eta),$$

let the function g be Lipschitz, g be metrically Fréchet differentiable at a, g be w^*-Fréchet differentiable at a,

$$MD(g,a)(w) = \|wd(g,x)(w)\| \quad \textit{for all } w \in \mathbb{R}^n, \tag{2.7}$$

and suppose that the set $\mathbb{R}^n \setminus C$ has $\mathcal{L}^n$-density 0 at a. Then f is metrically Fréchet differentiable at a, f is w^*-Fréchet differentiable at a, and

$$MD(f,a)(w) = \|wd(f,a)(w)\| \quad \textit{for all } w \in \mathbb{R}^n. \tag{2.8}$$

Proof. Without any loss of generality, we can (and do) assume that g is M-Lipschitz (by possibly enlarging M), $a = 0$, $f(a) = g(a) = 0$, and $M = 1$ (by rescaling). Note that Lemma 2.1 (i) implies that f is directionally metrically differentiable with $MD(f,a)(v) = MD(g,a)(v)$ for all $v \in \mathbb{R}^n$.

We see that $MD(f,a)(\cdot)$ is a seminorm (as it is equal to $MD(g,a)(\cdot)$). For a contradiction, suppose that condition (1.2) fails at a. Then there are $\varepsilon > 0$, $u_m \in \mathbb{R}^n$ with $\|u_m\| > 0$, $\lim_m u_m = 0$, and

$$\big|\|f(u_m)\| - MD(f,0)(u_m)\big| > \varepsilon\|u_m\|. \tag{2.9}$$

For large enough m we obtain (by the density of C at 0 and by (1.2) for g at 0)

$$\mathcal{L}^n(B(0,2\|u_m\|) \setminus C) < \left(\tfrac{\varepsilon}{32}\right)^n \mathcal{L}^n(B(0,2\|u_m\|)), \tag{2.10}$$

$B(0,2\|u_m\|) \subset B(0,\eta)$, and $\big|\|g(w)\| - MD(g,0)(w)\big| \le \frac{\varepsilon}{8}\|w\|$, for all $\|w\| \le 2\,\|u_m\|$. Thus by (2.10) there exist $u \in C \cap B(0,2\|u_m\|)$ with $\|u - u_m\| \le \frac{\varepsilon}{16}\|u_m\|$. Now estimate

$$\begin{aligned}
\big|\|f(u_m)\| - MD(f,0)(u_m)\big| &\le \|f(u_m) - f(u)\| \\
&\quad + \big|\|f(u)\| - MD(g,0)(u)\big| + MD(f,0)(u_m - u) \\
&\le \tfrac{\varepsilon}{16}\,\|u_m\| + \tfrac{\varepsilon}{8}\,\|u\| + \|u - u_m\| \\
&\le \tfrac{\varepsilon}{8}\,\|u_m\| + \tfrac{\varepsilon}{4}\,\|u_m\| \\
&\le \tfrac{\varepsilon}{2}\,\|u_m\|,
\end{aligned}$$

and that is a contradiction with (2.9). Thus (1.2) holds for f at a, and f is Fréchet metrically differentiable at a.

To finish the proof, it is enough to establish that $wd(g,0)(\cdot)$ is the w^*-Fréchet differential of f at 0. Then (2.8) follows from (2.7), and from the fact that we have the equality $MD(f,0)(\cdot) = MD(g,0)(\cdot)$. For a contradiction, suppose that f fails the condition (1.4) at $a = 0$ (with $wd(f,0)(\cdot) := wd(g,0)(\cdot)$). Then there are $y \in Y$, $\varepsilon > 0$, $u_m \in \mathbb{R}^n$ with $\|u_m\| > 0$, $\lim_m u_m = 0$, and

$$\big|\langle f(u_m) - wd(g,0)(u_m), y\rangle\big| > \varepsilon\|u_m\|. \tag{2.11}$$

For large enough m we obtain (by the density of C at 0 and by (1.4) for g at 0)

$$\mathcal{L}^n(B(0,2\|u_m\|) \setminus C) < \left(\tfrac{\varepsilon}{16}\|y\|\right)^n \mathcal{L}^n(B(0,2\|u_m\|)), \tag{2.12}$$

$B(0,2\|u_m\|) \subset B(0,\eta)$, and $\big|\langle g(w) - wd(g,0)(w), y\rangle\big| \le \frac{\varepsilon}{8}\,\|w\|$, for all $\|w\| \le 2\,\|u_m\|$. Thus by (2.12) there exists $u \in C \cap B(0,2\|u_m\|)$ with $\|u - u_m\| \le \left(\frac{\varepsilon}{8}\,\|y\|\right)\|u_m\|$. Now estimate

$$\begin{aligned}
\big|\langle f(u_m) - wd(g,0)(u_m), y\rangle\big| &\le \|y\|\,\|f(u_m) - f(u)\| + \big|\langle f(u) - wd(g,0)(u), y\rangle\big| \\
&\quad + \|y\|\,\|wd(g,0)(u - u_m)\| \\
&\le \tfrac{\varepsilon}{8}\,\|u_m\| + \tfrac{\varepsilon}{8}\,\|u\| + \|y\|\,\|u - u_m\| \\
&\le \tfrac{\varepsilon}{4}\,\|u_m\| + \tfrac{\varepsilon}{4}\,\|u_m\| \\
&\le \tfrac{\varepsilon}{2}\,\|u_m\|,
\end{aligned}$$

and that is a contradiction with (2.11). Thus (1.4) holds for f at a, and f is w^*-Fréchet differentiable at a. □

Definition 2.3. Let X, Z be Banach spaces, let $\Omega \subset X$ be open, and let $f : \Omega \mapsto Z$. By $S(f)$ denote the set of points, where f is pointwise Lipschitz.

Lemma 2.4. *Let X, Z be Banach spaces with X separable, let $\Omega \subset X$ be open, and let $f : \Omega \mapsto Z$. Then there exist Borel $A_{m,k} \subset \Omega$ ($m, k \in \mathbb{N}$) such that:*

- $A_{m,k} \cap A_{m',k'} = \emptyset$ *for* $(m, k) \neq (m', k')$;
- $\bigcup_{m,k} A_{m,k} = S(f)$;
- $f|_{A_{m,k}}$ *is* m*-Lipschitz;*
- $\|f(x)-f(w)\| \leq m\,\|x-w\|$ *for all* $x \in A_{m,k}$, *and* $w \in \Omega$ *with* $\|x-w\| < \frac{1}{m}$.

Proof. Write

$$A_m = \left\{x \in \Omega : \|f(x) - f(z)\| \leq m\|x - z\| \text{ when } z \in \Omega \text{ with } \|x - z\| < \tfrac{1}{m}\right\}.$$

Then A_m is easily seen to be closed in Ω (see e.g. [5, Lemma 1]), and $S(f) = \bigcup_m A_m$. For each $m \in \mathbb{N}$ let B_k^m be a sequence of closed balls, such that $\operatorname{diam}(B_k^m) = \frac{1}{m+1}$, and $\Omega \subset \bigcup_k B_k^m$. Define $A'_1 = A_1$, and for $m > 1$ let $A'_m := A_m \setminus \left(\bigcup_{j<m} A_j\right)$. Further define $A_{m,k} := A'_m \cap \left(B_k^m \setminus \bigcup_{j<k} B_j^m\right)$. Then $f|_{A_{m,k}}$ is m-Lipschitz for all $m, k \in \mathbb{N}$, $A_{m,k}$ are Borel, pairwise disjoint (by definition), and obviously $S(f) = \bigcup_{m,k} A_{m,k}$. The last condition follows from the definition of $A_{m,k}$ (resp. A_m). □

We have the following version of Theorem 1.1 for pointwise Lipschitz mappings. We follow a similar argument as in the proof of Stepanoff's theorem; see [12, Theorem 3.1.9].

Theorem 2.5. *Let X be a Banach space, and let $f : \mathbb{R}^n \mapsto (X, \|\cdot\|)$ be a function. Then, for almost every $x \in S(f)$, we have that f is metrically differentiable at x.*

Proof. Let $A_{m,k}$ be given by Lemma 2.4. Then $f|_{A_{m,k}}$ is m-Lipschitz, and thus it can be extended (see [15, Theorem 2]) to $C{\cdot}m$-Lipschitz function $f_{m,k}$ on $\mathbb{R}^n$ with $C > 0$ depending only on n. By Theorem 1.1, for almost all $x \in A_{m,k}$, we have that $f_{m,k}$ is metrically differentiable at x. By Lemma 2.1, because almost all points of $A_{m,k}$ are points of density of that set, we obtain that the function f is metrically differentiable at almost all points of $A_{m,k}$, and thus at almost all points of $S(f)$. □

We have the following version of Theorem 1.2. Again, we follow a similar argument as in the proof of Stepanoff's theorem; see [12, Theorem 3.1.9].

Theorem 2.6. *Let Y be a separable Banach space, and let $f : \mathbb{R}^n \mapsto (Y^*, \|\cdot\|)$ be a function. Then, for almost every $x \in S(f)$, we have that f is metrically Fréchet differentiable at x, w^*-Fréchet differentiable, and $MD(f,x)(v) = \|wd(f,x)(v)\|$ for all $v \in \mathbb{R}^n$.*

Proof. Let $A_{m,k}$ be from Lemma 2.4. Then $f|_{A_{m,k}}$ is m-Lipschitz, and thus it can be extended (see [15, Theorem 2]) to $C\cdot m$-Lipschitz function $f_{m,k}$ on $\mathbb{R}^n$ with $C > 0$ depending only on n. By $B_{m,k}$ denote the set of all $x \in A_{m,k}$, such that $f_{m,k}$ is metrically Fréchet differentiable at x, w^*-Fréchet differentiable, and $MD(f_{m,k},x)(v) = \|wd(f_{m,k},x)(v)\|$ for all $v \in \mathbb{R}^n$. By Theorem 1.2 we have that $\mathcal{L}^n(A_{m,k} \setminus B_{m,k}) = 0$. Let $x \in B_{m,k}$ be a point of density of $B_{m,k}$. Then Lemma 2.2 implies that f is metrically Fréchet differentiable at x, w^*-Fréchet differentiable, and $MD(f,x)(v) = \|wd(f,x)(v)\|$ for all $v \in \mathbb{R}^n$. Thus the conclusion holds for almost all $x \in S(f)$. □

Theorems 2.5 and 2.6 have the following corollary.

Corollary 2.7. *Let Y be a separable Banach space, and let $f : \mathbb{R}^n \mapsto Y^*$ be an arbitrary function. Then, for almost every $x \in S(f)$, we have that f is metrically differentiable at x, w^*-Fréchet differentiable at x, and $MD(f,x)(v) = \|wd(f,x)(v)\|$ for all $v \in \mathbb{R}^n$.*

Let X be a Banach space, $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \mapsto X$. We say that f *satisfies the condition* (RR) (the *spherical condition of Rado and Reichelderfer*) with weight $\theta \in L^1_{\text{loc}}(\Omega)$ provided

$$\big(\operatorname{diam}(u(B))\big)^n \le \int_B \theta(x)\,dx, \tag{2.13}$$

for each ball $B \subset \Omega$. Condition (RR) was used by Rado and Reichelderfer [22] as a sufficient condition for almost everywhere differentiability and area formula. Note that if f satisfies (RR)-condition, it is necessarily continuous. If $MD(f,x)(\cdot)$ is a seminorm, we define $\|MD(f,x)\| := \sup_{\|u\|=1} MD(f,x)(u)$. Theorem 2.5 has the following interesting corollary.

Corollary 2.8. *Let X be a Banach space, $\Omega \subset \mathbb{R}^n$ open, and $f : \Omega \mapsto X$. Let f satisfy the (RR)-condition with $L^1(\Omega)$-weight θ. Then:*

(i) *For almost all $x \in \Omega$, we have that the function f is metrically differentiable at x.*

(ii) *If $X = Y^*$, where Y is separable, then for almost all $x \in \Omega$ we have that f is metrically differentiable at x, w^*-Fréchet differentiable at x, and $MD(f,x)(v) = \|wd(f,x)(v)\|$ for all $v \in \mathbb{R}^n$.*

Further $\|MD(f,\cdot)\| \in L^n(\Omega)$.

Proof. If x is a Lebesgue point of θ, then

$$\left(\limsup_{y\to x}\frac{\|f(x)-f(y)\|}{\|x-y\|}\right)^n \le C\,\theta(x)<\infty.$$

To prove (i), Theorem 2.5 (extend f to $\mathbb{R}^n\setminus\Omega$ by 0) implies that f is metrically differentiable at almost all $x\in\Omega$. Part (ii) follows from Corollary 2.7.

At the Lebesgue points $x\in\Omega$ of θ, where $MD(f,x)(\cdot)$ is a seminorm, we have that

$$\|MD(f,x)\|^n \le \left(\limsup_{y\to x}\frac{\|f(x)-f(y)\|}{\|x-y\|}\right)^n \le C\,\theta(x).$$

Thus $\|MD(f,\cdot)\|\in L^n(\Omega)$. □

We say that a Banach space X has the RNP (or the *Radon–Nikodým property*), provided each absolutely continuous function $f:[0,1]\mapsto X$ is almost everywhere differentiable. For other equivalent definitions and properties of such spaces, see [4].

As an application of [5, Theorem 1], we obtain the following version of Stepanoff's theorem.

Theorem 2.9. *Let X be a Banach space with RNP, $\Omega\subset\mathbb{R}^n$ open, and $f:\Omega\mapsto X$. Then for almost all $x\in S(f)$, the function f is Fréchet differentiable at x.*

Proof. Let $A_{m,k}$ be from Lemma 2.4. Then $f|_{A_{m,k}}$ is m-Lipschitz, and thus it can be extended (see [15, Theorem 2]) to $C\cdot m$-Lipschitz function $f_{m,k}$ on $\mathbb{R}^n$, with $C>0$ depending only on n. By [5, Theorem 1], for almost all $x\in\mathbb{R}^n$ we have that $f_{m,k}$ is Gâteaux differentiable at x. The remark after the definition of Gâteaux differentiability implies that $f_{m,k}$ is in fact Fréchet differentiable at almost all $x\in\mathbb{R}^n$. By a version of [12, Lemma 3.1.5] for f with values in a Banach space (which holds with the same proof), because almost all points of $A_{m,k}$ are points of density of that set, we obtain that f is Fréchet differentiable at almost all points of $A_{m,k}$, and thus at almost all points of $S(f)$. □

We have the following corollary.

Corollary 2.10. *Let X be a Banach space with RNP, $\Omega\subset\mathbb{R}^n$ be an open set, and $f:\Omega\mapsto X$. Suppose that f satisfies the (RR)-condition with $L^1(\Omega)$-weight θ. Then for almost all $x\in\Omega$, the function f is Fréchet differentiable at x. Further $\|f'(x)\|\in L^n(\Omega)$.*

Proof. If x is a Lebesgue point of θ, then

$$\left(\limsup_{y\to x}\frac{\|f(x)-f(y)\|}{\|x-y\|}\right)^n \le C\,\theta(x)<\infty.$$

Thus $\mathcal{L}^n(\Omega \setminus S(f)) = 0$. By Theorem 2.9, for almost all $x \in S(f) \cap \Omega$, we have that f is Fréchet differentiable at x.

At the Lebesgue points $x \in \Omega$ of θ, where $f'(x)$ exists, we have that

$$\|f'(x)\|^n \le \left(\limsup_{y \to x} \frac{\|f(x) - f(y)\|}{\|x - y\|}\right)^n \le C\,\theta(x).$$

Thus $\|f'(\cdot)\| \in L^n(\Omega)$. □

Another corollary to the method of proof of Theorem 2.5 is the area formula for pointwise Lipschitz mappings, which generalizes Theorem 7 and Corollary 8 from [16]. Let s be a seminorm on $\mathbb{R}^n$. Following [16], we define the *Jacobian of s* by

$$\mathcal{J}(s) = \frac{\alpha(n)\,n}{\int_{\mathbb{S}^{n-1}} (s(x))^{-n}\, d\mathcal{H}^{n-1}(x)},$$

where $\alpha_n = \frac{\Gamma(1/2)^n}{\Gamma(n/2+1)} = \mathcal{L}^n(B(0,1))$. We have the following theorem (the case when $\Omega = \mathbb{R}^n$, and f is Lipschitz is proved in [16, Theorem 7]).

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^n$ be open, let $f : \Omega \mapsto (X, \cdot)$ be arbitrary, and let $A \subset S(f)$ be Lebesgue measurable. Then*

$$\int_A \mathcal{J}(MD(f,x))\, d\mathcal{L}^n(x) = \int_X N(f|_A, x)\, d\mathcal{H}^n_{\|\cdot\|}(x),$$

where $N(f|_A, x) = \operatorname{card}(A \cap f^{-1}(x))$.

Proof. Let $A_{m,k}$ be from Lemma 2.4. Let $F_{m,k}$ be the $C{\cdot}m$-Lipschitz extensions of $f|_{A_{m,k}}$ to $\mathbb{R}^n$ (see [15, Theorem 2]). For each $m, k \in \mathbb{N}$ we obtain that

$$\begin{aligned}\int_{A \cap A_{m,k}} \mathcal{J}(MD(f,x))\, \mathcal{L}^n(x) &= \int_{A \cap A_{m,k}} \mathcal{J}(MD(F_{m,k},x))\, \mathcal{L}^n(x) \\ &= \int_X N(f|_{A \cap A_{m,k}}, x)\, d\mathcal{H}^n_{\|\cdot\|}(x),\end{aligned}$$

where the first equality follows from the fact that $MD(f,x) = MD(F_{m,k},x)$ at all points x, which are points of density of $A \cap A_{m,k}$ (see Lemma 2.1), and the second equality follows from [16, Theorem 7]. By adding these for all $m, k \in \mathbb{N}$, we get the conclusion of the theorem. □

As a consequence (by standard approximation procedures), we obtain the following corollary.

Corollary 2.12. *Let $\Omega \subset \mathbb{R}^n$ be open, and $f : \Omega \mapsto (X, \|\cdot\|)$ be pointwise Lipschitz.*

(i) *If* $g : \Omega \mapsto \overline{\mathbb{R}}$ *is Lebesgue integrable, then*

$$\int_\Omega g(x)\, \mathcal{J}(MD(f,x))\, d\mathcal{L}^n(x) = \int_X \Bigg(\sum_{x \in f^{-1}(y) \cap \Omega} g(x) \Bigg)\, d\mathcal{H}^n_{\|\cdot\|}(y).$$

(ii) *If* $g : X \mapsto \overline{\mathbb{R}}$ *is* $\mathcal{H}^n_{\|\cdot\|}$*-measurable, and* $A \subset \Omega$ *is* $\mathcal{L}^n$*-measurable, then*

$$\int_\Omega g(f(x))\, \mathcal{J}(MD(f,x))\, d\mathcal{L}^n(x) = \int_X g(x)\, N(f|_A, y)\, d\mathcal{H}^n_{\|\cdot\|}(y).$$

3. Absolutely continuous functions

Let X be a real Banach space, let Ω be an open subset of $\mathbb{R}^n$. Let $0 < \lambda \le 1$. We say that a function $f : \Omega \mapsto X$ is (n,λ)*-absolutely continuous* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_i \mathcal{L}^n(B(x_i, r_i)) < \delta \implies \sum_i \big(\operatorname{Osc}_{B(x_i, \lambda r_i)} f \big)^n < \varepsilon,$$

for any disjoint sequence of balls $(B(x_i, r_i))_i$ in Ω. We say that f is in the class $\mathrm{BV}^n_\lambda(\Omega)$ provided $V^n_\lambda(f,\Omega) < \infty$, where the (n,λ)-variation $V^n_\lambda(f,A)$ of f on A is defined as

$$\sup \Big\{ \sum_i \big(\operatorname{Osc}_{B(x_i, \lambda r_i)} f \big)^n : \{B(x_i, r_i)\} \text{ is a disjoint family of balls in } A \Big\}.$$

We define the space $\mathrm{AC}^n_\lambda(\Omega)$ as the family of all (n,λ)-absolutely continuous functions in the space $\mathrm{BV}^n_\lambda(\Omega)$. Note that obviously

$$f \text{ is } (n,\lambda)\text{-absolutely continuous} \implies f \in \mathrm{BV}^n_{\lambda,\mathrm{loc}}(\Omega). \tag{3.1}$$

Hencl [14, Theorem 3.1] proved the following theorem for $\mathbb{R}^m$-valued functions. It holds with the same proof for X-valued (n,λ)-absolutely continuous functions (resp. BV^n_λ functions).

Theorem 3.1. *Let* $0 < \lambda_1 < \lambda_2 < 1$ *and* $f : \Omega \mapsto X$*. Then*

(i) f *is* (n,λ_1)*-absolutely continuous if and only if* f *is* (n,λ_2)*-absolutely continuous,*

(ii) $\mathrm{BV}^n_{\lambda_1}(\Omega) = \mathrm{BV}^n_{\lambda_2}(\Omega)$,

(iii) $\mathrm{AC}^n_{\lambda_1}(\Omega) = \mathrm{AC}^n_{\lambda_2}(\Omega)$.

The $\mathbb{R}^m$-valued (n,λ)-absolutely continuous functions are Fréchet differentiable almost everywhere (see Malý [18] for the case $\lambda = 1$, and the remark in [14] which asserts that the same proof works also for the case $0 < \lambda < 1$; in fact it is enough to replace r by λr and r_i by λr_i in Malý's proof).

Theorem 2 in [16] and Lemma 2.1 have the following corollary:

Theorem 3.2. *Let* $f : \Omega \mapsto X$ *be such that* $f \in \mathrm{BV}^n_{\lambda,loc}(\Omega)$. *Then:*

(i) f *is metrically differentiable at almost all* $x \in \Omega$.

(ii) *If* $X = Y^*$, *where* Y *is separable, then for almost all* $x \in \Omega$ *we have that* f *is metrically differentiable at* x, w^**-Fréchet differentiable at* x, *and* $MD(f,x)(v) = \|wd(f,x)(v)\|$ *for all* $v \in \mathbb{R}^n$.

Proof. The proof of Theorem 3.3 from [18] shows that $\operatorname{lip}(f,x) < \infty$ for almost all $x \in \Omega$. Thus Theorem 2.5 implies part (i). Part (ii) follows from Corollary 2.7. □

Remark 3.3. The inclusion (3.1) implies that Theorem 3.2 holds also for (n,λ)-absolutely continuous functions.

Theorem 2.9 has the following consequence:

Corollary 3.4. *Let* X *be a Banach space satisfying the RNP,* $f : \Omega \mapsto X$, *and* $0 < \lambda \leq 1$. *Then:*

(i) *if* $f \in BV^n_\lambda(\Omega)$, *then* $f'(x)$ *exists for almost every* $x \in \Omega$;

(ii) *if* f *is* (n,λ)*-absolutely continuous, then* $f'(x)$ *exists for almost each* $x \in \Omega$.

Proof. Because of (3.1), part (ii) easily follows from part (i). To prove (i), note that the proof of Theorem 3.3 from [18] shows that $\operatorname{lip}(f,x) < \infty$ for almost all $x \in \Omega$. Thus $\mathcal{L}^n(\Omega \setminus S(f)) = 0$. By Theorem 2.9, for almost all $x \in S(f) \cap \Omega$, we have that f is Fréchet differentiable at x. □

Kirchheim [16] proved an area formula for Lipschitz mappings from $\mathbb{R}^n$ to an arbitrary Banach space. Using the ideas from the proof of Theorem 2.5 we can prove an area formula for (n,λ)-absolutely continuous mappings with values in any Banach space. We have the following result:

Theorem 3.5. *Let* $f : \Omega \mapsto X$ *be* (n,λ)*-absolutely continuous so that* $\dim(X) \geq n$, *and let* $A \subset \Omega$ *be Lebesgue measurable. Then*

$$\int_A \mathcal{J}(MD(f,x))\, d\mathcal{L}^n(x) = \int_X N(f|_A, x)\, d\mathcal{H}^n_{\|\cdot\|}(x). \tag{3.2}$$

Proof. The proof of Theorem 3.3 from [18] shows that $\operatorname{lip}(f,x) < \infty$ for all $x \in \Omega \setminus N$ with $\mathcal{L}^n(N) = 0$. Apply Lemma 2.4 to obtain $A_{m,k}$. By [15, Theorem 2] extend $f|_{A_{m,k}}$ to $C \cdot m$-Lipschitz mappings $F_{m,k}$ on $\mathbb{R}^n$, and by Theorem 1.1 we have that $F_{m,k}$ are metrically differentiable almost everywhere in $\mathbb{R}^n$. Let $B_{m,k}$ be the set of density points of $A_{m,k}$ where $F_{m,k}$ is metrically differentiable. Then for each $x \in B_{m,k}$, we have that

$$F_{m,k}(x) = f(x) \quad \text{and} \quad MD(F_{m,k},x) = MD(f,x). \tag{3.3}$$

Thus by [16, Theorem 7] (applied to $F_{m,k}$ and $A \cap B_{m,k}$) and by (3.3) we have that
$$\int_{A\cap B_{m,k}} \mathcal{J}(MD(f,x))\, d\mathcal{L}^n(x) = \int_X N(f|_{A\cap B_{m,k}}, x)\, d\mathcal{H}^n_{\|\cdot\|}(x).$$
By adding these up for all $m, k \in \mathbb{N}$, it is easy to see that it is enough to prove (3.2) for $A \subset \Omega$ with $\mathcal{L}^n(A) = 0$.

We shall modify the proof of [14, Theorem 3.4]. By Theorem 3.1, we can assume that $\lambda = \frac{1}{5}$. Let
$$E_1 = \left\{ x \in A : \liminf_{r\to 0} \frac{\operatorname{Osc}_{B(x,r)} f}{\operatorname{Osc}_{B(x,r/5)} f} \le 10 \right\},$$
and
$$E_2 = \left\{ x \in A : \liminf_{r\to 0} \frac{\operatorname{Osc}_{B(x,r)} f}{\operatorname{Osc}_{B(x,r/5)} f} > 10 \right\}.$$
Choose $\varepsilon > 0$ and find $0 < \delta < 1$ from the definition of (n, λ)-absolute continuity of f. Let $G \subset \Omega$ be an open set containing A with $\mathcal{L}^n(G) < \delta$. For each $x \in E_1$ there is $r(x) > 0$ such that
$$B(x, r(x)) \subset G \quad \text{and} \quad \operatorname{Osc}_{B(x,r(x))} f < 11 \operatorname{Osc}_{B\left(x, \frac{r(x)}{5}\right)} f < \varepsilon. \tag{3.4}$$
For each $x \in E_2$ there is $w(x) > 0$ such that for all $r \in (0, w(x))$ we have $\frac{\operatorname{Osc}_{B(x,r)} f}{r} > 2\frac{\operatorname{Osc}_{B(x,r/5)} f}{\frac{r}{5}}$. Thus $\lim_{r\to 0} \frac{\operatorname{Osc}_{B(x,r)} f}{r} = 0$. Hence for each $x \in E_2$ we can find $r(x) > 0$ such that
$$B(x, r(x)) \subset G \quad \text{and} \quad \operatorname{Osc}_{B(x,r(x))} f < \varepsilon^{\frac{1}{n}}\, r(x) < \varepsilon. \tag{3.5}$$
Because $E = E_1 \cup E_2$, by Vitali's covering theorem, we can find a disjoint system
$$\left\{ B\left(x_i, \frac{r_i}{5}\right) : i \in \mathbb{N} \right\} \subset \left\{ B\left(x, \frac{r(x)}{5}\right) : x \in E \right\},$$
such that $A \subset \bigcup_i B(x_i, r_i)$. Let $\mathcal{I}_j = \{ i \in \mathbb{N} : x_i \in E_j \}$ for $j = 1, 2$. Now by (3.4) and (3.5) we get
$$\mathcal{H}^n_\varepsilon(f(A)) \le C \sum_i \big(\operatorname{diam}(f(B(x_i, r_i)))\big)^n = C \sum_i \big(\operatorname{Osc}_{B(x_i,r_i)} f\big)^n$$
and hence
$$\begin{aligned}
\mathcal{H}^n_\varepsilon(f(A)) &\le C\bigg(11^n \sum_{i\in\mathcal{I}_1} \big(\operatorname{Osc}_{B(x_i,r_i/5)} f\big)^n + \varepsilon \sum_{i\in\mathcal{I}_2} (r_i)^n \bigg) \\
&\le C\left(11^n\, \varepsilon + 5^n \varepsilon \sum_{i\in\mathcal{I}_2} \left(\frac{r_i}{5}\right)^n \right) \\
&\le C\, \varepsilon\, (11^n + 5^n \cdot \delta) \le C\, \varepsilon.
\end{aligned}$$
Now let $\varepsilon \to 0$ to obtain $\mathcal{H}^n(f(A)) = 0$. ☐

4. Auxiliary results

Let X be a separable Banach space, $A \subset X$, and let $0 \neq u \in X$. We say that $A \in \mathcal{A}(u)$ provided A is Borel and $\mathcal{L}^1(\{\lambda \in \mathbb{R} : x + \lambda u \in A\}) = 0$, for all $x \in X$. For a sequence $\{u_n\} \subset X$ we define

$$\mathcal{A}(\{u_n\}) = \left\{ E \in X : E = \bigcup_n E_n \text{ with } E_n \in \mathcal{A}(u_n) \right\}.$$

Finally, we say that A *is Aronszajn null* provided A is Borel and for each complete sequence $\{u_n\} \subset X$ we have $A \in \mathcal{A}(\{u_n\})$ (a sequence $\{u_n\}$ is *complete* provided $X = \overline{\text{span}}(\{u_n\})$). For more information about Aronszajn null sets, see [4].

We will need the following lemma:

Lemma 4.1. *Let X, Y be separable, and let $f : X \mapsto Y^*$ be Lipschitz. If V is a countable dense subset of X closed under linear combinations with rational coefficients, then:*

(i) *If $wd(f,x)(v)$ exists for all $v \in V$, then $wd(f,x)(w)$ exists for all $w \in X$. If $wd(f,x)(\cdot)$ is linear on V, then $wd(f,x)(\cdot) : X \mapsto Y^*$ is a bounded linear operator.*

(ii) *If $MD(f,x)(v)$ exists for all $v \in V$, then $MD(f,x)(w)$ exists for all $w \in X$. If $MD(f,x)(\cdot)$ is a seminorm on V, then $MD(f,x)(\cdot)$ is a seminorm on X.*

(iii) *If $MD(f,x)(v) = \|wd(f,x)(v)\|$ for all $v \in V$, then we have the equality $MD(f,x)(w) = \|wd(f,x)(w)\|$ for all $w \in X$.*

Proof. The proof of the lemma is standard so we omit it. Let us only remark that the main ingredience used is the fact that the difference quotients of a Lipschitz function are uniformly Lipschitz as a function of the direction. □

The proof of the following lemma is standard, and so we omit it.

Lemma 4.2. *Let X, Y be separable Banach spaces, $f : X \mapsto Y^*$ be Lipschitz, and $0 \neq v \in X$. Then the following sets are Borel:*

- $A = \{x \in X : MD(f,x)(v) \textit{ exists}\}$
- $B = \{x \in X : wd(f,x)(v) \textit{ exists}\}$
- $\{x \in A \cap B : MD(f,x)(v) \neq \|wd(f,x)(v)\|\}$
- $\{x \in X : wd(f,x)(w) \textit{ exists for all } w \in X \textit{ but is not linear in } w\}$
- $\{x \in X : MD(f,x)(w) \textit{ exists for all } w \in X; \textit{it is not a seminorm in } w\}$.

We will need the following theorem, which is a consequence of [2, Theorem 3.5].

Theorem 4.3. *Let X, Y be separable Banach space, let $f : X \mapsto Y^*$ be Lipschitz. Then the complement of the set of points $x \in X$, where f is w^*-Gâteaux differentiable at x, f is metrically Gâteaux differentiable at x, and*

$$MD(f,x)(w) = \|wd(f,x)(w)\| \tag{4.1}$$

for all $w \in X$, is Aronszajn null.

Proof. Let E be the set of $x \in X$ such that f is w^*-Gâteaux differentiable at x, f is directionally metrically differentiable at x, $MD(f,x)(\cdot)$ is a seminorm, and (4.1) holds for all $w \in X$. By Lemmata 4.1 and 4.2, it follows that the set E is Borel.

Let $\{v_n\}$ be any complete sequence in X, and put $V_n = \operatorname{span}\{v_k : k \leq n\}$. Let D_n be the set of those $x \in X$ such that $g_x^n(\cdot) = f(\cdot - x)|_{V_n}$ is metrically Fréchet differentiable at 0, g_x^n is w^*-Fréchet differentiable at 0, and $MD(g_x^n, 0)(w) = \|wd(g_x^n, 0)(w)\|$ for all $w \in V_n$. By [2, Theorem 3.5] we have that $\mathcal{L}^n(((X \setminus D_n) + y) \cap V_n) = 0$ for all $y \in X$. Thus by [4, Proposition 6.29] we have that $X \setminus D_n$ belongs to $\mathcal{A}(\{v_k : k \leq n\})$. By Lemma 4.1 the function f satisfies the conclusion of the theorem for all $x \in \bigcap_n D_n$. □

Lemma 4.4. *Let X, Z be Banach spaces, with X separable, let $f : X \mapsto Z$ be a mapping, $\tilde{f} : X \mapsto \ell_\infty$ be a Lipschitz mapping, $G \subset X$ Borel, $D \subset G$ be such that* $\operatorname{dist}(\cdot, G)$ *is Gâteaux differentiable on D, there exists $L, \delta > 0$ such that for all $z \in G$ we have that*

$$\|f(z) - f(w)\| \leq L\|z - w\| \quad \text{whenever } \|z - w\| < \delta. \tag{4.2}$$

Let $\Psi : f(G) \mapsto \ell_\infty$ be an isometric embedding such that $\Psi \circ f = \tilde{f}$ on G. Then:

- (i) *If $x \in D$ is such that $\tilde{f}$ is metrically Gâteaux differentiable at x, then f metrically Gâteaux differentiable at x.*
- (ii) *If $Z = Y^*$ with Y separable, $\Psi^*|_{\ell_1}$ is a quotient mapping from ℓ_1 onto Y, $x \in D$ is such that $\tilde{f}$ is metrically Gâteaux differentiable at x, w^*-Gâteaux differentiable at x, and*

$$MD(\tilde{f}, x)(w) = \|wd(\tilde{f}, x)(w)\| \quad \text{for all } w \in X, \tag{4.3}$$

then f is metrically Gâteaux differentiable at x, f is w^*-Gâteaux differentiable at x, and

$$MD(f, x)(w) = \|wd(f, x)(w)\| \quad \text{for all } w \in X. \tag{4.4}$$

Proof. We will only prove the part (ii), as the proof of (i) is similar. Without any loss of generality, we can assume that $\tilde{f}$ is L-Lipschitz, and $L = 1$. First, we will prove that f is directionally metrically differentiable at x (such that

$MD(f,x)(w) = MD(\tilde{f},x)(w))$, and that $wd(f,x)(w) = wd(\tilde{f},x)(w)$. Let $w \in X$, and fix $y \in Y$. As $\Psi^*|_{\ell_1}$ is a quotient mapping from ℓ_1 onto Y, and thus there exists $\mu = (\mu_i) \in \ell_1$ such that $\Psi^*(\mu) = y$ and $\|\mu\| \le C\|y\|$. Given $\varepsilon > 0$, by the existence of $MD(\tilde{f},x)(w)$, $wd(\tilde{f},x)(w)$, and by the differentiability of the distance function $\operatorname{dist}(\cdot, G)$ at the point x, there exists $\tau_\varepsilon > 0$ such that

$$\left| |t|^{-1} \|\tilde{f}(x+tw) - \tilde{f}(x)\| - MD(\tilde{f},x)(w) \right| < \frac{\varepsilon}{3} \tag{4.5}$$

$$\left| \left\langle t^{-1}(\tilde{f}(x+tw) - \tilde{f}(x)) - wd(\tilde{f},x)(w), \mu \right\rangle \right| < \frac{\varepsilon}{10}, \tag{4.6}$$

and $\operatorname{dist}(x+tw, G) < \frac{\varepsilon}{5\max(\|\mu\|,1)}|t|$, for each $0 < |t| < \tau_\varepsilon$.

Let $0 < |t| < \min\left(\tau_\varepsilon, 5\max(\|\mu\|,1)\frac{\delta}{2\varepsilon}\right)$ and let $z_t \in G$ be such that $\|x + tw - z_t\| < \frac{\varepsilon}{5\max(\|\mu\|,1)}|t|$. Now by (4.5) it follows that

$$\begin{aligned} \big| \|f(x+tw) - f(x)\| &- MD(\tilde{f},x)(tw) \big| \\ &\le \big| \|\tilde{f}(x+tw) - \tilde{f}(x)\| - MD(\tilde{f},x)(tw) \big| \\ &\quad + \|\tilde{f}(x+tw) - \tilde{f}(z_t)\| + \|f(x+tw) - f(z_t)\| \\ &\le \frac{\varepsilon|t|}{3} + \frac{\varepsilon|t|}{3} + \frac{\varepsilon|t|}{3} = \varepsilon|t|, \end{aligned}$$

(we used that $\Psi(f(u)) = \tilde{f}(u)$ for $u \in G$). This proves that $MD(f,x)(w)$ exists and f is directionally metrically differentiable at x. Because of (4.3), we easily see that $MD(f,x)(\cdot)$ is a continuous seminorm. Similarly, for an $0 < |s| < \min\left(\tau_\varepsilon, 5\max(\|\mu\|,1)\frac{\delta}{2\varepsilon}\right)$ there exists $z_s \in G$ satisfying $\|x + sw - z_s\| < \frac{\varepsilon}{5\max(\|\mu\|,1)}|s|$. Thus by (4.6) we can estimate

$$\begin{aligned} &\left| \left\langle \frac{f(x+tw) - f(x)}{t} - \frac{f(x+sw) - f(x)}{s}, y \right\rangle \right| \\ &\quad = \left| \left\langle \frac{f(x+tw) - f(x)}{t} - \frac{f(x+sw) - f(x)}{s}, \Psi^*(\mu) \right\rangle \right| \\ &\quad = \left| \left\langle t^{-1}\big(\Psi(f(x+tw)) - \Psi(f(x))\big) - s^{-1}\big(\Psi(f(x+sw)) - \Psi(f(x))\big), \mu \right\rangle \right| \\ &\quad \le \left| \left\langle t^{-1}\big(\tilde{f}(x+tw) - \tilde{f}(x)\big) - s^{-1}\big(\tilde{f}(x+sw) - \tilde{f}(x)\big), \mu \right\rangle \right| \\ &\qquad + \|\mu\|\,|t|^{-1}\big\|\tilde{f}(x+tw) - \tilde{f}(z_t)\big\| + \|\mu\|\,|t|^{-1}\|\Psi(f(x+tw)) - \Psi(f(z_t))\| \\ &\qquad + \|\mu\|\,|s|^{-1}\big\|\tilde{f}(x+sw) - \tilde{f}(z_s)\big\| + \|\mu\|\,|s|^{-1}\|\Psi(f(x+sw)) - \Psi(f(z_s))\| \\ &\quad \le \varepsilon, \end{aligned}$$

(we used that $\Psi(f(u)) = \tilde{f}(u)$ for $u \in G$). This proves that $wd(f,x)(w)$ exists.

Now we will prove that f is w^*-Gâteaux differentiable at x. By the last paragraph, we have that $g_w(y) := \lim_{t\to 0}\langle t^{-1}(f(x+tw) - f(x)), y\rangle$ exists for all

$y \in Y$. It is easy to see that $g_w(\cdot) \in Y^*$. Now, by a similar argument as above, it is easy to see that $g_{(av+bw)} = ag_v + bg_w$ for all $a, b \in \mathbb{R}$, and $v, w \in X$; the boundedness of g_w in w follows from (4.2).

Note that

$$\Psi(wd(f,x)(w)) = wd(\tilde{f},x)(w), \tag{4.7}$$

as for any $\nu \in \ell_1$ we have

$$\begin{aligned}
&\left|\left\langle \frac{\tilde{f}(x+tw)-\tilde{f}(x)}{t} - \Psi(wd(f,x)(w)), \nu \right\rangle\right| \\
&\quad \le \left|\left\langle t^{-1}\big(\Psi(f(x+tw)-f(x))\big) - \Psi(wd(f,x)(w)), \nu\right\rangle\right| \\
&\qquad + \|\nu\|\,|t|^{-1}\big\|\tilde{f}(x+tw)-\tilde{f}(z_t)\big\| + \|\nu\|\,|t|^{-1}\|\Psi(f(x+tw))-\Psi(f(z_t))\| \\
&\quad \le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

for each $0 < |t| < \tau$, where $\tau > 0$ is small enough (we used that $\Psi(f(w)) = \tilde{f}(w)$ for $w \in G$, and that $\Psi^*(\ell_1) \subset Y$).

Equality (4.4) follows from (4.3), (4.7), and from the equality $MD(f,x)(\cdot) = MD(\tilde{f},x)(\cdot)$. □

5. Metric and w^*-differentiability of mappings with infinite-dimensional domains

The following example shows that there is no hope to obtain (1.3) in the case when X is infinite-dimensional.

Example 5.1. There exists a 1-Lipschitz mapping $f : \ell_2 \mapsto \ell_2$ such that f is everywhere directionally metrically differentiable, and $MD(f,x)(\cdot)$ is a seminorm for all $x \in \ell_2$, but (1.3) doesn't hold for any $x \in \ell_2$.

Proof. Consider the mapping $f : \ell_2 \mapsto \ell_2$ defined as $f((x_i)_{i\in\mathbb{N}}) = (|x_i|)_{i\in\mathbb{N}}$. Then f is everywhere directionally metrically differentiable with $MD(f,x)(u) = \|u\|$. Take $x \in \ell_2$, fix $\varepsilon > 0$, and find $n \in \mathbb{N}$ such that $|x_n| < \frac{\varepsilon}{2}$. Then for vectors $y = x + \varepsilon \cdot e_n, z = x - \varepsilon \cdot e_n$, we obtain

$$\big|\|f(z)-f(y)\| - MD(f,x)(z-y)\big| \ge 2\varepsilon + |x_n+\varepsilon| - |x_n-\varepsilon| \ge \varepsilon,$$

and the condition (1.3) is violated (because $\|y-x\| = \|z-x\| = \varepsilon$). □

The next example shows that also condition (1.2) can fail outside a set which is Aronszajn null even if we consider only real valued functions defined on ℓ_1 (or in other words, f is not metrically Fréchet differentiable at almost all x).

Example 5.2. There exists a Lipschitz function $f : \ell_1 \mapsto \mathbb{R}$ such that (1.2) does not hold for almost all $x \in \ell_1$ (i.e., outside an Aronszajn null set).

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = \max(0, x)$. Define $f : \ell_1 \mapsto \mathbb{R}$ as $f\big((x_i)_{i\in\mathbb{N}}\big) = \sum_i g(x_i)$. It is easy to see that f is well defined, 1-Lipschitz, and $MD(f,x)(\cdot)$ exists and is a seminorm for $x \in \ell_1$ with $x_i \neq 0$ for all $i \in \mathbb{N}$; call this set P. We see that $\ell_1 \setminus P$ is a countable union of hyperplanes $\{x_i = 0\}$, and thus it is Aronszajn null. Let $x \in P$. It is easily seen that

$$MD(f,x)(u) = \sum_i \big|g(\operatorname{sign}(x_i)) \cdot u_i\big| \qquad (u \in \ell_1).$$

If there exists an infinite $A = A_x \subset \mathbb{N}$ such that $x_i < 0$ for all $i \in A$, then note that for $i \in A$ and $y = x + 2|x_i|e_i$ we have

$$\big|f(y) - f(x)\big| - MD(f,x)(2|x_i|e_i) = \big|g(x_i + 2|x_i|)\big| = |x_i| = \frac{\|y-x\|}{2}.$$

If $x_i > 0$ for all $i > i_0$ for some $i_0 \in \mathbb{N}$, then for $y = x - 2|x_i|e_i$ we have

$$\big|f(y) - f(x)\big| - MD(f,x)(2|x_i|e_i) = \big|g(x_i - 2|x_i|) - g(x_i)\big| - 2|x_i| = \frac{\|y-x\|}{2}.$$

Thus (1.2) does not hold for any $x \in P$. □

We obtain the following theorem.

Theorem 5.3. *Let X, Y be separable Banach spaces, and let $f : X \mapsto Y^*$ be a mapping. Let G be the set of all points $x \in X$ at which f is Lipschitz. Then there exists a set $E \in \mathcal{A}$ such that for each $x \in G\setminus E$ we have that f is metrically Gâteaux differentiable at x, f is w^*-Gâteaux differentiable at x, and*

$$MD(f,x)(w) = \|wd(f,x)(w)\| \quad \textit{for all } w \in X. \tag{5.1}$$

Proof. Note that there exists an isometric embedding Ψ of Y^* to ℓ_∞ (take a dense sequence $\{y_i\} \subset S_Y$ and define $\Psi(y^*) = \{\langle y^*, y_i\rangle\}$). It is easy to see that $\Psi^*|_{\ell_1}$ is a quotient mapping from ℓ_1 onto Y. Let $A_{m,k}$ be from Lemma 2.4. Order the sequence $\{A_{m,k}\}$ into a single sequence; call it $\{G_j\}$. Then there are $L_j > 0$ such that $f|_{G_j}$ is L_j-Lipschitz, and $\delta_j > 0$ such that $\|f(x) - f(w)\| \le L_j\|x - w\|$ for all $x \in G_j$ and $w \in X$ with $\|x - w\| < \delta_j$. Since the distance function $\operatorname{dist}(\cdot, G_j)$ is 1-Lipschitz on X, by Aronszajn's theorem (see [3, Theorem 1]), there exists a Borel set D_j such that $X \setminus D_j \in \mathcal{A}$ and $\operatorname{dist}(\cdot, G_j)$ is Gâteaux differentiable on D_j. Then $G_j \setminus D_j \in \mathcal{A}$. Fix $j \in \mathbb{N}$. By [4, Lemma 1.1 (ii)], we can extend $\Psi \circ f|_{G_j}$ to L_j-Lipschitz mapping $f_j : X \mapsto \ell_\infty$. By Theorem 4.3 there exist $E_j \in \mathcal{A}$ such that f_j is metrically Gâteaux differentiable at x, f_j is w^*-Gâteaux differentiable at x, and (5.1) holds for $x \in G_j \setminus E_j$. If $x \in (D_j \cap$

$G_j) \setminus E_j$, then by Lemma 4.4 (ii) (applied to f, $\tilde{f} = f_j$, $G = G_j$, $D = G_j \cap D_j$, $L = L_j$, and $\delta = \delta_j$) we obtain that f is metrically Gâteaux differentiable at x, f is w^*-Gâteaux differentiable at x, and (5.1) holds.

Now define $E = \bigcup_j (E_j \cap G_j) \cup \bigcup_j (G_j \setminus D_j)$. If $x \in S(f) \setminus E$, then there exists $j \in \mathbb{N}$ such that $x \in (D_j \cap G_j) \setminus E_j$ and the conclusion follows. □

As a consequence of Theorem 4.3, we obtain the following theorem. Its proof follows the same lines as the proof of Theorem 5.3 (this time using part (i) of Lemma 4.4) and thus we omit it.

Theorem 5.4. *Let X, Y be Banach spaces with X separable, and let $f : X \mapsto Y$ be a mapping. Let G be the set of all points $x \in X$ at which f is Lipschitz. Then there exists a set $E \in \mathcal{A}$ such that for each $x \in G \setminus E$ we have that f is metrically Gâteaux differentiable at x.*

Let us finish with a lemma and two propositions that suggest an application of the metric differential. Similar reasoning was used e.g. in [11]. For $f : X \mapsto Y$ and $x, u \in X$ define *the tangent of f at x in the direction u* as

$$\tau(f, x)(u) = \lim_{t \to 0} \operatorname{sign}(t) \frac{f(x + tu) - f(x)}{\|f(x + tu) - f(x)\|}.$$

If $X = \mathbb{R}$, put $\tau(f, x) = \tau(f, x)(1)$ and $md(f, x) = MD(f, x)(1)$.

Lemma 5.5. *Let X, Y be Banach spaces. Let $f : X \mapsto Y$ be such that f is directionally metrically differentiable at x and for each $u \in X$ either $MD(f, x)(u) = 0$ or $\tau(f, x)(u)$ exists. Then f is directionally differentiable at x.*

Proof. If $MD(f, x)(u) = 0$, then obviously $D(f, x)(u) = 0$, otherwise we have

$$D(f, x)(u) = \lim_{t \to 0} t^{-1}(f(x + tu) - f(x)) = MD(f, x)(u) \cdot \tau(f, x)(u). \qquad \square$$

Proposition 5.6. *Let Y be a Banach space, and let $f : [0, 1] \mapsto Y$ be a pointwise-Lipschitz mapping such that $f'(x)$ doesn't exist for any $x \in [0, 1]$.*[2] *Then if $md(f, x)$ exists, then $md(f, x) > 0$, and thus for almost all $x \in [0, 1]$ we have that $\tau(f, x)$ doesn't exist.*

Proof. By Theorem 2.5, $md(f, x)$ exists for almost all $x \in [0, 1]$. If $\tau(f, x)$ exists, then we have a contradiction with Lemma 5.5 and our assumption. □

We have the following characterization of RNP in terms of existence of tangents.

[2]Existence of such a Lipschitz function is equivalent to Y failing the RNP; see, e.g., [4].

Proposition 5.7. *Let X be a Banach space. Then X has the RNP if and only if for every non-constant Lipschitz function $f : [0,1] \mapsto X$ there exists a set $A \subset [0,1]$ with $\lambda(A) > 0$ and such that $\tau(f,x)$ exists for all $x \in A$.*

Proof. If X has the RNP, then every Lipschitz function $f : [0,1] \mapsto X$ is almost everywhere differentiable. Because f is non-constant, we have that $f'(x) \neq 0$ for all x in a set of positive measure; call it A (otherwise $f \equiv 0$ in $[0,1]$ by [10, Lemma 2.1]). Thus $\tau(f,x) = \frac{f'(x)}{\|f'(x)\|}$ exists for all $x \in A$.

Suppose that X satisfies the condition. Let $f : [0,1] \mapsto X$ be a non-constant Lipschitz function. By Theorem 1.1 we have that $md(f,x)$ exists for almost all $x \in [0,1]$. Thus there exists $x \in A$ such that $md(f,x)$ exists. Now by Lemma 5.5 we obtain that $f'(x)$ exists. Thus X has the RNP by [4, Theorem 5.21]. □

Acknowledgements. The author would like to thank Professor Luděk Zajíček for suggesting the problem, and many valuable conversations. The author would also like to thank Professor David Preiss for a valuable discussion that led to strengthening of the results and simplification of the proofs of the main theorems. The author was partially supported by ISF.

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Received July 11, 2005