Asymptotically Almost Periodic and Almost Periodic Solutions for Partial Neutral Integrodifferential Equations

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Abstract. In this paper we study the existence of almost periodic and asymptotically almost periodic solutions for a class of partial neutral functional integro-differential equation with unbounded delay.

1. Introduction

In this work we study the existence of asymptotically almost periodic and almost periodic solutions for a class of partial neutral integro-differential equation with unbounded delay modelled in the form

$$\frac{d}{dt}D(t,u_t) = AD(t,u_t) + \int_0^t B(t-s)D(s,u_s)ds + g(t,u_t), \qquad (1)$$

where $A : D(A) \subset X \mapsto X$, $B(t) : D(B(t)) \subset X \mapsto X$, $t \geq 0$, are linear closed and densely defined operators on a Banach space X; $D(B(t)) \supset D(A)$ for every $t \geq 0$; the history $x_t : (-\infty, 0] \mapsto X$, $x_t(\theta) = x(t+\theta)$, belongs to some

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abstract space $\mathcal{B}$ described axiomatically; $D(t, \varphi) = \varphi(0) + f(t, \varphi)$ and $f(\cdot), g(\cdot)$ are appropriate functions.

The existence of almost periodic and asymptotically almost periodic solutions is one of the most attracting topics in the qualitative theory of differential equations due to their significance in physical sciences. The existence of these type of solution for the case which $f \equiv 0$ is studied in [5, 11, 16] among other papers. The cases of ordinary neutral differential equations ($A \equiv 0$) and abstract partial neutral differential equations ($B \equiv 0$) have been treated recently in [21, 24] and [15], respectively. To the best of our knowledge, nothing has been done in terms of "partial" neutral integro-differential equations. This fact is the main motivation of this work.

Neutral differential equations arise in many areas of applied mathematics and for this reason, this type of equation has received much attention in recent years. The literature relative to ordinary neutral differential equations is very extensive, and we suggest to the reader the Hale & Lunel book [9] concerning this matter. Referring to partial neutral functional differential equations, we cite the pioneer Hale paper [10] and Wu [27, 28] for finite delay equations and Hernández & Henriquez [12, 13], Hernández [14] for the unbounded delay one.

The system (1) permits the abstract formulation of some integro-differential system which arises, for instance, in the theory development in Gurtin & Pipkin [8] and Nunziato [22] for the description of heat conduction in materials with fading memory. In the classic heat conduction theory, it is assumed that the internal energy and the heat flux depends linearly on the temperature $u(\cdot)$ and on $\nabla u(\cdot)$. Under these conditions, the classic heat equation describe sufficiently well the evolution of the temperature in different type of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [8, 22], the internal energy and the heat flux are described as functionals of u and u_x. The next system has been frequently used to describe this phenomena (see, for instance, [1, 4, 19, 25])

$$\frac{d}{dt}\left[c_0 u(t,x) + \int_{-\infty}^{t} a_1(t-s)u(s,x)ds\right] = c_1 \triangle u(t,x) + \int_{-\infty}^{t} a_2(t-s)\triangle u(s,x)ds$$

$$u(t,x) = 0, \quad x \in \partial\Omega.$$

In this system, Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary; $(t, x) \in [0, \infty) \times \Omega$; $u(t, x)$ represents the temperature in x at the time t; c_0, c_1 are physical constants and $a_i : \mathbb{R} \mapsto \mathbb{R}$, $i = 1, 2$, are the internal energy and the heat flux relaxation, respectively. By assuming that the solution $u(\cdot)$ is known on $(-\infty, 0]$ and that $a_1 = a_2$, we can transform this system into the neutral system (1) by defining $B(t) = 0$ for $t \geq 0$. For additional details related abstract partial integro-differential equations we cite [1, 3, 4, 6, 7, 18, 23].

This paper has four sections. In Section 2 we mention some concepts, notations and results referents resolvent of operators, asymptotically almost periodic functions and almost periodic functions needed to establish our results. The existence of asymptotically almost periodic and almost periodic solutions for system (1) is discussed in Section 3. In Section 4 one example is considered.

2. Preliminaries

In this paper, $(X, \|\cdot\|)$ is an abstract Banach space; $A : D(A) \subset X \mapsto X$ and $B(t) : D(B(t)) \subset X \mapsto X$, $t \geq 0$, are linear, closed and densely defined operators on X with $D(B(t)) \supset D(A)$ for each $t \geq 0$. To obtain our results we always assume that the integro-differential abstract Cauchy problem

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds, \quad t \geq 0, \tag{2}$$

$$x(0) = x_0 \in X, \tag{3}$$

has an associated resolvent family of bounded linear operators $(R(t))_{t \geq 0}$ on X.

Definition 2.1. A one parameter family $(R(t))_{t \geq 0}$ of bounded linear operators from X into X is called an *strongly continuous resolvent operator* for (2)–(3) if the following conditions are satisfied:

(i) $R(0) = I_d$, and the function $R(t)x$ is continuous on $[0, \infty)$ for every $x \in X$;

(ii) $R(t)D(A) \subset D(A)$ for all $t \geq 0$ and for $x \in D(A)$, $AR(t)x$ is continuous on $[0, \infty)$ and $R(t)x$ is continuously differentiable on $[0, \infty)$;

(iii) for $x \in D(A)$, the next resolvent equations are satisfied:

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)x ds, \quad t \geq 0$$

$$R'(t)x = R(t)Ax + \int_0^t R(t-s)B(s)x ds, \quad t \geq 0.$$

In this paper, we always assume that the resolvent family $(R(t))_{t \geq 0}$ is uniformly exponentially stable and that $\widetilde{M}, \delta$ are positive constants such that $\|R(t)\| \leq \widetilde{M}e^{-\delta t}$ for every $t \geq 0$. For a complementary literature related to partial integro-differential equations and the theory of resolvent of operators, we cite the papers [6, 7].

In this work, the phase space $\mathcal{B}$ is defined axiomatically. Specifically, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms:

(A) If $x : (-\infty, \sigma + a) \mapsto X$, $a > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$;

(ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$;

(iii) $\|x_t\|_{\mathcal{B}} \le K(t - \sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \mapsto [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

(C2) If $(\varphi^n)_{n \in \mathbb{N}}$ is an uniformly bounded sequence in $C((-\infty, 0]; X)$ formed by functions with compact support and $\varphi^n \to \varphi$ uniformly on compact subset of $(-\infty, 0]$, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Definition 2.2. Let $S(t) : \mathcal{B} \mapsto \mathcal{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ for $\theta \in [-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t + \theta)$ for $\theta \in (-\infty, -t]$. The space $\mathcal{B}$ is called a *fading memory* if $\|S(t)\varphi\|_{\mathcal{B}} \to 0$ as $t \to \infty$ for every $\varphi \in \mathcal{B}$ such that $\varphi(0) = 0$.

Remark 2.3. In this paper, $\mathfrak{L} > 0$ is such that $\|\varphi\|_{\mathcal{B}} \le \mathfrak{L}\sup_{\theta \le 0}\|\varphi(\theta)\|$ for every $\varphi \in \mathcal{B}$ continuous and bounded. Moreover, for the case in which $\mathcal{B}$ is a fading memory, we will assume that $\mathfrak{K}$ is a constant such that $\max\{K(t), M(t)\} \le \mathfrak{K}$ for every $t \ge 0$. For details with respect to these assumptions see [17, Proposition 7.1.1] and [17, Proposition 7.1.5].

For additional literature concerning abstract phase space, we refer to the reader the book Hino, Murakami & Naito [17].

Next we mention a few results, definitions and notations related to asymptotically almost periodic and almost periodic functions. Next, $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$ are Banach spaces and $C_0([0, \infty); Z)$ is the subspace of $C([0, \infty); Z)$ formed by the functions that vanishes at infinity.

Definition 2.4. A function $F \in C(\mathbb{R}; Z)$ is called *almost periodic* (a.p.) if for every $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, F, Z)$, such that

$$\|F(t + \xi) - F(t)\|_Z < \varepsilon, \quad t \in \mathbb{R}, \; \xi \in \mathcal{H}(\varepsilon, F, Z).$$

Definition 2.5. A function $F \in C([0, \infty); Z)$ is called *asymptotically almost periodic* (a.a.p.) if there exists $w \in C_0([0, \infty); Z)$ and an almost periodic function $g(\cdot)$ such that $F(\cdot) = g(\cdot) + w(\cdot)$.

The next Lemmas are useful characterizations of a.p. and a.a.p. functions.

Lemma 2.6 ([29, p. 25]). *A function $f \in C(\mathbb{R}; Z)$ is almost periodic if, and only if, the set of functions $\{H_t f : t \in \mathbb{R}\}$, where $(H_t f)(s) = f(s + t)$, is relatively compact in $C(\mathbb{R}; Z)$.*

Lemma 2.7 ([29, Theorem 5.5]). *A function* $F \in C([0, \infty); Z)$ *is asymptotically almost periodic if and only if, for every* $\varepsilon > 0$ *there exists* $L(\varepsilon, F, Z) > 0$ *and a relatively dense subset of* $[0, \infty)$, *denoted by* $\mathcal{T}(\varepsilon, F, Z)$, *such that*

$$\|F(t + \xi) - F(t)\|_Z < \varepsilon, \quad t \geq L(\varepsilon, F, Z), \xi \in \mathcal{T}(\varepsilon, F, Z).$$

In this paper, $AP(Z)$ and $AAP(Z)$ are the spaces

$$AP(Z) = \{F \in C(\mathbb{R}; Z) : F \text{ is a.p.}\}$$
$$AAP(Z) = \{F \in C([0, \infty); Z) : F \text{ is a.a.p.}\},$$

endowed with the norms $\||u|\|_Z = \sup_{s \in \mathbb{R}} \|u(s)\|$ and $\|u\|_Z = \sup_{s \geq 0} \|u(s)\|$, respectively. We know from [29] that $AP(Z)$ and $AAP(Z)$ are Banach spaces.

Definition 2.8. Let Ω be an open subset of W.
(a) A function $F \in C(\mathbb{R} \times \Omega; Z)$ is called *pointwise almost periodic* (p.a.p.) if $F(\cdot, x) \in AP(Z)$ for every $x \in \Omega$.
(b) A function $F \in C([0, \infty) \times \Omega; Z)$ is called *pointwise asymptotically almost periodic* (p.a.a.p.) if $F(\cdot, x) \in AAP(Z)$ for every $x \in \Omega$.
(c) A function $F \in C(\mathbb{R} \times \Omega; Z)$ is called *uniformly almost periodic* (u.a.p.), if for every $\varepsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, F, K, Z)$, such that

$$\|F(t + \xi, y) - F(t, y)\|_Z \leq \varepsilon, \quad (t, \xi, y) \in \mathbb{R} \times \mathcal{H}(\varepsilon, F, K, Z) \times K.$$

(d) A function $F : C([0, \infty) \times \Omega; Z)$ is called *uniformly asymptotically almost periodic* (u.a.a.p.), if for every $\varepsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $[0, \infty)$, denoted by $\mathcal{T}(\varepsilon, F, K, Z)$, and $L(\varepsilon, F, K, Z) > 0$ such that

$$\|F(t+\xi, y) - F(t, y)\|_Z \leq \varepsilon, \quad t \geq L(\varepsilon, F, K, Z), (\xi, y) \in \mathcal{T}(\varepsilon, F, K, Z) \times K.$$

The next lemma summarizes some properties which are fundamental to obtain our results. This result can be obtained from [26, Theorem 1.2.7] and [17, Proposition 7.1.3].

Lemma 2.9. *Let* $\Omega \subset W$ *be an open set. Then the following properties hold:*
(a) *If* $F \in C(\mathbb{R} \times \Omega; Z)$ *is pointwise almost periodic and satisfies a local Lipschitz condition at* $x \in \Omega$, *uniformly at* t, *then* F *is pointwise almost periodic.*
(b) *If* $F \in C([0, \infty) \times \Omega; Z)$ *is pointwise asymptotically almost periodic and satisfies a local Lipschitz condition at* $x \in \Omega$, *uniformly at* t, *then* F *is pointwise asymptotically almost periodic.*

(c) *If $x \in AP(X)$, then $t \to x_t \in AP(\mathcal{B})$. Moreover, if $\mathcal{B}$ is a fading memory space and $z \in C(\mathbb{R}; X)$ is such that $z_0 \in \mathcal{B}$ and $z|_{[0,\infty)} \in AAP(X)$, then $t \mapsto z_t \in AAP(\mathcal{B})$.*

(d) *If $F \in C(\mathbb{R} \times \Omega; Z)$ is uniformly almost periodic and $y \in AP(W)$ is such that $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$, then $F(t, y(t)) \in AP(Z)$.*

(e) *If $F \in C([0,\infty) \times \Omega; Z)$ is asymptotically almost periodic, $y \in AAP(W)$ and $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$, then $F(t, y(t)) \in AAP(Z)$.*

The terminology and notations in this paper are those generally used in functional analysis. In particular, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W, and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. Moreover $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z.

3.   Existence Results

In this section we study the existence of asymptotically almost periodic and almost periodic solutions of (1). The next result is proved using the ideas and estimates in [29, Example 2.2] and we include the proof by completeness.

Lemma 3.1. *Let $v \in AAP(X)$ and $u : [0,\infty) \mapsto X$ be the function defined by $u(t) = \int_0^t R(t-s)v(s)ds, t \geq 0$. Then $u \in AAP(X)$.*

Proof. It's clear that $u(\cdot)$ is well defined and continuous. Let $\varepsilon > 0$ be given and $\eta = \int_0^\infty \widetilde{M}e^{-\delta s}ds$. Let $\mathcal{T}(\frac{\varepsilon}{3}\eta^{-1}, v, X_\mu)$, $L = L(\frac{\varepsilon}{3}\eta^{-1}, v, X)$ be as in Lemma 2.7, and let $L_1 > 0$ be such that $2\|v\|_X \eta e^{\delta L} e^{-\delta L_1} \leq \frac{\varepsilon}{3}$. For $t \geq L + L_1$ and $\xi \in \mathcal{T}(\frac{\varepsilon}{3}\eta^{-1}, v, X)$ we get

$$\|u(t+\xi) - u(t)\|$$
$$\leq \int_0^\xi \|R(t+\xi-s)v(s)\|ds + \int_0^t \|R(t-s)(v(s+\xi) - v(s))\|ds$$
$$\leq e^{-\delta t}\|v\|_X \int_0^\xi \widetilde{M}e^{-\delta(\xi-s)}ds + \int_0^L \|R(t-s)\|\|v(s+\xi) - v(s)\|ds$$
$$+ \int_L^t \|R(t-s)\|\|v(s+\xi) - v(s)\|ds$$
$$\leq e^{-\delta t}\|v\|_X \eta + 2\|v\|_X e^{-\delta(t-L)} \int_0^L \widetilde{M}e^{-\delta(L-s)}ds + \frac{\varepsilon}{3\eta} \int_0^\infty \widetilde{M}e^{-\delta s}ds$$
$$\leq e^{-\delta t}\|v\|_X \eta + 2\|v\|_X e^{\delta L}\eta e^{-\delta t} + \frac{\varepsilon}{3},$$

which implies that

$$\|u(t+\xi) - u(t)\| \leq \varepsilon, \quad t \geq L\left(\tfrac{\varepsilon}{3}\eta^{-1}, v, X\right) + L_1, \ \xi \in \mathcal{T}\left(\tfrac{\varepsilon}{3}\eta^{-1}, v, X\right).$$

This inequality and Lemma 2.7 permits to conclude that $u(\cdot)$ is a.a.p. The proof is now completed. □

To prove our existence results we always assume that the next condition holds.

(H$_1$) The functions $f, g : \mathbb{R} \times \mathcal{B} \mapsto X$ are continuous and there are continuous and nondecreasing functions $L_g, L_f : [0, \infty) \mapsto [0, \infty)$ such that

$$\|f(t, \psi_1) - f(t, \psi_2)\| \le L_f(r)\|\psi_1 - \psi_2\|_{\mathcal{B}}$$
$$\|g(t, \psi_1) - g(t, \psi_2)\| \le L_g(r)\|\psi_1 - \psi_2\|_{\mathcal{B}},$$

for every $t \in \mathbb{R}$ and every $\psi_i \in \mathcal{B}$ such that $\|\psi_i\|_{\mathcal{B}} \le r$.

From the theory of resolvent operators, see [7], we introduce the following concept of mild solution of (1).

Definition 3.2. A function $u : (-\infty, \sigma + a) \mapsto X$, $a > 0$, is called a *mild solution* of the neutral integro-differential system (1) on $[\sigma, \sigma + a)$, if $u_\sigma \in \mathcal{B}$, $u \mid_{[\sigma, \sigma + a)}$ is continuous and

$$u(t) = R(t)(\varphi(0) + f(\sigma, \varphi)) - f(t, u_t) + \int_\sigma^t R(t - s)g(s, u_s)ds, \quad t \in [\sigma, \sigma + a).$$

Now, we can establish our first existence result.

Theorem 3.3. *Assume that $\mathcal{B}$ is a fading memory space, $f(\cdot)$ and $g(\cdot)$ are p.a.a.p. and that $f(t, 0) = g(t, 0) = 0$ for every $t \in \mathbb{R}$. If $\left[L_f(0) + \frac{\widetilde{M}L_g(0)}{\delta}\right]\mathfrak{K} < 1$, where $\mathfrak{K}$ is the constant introduced in Remark 2.3, then there exists $\varepsilon > 0$ such that for each $\varphi \in B_\varepsilon(0, \mathcal{B})$ there exists a mild solution $u(\cdot, \varphi)$ of (1) on $[0, \infty)$ such that $u(\cdot, \varphi) \in AAP(X)$ and $u_0(\cdot, \varphi) = \varphi$.*

Proof. Let $r > 0$ and $0 < \lambda < 1$ be such that

$$\Theta = \widetilde{M}(H + L_f(\lambda r))\lambda + \left[L_f((\lambda + 1)\mathfrak{K}r) + \frac{\widetilde{M}L_g((\lambda + 1)\mathfrak{K}r)}{\delta}\right](\lambda + 1)\mathfrak{K} < 1.$$

We affirm that the assertion holds for $\varepsilon = \lambda r$. Let $\varphi \in B_\varepsilon(0, \mathcal{B})$. On the space

$$\mathfrak{D} = \{u \in AAP(X) : u(0) = \varphi(0), \|u(t)\| \le r, t \ge 0 \}$$

endowed with the metric $d(u, v) = \|u - v\|_X$, we define the operator $\Gamma : \mathfrak{D} \mapsto C([0, \infty); X)$ by

$$\Gamma u(t) = R(t)(\varphi(0) + f(0, \varphi)) - f(t, \widetilde{u}_t) + \int_0^t R(t - s)g(s, \widetilde{u}_s)ds, \quad t \ge 0,$$

where $\widetilde{u} : \mathbb{R} \mapsto X$ is such that $\widetilde{u}_0 = \varphi$ on $(-\infty, 0)$ and $\widetilde{u} = u$ on $[0, \infty)$. From the properties of $(R(t))_{t \geq 0}$ and $f(\cdot), g(\cdot)$, we infer that $\Gamma u(\cdot)$ is well defined and that $\Gamma u \in C([0, \infty); X)$. Moreover, from Lemmas 2.9 and 3.1 it follows that $\Gamma u \in AAP(X)$.

Next, we prove that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. Let $u \in \mathfrak{D}$ and $t \geq 0$. From the inequality $\|\widetilde{u}_t\| \leq (\lambda + 1)\mathfrak{K}r$, which follows from the space axioms, we get

$$
\begin{aligned}
\|\Gamma u(t)\| &\leq \widetilde{M}(H\lambda r + L_f(\lambda r)\lambda r) + L_f((\lambda + 1)\mathfrak{K}r)(\lambda + 1)\mathfrak{K}r \\
&\quad + \int_0^t \widetilde{M} e^{-\delta(t-s)} L_g((\lambda + 1)\mathfrak{K}r)(\lambda + 1)\mathfrak{K}r \, ds \\
&\leq \widetilde{M}(H + L_f(\lambda r))\lambda r + L_f((\lambda + 1)\mathfrak{K}r)(\lambda + 1)\mathfrak{K}r \\
&\quad + \frac{\widetilde{M} L_g((\lambda + 1)\mathfrak{K}r)(\lambda + 1)\mathfrak{K}r}{\delta} \\
&\leq \Theta r,
\end{aligned}
$$

which implies that $\Gamma(\mathfrak{D}) \subset \mathfrak{D}$. On the other hand, for $u, v \in \mathfrak{D}$ we see that

$$
\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\| &\leq \|f(t, \widetilde{u}_t) - f(t, \widetilde{v}_t)\| + \int_0^t \widetilde{M} \|g(s, \widetilde{u}_s) - g(s, \widetilde{v}_s)\| ds \\
&\leq L_f((\lambda + 1)\mathfrak{K}r)\|\widetilde{u}_t - \widetilde{v}_t\|_X \\
&\quad + \int_0^t \widetilde{M} L_g((\lambda + 1)\mathfrak{K}r) e^{-\delta(t-s)} \|\widetilde{u}_s - \widetilde{v}_s\|_{\mathcal{B}} ds \\
&\leq \left[L_f((\lambda + 1)\mathfrak{K}r) + \frac{\widetilde{M} L_g((\lambda + 1)\mathfrak{K}r)}{\delta} \right] \mathfrak{K} \|u - v\|_X,
\end{aligned}
$$

which shows that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$.

The existence of a mild solution with the required properties is now a consequence of the contraction mapping principle. The proof is completed. $\qquad \square$

The next result is proved using the ideas and estimates in the proof of the previous theorem and because of this we choose to omit the proof.

Theorem 3.4. *If $\mathcal{B}$ is a fading memory space; $f(\cdot)$ and $g(\cdot)$ are p.a.a.p; $L_f(t) = L_f$ and $L_g(t) = L_g$ for all $t \geq 0$ and $\left[L_f + \frac{\widetilde{M} L_g}{\delta} \right] \mathfrak{K} < 1$, then for every $\varphi \in \mathcal{B}$ there exists a unique mild solution, $u(\cdot, \varphi)$, of (1) on $[0, \infty)$ such that $u(\cdot, \varphi) \in AAP(X)$ and $u_0(\cdot, \varphi) = \varphi$.*

Now we discuss the existence of almost periodic solutions for (1).

Theorem 3.5. *Assume that $f(\cdot)$ and $g(\cdot)$ are p.a.p. functions. If $L_f(t) = L_f$ and $L_g(t) = L_g$ for all $t \geq 0$ and $\mathfrak{L}\left[L_f + \frac{\widetilde{M} L_g}{\delta} \right] < 1$, where $\mathfrak{L}$ is the constant in Remark 2.3, then there exists a unique $u \in AP(X)$ such that $u(\cdot)$ is a mild solution of (1) on every interval $[\sigma, \sigma + a)$.*

Proof. Let $\Gamma : AP(X) \mapsto C(\mathbb{R}; X)$ be the map defined by

$$\Gamma u(t) = -f(t, u_t) + \int_{-\infty}^{t} R(t - s)g(s, u_s)ds, \quad t \in \mathbb{R}.$$

From the assumption, it is easy to see that $\Gamma u(\cdot)$ is continuous and from Lemmas 2.6 and 2.9 we infer that $v(t) = (f(t, u_t), g(t, u_t)) \in AP(X \times X)$. If $t \in \mathbb{R}$ and $\xi \in \mathcal{H}(\varepsilon, v, X \times X)$ we get

$$\|\Gamma u(t + \xi) - \Gamma u(t)\| \leq \|f(t + \xi, u_{t+\xi}) - f(t, u_t)\|$$
$$+ \int_{-\infty}^{t} \widetilde{M}e^{-\delta(t-s)}\|g(s + \xi, u_{s+\xi}) - g(s, u_s)\|ds$$
$$\leq \varepsilon + \int_{-\infty}^{t} \widetilde{M}e^{-\delta(t-s)}\varepsilon \ ds$$
$$\leq \varepsilon\left(1 + \frac{\widetilde{M}}{\delta}\right),$$

which implies that $\Gamma u \in AP(X)$. Thus, $\Gamma(\cdot)$ is well defined and with values in $AP(X)$. Moreover, for $u, v \in AP(X)$ we find that

$$\|\Gamma u(t) - \Gamma v(t)\| \leq L_f \|u_t - v_t\|_{\mathcal{B}} + \int_{-\infty}^{t} \widetilde{M}e^{-\delta(t-s)}L_g\|u_s - v_s\|_{\mathcal{B}}ds$$
$$\leq \mathfrak{L}\left[L_f + \frac{\widetilde{M}L_g}{\delta}\right]\||u - v\||_X,$$

which permits conclude that $\Gamma(\cdot)$ is a contraction. Now, the assertion is a consequence of the contraction mapping principle. $\square$

4. Examples

In this section we apply our result to establish the existence of asymptotically almost periodic and almost periodic solutions for a concrete partial functional differential equation with unbounded delay. We have already introduced the required technical framework in Section 3.

Let $h : (-\infty, -r) \mapsto \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and bounded function γ on $(-\infty, 0]$ such that $h(\xi + \theta) \leq \gamma(\xi)h(\theta)$, for all $\xi \leq 0$ and every $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $\mathcal{B} = C_r \times L^p(h; X)$ consists of all classes of functions $\varphi : (-\infty, 0] \mapsto X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable and $h\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $\mathcal{B}$, denoted by $\| \cdot \|_{\mathcal{B}}$, is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} h(\theta)\|\varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

Assume that $h(\cdot)$ verifies the conditions (g-5), (g-6) and (g-7) in the nomenclature of [17]. In this conditions, $\mathcal{B}$ is a fading memory space which verifies axioms (A), (A-1), (B) and (C2), see [17, Theorem 1.3.8] and [17, Example 7.1.7] for details. Moreover, when $r = 0$ and $p = 2$, we have that $H = 1$, $M(t) = \gamma(-t)^{\frac{1}{2}}$, $K(t) = 1 + \left(\int_{-t}^{0} h(\theta)\, d\theta \right)^{\frac{1}{2}}$ for $t \geq 0$, $\mathfrak{L} = \left(\int_{-\infty}^{0} h(s)\, ds \right)^{\frac{1}{2}}$, and $\mathfrak{K} = \left(\sup_{s \leq 0} |\gamma(s)^{\frac{1}{2}}| + 1 + \left(\int_{-\infty}^{0} h(\theta) d\theta \right)^{\frac{1}{2}} \right)$.

Let $X = H_0^1(\Omega) \times L^2(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is an open set with smooth boundary of class C^∞; $\alpha(\cdot), \beta(\cdot)$ are $\mathbb{R}$-valued functions of class C^2 on $[0, \infty)$ with $\alpha(0) > 0$, $\beta(0) > 0$ and $A : D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \mapsto X$ be the operator defined by

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \alpha(0)x'' - \beta(0)y \end{pmatrix}.$$

We know from Chen [2], that A is the infinitesimal generator of a uniformly exponentially stable C_0-semigroup $(T(t))_{t \geq 0}$ on X. In the sequel, we will assume that $\widetilde{M}, \gamma$ are positive constants such that $\|T(t)\| \leq \widetilde{M}e^{-\gamma t}$ for all $t \geq 0$.

Let $B(t) = AF(t)$ where $F(t) : X \mapsto X$, $t \geq 0$, is defined by

$$F = (F_{ij}) = \begin{bmatrix} 0 & 0 \\ -\beta'(t) + \beta(0)\dfrac{\alpha'(t)}{\alpha(0)} & \dfrac{\alpha'(t)}{\alpha(0)} \end{bmatrix}.$$

Assume functions $\alpha^{(i)}(\cdot), \beta^{(i)}(\cdot)$, $i = 1, 2$, be bounded, uniformly continuous and that $\max\{\|F_{22}(t)\|, \|F_{21}(t)\|\} \leq \frac{\gamma e^{-\gamma t}}{2M}$ and $\max\{\|F'_{22}(t)\|, \|F'_{21}(t)\|\} \leq \frac{\gamma^2 e^{-\gamma t}}{4M^2}$, for $t \geq 0$. Under these conditions, the abstract integro-differential system

$$x'(t) = Ax(t) + \int_0^t AF(t-s)x(s)ds,$$

has associated a resolvent of operator $(R(t))_{t \geq 0}$ on X such that $\|R(t)\| \leq \widetilde{M}e^{\frac{-\gamma t}{2}}$ for $t \geq 0$, see Grimmer [6, p. 343] for details.

Motivated by the abstract systems studied in [1, 4, 19, 25], we consider the neutral integro-differential system

$$\begin{aligned}
\frac{\partial}{\partial t} &\left[u(t, \xi) + \int_{-\infty}^t a_1(t-s)u(s, \xi)ds \right] \\
&= A \left(u(t, \xi) + \int_{-\infty}^t a_1(t-s)u(s, \xi)ds \right) \\
&\quad + \int_0^t AF(t-s) \left(u(s, \xi) + \int_{-\infty}^s a_1(s-\tau)u(\tau, \xi)d\tau \right) ds \\
&\quad + \int_{-\infty}^t a_2(t-s)u(s, \xi)ds,
\end{aligned} \tag{4}$$

where $a_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are continuous and $L_i = \left(\int_{-\infty}^0 \frac{a_i^2(s)}{h(s)} ds \right)^{\frac{1}{2}} < \infty$ for $i = 1, 2$. By assuming that $u(\cdot)$ is known on $(-\infty, 0]$, we can transform this system into a delayed system. In fact, if $\mathcal{B} := C_0 \times L^p(h; X)$ and $f, g : \mathbb{R} \times \mathcal{B} \mapsto X$ are the operators defined by

$$f(t, \varphi)(\xi) = \int_{-\infty}^0 a_1(-s)\varphi(s)(\xi) \, ds$$

$$D(t, \varphi)(\xi) = \varphi(0)(\xi) + f(t, \varphi)(\xi)$$

$$g(t, \varphi)(\xi) = \int_{-\infty}^0 a_2(-s)\varphi(s)(\xi) \, ds,$$

then the system (4) can be rewritten as an abstract system of the form (1). Moreover, the functions $f(t, \cdot)$, $g(t, \cdot)$ are bounded linear operator with $\|f(t, \cdot)\|_{\mathcal{L}(\mathcal{B}, X)} \leq L_1$ and $\|g(t, \cdot)\|_{\mathcal{L}(\mathcal{B}, X)} \leq L_2$ for every $t \geq 0$.

The next result follows from Theorems 3.3 and 3.5. We will omit the proof.

Proposition 4.1. *Let* $\mathfrak{K} = \left(\sup_{s \leq 0} |\gamma(s)^{\frac{1}{2}}| + 1 + \left(\int_{-\infty}^0 h(\theta) d\theta \right)^{\frac{1}{2}} \right)$. *Assume that the previous conditions are verified. If*

$$\mathfrak{K} \cdot \left[\left(\int_{-\infty}^0 \frac{a_1^2(s)}{h(s)} ds \right)^{\frac{1}{2}} + \frac{\widetilde{M}}{\gamma} \left(\int_{-\infty}^0 \frac{a_2^2(s)}{h(s)} ds \right)^{\frac{1}{2}} \right] < 1,$$

then there exists $u \in AAP(X)$ *and* $v \in AP(X)$ *such that* $u(\cdot)$ *is a mild solution of* (4) *on* $[0, \infty)$ *and* $v(\cdot)$ *is a mild solution of* (4) *on each interval of the form* $[\sigma, \sigma + a)$ *with* $a > 0$ *and* $\sigma \in \mathbb{R}$.

References

[1] Cannarsa, P. and Sforza, D., Global solutions of abstract semilinear parabolic equations with memory terms. *NoDEA Nonlin. Diff. Eqs. Appl.* 10 (2003)(4), 399 – 430.

[2] Chen, G., Control and stabilization for the wave equation in a bounded domain. *SIAM J. Control Optim.* 17 (1979)(1), 66 – 81.

[3] Clément, Ph. and Da Prato, G., Some results on nonlinear heat equations for materials of fading memory type. *J. Integral Eqs. Appl.* 2 (1990)(3), 375 – 391.

[4] Clément, Ph. and Prüss, J., Global existence for a semilinear parabolic Volterra equation. *Math. Z.* 209 (1992)(1), 17 – 26.

[5] Furumochi, Tetsuo; Naito, Toshiki and Minh, Nguyen Van, Boundedness and almost periodicity of solutions of partial functional differential equations. *J. Diff. Eqs.* 180 (2002)(1), 125 – 152.

[6] Grimmer, R. C., Resolvent operators for integral equations in a Banach space. *Trans. Amer. Math. Soc.* 273 (1982)(1), 333 – 349.

[7] Grimmer, R. and Prüss, J., On linear Volterra equations in Banach spaces. Hyperbolic partial differential equations II. *Comput. Math. Appl.* 11 (1985)(1–3), 189 – 205.

[8] Gurtin, M. E. and Pipkin, A. C., A general theory of heat conduction with finite wave speed. *Arch. Rat. Mech. Anal.* 31 (1968), 113 – 126.

[9] Hale, Jack K.; Verduyn Lunel, Sjoerd M., *Introduction to functional-differential equations.* Appl. Math. Sciences 99. New York: Springer 1993.

[10] Hale, Jack K., Partial neutral functional-differential equations. *Rev. Roumaine Math. Pures Appl.* 39 (1994)(4), 339 – 344

[11] Henríquez, H. R. and Vásquez, C. H., Almost periodic solutions of abstract retarded functional-differential equations with unbounded delay. *Acta Appl. Math.* 57 (1999)(2), 105 – 132.

[12] Hernández, E. and Henríquez, H. R., Existence results for partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* 221 (1998)(2), 452 – 475.

[13] Hernández, E. and Henríquez, H. R., Existence of periodic solutions of partial neutral functional-differential equations with unbounded delay. *J. Math. Anal. Appl.* 221 (1998)(2), 499 – 522.

[14] Hernández, E., Existence results for partial neutral functional integrodifferential equations with unbounded delay. *J. Math. Anal. Appl.* 292 (2004)(1), 194 – 210.

[15] Hernández M., E. and Pelicer, M. L., Asymptotically almost periodic and almost periodic solutions for partial neutral differential equations. *Appl. Math. Lett.* 18 (2005)(11), 1265 – 1272.

[16] Hino, Yoshiyuki and Murakami, Satoru, Limiting equations and some stability properties for asymptotically almost periodic functional differential equations with infinite delay. *Tohoku Math. J.* (2) 54 (2002)(2), 239 – 257.

[17] Hino, Yoshiyuki; Murakami, Satoru and Naito, Toshiki, *Functional-Differential Equations with Infinite Delay.* Lecture Notes Math. 1473. Berlin: Springer 1991.

[18] Lunardi, A., Laplace transform method in integrodifferential equations. *J. Integral Eqs. Appl.* 10 (1985), 185 – 211.

[19] Lunardi, A., On the linear heat equation with fading memory. *SIAM J. Math. Anal.* 21 (1990)(5), 1213 – 1224.

[20] Miller, R. K., An integro-differential equation for rigid heat conductors with memory. *J. Math. Anal. Appl.* 66 (1978)(2), 313 – 332.

[21] Nguyen Minh Man and Nguyen Van Minh, On the existence of quasi periodic and almost periodic solutions of neutral functional differential equations. *Commun. Pure Appl. Anal.* 3 (2004)(2), 291 – 300.

[22] Nunziato, J. W., On heat conduction in materials with memory. *Quart. Appl. Math.* 29 (1971), 187 – 204.

[23] Prüss, J., *Evolutionary Integral Equations and Applications.* Monographs Math. 87. Basel: Birkhäuser 1993.

[24] Yuan, Rong, Existence of almost periodic solutions of neutral functional-differential equations via Liapunov–Razumikhin function. *Z. Angew. Math. Phys.* 49 (1998)(1), 113 – 136.

[25] Sforza, D., Existence in the large for a semilinear integrodifferential equation with infinite delay. *J. Diff. Eqs.* 120 (1995)(2), 289 – 303.

[26] Yoshizawa, T., *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions.* Appl. Math. Sciences 14. New York: Springer 1975.

[27] Wu, Jianhong and Xia, Huaxing, Self-sustained oscillations in a ring array of coupled lossless transmission lines. *J. Diff. Eqs.* 124 (1996)(1), 247 – 278.

[28] Wu, Jianhong, *Theory and Applications of Partial Functional-Differential Equations.* Appl. Math. Sciences 119. New York: Springer 1996.

[29] Zaidman, S., *Almost-Periodic Functions in Abstract Spaces.* Research Notes Math. 126. Boston (MA): Pitman 1985.