Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 26 (2007), 377–390

Nontangential Limits of Poisson Integrals Associated to Dunkl Operators for Dihedral Groups

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Abstract. In this paper we study the differentiation and maximal functions of complex Borel measures on the unit circle of $\mathbb{C}$ with respect to the measures associated to Dunkl differential-difference operators for dihedral groups. We prove that the Poisson integrals corresponding to these differential-difference operators have nontangential limits almost everywhere. Our approach relies on the proof of the doubling condition to obtain an appropriate covering lemma.

Keywords. Dunkl operators, covering lemma, maximal function, Poisson integrals, nontangential limits

Mathematics Subject Classification (2000). Primary 31A20, secondary 30E25, 42B25

1. Notation and statement of the main results

The main purpose of this paper is to study the nontangential boundary behavior of Poisson integrals associated to Dunkl operators for dihedral groups. The corresponding Dirichlet problem has been studied previously by the author in [7]. Problems of this kind were considered for Poisson integrals associated to ultraspherical expansions in [4]. For more information about Dunkl operators, see [5].

For every integer q such that $q \geq 1$, let D_q be the dihedral group of order $2q$, that is, D_q consists of the rotations $z \mapsto ze^{\frac{2\pi il}{q}}$ and the reflections $z \mapsto \overline{z}\, e^{\frac{2\pi il}{q}}$, $0 \leq l \leq q-1$, $z \in \mathbb{C}$.

Fix an integer $k \geq 1$ and real numbers $\alpha, \beta > 0$, or $\beta \geq 0$ when k is odd, and consider the weight function h defined by

$$h(z) = \left| \frac{z^k - \overline{z}^k}{2i} \right|^\alpha \left| \frac{z^k + \overline{z}^k}{2} \right|^\beta,$$

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which is a product of powers of the linear functions on $\mathbb{R}^2 \cong \mathbb{C}$ whose zero-sets are the mirrors of the reflections in D_k if $\alpha > 0$, $\beta = 0$, and D_{2k} if α, $\beta > 0$ (see [1]).

Consider the measure $dm(e^{i\theta}) = c_{\alpha,\beta} h(e^{i\theta})^2 d\theta$ on the unit circle with

$$c_{\alpha,\beta} = \left(\int_{-\pi}^{\pi} h(e^{i\theta})^2 d\theta \right)^{-1} = \left(2\,\mathcal{B}\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2} \right) \right)^{-1},$$

where $\mathcal{B}$ denotes the beta function.

The complex Dunkl operators are defined for a complex-valued function f of class C^1 on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by

$$T_h f(z) = \frac{\partial f(z)}{\partial z} + \alpha \sum_{l=0}^{k-1} \frac{f(z) - f(\overline{z}\omega^{2l})}{z - \overline{z}\omega^{2l}} + \beta \sum_{l=0}^{k-1} \frac{f(z) - f(\overline{z}\omega^{2l+1})}{z - \overline{z}\omega^{2l+1}}$$

and

$$\overline{T}_h f(z) = \frac{\partial f(z)}{\partial \overline{z}} - \alpha \sum_{l=0}^{k-1} \frac{f(z) - f(\overline{z}\omega^{2l})}{z - \overline{z}\omega^{2l}} \omega^{2l} - \beta \sum_{l=0}^{k-1} \frac{f(z) - f(\overline{z}\omega^{2l+1})}{z - \overline{z}\omega^{2l+1}} \omega^{2l+1},$$

where $\omega = e^{\frac{\pi i}{k}}$ (see [2], [3]).

We write S^1 for the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and $B(\zeta, \delta)$ for the arc $\{z \in S^1 : |z - \zeta| < \delta\}$, where $\zeta \in S^1$ and $\delta > 0$.

The total variation measure of a complex Borel measure μ on S^1 is denoted by $|\mu|$, and $\mu \perp m$ means that μ is singular with respect to m, i.e., that there is a Borel set $E \subset S^1$ such that $m(E) = 0$ and $|\mu|(E) = \|\mu\|$, where $\|\mu\| = |\mu|(S^1)$.

Define the maximal function $M\mu$ of a complex Borel measure μ on S^1 by

$$(M\mu)(\zeta) = \sup_{\delta > 0} \frac{|\mu|(B(\zeta, \delta))}{m(B(\zeta, \delta))} \tag{1}$$

for $\zeta \in S^1$. In Theorem 2.4 below we establish an estimate for the maximal function, that will be used to prove the following

Theorem 1.1. *If $f \in L^1(m)$, then*

$$\lim_{\delta \to 0} \frac{1}{m(B(\zeta, \delta))} \int_{B(\zeta, \delta)} |f - f(\zeta)|\, dm = 0$$

for almost every $\zeta \in S^1$. Hence

$$f(\zeta) = \lim_{\delta \to 0} \frac{1}{m(B(\zeta, \delta))} \int_{B(\zeta, \delta)} f\, dm$$

almost everywhere.

Theorem 1.2. *If μ is a complex Borel measure on S^1 and $\mu \perp m$, then*

$$\lim_{\delta \to 0} \frac{\mu}{m}(B(\zeta, \delta)) = 0$$

almost everywhere with respect to m.

Combining these two theorems, we obtain the following corollary:

Corollary 1.3. *If μ is a complex Borel measure on S^1, then its derivative*

$$(\mathcal{D}\mu)(\zeta) = \lim_{\delta \to 0} \frac{\mu}{m}(B(\zeta, \delta))$$

exists almost everywhere with respect to m; if $d\mu = f\, dm + d\mu_s$ with $f \in L^1(m)$ and $\mu_s \perp m$, then

$$(\mathcal{D}\mu)(\zeta) = f(\zeta)$$

almost everywhere with respect to m.

In Section 3 below, we shall study the connection between the maximal function $M\mu$ and the class $L \log L$ of all Borel functions f on S^1 that satisfy

$$\int_{S^1} |f| \log^+ |f|\, dm < \infty,$$

where $\log^+ |f(z)| = \max(\log |f(z)|, 0)$ for $z \in S^1$. More precisely, we shall prove the following

Theorem 1.4. *If μ is a complex Borel measure on S^1 for which $M\mu \in L^1(m)$, then there is an $f \in L \log L$ such that $d\mu = f\, dm$.*

In Section 4, we assume $\beta = 0$ and consider the Poisson kernel P associated to the h-Laplacian operator $\Delta_h = 4T_h\overline{T}_h$. It is given for $z, w \in \mathbb{C}$ such that $|zw| < 1$ by

$$P(z, w) = \frac{1 - |z|^2|w|^2}{\mathcal{B}(\alpha, \alpha + 1)\, |1 - z\overline{w}|^2} \int_0^1 \frac{u^{\alpha-1}(1-u)^\alpha du}{\left[(1-u)|1 - z^k w^k|^2 + u|1 - z^k \overline{w}^k|^2\right]^\alpha} \quad (2)$$

(see [2, Theorems 1.3 and 2.1]), and it satisfies $\Delta_h P(., w) = 0$ on $\mathbb{D}$, for fixed $w \in S^1$.

For $\gamma > 1$, we consider the nontangential approach region defined at a boundary point $\zeta \in S^1$ by

$$\Omega_\gamma(\zeta) = \{z \in \mathbb{D} : |z - \zeta| < \gamma(1 - |z|)\}.$$

We shall establish the following result:

Theorem 1.5. *Assume $\beta = 0$. To every $\gamma > 1$ it corresponds a constant $C(\gamma) < \infty$ such that*

$$\sup_{z\in\Omega_\gamma(\zeta)} \left| \int_{S^1} P(z,w)h(w)^2 d\mu(w) \right| \leq C(\gamma)(M\mu)(\zeta) \quad (\zeta \in S^1) \tag{3}$$

for all complex Borel measures μ on S^1.

The Poisson integral $P[f]$ of a function $f \in L^1(m)$ is defined as in [7] by

$$P[f](z) = \int_{S^1} f(w)P(z,w)dm(w)$$

for $z \in \mathbb{D}$. Theorem 1.5 will be combined with Theorem 1.1 and the results in Section 2 to prove the following

Theorem 1.6. *Assume $\beta = 0$. If $f \in L^1(d\theta)$, then for almost every $\zeta \in S^1$,*

$$\lim_{z\to\zeta,\ z\in\Omega_\gamma(\zeta)} P[f](z) = f(\zeta)$$

for all $\gamma > 1$.

2. Differentiation of measures on S^1

In what follows, the symbol B always denotes an arc $B(\zeta,\delta)$, where $\zeta \in S^1$ and $\delta > 0$. If $B = B(\zeta,\delta)$ and $\tau > 0$, we write τB in place of $B(\zeta,\tau\delta)$.

To prove the results of Sections 2 and 3, it is natural to follow the ideas used in the classical case by Rudin [6] for the normalized rotation-invariant surface measure on the unit sphere of $\mathbb{C}^N$. However, some difficulties arise in proving the finiteness of the constants

$$A(\tau) := \sup_B \frac{m(\tau B)}{m(B)} \quad (\tau > 1).$$

This is established in Lemma 2.1 below, and is trivial when m is replaced by the measure $d\theta$ on S^1.

Lemma 2.1. *For any real number $\tau > 1$,*

$$A(\tau) = \sup_B \frac{m(\tau B)}{m(B)} < \infty.$$

Proof. Fix $\tau > 1$. Since S^1 is compact, it is enough to show that for every $\zeta_0 \in S^1$, there exist a neighborhood $U(\zeta_0)$ of ζ_0 in S^1, $\delta_{\zeta_0} > 0$, and $C_{\zeta_0} \geq 0$ such that

$$\frac{m(B(\zeta,\tau\delta))}{m(B(\zeta,\delta))} \leq C_{\zeta_0}, \quad \text{for all } \zeta \in U(\zeta_0),\ \delta \in (0,\delta_{\zeta_0}]. \tag{4}$$

For $\zeta = e^{i\varphi} \in S^1$ and $\delta \in (0,2]$, we have $m(B(\zeta,\delta)) = G\left(\varphi, 2\arcsin\frac{\delta}{2}\right)$, where

$$G(\varphi,\theta) = c_{\alpha,\beta}\int_{\varphi-\theta}^{\varphi+\theta} (\sin^2 kt)^\alpha (\cos^2 kt)^\beta dt \quad (\varphi \in \mathbb{R},\ \theta \in (0,\pi]).$$

Fix $\zeta_0 = e^{i\varphi_0} \in S^1$. If $\zeta_0^{2k} \notin \{-1,1\}$, we have $G(\varphi,\theta) \sim 2c_{\alpha,\beta}\kappa_{\zeta_0}\theta$ as $(\varphi,\theta) \to (\varphi_0,0)$, where $\kappa_{\zeta_0} = (\sin^2 k\varphi_0)^\alpha(\cos^2 k\varphi_0)^\beta \neq 0$, so that there are a neighborhood $U(\zeta_0)$ of ζ_0 in S^1 and a number $\delta_{\zeta_0} > 0$ such that $U(\zeta_0)$, δ_{ζ_0} and $C_{\zeta_0} = \tau + 1$ satisfy (4).

We now consider the case when $\zeta_0^{2k} = \pm 1$. We may assume $\zeta_0 \in \{1, e^{\frac{i\pi}{2k}}\}$, because of the periodicity of $t \mapsto (\sin^2 kt)^\alpha(\cos^2 kt)^\beta$. For $\eta \geq 0$, let f_η be the function defined by $f_\eta(t) = (t^2)^\eta t$ for any real number t (with the understanding that $f_0(t) := t$). Since

$$G(\varphi,\theta) \sim \frac{c_{\alpha,\beta}k^{2\alpha}}{2\alpha+1}[f_\alpha(\varphi+\theta) - f_\alpha(\varphi-\theta)]$$

and

$$\begin{aligned} G\left(\frac{\pi}{2k}+\varphi,\theta\right) &= c_{\alpha,\beta}\int_{\varphi-\theta}^{\varphi+\theta}(\sin^2 kt)^\beta(\cos^2 kt)^\alpha dt \\ &\sim \frac{c_{\alpha,\beta}k^{2\beta}}{2\beta+1}[f_\beta(\varphi+\theta) - f_\beta(\varphi-\theta)] \end{aligned}$$

as $(\varphi,\theta) \to (0,0)$, it is enough to see that for fixed $\eta \geq 0$, there is $\tilde\delta > 0$ such that the function F_η defined by

$$F_\eta(\varphi,\delta) = \frac{f_\eta(\varphi+\Theta(\tau\delta)) - f_\eta(\varphi-\Theta(\tau\delta))}{f_\eta(\varphi+\Theta(\delta)) - f_\eta(\varphi-\Theta(\delta))}$$

is bounded on $\mathbb{R} \times (0,\tilde\delta]$, where $\Theta(\delta) = 2\arcsin(\frac{\delta}{2})$.

If $0 < \delta < \frac{2}{\tau}$, so that $\frac{\Theta(\delta)}{\Theta(\tau\delta)} \leq \frac{1}{\tau}$, and if $|\varphi| > \Theta(\tau\delta)$, then

$$F_\eta(\varphi,\delta) = \frac{f_\eta\left(1+\frac{\Theta(\tau\delta)}{|\varphi|}\right) - f_\eta\left(1-\frac{\Theta(\tau\delta)}{|\varphi|}\right)}{f_\eta\left(1+\frac{\Theta(\delta)}{|\varphi|}\right) - f_\eta\left(1-\frac{\Theta(\delta)}{|\varphi|}\right)} = \frac{\Theta(\tau\delta)}{\Theta(\delta)}\left(\frac{r_{\varphi,\delta}}{s_{\varphi,\delta}}\right)^{2\eta},$$

where $r_{\varphi,\delta}$ and $s_{\varphi,\delta}$ are real numbers such that

$$|r_{\varphi,\delta} - 1| < \frac{\Theta(\tau\delta)}{|\varphi|} \le 1 \qquad \text{and} \qquad |s_{\varphi,\delta} - 1| < \frac{\Theta(\delta)}{|\varphi|} \le \frac{1}{\tau},$$

so that we get

$$F_\eta(\varphi, \delta) \le \frac{\Theta(\tau\delta)}{\Theta(\delta)} \left(\frac{2\tau}{\tau - 1}\right)^{2\eta}$$

for $0 < \delta < \frac{2}{\tau}$ and $|\varphi| > \Theta(\tau\delta)$. Since we also have

$$F_\eta(\varphi, \delta) \le \frac{f_\eta(2\,\Theta(\tau\delta))}{2f_\eta(\Theta(\delta))} = 4^\eta \left(\frac{\Theta(\tau\delta)}{\Theta(\delta)}\right)^{2\eta+1}$$

if $|\varphi| \le \Theta(\tau\delta)$, the desired conclusion follows easily. □

The following covering lemma will be used in the proofs of Theorems 2.4 and 3.1.

Lemma 2.2. *If E is the union of a finite collection Φ of arcs $B \subset S^1$, then Φ has a disjoint subcollection Γ satisfying*

$$E \subset \bigcup_\Gamma 3B \tag{5}$$

and

$$m(E) \le A(3) \sum_\Gamma m(B). \tag{6}$$

Proof. Write the members B_n of Φ as $B(\zeta_n, \delta_n)$ and index them so that $\delta_n \ge \delta_{n+1}$. Set $n_1 = 1$. Assume that $p \ge 1$, $n_1 < \ldots < n_p$ are chosen and $B_{n_1}, \ldots, B_{n_p}$ are pairwise disjoint. If B_n intersects $\cup_{l=1}^p B_{n_l}$ for any $n > n_p$, stop; if not, let n_{p+1} be the first index such that $B_{n_{p+1}}$ is disjoint from $\cup_{l=1}^p B_{n_l}$. This process stops because Φ is finite, and we thus get a disjoint subcollection $\Gamma = \{B_{n_1}, \ldots, B_{n_q}\}$ of Φ.

Now consider an arc $B_n \in \Phi$. Let p be the largest integer between 1 and q such that $n \ge n_p$. Then there is $l \in \{1, \ldots, p\}$ such that B_n intersects B_{n_l}, and $\delta_n \le \delta_{n_l}$, so that $B_n \subset 3B_{n_l}$. Hence (5) holds, and, combined with the definition of $A(3)$, gives (6). □

Remark 2.3. The maximal function $M\mu$ of a complex Borel measure μ on S^1 is lower semicontinuous. Indeed, for fixed $\delta > 0$, $\zeta \mapsto m(B(\zeta, \delta))$ is continuous on S^1 by the definition of m, and $\zeta \mapsto |\mu|(B(\zeta, \delta))$ is lower semicontinuous because if $\zeta_0 \in S^1$, $\delta' \in (0, \delta)$ and $\zeta \in B(\zeta_0, \delta')$, we have $|\mu|(B(\zeta, \delta)) \ge |\mu|(B(\zeta_0, \delta)) - |\mu|(B(\zeta_0, \delta) \backslash B(\zeta_0, \delta - \delta'))$. It follows that for fixed $\delta > 0$, the quotient in formula (1) is a lower semicontinuous function of ζ, and so is $M\mu$, as the supremum of a collection of lower semicontinuous functions.

Theorem 2.4. *If μ is a complex Borel measure on S^1, then for any $t > 0$,*

$$m\big(\{\zeta \in S^1 : (M\mu)(\zeta) > t\}\big) \le A(3)\, t^{-1}\|\mu\|. \tag{7}$$

Proof. Fix μ and t. Write U_t for the set $\{\zeta \in S^1 : (M\mu)(\zeta) > t\}$, which is open by Remark 2.3, and let K be a compact subset of U_t. For each $\zeta \in K$ we have an arc B centered at ζ such that

$$|\mu|(B) > t\, m(B), \tag{8}$$

by the definition of $M\mu$. Cover K by the union of a finite collection Φ of arcs B that satisfy (8). Applying Lemma 2.2 to Φ, we obtain a collection Γ of pairwise disjoint arcs B satisfying (8), such that $m(\cup_\Phi B) \le A(3)\sum_\Gamma m(B)$. It follows that

$$m(K) \le A(3)\sum_\Gamma m(B) < A(3)\, t^{-1}\sum_\Gamma |\mu|(B) \le A(3)\, t^{-1}\|\mu\|.$$

Taking the supremum over all compacts $K \subset U_t$ gives us (7). □

Define the maximal function Mf of a function $f \in L^1(m)$ to be the maximal function of the measure $f\,dm$, that is,

$$(Mf)(\zeta) = \sup_{\delta>0} \frac{1}{m(B(\zeta,\delta))}\int_{B(\zeta,\delta)} |f|\, dm \quad (\zeta \in S^1).$$

Proof of Theorem 1.1. Set

$$\mathcal{L}_f(\zeta) = \limsup_{\delta\to 0} \frac{1}{m(B(\zeta,\delta))}\int_{B(\zeta,\delta)} |f - f(\zeta)|\, dm$$

for $\zeta \in S^1$. Let $t > 0$ and let $\varepsilon > 0$. There is a continuous function u on S^1 such that $\|f-u\|_{L^1(m)} < \varepsilon$. Setting $v = f-u$, we obviously have $\mathcal{L}_f \le \mathcal{L}_u + \mathcal{L}_v$, and $\mathcal{L}_v \le |v| + Mv$. Moreover, $\mathcal{L}_u = 0$ because u is continuous, so that we get $\mathcal{L}_f \le |v| + Mv$, which implies that $\{\zeta \in S^1 : \mathcal{L}_f(\zeta) > t\} \subset E_{t,\varepsilon}$, where

$$E_{t,\varepsilon} = \left\{\zeta \in S^1 : |v(\zeta)| > \frac{t}{2}\right\} \cup \left\{\zeta \in S^1 : (Mv)(\zeta) > \frac{t}{2}\right\}.$$

It follows from Theorem 2.4 that

$$m(E_{t,\varepsilon}) \le \frac{2}{t}\,\|v\|_{L^1(m)} + A(3)\frac{2}{t}\,\|v\|_{L^1(m)} < 2(1+A(3))t^{-1}\varepsilon.$$

We thus have

$$\{\zeta \in S^1 : \mathcal{L}_f(\zeta) > 0\} \subset \bigcup_{p=1}^{\infty}\bigcap_{n=1}^{\infty} E_{\frac{1}{p},\frac{1}{n}}$$

with $m\big(\bigcup_{p=1}^{\infty}\bigcap_{n=1}^{\infty} E_{\frac{1}{p},\frac{1}{n}}\big) = 0$, so that $\mathcal{L}_f(\zeta) = 0$ almost everywhere, which concludes the proof. □

Proof of Theorem 1.2. Since $\mu \perp m$ if and only if $|\mu| \perp m$, we may assume that $\mu \geq 0$. Choose $t > 0$ and $\varepsilon > 0$. There is a Borel set $E \subset S^1$ such that $m(E) = 0$ and $\mu(E) = \|\mu\|$. Choose a compact $K \subset E$ satisfying $\mu(K) > \mu(E) - \varepsilon$. Let μ_1 denote the restriction of μ to K, and put $\mu_2 = \mu - \mu_1$. Then

$$\|\mu_2\| < \varepsilon. \tag{9}$$

Set

$$(\overline{\mathcal{D}}\mu)(\zeta) = \limsup_{\delta \to 0} \frac{\mu}{m}(B(\zeta, \delta)) \quad (\zeta \in S^1).$$

It follows from the definition of μ_1 that if $\zeta \notin K$, then $\frac{\mu_1}{m}(B(\zeta, \delta)) \to 0$ as $\delta \to 0$, which implies that $(\overline{\mathcal{D}}\mu)(\zeta) = (\overline{\mathcal{D}}\mu_2)(\zeta)$. We therefore have, writing $\mathcal{U}_t$ for the set $\{\zeta \in S^1 : (\overline{\mathcal{D}}\mu)(\zeta) > t\}$,

$$K \cup \mathcal{U}_t \subset K \cup \big\{\zeta \in S^1 : (\overline{\mathcal{D}}\mu_2)(\zeta) > t\big\} \subset K \cup \big\{\zeta \in S^1 : (M\mu_2)(\zeta) > t\big\},$$

and since $m(K) = 0$, we get, using Theorem 2.4 and (9),

$$m(\mathcal{U}_t) \leq m\big(\{\zeta \in S^1 : (M\mu_2)(\zeta) > t\}\big) \leq A(3)\, t^{-1}\|\mu_2\| < A(3)\, t^{-1}\varepsilon.$$

Letting ε tend to 0 then yields $m(\mathcal{U}_t) = 0$, and since t was arbitrary, this finally shows that $\lim_{\delta \to 0} \frac{\mu}{m}(B(\zeta, \delta)) = 0$ almost everywhere with respect to m. □

3. The maximal function and the class $L \log L$

We first prove the following result:

Theorem 3.1. *Let μ be a complex Borel measure on S^1. If $d\mu = f\,dm + d\mu_s$ with $f \in L^1(m)$ and $\mu_s \perp m$, then*

$$|\mu_s|(\{\zeta \in S^1 : (M\mu)(\zeta) < \infty\}) = 0. \tag{10}$$

Proof. Since $\{\zeta \in S^1 : (M\mu)(\zeta) < \infty\} = \bigcup_{n=1}^{\infty} E_n$ where $E_n = \{\zeta \in S^1 : (M\mu)(\zeta) \leq n\}$, (10) will be proved once we show that $|\mu_s|(E_n) = 0$ for every positive integer n.

Fix such an integer n. Since $\mu_s \perp m$, there is a Borel set $E \subset S^1$ such that $m(E) = 0$ and $|\mu_s|(E) = \|\mu_s\|$. We have $|\mu_s|(E_n) = |\mu_s|(E_n \cap E) = |\mu|(E_n \cap E)$, so that it is enough to prove that $|\mu|(K) = 0$ for any compact $K \subset E_n$ with $m(K) = 0$.

Consider such a compact K and let $\varepsilon > 0$. There is an open set $V \supset K$ satisfying $m(V) < \varepsilon$. Cover K by the union of a finite collection Φ of arcs $B \subset V$, with centers in K. For each arc $B \in \Phi$, $3B$ is an arc with center in $K \subset E_n$, so that the definition of $M\mu$ implies that $|\mu|(3B) \leq n\, m(3B)$. By the

Covering Lemma 2.2, Φ has a subcollection Γ of pairwise disjoint arcs such that $\bigcup_\Phi B \subset \bigcup_\Gamma 3B$. Consequently, we have

$$|\mu|(K) \leq \sum_\Gamma |\mu|(3B) \leq n \sum_\Gamma m(3B) \leq A(3)\, n \sum_\Gamma m(B) \leq A(3)\, n\, m(V) < A(3)\, n\, \varepsilon.$$

Since ε was arbitrary, we get $|\mu|(K) = 0$, which completes the proof. □

Theorem 3.1 will be needed in the proof of Theorem 1.4, as well as the following covering lemma.

Lemma 3.2. *Let μ be a complex Borel measure on S^1. If $t > \|\mu\|$, then there exist arcs B_n and pairwise disjoint Borel sets $V_n \subset B_n$ such that*

(i) $\{\zeta \in S^1 : (M\mu)(\zeta) > t\} \subset \bigcup_n B_n = \bigcup_n V_n$;
(ii) $m(B_n) \leq A(4)\, t^{-1} |\mu|(B_n)$;
(iii) $|\mu|(V_n) < A(4)\, t\, m(V_n)$.

Proof. Set $U_t = \{\zeta \in S^1 : (M\mu)(\zeta) > t\}$. For fixed $\zeta \in U_t$, the assumption that $\|\mu\| < t$ allows us to define $\delta = \max\{\varrho > 0 : |\mu|(B(\zeta, \varrho)) \geq t\, m(B(\zeta, \varrho))\}$. Setting $Q = B(\zeta, \delta)$, we thus have

$$|\mu|(Q) \geq t\, m(Q), \tag{11}$$

and

$$|\mu|(4Q) < t\, m(4Q). \tag{12}$$

One can therefore cover U_t by a collection Γ_1 of arcs Q satisfying (11) and (12).

Let $\rho_1 = \sup\{\rho(Q) : Q \in \Gamma_1\}$, where $\rho(Q)$ denotes the radius of the arc Q, and choose $Q_1 \in \Gamma_1$ such that $\rho(Q_1) > \frac{3}{4}\rho_1$. Let Γ_2 be the collection of the arcs $Q \in \Gamma_1$ that are disjoint from Q_1, set $\rho_2 = \sup\{\rho(Q) : Q \in \Gamma_2\}$, and choose $Q_2 \in \Gamma_2$ such that $\rho(Q_2) > \frac{3}{4}\rho_2$. Let Γ_3 be the collection of the arcs $Q \in \Gamma_2$ that are disjoint from Q_2, etc. If $\Gamma_n = \emptyset$ for some n, this process stops; if not, it continues through the natural numbers.

The arcs Q_n we thus obtain are pairwise disjoint. Set $B_n = 4Q_n$, and

$$V_n = Q_n \cup \left[B_n \backslash \left(\bigcup_{l<n} B_l \cup \bigcup_{l \neq n} Q_l \right) \right];$$

then it can easily be seen that the Borel sets V_n are pairwise disjoint, that for each n we have

$$Q_n \subset V_n \subset B_n, \tag{13}$$

and that $\bigcup_n V_n = \bigcup_n B_n$.

For each $Q \in \Gamma_1$, there is an index n such that $Q \in \Gamma_n \backslash \Gamma_{n+1}$ since otherwise there are necessarily infinitely many Γ_l, and $Q \in \Gamma_l$ for each l, so that $\rho(Q) \leq \rho_l$

for every integer $l \geq 1$; then the definition of Q_l implies that $\rho(Q_l) > \frac{3}{4}\rho(Q)$ for each $l \geq 1$, but this is impossible because of the disjointness of $\{Q_l\}$. If $Q \in \Gamma_n \backslash \Gamma_{n+1}$, then Q intersects Q_n, and $\rho(Q) < \frac{4}{3}\rho(Q_n)$, so that $Q \subset B_n$, since $1 + \frac{4}{3} + \frac{4}{3} < 4$. Hence $U_t \subset \cup_n B_n$, which completes the proof of (i).

It follows from the definition of $A(4)$ and (11) that

$$m(B_n) \leq A(4)\, m(Q_n) \leq A(4)\, t^{-1}|\mu|(Q_n),$$

which gives (ii). By (12) and (13), we have

$$|\mu|(V_n) \leq |\mu|(B_n) < t\, m(B_n) \leq t\, A(4)\, m(Q_n) \leq t\, A(4)\, m(V_n),$$

so that (iii) is proved. □

Proof of Theorem 1.4. Since $M\mu = M|\mu|$ and since there is a Borel function u with $|u| = 1$ such that $d\mu = u\, d|\mu|$, we can suppose that $\mu \geq 0$. We may also assume that $\|\mu\| = 1$.

For $t > 1$, let $U_t = \{\zeta \in S^1 : (M\mu)(\zeta) > t\}$. We are first going to show that

$$\mu(U_t) \leq A(4)\, t\, \psi(t) \quad (t > 1), \tag{14}$$

where $\psi(t) = m(\{\zeta \in S^1 : A(4)^2(M\mu)(\zeta) \geq t\})$. Fix $t > 1$ and choose B_n, V_n as in Lemma 3.2. Using parts (i) and (iii) of that lemma, and the disjointness of $\{V_n\}$, we get

$$\mu(U_t) \leq \sum_n \mu(V_n) < A(4)\, t \sum_n m(V_n) = A(4)\, t\, m\Big(\bigcup_n B_n\Big). \tag{15}$$

If $w \in B_n$ and if δ_n denotes the radius of B_n, we have $B_n \subset B(w, 2\delta_n) \subset 4B_n$, so that it follows from the definition of $A(4)$ and part (ii) of Lemma 3.2 that

$$(M\mu)(w) \geq \frac{\mu}{m}(B(w, 2\delta_n)) \geq \frac{\mu(B_n)}{A(4)\, m(B_n)} \geq \frac{t}{A(4)^2}.$$

Combined with (15), this proves (14).

We have

$$\begin{aligned} \int_0^\infty \psi(t)\, dt &= \int_0^\infty m\big(\{\zeta \in S^1 \colon A(4)^2(M\mu)(\zeta) \geq t\}\big)\, dt \\ &= A(4)^2 \int_{S^1} (M\mu)\, dm < \infty, \end{aligned} \tag{16}$$

and ψ is decreasing; this implies that $t\, \psi(t) \to 0$ as $t \to \infty$, and then

$$\mu(U_t) \to 0 \quad (t \to \infty), \tag{17}$$

by (14). Write $d\mu = f\,dm + d\mu_s$ with $f \in L^1(m)$ and $\mu_s \perp m$. Note that $f \geq 0$ and $\mu_s \geq 0$, since $\mu \geq 0$. By (17), $\mu_s\big(\{\zeta \in S^1 : (M\mu)(\zeta) = \infty\}\big) = \mu_s\big(\bigcap_{t>1} U_t\big) = 0$, and by Theorem 3.1, $\mu_s\big(\{\zeta \in S^1 : (M\mu)(\zeta) < \infty\}\big) = 0$. Hence $\mu_s = 0$, so that $d\mu = f\,dm$, and

$$Mf = M\mu. \tag{18}$$

It follows from Theorem 1.1 that $f \leq Mf$ almost everywhere with respect to m. Using (18), we thus get $f \leq t$ almost everywhere outside U_t, that is, $\{\zeta \in S^1 : f(\zeta) > t\} \subset U_t \cup W$ for some Borel set W with $m(W) = 0$. Consequently,

$$\int_{\{\zeta\in S^1\,:\,f(\zeta)>t\}} f\,dm \leq \mu(U_t) \leq A(4)\,t\,\psi(t) \quad (t > 1),$$

by (14). Applying Fubini's theorem, we therefore obtain

$$A(4)\int_1^\infty \psi(t)\,dt \geq \int_1^\infty \frac{1}{t}\left(\int_{\{\zeta\in S^1\,:\,f(\zeta)>t\}} f\,dm\right)dt = \int_{S^1} f\,\log^+ f\,dm,$$

and the proof is completed by using (16). □

4. Nontangential limits

The following lemma will be used in the proof of Theorem 1.5.

Lemma 4.1. *If $\zeta \in S^1$ and $\gamma > 1$, then for all $z \in \Omega_\gamma(\zeta)$ and all $w \in S^1$,*

$$P(z, w) \leq (k\gamma + 2)^{2\alpha+2} P(r\zeta, w),$$

where $r = |z|$.

Proof. If $z \in \Omega_\gamma(\zeta)$ and $r = |z|$, then $|w - r\zeta| \leq (\gamma + 2)|w - z|$ for all $w \in S^1$, and since we also have $z^k \in \Omega_{k\gamma}(\zeta^k)$, we get $|w^k - r^k\zeta^k| \leq (k\gamma + 2)|w^k - z^k|$ and $|\overline{w}^k - r^k\zeta^k| \leq (k\gamma+2)|\overline{w}^k - z^k|$ for all $w \in S^1$. Then the desired conclusion follows from formula (2). □

Proof of Theorem 1.5. Since $M\mu = M|\mu|$, it suffices to prove (3) for positive μ. By Lemma 4.1, it is then enough to show that there is a constant $C < \infty$ such that

$$\sup_{0\leq r<1} \int_{S^1} P(r\zeta, w)h(w)^2 d\mu(w) \leq C(M\mu)(\zeta)$$

for every finite positive Borel measure μ on S^1 and every $\zeta \in S^1$.

Fix μ, ζ, and r, $0 \leq r < 1$. For $w \in S^1$, set

$$a(w) = |1 - r^k\zeta^k w^k|^2 \qquad \text{and} \qquad b(w) = |1 - r^k\zeta^k\overline{w}^k|^2.$$

We consider the integral $\int_{S^1} P(r\zeta,w)h(w)^2 d\mu(w)$ in several pieces: put $s=1-r$, set

$$\mathbb{V}_1=\{w\in S^1 : b(w)\le 2\,a(w)\},$$

denote by $\mathbb{V}_2$ the complement of $\mathbb{V}_1$, and set

$$\mathbb{W}_0=B(\zeta,s),\qquad \mathbb{W}_l=\{w\in S^1 : 2^{l-1}s\le |w-\zeta|<2^l s\},$$

for $l=1,2,\dots$, until $2^l s>2$; then

$$\int_{S^1} P(r\zeta,w)h(w)^2 d\mu(w)=\sum_{j=1}^{2}\sum_{l\ge 0} I_{j,l}$$

where

$$I_{j,l}=\int_{\mathbb{V}_j\cap\mathbb{W}_l}\frac{1-r^2}{\mathcal{B}(\alpha,\alpha+1)|1-r\zeta\overline{w}|^2}\int_0^1\frac{u^{\alpha-1}(1-u)^\alpha du}{[(1-u)a(w)+u\,b(w)]^\alpha}\,h(w)^2\,d\mu(w).$$

For any $w\in S^1$, we have $|w^k-\overline{w}^k|\le\sqrt{a(w)}+\sqrt{b(w)}$, so that if $w\in\mathbb{V}_1$, we get $h(w)^2\le\big(\frac{1+\sqrt{2}}{2}\big)^{2\alpha}a(w)^\alpha$. Set $\kappa=\frac{1}{\mathcal{B}(\alpha,\alpha+1)}\big(\frac{1+\sqrt{2}}{2}\big)^{2\alpha}$. Since $\frac{1-r^2}{|1-r\zeta\overline{w}|^2}\le\frac{2}{s}$,

$$I_{1,0}\le\frac{2\kappa}{\alpha s}\,\mu(B(\zeta,s)).$$

Moreover, $m(B(\zeta,s))\le c_{\alpha,0}\pi s$ since $(\sin^2 kt)^\alpha\le 1$, so that $\mu(B(\zeta,s))\le (M\mu)(\zeta)m(B(\zeta,s))\le c_{\alpha,0}\,\pi(M\mu)(\zeta)s$. Thus

$$I_{1,0}\le\frac{2\kappa\,c_{\alpha,0}\,\pi}{\alpha}\,(M\mu)(\zeta).$$

Since $|w-\zeta|\le 2|w-r\zeta|$ for any $w\in S^1$, we have, for $l\ge 1$,

$$I_{1,l}\le\frac{8\kappa s}{\alpha\,(2^{l-1}s)^2}\,\mu(B(\zeta,2^l s))\le\frac{32\kappa\,c_{\alpha,0}\,\pi}{\alpha}\frac{(M\mu)(\zeta)}{2^l}.$$

If $w\in\mathbb{V}_2$, then $h(w)^2\le\big(\frac{1+\frac{1}{\sqrt{2}}}{2}\big)^{2\alpha}b(w)^\alpha$, and for $u\in[0,1]$, $(1-u)a(w)+u\,b(w)\ge a(w)+u\frac{b(w)}{2}$, so that

$$\begin{aligned}\int_0^1\frac{u^{\alpha-1}(1-u)^\alpha du}{[(1-u)a(w)+u\,b(w)]^\alpha}\,h(w)^2&\le\left(\frac{1+\frac{1}{\sqrt{2}}}{2}\right)^{2\alpha}\frac{b(w)^\alpha}{a(w)^\alpha}\int_0^1\frac{u^{\alpha-1}}{[1+u\,c(w)]^\alpha}\,du\\&=\left(\frac{1+\sqrt{2}}{2}\right)^{2\alpha}\int_0^{c(w)}\frac{t^{\alpha-1}}{(1+t)^\alpha}\,dt\,,\end{aligned}$$

where $c(w)=\frac{b(w)}{2a(w)}>1$.

If $l \geq 0$ and $w \in B(\zeta, 2^l s)$, we have $\sqrt{b(w)} \leq k|w - r\zeta| \leq k(2^l+1)s \leq 2^{l+1}ks$ and $\sqrt{a(w)} \geq 1 - r^k \geq s$, so that $c(w) \leq 2^{2l+1}k^2$. It follows that

$$I_{2,0} \leq \frac{2\kappa}{s}\,\mu(B(\zeta,s)) \int_0^{2k^2} \frac{t^{\alpha-1}}{(1+t)^\alpha}\,dt \leq 2\kappa\, c_{\alpha,0}\,\pi(M\mu)(\zeta) \int_0^{2k^2} \frac{t^{\alpha-1}}{(1+t)^\alpha}\,dt,$$

and since $\int_1^{c(w)} \frac{t^{\alpha-1}}{(1+t)^\alpha}\,dt \leq \ln(c(w))$ for any $w \in \mathbb{V}_2$,

$$\begin{aligned} I_{2,l} &\leq \frac{8\kappa s}{(2^{l-1}s)^2}\left(\int_0^1 \frac{t^{\alpha-1}}{(1+t)^\alpha}\,dt + (2l+1)\ln 2 + 2\ln k\right)\mu(B(\zeta, 2^l s)) \\ &\leq 32\kappa\, c_{\alpha,0}\,\pi \left(\int_0^1 \frac{t^{\alpha-1}}{(1+t)^\alpha}\,dt + (2l+1)\ln 2 + 2\ln k\right)\frac{(M\mu)(\zeta)}{2^l} \end{aligned}$$

for $l \geq 1$, which completes the proof of the theorem. □

Proof of Theorem 1.6. If $f \in L^1(d\theta)$ is given, it suffices to show that for any $\varepsilon > 0$, there exists a Borel set $E_\varepsilon \subset S^1$ satisfying $m(E_\varepsilon) \leq \varepsilon$, such that for every $\zeta \in S^1 \backslash E_\varepsilon$ and every $\gamma > 1$, $\limsup_{z\to\zeta,\ z\in\Omega_\gamma(\zeta)} |P[f](z) - f(\zeta)| \leq D(\gamma)\varepsilon$, where $D(\gamma)$ is a finite constant depending only on γ.

Let $\varepsilon > 0$. Put $g = \frac{f}{h^2}$, and write $g = g_1 + g_2$, where g_1 is continuous on S^1 and $\|g_2\|_{L^1(m)} \leq \frac{\varepsilon^2}{A(3)}$. It follows from Theorem 1.1 that $|g_2(\zeta)| \leq (Mg_2)(\zeta)$ for almost every $\zeta \in S^1$. By Theorem 2.4,

$$m(\{\zeta \in S^1 : (Mg_2)(\zeta) > \varepsilon\}) \leq A(3)\,\varepsilon^{-1}\|g_2\|_{L^1(m)} \leq \varepsilon.$$

Consequently, there is $E_\varepsilon \subset S^1$ with $m(E_\varepsilon) \leq \varepsilon$, such that if $\zeta \in S^1 \backslash E_\varepsilon$ and $\gamma > 1$, we have for any $z \in \Omega_\gamma(\zeta)$,

$$\begin{aligned} |P[f](z) - f(\zeta)| &\leq |P[g_1h^2](z) - (g_1h^2)(\zeta)| + |P[g_2h^2](z)| + |(g_2h^2)(\zeta)| \\ &\leq |P[g_1h^2](z) - (g_1h^2)(\zeta)| + C(\gamma)(Mg_2)(\zeta) + (Mg_2)(\zeta) \\ &\leq |P[g_1h^2](z) - (g_1h^2)(\zeta)| + (C(\gamma)+1)\varepsilon, \end{aligned}$$

where the constant $C(\gamma)$ is given by Theorem 1.5. By Theorem 1.1 in [7],

$$\lim_{z\to\zeta} P[g_1h^2](z) = (g_1h^2)(\zeta),$$

so that we obtain $\limsup_{z\to\zeta,\ z\in\Omega_\gamma(\zeta)} |P[f](z) - f(\zeta)| \leq (C(\gamma)+1)\varepsilon$, and the proof is complete. □

Acknowledgement. I would like to thank Professor E. H. Youssfi for proposing to me this problem as well as his advice and suggestions during the preparation of this paper.

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Received September 1, 2005