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Asymptotic Self-Similarity for Solutions of Partial Integro-Differential Equations

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Abstract. The question is studied whether weak solutions of linear partial integro-differential equations approach a constant spatial profile after rescaling, as time goes to infinity. The possible limits and corresponding scaling functions are identified and are shown to actually occur. The limiting equations are fractional diffusion equations which are known to have self-similar fundamental solutions. For an important special case, is is shown that the asymptotic profile is Gaussian and convergence holds in L^2, that is, solutions behave like fundamental solutions of the heat equation to leading order. Systems of integro-differential equations occurring in viscoelasticity are also discussed, and their solutions are shown to behave like fundamental solutions of a related Stokes system. The main assumption is that the integral kernel in the equation is regularly varying in the sense of Karamata.

Keywords. Self-similar solution, intermediate asymptotics, integro-differential equation, fractional diffusion, regular variation

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1. Introduction

Consider the linear heat equation $u_t = \Delta u$ in $\mathbb{R}^n$, with fundamental solution $U(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-|x|^2/4t}$. It is well-known and easy to see from the solution formula that as $t \to \infty$, $u(x,t) = U_0 U(x,t) + o(t^{-\frac{n}{2}})$, where $U_0 = \int_{\mathbb{R}^n} u(\cdot, 0)$ is the initial mass of the solution, assumed to be finite. Thus, $t^{\frac{n}{2}} u(x\sqrt{t}, t) \to U_0 U(x,1)$. Similarly, for solutions of the wave equation $u_{tt} = u_{xx}$ on $\mathbb{R} \times [0,\infty)$ with initial data $u(\cdot,0) = u_0$, $u_t(\cdot,0) = 0$, the well-known solution formula $u(x,t) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right)$ implies that $tu(xt,t) \sim \frac{1}{2} U_0 \left(\delta_{-1} + \delta_1\right)$ as $t \to \infty$, in this case in the sense of distributions. For the case of the wave equation on $\mathbb{R}^n$, the solution formulae that use spherical means imply that $t^n u(xt,t) \to U_0 w$, where w is a distribution of dimension dependent order that is supported on the unit sphere in $\mathbb{R}^n$. In the case of the heat equation, the

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solution depends (up to a multiplicative factor) asymptotically on the similarity variable $\xi = \frac{|x|}{\sqrt{t}}$, and convergence is uniform. In the case of the wave equation, the similarity variable is $\xi = \frac{|x|}{t}$, with convergence in a space of distributions.

In this paper, I investigate whether solutions of integro-differential equations

$$u_t(\cdot,t) = a_0 \Delta u(\cdot,t) + \int_0^t a(t-s)\Delta u(\cdot,s)ds$$

in $\mathbb{R}^n$ have similar properties. Here $\mathbb{R} \ni a_0 \geq 0$ is a scalar, and $a : [0,\infty) \to \mathbb{R}$ is a scalar kernel. The special cases $a_0 = 1$, $a = 0$ and $a_0 = 0$, $a = 1$ correspond to the heat equation and the wave equation, respectively, and therefore are included. Thus the question is whether $m(t)u(k(t)x,t) \sim u_\infty$ as $t \to \infty$, in a suitable sense.

Since all equations in this class are of the form $u_t + \nabla \cdot q = 0$ for some flux q, the L^1-integral of solutions is formally preserved as t varies, $\int_{\mathbb{R}^n} u(\cdot,t) = \int_{\mathbb{R}^n} u_0 = U_0$. One therefore expects that $m(t) = k(t)^n$, that is $k(t)^n u(xk(t),t) \sim U_0 w_\infty$ for a suitable function $k(\cdot)$ as $t \to \infty$ in a suitable distributional sense, for a limiting distribution w_∞. Then the "trivial" behavior $w_\infty = \delta_0$ can always be achieved by letting k grow to ∞ very rapidly. This trivial behavior must therefore be excluded. Also, solutions are expected to go to zero locally, so the trivial case $w_\infty = 0$ is possible if k goes to ∞ too slowly and must also be excluded. With the right choice of k one hopes to obtain a nontrivial limit w_∞.

It turns out that for a large class of such integro-differential equations there is a choice of k (unique up to an asymptotically constant factor) for which the limit w_∞ is indeed non-trivial. It will be shown that the correct choice is

$$k(t) = \sqrt{t\left(a_0 + \int_0^t a(s)ds\right)}\,.$$

The limiting distributions w_∞ will also be identified. They turn out to belong to a one parameter family, parametrized by $\beta \in (-1,1]$, with $\beta = 0$ corresponding to the heat equation, $\beta = 1$ corresponding to the wave equation, and the cases of non-integer β corresponding to fundamental solutions of *fractional diffusion* equations. The main assumption is that the integrated kernel $A(t) = a_0 + \int_0^t a(s)ds$ should be regularly varying in the sense of KARAMATA ([1]), and the index of variation β then determines the limiting distribution w_∞. As an aside, it should be noted that for the same w_∞, there are many types of scaling functions k possible that are not asymptotically equivalent. It will be shown that all these possible limiting distributions are actually attained (in the sense of distribution, or in an important special case in L^2).

The literature on self-similar asymptotics is huge, so I only mention the book [2] by G. Barenblatt. Fractional diffusion equations were discussed in [16] and [6], with systematic studies carried out in [5, 7, 11, 12]. Various physical models leading to fractional diffusion equations are discussed in [9, 10, 18]. The main reference for integro-differential equations of the type discussed here is the book [15] by J. Prüss. The idea that regularly varying integral kernels lead to asymptotically self-similar wave profiles for problems in viscoelasticity is exploited in [14], for the case of the signalling problem. A related asymptotic concept is equipartition of energy, discussed for a class of exponential kernels corresponding to $\beta = 1$ in [4].

The plan of this paper is as follows. In the following Section 2, two types of scaling (introduced at the end of this section) and the relations between them are discussed. All possible distributional limits and their corresponding scaling functions k are identified in Section 3. Section 4 is devoted to giving sufficient conditions such that these distributional limits are actually attained. In Section 5, the question of asymptotic self-similarity is studied in L^2, leading to results about the time-asymptotic behavior of solutions that are sharp to leading order. The same question is taken up for three-dimensional linear homogeneous isotropic viscoelasticity in Section 6. Appendix A contains two important technical results for families for scalar integral equations, and two explicit examples for asymptotic behavior outside the theory developed in this paper are presented in Appendix B.

The notation $\langle u, \varphi \rangle$ will be used for the result of applying a distribution $u \in \mathcal{D}'$ to a test function $\varphi \in \mathcal{D} = C_0^\infty(\mathbb{R}^n)$. The pairing between test functions $\Phi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$ and distributions U on $\mathbb{R}^n \times [0,\infty)$ is denoted by $\langle\langle U, \Phi \rangle\rangle$. In particular, if $U(\cdot,t)_{t\geq 0}$ is a family of distributions in $\mathcal{D}'$ that is measurable and bounded with respect to the system of seminorms defining the usual topology on $\mathcal{D}'$, one can write

$$\langle\langle U, \Phi \rangle\rangle = \int_0^\infty \langle U(\cdot,t), \Phi(\cdot,t)\rangle dt\,.$$

Convolution with respect to $t \in \mathbb{R}$ is denoted by an asterisk, $u * v(t) = \int_0^t u(t-s)v(s)\,ds$ if u and v are both supported on the positive half axis. The Fourier transform of a function $f \in \mathcal{D}$ is denoted by $\hat{f}$, $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx$, and this is extended in the usual way to functions in L^1 or in L^2 or to distributions. The Laplace transform of a function $a : [0,\infty) \to \mathbb{R}$ is denoted by $\tilde{a}(s) = \int_0^\infty e^{-st} a(t) dt$ if defined, i.e. for $s \in \mathbb{C}$ such that $\Re s > \alpha$ for some $\alpha \in [-\infty, +\infty)$. The usual notation for Lebesgue spaces $L^p(\Omega)$ and for the Sobolev spaces $H^s(\mathbb{R}^n)$ is employed, $1 \leq p \leq \infty$, $-\infty < s < \infty$. Vector-valued Lebesgue and Sobolev spaces are denoted in the usual way, e.g. $L^p([0,T], H^1(\mathbb{R}^n))$ or $H^s(\mathbb{R}^3, \mathbb{R}^3)$.

Let $\mathbb{R}^+ \ni t \mapsto u(\cdot,t)$ be a measurable and locally bounded family of distributions on $\mathbb{R}^n$, and let $k : [0,\infty) \to \mathbb{R}^+$ be continuous and increasing to ∞. In this paper, the scaled version of u is denoted by u_k, defined by

$$\langle u_k(\cdot,t),\varphi\rangle = \langle u(\cdot,t),\varphi(k(t)^{-1}\cdot)\rangle$$

or, in case $u(\cdot,t)$ is a function for almost all t,

$$u_k(x,t) = k(t)^n u(k(t)x,t)\,.$$

Thus if $u(\cdot,t)\in L^1(\mathbb{R}^n)$, then also $u_k(\cdot,t)\in L^1(\mathbb{R}^n)$, and the L^1 integral is unchanged.

There is an alternative scaling $u_{T,k}$, defined by

$$\langle u_{T,k}(\cdot,t),\varphi\rangle = \langle u(\cdot,Tt),\varphi(k(T)^{-1}\cdot)\rangle$$

or if $u(\cdot,t)$ is a function for almost all t,

$$u_{T,k}(x,t) = k(T)^n u(k(T)x,Tt)$$

for $x\in\mathbb{R}^n$ and $t>0$. The result now depends on k and the parameter $T>0$. This scaling again preserves the spatial L^1-integral.

2. Scaling

Let $t\mapsto u(\cdot,t)$ be a measurable and locally bounded (with respect to the usual system of seminorms) family of tempered distributions on $\mathbb{R}^n$, and let $k:[0,\infty)\to\mathbb{R}^+$ be continuous and increasing to ∞. Assume that the scaled family $u_k(\cdot,t)$ converges in $\mathcal{D}'$ to some $U\in\mathcal{D}'$ as $t\to\infty$, that is,

$$\langle u_k(\cdot,t),\varphi(\cdot)\rangle \to \langle U,\varphi\rangle \quad \forall\varphi\in C_0^\infty(\mathbb{R}^n)\,. \tag{1}$$

It is necessary to relate this property to the behavior of the family $u_{T,k}$ as $T\to\infty$. Using a tensor product argument, one easily sees that for all test functions $\Phi\in C_0^\infty(\mathbb{R}^n\times(0,\infty))$

$$\int_0^\infty \langle u_{T,k}(\cdot,\tau),\Phi(\cdot,\tau)\rangle d\tau = \int_0^\infty \left\langle u_k(\cdot,T\tau),\Phi\left(\frac{k(T\tau)}{k(T)}\cdot,\tau\right)\right\rangle d\tau\,. \tag{2}$$

Assuming that (1) holds, the goal is to obtain a nontrivial limit in (2) as $T\to\infty$. Then a natural assumption is that

$$\lim_{T\to\infty}\frac{k(T\tau)}{k(T)} = p(\tau) \tag{3}$$

exists for all $\tau\in\mathbb{R}^+$, since in this case $\Phi\big(\frac{k(T\tau)}{k(T)}x,\tau\big)\to\Phi(p(\tau)x,\tau)$ uniformly with all derivatives in x, boundedly in τ, and thus

$$\int_0^\infty \langle u_{T,k}(\cdot,\tau),\Phi(\cdot,\tau)\rangle d\tau \to \int_0^\infty \langle U,\Phi(p(\tau)\cdot,\tau)\rangle\, d\tau.$$

A function k that is eventually positive and for which (3) holds for some function p, for all $\tau \in (0,\infty)$ is called *regularly varying* ([1]). It is known that in this case $p(\tau) = \tau^\alpha$ for some $\alpha \in \mathbb{R}$ which is called the *index*. An equivalent condition is

$$\lim_{T\to\infty} \frac{k(T\tau)}{b(T)} = p_1(\tau)$$

for some b, p_1 and all τ in a neighborhood of $\tau = 1$. In this case, necessarily $p_1(\tau) = C\tau^\alpha$ for some $C = p_1(1) > 0$, and one may choose $b(t) = k(t)$.

Since k is increasing by assumption, α must be non-negative. If (3) holds merely for τ in a set of positive Lebesgue measure, it must already hold for all positive τ, and the limit is uniform on closed subintervals of $(0,\infty)$ (see [1]). Moreover, α can be recovered from the limit (which always exists)

$$\alpha = \lim_{t\to\infty} \frac{\log\, k(t)}{\log t}\,.$$

Returning to (2) and assuming now that k is regularly varying with index $\alpha \geq 0$, one obtains

$$\int_0^\infty \langle u_{T,k}(\cdot,\tau), \Phi(\cdot,\tau)\rangle d\tau \;\to\; \int_0^\infty \langle U, \Phi_\tau(\cdot,\tau)\rangle d\tau$$

where $\Phi_\tau(x,\tau) = \Phi(x\tau^\alpha,\tau)$. If U is a locally integrable function, this means

$$u_{T,k}(x,\tau) \sim \tau^{-n\alpha} U(x\tau^{-\alpha})\,.$$

in a suitable sense (e.g. pointwise a.e. in (x,τ)), as $T\to\infty$. All this proves the first statement of the following proposition.

Proposition 2.1. *Let $t \mapsto u(\cdot,t)$ be a measurable locally bounded family of tempered distributions on $\mathbb{R}^n$. Let $k : [0,\infty) \to \mathbb{R}^+$ be continuous and increasing to ∞, and regularly varying with index $\alpha \geq 0$. Define u_k and $u_{T,k}$ as above.*

a) *If U is a distribution on $\mathbb{R}^n$ such that $u_k(\cdot,\tau) \to U$, then $u_{T,k}(\cdot,\tau) \to \bar U(\cdot,\tau)$ for a.e. τ in $\mathcal{D}'$ as $T\to\infty$, where*

$$\langle \bar U(\cdot,\tau), \varphi\rangle = \langle U, \varphi(\tau^\alpha\cdot)\rangle\,. \tag{4}$$

b) *Assume that $U \in H^s(\mathbb{R}^n)$ for some $s\in\mathbb{R}$ and that $u_{T,k}$ converges to $\bar U$, defined in (4), locally uniformly in τ as an H^s-valued function. Then also $u_k(\cdot,t) \to U$ in $H^s(\mathbb{R}^n)$, as $t\to\infty$.*

c) *Suppose $U \in L^r(\mathbb{R}^n)$ for some $r\in[1,\infty]$ and $u_k(\cdot,t) \to U$ in $L^r(\mathbb{R}^n)$. Set $w(x,t) = k(t)^{-n}U(xk(t)^{-1})$, then*

$$\|u(\cdot,t) - w(\cdot,t)\|_{L^r} = o\big(k(t)^{n(r^{-1}-1)}\big)\,.$$

Moreover, if $D^mU \in L^r(\mathbb{R}^n)$ and $D^mu_k(\cdot,t) \to D^mU$ in $L^r(\mathbb{R}^n)$ for some partial derivative D^m of order m, then

$$\|D^mu(\cdot,t) - D^mw(\cdot,t)\|_{L^r} = o\big(k(t)^{n(r^{-1}-1)-m}\big)\,.$$

Proposition 2.2. *Let k, l be regularly varying functions with index $\alpha \geq 0$ such that $\lim_{t\to\infty} \frac{l(t)}{k(t)} = C \in (0,\infty)$, and let $U \in \mathcal{D}'$. Suppose that $u_{T,k}(\cdot,t) \to \bar{U}(\cdot,t)$, defined as in* (4)*, locally uniformly in some H^s with $s \in \mathbb{R}$. Then $u_{T,l}(\cdot,t) \to \bar{V}(\cdot,t)$ in H^s, locally uniformly in t, where for $\varphi \in \mathcal{D}$*

$$\langle \bar{V}(\cdot,\tau), \varphi \rangle = \langle U, \varphi(C^{-1}\tau^{\alpha}\cdot) \rangle .$$

Thus if U is a function, then

$$u_{T,l}(x,\tau) \sim C^n \tau^{-n\alpha} U(Cx\tau^{-\alpha})$$

in a suitable sense, and the estimates of part c) *of the previous proposition hold.*

The easy proofs of the remaining parts of Proposition 2.1 and of Proposition 2.2 are left to the reader.

If $u(\cdot,t) \to 0$ in some weak sense as $t \to \infty$, then it is possible to find $k(\cdot)$, going to ∞, such that $u_{T,k} \to 0$ as a distribution on $\mathbb{R}^n \times (0,\infty)$. Similarly, if $u(\cdot,t) \in L^1$ for all t and $\int_{\mathbb{R}^n} u(\cdot,t) = C$ is constant, then one expects that $u_{T,k}(\cdot,t) \to C\delta_0$ if $k(\cdot)$ goes to ∞ sufficiently rapidly. The limiting cases $U = 0$ and $U(\cdot,t) = C\delta_0$ should be excluded and will be called *trivial.* Non-trivial limiting distributions U should be neither zero nor supported on $\{0\} \times [0,\infty) \subset \mathbb{R}^n \times [0,\infty)$.

3. Identifying asymptotic limits

In this section, I shall classify the types of limiting behavior that are possible for distributional solutions of the partial integro-differential equation

$$u_t(x,t) = a_0 \Delta u(x,t) + \int_0^t a(t-s)\Delta u(x,s)ds \tag{5}$$

or more shortly $u_t = a_0\Delta u + a * \Delta u$ for $x \in \mathbb{R}^n$, $t > 0$, with initial data $u(\cdot,0) = u_0$. It will be assumed throughout that $a_0 \geq 0$, the integral kernel a is bounded on any set $[\epsilon,\infty)$ and integrable on $(0,1)$, and $u_0 \in L^1(\mathbb{R}^n)$. Let us begin by describing the limiting equations and their solutions. For $\alpha > 0$, the *Mittag-Leffler* function E_α is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \tag{6}$$

see [3]. This is an entire function for any $\alpha > 0$. Special cases include $E_1(z) = e^z$, $E_2(z^2) = \cosh(z)$, and $E_{\frac{1}{2}}(z) = e^{z^2} erfc(-z)$, where $erfc$ is the complementary error function. For $0 < \alpha < 2$, $\alpha \neq 1$, there is an asymptotic expansion

$$E_\alpha(z) = \sum_{n=1}^{N} \frac{z^{-n}}{\Gamma(1-\alpha n)} + O\big(|z|^{-N-1}\big)$$

as $z \to \infty$ in a sector about the negative real axis, where the reciprocal of the Γ-function is extended as zero at the poles of Γ. In particular, E_α is bounded on the negative real axis for all $\alpha \leq 2$; see [3] for details and other properties.

For $\alpha > 0$, $\lambda \in \mathbb{R}$, the function $u(t) = E_\alpha(-\lambda t^\alpha)$ is the solution of the scalar integral equation

$$u(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds = 1 \tag{7}$$

as a direct calculation shows.

Let $0 < \alpha \leq 2$. Define $w_\alpha(\cdot, t)$ as the tempered distribution on $\mathbb{R}^n$ whose spatial Fourier transform is

$$\hat{w}_\alpha(\xi, t) = E_\alpha\big(-|\xi|^2 t^\alpha\big) \quad (\xi \in \mathbb{R}^n,\ t \geq 0) \tag{8}$$

for $t > 0$. If $\alpha < 2$, the asymptotic behavior of E_α implies that $w_\alpha(\cdot, t) \in H^s(\mathbb{R}^n)$ for $s < 2 - \frac{n}{2}$. In particular, if $\alpha < 2$, with the exception $\alpha = 1$, then $w_\alpha(\cdot, t) \in L^2(\mathbb{R}^n)$ if and only if $n \leq 3$ and $w_\alpha(\cdot, t) \in L^\infty(\mathbb{R}^n)$ if and only if $n = 1$. For $\alpha = 2$, one has $\hat{w}_2(\xi, t) = \cos(|\xi| t)$ and thus $w_2(\cdot, t) \in H^s(\mathbb{R}^n)$ for all $s < -\frac{n}{2}$. If $\alpha = 1$, then $\hat{w}_1(\xi, t) = e^{-|\xi|^2 t}$, and w_1 is the well-known fundamental solution of the heat equation. Also, $w_\alpha(\cdot, t) \to \delta_0$ as $t \to 0$ for all α, in the sense of distributions.

The distribution w_α solves the integro-differential equation

$$w(\cdot, t) - \Delta\left(\int_0^t \Gamma(\alpha)^{-1} (t-s)^{\alpha-1} w(\cdot, s)\, ds\right) = \delta_0\,.$$

This follows immediately from (7) and (8). If $\alpha = 1$, this is the heat equation, and if $\alpha = 2$, this is the wave equation. For $1 < \alpha < 2$, equation (12) can be differentiated formally, resulting in the *fractional heat equation*

$$w_t(\cdot, t) = \Delta\left(\int_0^t \Gamma(\alpha-1)^{-1} (t-s)^{\alpha-2} w(\cdot, s)\, ds\right). \tag{9}$$

These equations have been studied in [6, 7, 10, 11, 12, 16].

Returning to solutions of (5), let us assume that $t \mapsto u(\cdot, t)$ is a function with values in the set of tempered distributions such that for all test functions $\Phi \in C_0^\infty(\mathbb{R}^n \times [0, \infty))$ the equation holds

$$\int_0^\infty \left\langle u(\cdot, s), \Phi_t(\cdot, s) + a_0 \Delta\Phi(\cdot, s) + \int_s^\infty a(t-s)\Phi(\cdot, t)\, dt \right\rangle ds + \langle u(\cdot, 0), \Phi(\cdot, 0)\rangle = 0.$$

With $A(t) = a_0 + \int_0^t a(s)$, one can write equivalently

$$u(\cdot,t) = u(\cdot,0) + \Delta\left(\int_0^t A(t-s)u(\cdot,s)\,ds\right)$$

in the sense of distributions, or with $\Phi_1(x,s) = \int_s^\infty A(t-s)\Phi(x,t)\,dt$

$$\int_0^\infty \langle u(\cdot,s), \Phi(\cdot,s) - \Delta\Phi_1(\cdot,s)\rangle\,ds = \Big\langle u(\cdot,0), \int_0^\infty \Phi(\cdot,t)dt\Big\rangle.$$

Let us consider solutions of (5) with the scaling

$$u_{T,k}(x,\tau) = k(T)^n u(k(T)x, T\tau)$$

introduced earlier, where k is left unspecified for now. Set $K = k(T)$, then $v = u_{T,k}$ is seen to be a distributional solution of the problem

$$v = u_{0,K} + \frac{T}{K^2} A_T * \Delta v$$

where $u_{0,K} = K^n u_0(Kx)$ and $A_T(t) = A(Tt) = a_0 + \int_0^{Tt} a(s)\,ds$. Also, the spatial Fourier transform $\hat{u}_{T,k}$ solves

$$\hat{u}_{T,k}(\xi,t) + |\xi|^2 \frac{T}{K^2} A_T * \hat{u}_{T,k}(\xi,t) = \hat{u}_0\Big(\frac{\xi}{K}\Big) \tag{10}$$

in the sense of distributions. If one wishes to obtain a limiting equation of a similar form, one is led to assume that there exist functions p, A_∞ such

$$\lim_{T\to\infty} \frac{A_T(t)}{p(T)} = A_\infty(t)\,.$$

Let us also assume that A is eventually positive (not necessarily bounded away from zeros). As explained in the previous section, this implies that A is regularly varying and that one may choose $p(T) = cA(T) = cA_T(1), A_\infty(t) = c^{-1}t^\beta$ for some $\beta \in \mathbb{R}$ and any constant $c > 0$. If the kernel A_∞ is to be integrable at $t = 0$, then one should require $\beta > -1$. Since $A' = a$ was assumed to be bounded on $(1,\infty)$, necessarily $\beta \le 1$. This motivates the main assumption in the following result.

Theorem 3.1. *Let u be a solution of* (5) *in the sense described above, and let A be eventually positive and regularly varying with index $\beta \in (-1,1]$. Assume that there exist a non-decreasing function $k_0 : [0,\infty) \to [0,\infty)$ with $k_0(\infty) = \infty$, a sequence $T_n \to \infty$ and a non-trivial limiting distribution u_∞ on $\mathbb{R}^n \times [0,\infty)$, $u_\infty(\cdot,t) \in H^s(\mathbb{R}^n)$ for a.e. t for some fixed $s > -\infty$ such that*

$u_{T_n,k_0} \to u_\infty$ *as* $n \to \infty$, *a.e. boundedly in* $H^s(\mathbb{R}^n)$. *Then one may choose* $k(t) = \sqrt{tA(t)\Gamma(1+\beta)}$, *and with this choice*

$$u_{T,k} \to U_0 w_{1+\beta} = u_\infty$$

where $U_0 = \int_{\mathbb{R}^n} u_0(x)\,dx$ *and* $w_{1+\beta}$ *is the distributional solution of the integro-differential equation*

$$w(\cdot,t) - \Delta\left(\int_0^t \Gamma(1+\beta)^{-1}(t-s)^\beta w(\cdot,s)\,ds\right) = \delta_0$$

defined in (8).

Proof. It should be noted that A is regularly varying with index β iff k is regularly varying with index $\frac{1+\beta}{2}$. The assumptions for A imply that

$$\frac{A_T(t)}{A(T)\Gamma(1+\beta)} \to A_\infty(t) = \frac{t^\beta}{\Gamma(1+\beta)}$$

as $T \to \infty$. Let $\Phi \in C_0^\infty(\mathbb{R}^n, \times(0,\infty))$ be a test function such that

$$\langle\langle \Delta\Phi_\infty, u_\infty\rangle\rangle \neq 0, \qquad \langle\langle u_\infty, \Phi\rangle\rangle \neq U_0 \int_0^\infty \Phi(0,t)\,dx\,dt,$$

where $\Phi_\infty(x,s) = \int_t^\infty A_\infty(t-s)\Phi(x,t)\,dt$. This is possible since the limiting distribution is non-trivial (not identically equal to zero, not supported on a subset of $\mathbb{R}^n \times [0,\infty)$ and in some fixed H^s for a.e. t. Set $\Phi_n(x,s) = \int_s^\infty \frac{A(T_n(t-s))}{A(T_n)\Gamma(1+\beta)}\Phi(x,t)\,dt$. Then $\Phi_n \to \Phi_\infty$ together with all derivatives.

Let us first show that k_0 may be replaced with k, i.e., $u_{T_n,k} \to u_\infty$. Indeed, since

$$\langle\langle \Phi, u_{T_n,k_0}\rangle\rangle = \int_0^\infty \langle \Phi(\cdot,t), u_{0,k(T_n)}\rangle\,dt + \frac{\Gamma(1+\beta)T_nA(T_n)}{k_0(T_n)^2}\langle\langle \Delta\Phi_n, u_{T_n,k_0}\rangle\rangle$$

and all terms except the fraction have limits as $n \to \infty$, it follows that

$$L = \lim_{n\to\infty} \frac{\Gamma(1+\beta)T_nA(T_n)}{k_0(T_n)^2}$$

exists. Of course, $L > 0$, and after replacing k_0 with $\sqrt{L}k_0$, one may assume without loss of generality that $L = 1$. One can therefore replace k_0 with $k(t) = \sqrt{\Gamma(1+\beta)tA(t)}$ and obtain that $u_{T_n,k} \to u_\infty$.

Observe next that (10) now takes the form

$$\hat{u}_{T,k}(\xi,t) + \frac{|\xi|^2}{A(T)\Gamma(1+\beta)} A_T * \hat{u}_{T,k}(\xi,t) = \hat{u}_0\Big(\frac{\xi}{K}\Big)$$

for all ξ. Thus one can write $\hat{u}_{T,k}(\xi,t) = z_T(|\xi|^2,t)\hat{u}_0\left(\frac{\xi}{K}\right)$, where $z_T(\rho,\cdot)$ solves the equation

$$z_T(\rho,\cdot) + \frac{\rho}{A(T)\Gamma(1+\beta)} A_T * z_T(\rho,\cdot) = 1.$$

Note that $z_T(\rho,t) = z(\lambda, Tt)$, where $z(\lambda,\cdot)$ solves

$$z(\lambda,\cdot) + \lambda A * z(\lambda,\cdot) = 1 \tag{11}$$

with $\lambda = \frac{\rho}{TA(T)\Gamma(1+\beta)}$.

By Lemma A.2, as $T \to \infty$, $z_T(\rho,t)$ converges to $E_{1+\beta}(-\rho t^{1+\beta})$, locally uniformly in ρ and $t \geq 0$, and thus $\hat{u}_{T,k}$ converges pointwise in ξ, locally uniformly in t, to $\hat{v}(\xi,t) = U_0 E_{1+\beta}(-|\xi|^2 t^{1+\beta}) = U_0 w_{1+\beta}(\xi,t)$, that is, a solution of the limiting equation

$$\hat{v}(\xi,t) + |\xi|^2 \int_0^t \frac{(t-s)^\beta}{\Gamma(1+\beta)} \hat{v}(\xi,s)\, ds = U_0\,.$$

But for the subsequence T_n, the limit is $\hat{u}_\infty$. Therefore, convergence holds along the full sequence $T \to \infty$, and the limit is $U_0 \hat{w}_{1+\beta}$. By Parseval's identity, the theorem follows. □

To sum up, the possibilities for limiting behavior identified in this result are the following:

1. Behavior like the fundamental solution of the wave equation ($\beta = 1$), expected if e.g. $a(\cdot) \sim c > 0$ or $a(t) \sim (\log t)^m$ as $t \to \infty$ for some real number m. In this case, $k(t) \sim \sqrt{c}t$ or $k(t) \sim t(\log t)^{\frac{m}{2}}$.

2. Behavior like the fundamental solution of the heat equation ($\beta = 0$), expected if e.g. $a(\cdot)$ is integrable and either $a \geq 0$ or $\int_0^\infty a(s)ds + a_0 > 0$, but also if e.g. $a(t) \sim t^{-1}$. If a is integrable, then $k(t) \sim \sqrt{At}$, where $A = a_0 + \int_0^\infty a(s)ds$, while if e.g. $a(t) \sim t^{-1}$, then $k(t) \sim \sqrt{t \log t}$.

3. Behavior like the fundamental solution of a fractional integro-differential equation of order $1+\beta$. If $0 < \beta < 1$, this is expected if e.g. $a(t) \sim t^{\beta-1}$. If $-1 < \beta < 0$, this may occur if e.g. a is negative and integrable, $\int_0^\infty a(s)ds = -a_0$, and $a(t) \sim -t^{\beta-1}$. In each case, $k(t) \sim t^{\frac{1+\beta}{2}}$. Of course, the behavior of k may be modified by additional logarithmic factors also in this case.

4. Asymptotic distributional limits

The purpose of this section is to demonstrate that under mild assumptions, solutions of (7) do converge to limiting solutions under the scaling $u \rightsquigarrow u_{T,k}$. I shall start by stating a general existence result for strong solutions that is essentially well-known for the Hilbert space case; see [15].

Proposition 4.1. *Let $u_0 \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$ and assume that*

$$\lim_{h \downarrow 0}(a_0 + \Re\tilde{a}(i\omega + h)) \geq 0$$

for all $\omega \in \mathbb{R}$. Then there exists a unique function $u \in C([0,\infty), H^s(\mathbb{R}^n))$ that solves (5) *in the sense of distributions and for which $u(\cdot, 0) = u_0$. The Fourier transform $\hat{u}$ is given by*

$$\hat{u}(\xi, t) = z(|\xi|^2, t)\hat{u}_0(\xi) \tag{12}$$

where $z(\lambda, \cdot)$ is the solution of (11).

The condition for the kernel a is equivalent to the requirement that its cosine transform is bounded below by $-a_0$ in the sense of measures. Equivalently, the measure $a_0\delta_0 + a(t)dt$ is required to be positive definite (see [8]). We do not require that $u_0 \in L^1$ and cannot assert that $u(\cdot, t) \in L^1$ for $t > 0$.

Proof. Consider the integral equations (11) for $\lambda \geq 0$. By Lemma A.1, the estimate $|z(\lambda, t)| \leq 1$ holds for all λ and t. Then define $u(\cdot, t)$ as in (12). Using Parseval's identity and the bound for $z(\lambda, \cdot)$, one obtains $\|u(\cdot, t)\|_{H^s} \leq \|u_0\|_{H^s}$ for all t. Also, $\hat{u}(\xi, t) \to \hat{u}_0(\xi)$ pointwise a.e., and by construction

$$\frac{\partial}{\partial t}\hat{u}(\xi, t) + |\xi|^2 \left(a_0\hat{u}(\xi, t) + a * \hat{u}(\xi, t)\right) = 0$$

for almost all ξ. Since $z(\lambda, \cdot)$ is continuous, locally uniformly in λ, and uniformly bounded, $\hat{u}(\xi, \cdot)$ is also continuous with values in $H^s(\mathbb{R}^n)$. Thus u is a distributional solution of (5). Uniqueness follows by taking the Fourier transform of a solution in this class and recognizing that it must have the form (12). □

Such solutions converge to the limiting solutions identified in the previous section under the scaling introduced there, if the primitive A of a is regularly varying. As explained above, these are natural conditions. From now on u_0 will always be assumed to be integrable.

Theorem 4.2. *Assume that $u_0 \in L^1(\mathbb{R}^n)$ and that*

1. *the kernel satisfies $\lim_{h \downarrow 0}(a_0 + \Re\tilde{a}(i\omega + h)) \geq 0$ for all $\omega \in \mathbb{R}$,*
2. *the primitive $A(t) = a_0 + \int_0^t a(s)ds$ is eventually positive and regularly varying with index $\beta \in (-1, 1]$.*

Set $k(t) = \sqrt{tA(t)\Gamma(1+\beta)}$, then for the solution u of (5) *found in the previous proposition*

$$u_{T,k}(\cdot, t) \;\to\; U_0 u_\infty(\cdot, t)$$

in $H^s(\mathbb{R}^n)$, if $s < -\frac{n}{2}$. Here

$$\hat{u}_\infty(\xi, t) = E_{1+\beta}\big(-|\xi|^2 t^{1+\beta}\big), \quad U_0 = \int_{\mathbb{R}^n} u_0(x)dx.$$

The convergence is uniform on any interval $[c, d] \subset (0, \infty)$.

A few remarks can serve to put the result in perspective. First, if $\beta = 1$, then $\hat{u}_\infty(\xi, t) = \cos(|\xi|t)$, and thus $u_\infty(\cdot, t) \notin H^{-\frac{n}{2}}(\mathbb{R}^n)$. Thus for a result that covers the entire range $\beta \in (-1, 1]$, one cannot expect convergence in better spaces than $H^s(\mathbb{R}^n)$ with $s < -\frac{n}{2}$. Also, conditions 1 and 2 in the above result are independent. For example, the kernel $a(t) = \cos(t)$ with $A(t) = \sin(t)$ and $\tilde{a} = \frac{1}{2}(\delta_i + \delta_{-i})$ in the sense of measures on $i\mathbb{R}$ satisfies condition 1, but is not regularly varying. The kernel $a(t) = 1 - e^{-t}$ has the antiderivative $A(t) = a_0 + t - 1 + e^{-t}$ which is regularly varying with $\beta = 1$, but since $\Re\tilde{a}(i\omega) = -\frac{1}{1+\omega^2}$ for $\omega \neq 0$, condition 1 is not satisfied if $a_0 < 1$. Finally, it should be recalled that $L^1(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ for $s < -\frac{n}{2}$, but not for larger s.

Proof. Let u be the distributional solution in $C([0, \infty), H^s(\mathbb{R}^n))$ constructed in Proposition 4.1. Then $\hat{u}_{T,k}$ satisfies

$$\hat{u}_{T,k}(\xi, t) + \frac{|\xi|^2}{A(T)\Gamma(1 + \beta)} A_T * \hat{u}_{T,k}(\xi, t) = \hat{u}_0 \left(\frac{\xi}{k(T)} \right)$$

and therefore with $k(T) = \sqrt{TA(T)\Gamma(1 + \beta)}$,

$$\hat{u}_{T,k}(\xi, t) = z\left(\frac{|\xi|^2}{k^2(T)}, Tt \right) \hat{u}_0\left(\frac{\xi}{k(T)} \right)$$

where $z(\lambda, \cdot)$ solves (11) and therefore $v_\lambda(t) = z\big(\frac{\lambda}{TA(T)\Gamma(1+\beta)}, Tt\big)$ solves (22). By Lemma A.2, as $T \to \infty$,

$$v_\lambda(t) = z\left(\frac{\lambda}{TA(T)\Gamma(1 + \beta)}, Tt \right) \to E_{1+\beta}(-\lambda t^{1+\beta})$$

for all $\lambda > 0$. Consequently, $\hat{u}_{T,k}(\xi, t) \to \hat{u}_0(0) E_{1+\beta}(-|\xi|^2 t^{1+\beta})$ pointwise for all ξ, locally uniformly in t.

To prove that convergence holds in $H^s(\mathbb{R}^n)$ for $s < -\frac{n}{2}$, one invokes again Lemma A.1 to deduce that $|z(\lambda, t)| \leq 1$ and therefore

$$|\hat{u}_{T,k}(\xi, t)| \leq \left| \hat{u}_0\left(\frac{\xi}{k(T)} \right) \right| \leq C$$

for all ξ and T. Then for $t > 0$

$$\|u_{T,k}(\cdot, t) - U_0 u_\infty(\cdot, t)\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \big| \hat{u}_{T,k}(\xi, t) - U_0 E_{1+\beta}\big(- |\xi|^{2+2\beta} t \big) \big|^2 .$$

Since $2s < -n$, Lebesgue's dominated convergence theorem implies the conclusion. $\square$

5. Asymptotic limits in L^2

In this section, equation (5) will be considered under the scaling $u \rightsquigarrow u_{T,k}$, with the goal of proving that limiting solutions are attained in $L^\infty_{loc}(0,\infty;L^2(\mathbb{R}^n))$ or more generally $L^\infty_{loc}(0,\infty;H^s(\mathbb{R}^n))$ with $s \geq 0$, provided the initial data are in L^2 or H^s. The limiting solutions were identified in Section 2 and are unbounded in any $L^r(\mathbb{R}^n)$, $r > 1$ as $t \to 0$. Thus one cannot expect uniform convergence up to $t = 0$. A convergence result in L^2 or in a better space allows one to obtain the exact asymptotic behavior of solutions of (7) in this space to leading order, by Proposition 2.1(c).

I only have a result for the case where A is regularly varying with index $\beta = 0$, i.e. $0 < a_0 + \int_0^\infty a(t)dt < \infty$. The result is independent of the space dimension. In this case the limiting equation is the heat equation. Note that f $\beta \neq 0$, the limiting distributional solution is in $L^2(\mathbb{R}^n)$ for $t > 0$ if and only if $n \leq 3$. Thus a result that holds for all spatial dimensions cannot be expected if $\beta \neq 0$.

Theorem 5.1. *Assume that*

1. *the initial data satisfy $u_0 \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ for some $s \geq 0$,*
2. *for some $\alpha > 0$, $\tilde{a}$ can be extended to the half plane $\{s \in \mathbb{C} \,|\, \Re s \geq -\alpha\}$ and either $a_0 > 0$ and $a_0 + \tilde{a}(i\omega)) > 0$, or $\Re\tilde{a}(-\alpha + i\omega) \geq 0$ for all ω.*

Set $k(t) = \sqrt{tA(t)}$, then for the solution u of (5) *found in Proposition* 3.1

$$u_{T,k}(\cdot,t) \;\to\; U_0 u_\infty(\cdot,t)$$

in $H^s(\mathbb{R}^n)$, where $U_0 = \int_{\mathbb{R}^n} u_0(x)dx$. Here $\hat{u}_\infty(\xi,t) = e^{-|\xi|^2 t}$, that is, u_∞ is the fundamental solution of the heat equation. The convergence is uniform on any compact subinterval of $(0,\infty)$.

Proof. Note that assumption 2 implies that $0 < a_0 + \int_0^\infty a(t)dt < \infty$. Thus A is regularly varying with index $\beta = 0$. From the proof of Theorem 4.2 one sees that

$$\hat{u}_{T,k}(\xi,t) = z\left(\frac{|\xi|^2}{TA(T)}, Tt\right)\hat{u}_0\left(\frac{\xi}{k(T)}\right) = z(\lambda, Tt)\hat{u}_0\left(\frac{\xi}{k(T)}\right)$$

with $\lambda = \frac{|\xi|^2}{TA(T)}$. By Lemma A.1, there are therefore estimates, valid for all sufficiently large T,

$$\left|z\left(\frac{|\xi|^2}{TA(T)}, Tt\right)\right| \leq \begin{cases} 2e^{-c|\xi|^2 t} & \text{if } |\xi|^2 \leq dT \\ 2e^{-cdTt} & \text{if } |\xi|^2 \geq dT\,. \end{cases}$$

Here $c = \frac{\epsilon}{2A_\infty}$, $d = \frac{A_\infty L}{2}$ are positive constants, and ϵ, L are as in Lemma A.1. Now consider

$$
\begin{aligned}
&\|u_{T,k}(\cdot,t) - U_0 u_\infty(\cdot,t)\|_{H^s} \\
&\qquad = \left(\int \left(1+|\xi|^{2s}\right) \left| z(\lambda,Tt)\hat{u}_0\left(\frac{\xi}{k(T)}\right) - U_0 e^{-|\xi|^2 t} \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\qquad \le \left(\int_{|\xi|\le R} \left(1+|\xi|^{2s}\right) \left| z(\lambda,Tt)\hat{u}_0\left(\frac{\xi}{k(T)}\right) - U_0 e^{-|\xi|^2 t} \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\qquad\quad + \left(\int_{|\xi|\ge R} \left(1+|\xi|^{2s}\right) \left| z(\lambda,Tt)\hat{u}_0\left(\frac{\xi}{k(T)}\right) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\qquad\quad + \left(\int_{|\xi|\ge R} \left(1+|\xi|^{2s}\right) U_0^2 e^{-2|\xi|^2 t} d\xi \right)^{\frac{1}{2}}
\end{aligned}
$$

for arbitrary R, where as before $\lambda = \frac{|\xi|^2}{TA(T)}$. Then for given $\delta > 0$ and $[a,b] \subset (0,\infty)$, choose R so large that the last integral is less than δ, uniformly in $t \in [a,b]$, and then choose T large enough such that the first integral is also bounded by δ, uniformly in t. This is possible because $z\left(\frac{|\xi|^2}{TA(T)}, Tt\right) \to e^{-|\xi|^2 t}$ as $T \to \infty$ by Lemma A.2, locally uniformly in ξ, and because $\hat{u}_0$ is continuous. Then if $dT \ge R^2$, the second integral in the last expression can be estimated by

$$
\cdots \le \left(\int_{|\xi|\ge R} \left(1+|\xi|^{2s}\right) e^{-2cdTt} \left| \hat{u}_0\left(\frac{\xi}{k(T)}\right) \right|^2 d\xi \right)^{\frac{1}{2}} \le e^{-cdTt} k(T)^{\frac{n}{2}+s} \|u_0\|_{H^s}
$$

which is also smaller than δ, if T is chosen sufficiently large, locally uniformly in t. The proof of this theorem is therefore complete. □

Using Proposition 2.2, one obtains

Corollary 5.2. *Under the assumptions of Theorem* 5.1*, the solution u of* (5) *satisfies*

$$
\|u(\cdot,t) - w(\cdot,t)\|_{H^s} = o\left(t^{-\frac{n}{4}}\right)
$$

where $w(x,t) = U_0 \left(4A_\infty \pi t\right)^{-\frac{n}{2}} \exp\left(-|x|^2/(4A_\infty t)\right)$ is the solution of the heat equation

$$
w_t = A_\infty \Delta w, \quad w(\cdot,0) = U_0 \delta_0
$$

and $A_\infty = a_0 + \int_0^\infty a(s)\, ds$.

6. Linear viscoelasticity

Consider a viscoelastic material with mass density $\rho = 1$ occupying all $\mathbb{R}^3$, and denote the displacement of a material point at position x and time t by $u(x,t)$ and the velocity at this point by $v(x,t) = \partial_t u(x,t)$. Let us assume that the material is at rest for $t < 0$ and prescribe an initial velocity field $v(\cdot,0) = v_0$. The BOLTZMANN model for linear isotropic homogeneous viscoelasticity ([13]) leads to the equations of motion

$$v_t = a_0 \Delta v + \frac{a_0 + 2b_0}{3} \nabla\nabla \cdot v + a * \Delta v + \frac{a + 2b}{3} * \nabla\nabla \cdot v \,. \tag{13}$$

Here $a_0 \geq 0$, $b_0 \in \mathbb{R}$, and a, b are suitable scalar-valued functions that describe the stress response of the material under shear and compression, respectively.

The reader should recall the well-known decomposition into divergence free and gradient components, as follows. For $u \in H^s(\mathbb{R}^3, \mathbb{R}^3)$, let Pu, $Qu \in H^s$ be defined by

$$P\hat{u}(\xi) = \frac{\xi\xi^T}{|\xi|^2}\hat{u}(\xi), \qquad Q\hat{u}(\xi) = \left(1 - \frac{\xi\xi^T}{|\xi|^2}\right)\hat{u}(\xi)\,.$$

Then for an H^s-valued solution v of (13), one obtains that $p = Pv$ and $q = Qv$ satisfy the equations

$$\begin{aligned} p_t &= \beta_0 \Delta p + \beta * \Delta p \\ q_t &= a_0 \Delta q + a * \Delta q \end{aligned}$$

with $\beta_0 = \frac{4a_0 + 2b_0}{3}$ and $\beta(t) = \frac{4a(t) + 2b(t)}{3}$ and with initial data $p(\cdot,0) = p_0 = Pv_0$ and $q(\cdot,0) = q_0 = Qv_0 = v_0 - p_0$. In addition, $\nabla \times p = 0$ and $\nabla \cdot q = 0$ in the sense of distributions. Thus p and q satisfy scalar integro-differential equations, and as in Proposition 4.1 one obtains an existence result together with a representation formula for the solution, given next.

Proposition 6.1. *Let $v_0 \in H^s(\mathbb{R}^3, \mathbb{R}^3)$ for some $s \in \mathbb{R}$ and assume that*

$$\begin{aligned} &\lim_{h \downarrow 0}(a_0 + \Re\tilde{a}(i\omega + h)) \geq 0 \quad \forall \omega \in \mathbb{R} \\ &\lim_{h \downarrow 0}(\beta_0 + \Re\tilde{\beta}(i\omega + h)) \geq 0 \quad \forall \omega \in \mathbb{R}\,. \end{aligned}$$

Then there exists a unique function $v = p + q \in C([0,\infty), H^s(\mathbb{R}^3, \mathbb{R}^3))$ that solves (13) *in the sense of distributions and for which $v(\cdot,0) = v_0$. The Fourier transforms $\hat{p}$, $\hat{q}$ are given by*

$$\hat{p}(\xi,t) = z_1(|\xi|^2, t)\hat{p}_0(\xi) \tag{14}$$

$$\hat{q}(\xi,t) = z(|\xi|^2, t)\hat{q}_0(\xi)\,, \tag{15}$$

where $z(\lambda,\cdot)$ is the solution of (11) *and z_1 solves* (11) *with a_0, $a(\cdot)$ replaced by β_0, $\beta(\cdot)$.*

Let us now consider the case where $a, \beta \in L^1$, corresponding to a viscoelastic material with vanishing elastic equilibrium response, i.e., a liquid. In this case, the limiting behavior is expected to resemble the fundamental solution of the compressible Stokes system

$$w_t = A\Delta w + (B - A)\nabla\nabla \cdot w \tag{16}$$

where $A = a_0 + \int_0^\infty a(s)\,ds$ and $B = \beta_0 + \int_0^\infty \beta(s)\,ds$. The fundamental solution is known to be the matrix valued function $W(x,t) = U(x, Bt) + V(x, At)$, where

$$\hat{U}(\xi, t) = \frac{\xi\xi^T}{|\xi|^2} e^{-|\xi|^2 t}, \qquad \hat{V}(\xi, t) = \left(E - \frac{\xi\xi^T}{|\xi|^2}\right) e^{-|\xi|^2 t}$$

with E denoting the identity matrix. In real terms, U and V can be expressed in terms of error functions (Kummer functions, confluent hypergeometric functions), e.g.

$$U_{ij}(x) = \partial_i \partial_j \left(\frac{1}{4\pi|x|} Erf\left(\frac{|x|}{\sqrt{4t}} \right) \right) .$$

A recent derivation of these fundamental solutions in a more general situation may be found in [17]. Note that these functions are not integrable with respect to x, since their Fourier transforms are not continuous at $\xi = 0$; indeed they behave like $O(|x|^{-3})$ as $|x| \to \infty$ for fixed $t > 0$, due to well-known asymptotic results for the Kummer function. Using the arguments that led to the proof of Theorem 5.1, one can now describe the asymptotic behavior of solutions of (13) in terms of U and V. For this purpose, let us assume the following:

- For some $s \geq 0$, $v_0 \in L^1(\mathbb{R}^3, \mathbb{R}^3) \cap H^s(\mathbb{R}^3, \mathbb{R}^3)$.
- For some $\alpha > 0$, the Laplace transforms $\tilde{a}$, $\tilde{\beta}$ can be extended to the half plane $\{z \,|\, \Re z \geq -\alpha\}$
- Either $a_0 > 0$ and $a_0 + \tilde{a}(i\omega)) > 0$, or $\Re\tilde{a}(-\alpha + i\omega) \geq 0$ for all ω.
- The same assumption for β_0 and $\beta(\cdot)$.

As in the previous section, one can then use the representation formulae (14) and (15) together with the results of Appendix A to prove the following result.

Theorem 6.2. *Under these assumptions, the solution v of* (13) *satisfies*

$$\|v(\cdot, t) - V_0^T U(\cdot, Bt) - V_0^T V(\cdot, At)\|_{H^s} = o\big(t^{-\frac{n}{4}}\big),$$

where $V_0 = \int_{\mathbb{R}^3} v_0(x)\,dx \in \mathbb{R}^3$, U and V are the components of the fundamental solution of the compressible Stokes *system* (16)*, and*

$$A = a_0 + \int_0^\infty a(s)ds, \quad B = \frac{4}{3}A + \frac{2}{3}\left(b_0 + \int_0^\infty b(s)ds\right) .$$

The result shows that to leading order for large t, the solution $v(\cdot, t)$ of the Boltzmann system (13) behaves like the solution of the Stokes system (16) with distributional initial data $w(\cdot, 0) = V_0\delta_0$.

A. Scalar integral equations

In the following, let us assume that $z : [0, \infty) \to \mathbb{R}$ is a solution of the scalar integro-differential equation

$$z'(t) + \lambda \left(a_0 z(t) + a * z(t) \right) = 0, \; z(0) = 1 \tag{17}$$

where $a_0 \geq 0$, $\lambda \geq 0$, and $a \in L^1_{loc}(0, \infty; \mathbb{R}), a \in L^\infty(1, \infty; \mathbb{R})$. Let $\tilde{a}$ be the Laplace transform of a, defined for $\Re s > 0$. Recall that by Parseval's identity, $a_0 + \Re \tilde{a}(s) \geq c$ for all s in the right half plane, for some non-positive constant c, if and only if

$$\int_0^T u(t) \left(a_0 u(t) + a * u(t) \right) dt \geq c \int_0^T |u(t)|^2 dt$$

for all real-valued square integrable functions u and all T; see [8].

Lemma A.1.

1. *If* $\lim_{h \downarrow 0} (a_0 + \Re \tilde{a}(i\omega + h)) \geq 0$ *for all* $\omega \in \mathbb{R}$, *then* $|z(t)| \leq 1$ *for all* t *and all* $\lambda \geq 0$.
2. *Assume that for some* $\alpha > 0$, $\tilde{a}$ *can be extended to* $\{s \in \mathbb{C} | \Re s \geq -\alpha\}$ *and that either* $a_0 > 0$ *and* $\inf_{\omega \in \mathbb{R}} (a_0 + \tilde{a}(i\omega)) > 0$, *or that* $\Re \tilde{a}(-\alpha + i\omega) \geq 0$ *for all* ω. *Then there is a constant* $\epsilon > 0$ *such that for all* $t > 0$

$$|z(t)| \leq 2e^{-\epsilon \min(\lambda, 1) t}.$$

Proof. Let us consider the more general equation

$$z'(t) + \lambda \left(a_0 + a * z(t) \right) = f(t)$$

where $f \in L^1_{loc}(0, \infty; \mathbb{R})$. It will be shown that

a) under the assumptions of part 1, for $\lambda = 1$,

$$|z(t)| \leq 1 + \int_0^t |f(s)| ds \quad \forall t, \tag{18}$$

b) under the first set of assumptions in part 2, with $f = 0$,

$$|z(t)| \leq e^{-\delta t} \quad \forall t,$$

c) under the second set of assumptions and with $f = 0$,

$$|z(t)| \leq 2e^{-\delta t} \quad \forall t. \tag{19}$$

Here $\delta = \lambda \min\{\epsilon, 1\}$, and $\epsilon > 0$ depends only on a_0 and a. Together these assertions imply the lemma.

To prove part a), multiply (17) with $z(t)$ and integrate over $[0,T]$, resulting in the identity

$$\frac{1}{2}|z(T)|^2 + \int_0^T z(t)\left(a_0 z(t) + a * z(t)\right) dt = \frac{1}{2} + \int_0^T z(t) f(t) dt \,.$$

Since $a_0 + \tilde{a}(i\omega) \geq 0$, the integral on the left is non-negative, and the inequality $\frac{1}{2}|z(T)|^2 \leq \frac{1}{2} + \int_0^T |z(t)||f(t)|dt$ follows for all T. Bihari's theorem now implies (18). Evidently this estimate is independent of $\lambda \geq 0$.

For the proof of b), set $z_\delta(t) = e^{\delta t} z(t)$ and $a_\delta(t) = e^{\delta t} a(t)$, where $\delta > 0$ will be fixed later. Then z_δ satisfies

$$z_\delta'(t) + \lambda\left((a_0 - \delta\lambda^{-1}) z_\delta(t) + a_\delta * z_\delta(t)\right) = 0 \tag{20}$$

and $z_\delta(0) = 1$. Note that $\tilde{a}_\delta(s) = \tilde{a}(s - \delta)$ whenever $\delta < \alpha$. Since $\Re\tilde{a}(s)$ is bounded on any vertical line $\Re s = \beta$ with $\beta > -\alpha$, harmonic to the right of any such line, and bounded away from 0 near $\Re s = -\alpha$ one can find $\epsilon > 0$ such that $a_0 + \Re\tilde{a}(-\epsilon + i\omega) \geq \epsilon$ for all $\omega \in \mathbb{R}$. Then also $a_0 + \Re\tilde{a}(s) \geq \epsilon$ whenever $\Re s > -\epsilon$.

Let now $\lambda \leq 1$. Set $\delta = \lambda\epsilon$, then obviously

$$a_0 - \delta\lambda^{-1} + \Re\tilde{a}_\delta(i\omega) = a_0 - \epsilon + \Re\tilde{a}(-\delta + i\omega) \geq 0 \,.$$

If $\lambda > 1$, one sets $\delta = \epsilon$ and obtains

$$a_0 - \delta\lambda^{-1} + \Re\tilde{a}_\delta(i\omega) \geq a_0 - \epsilon + \Re\tilde{a}(-\epsilon + i\omega) \geq 0 \,.$$

Part a), applied to (20), implies the desired estimate in both cases.

To prove c), note that

$$\Re\tilde{a}(i\omega) \geq \frac{c_1}{\omega^2 + \alpha^2} \tag{21}$$

for all ω, for some $c_1 > 0$, since $\Re\tilde{a}$ is harmonic and positive for $\Re s \geq -\alpha$. Also, $\arg\tilde{a}(z) = 0$ for $\Re z = 0$ and $\arg\tilde{a}(-\alpha + i\omega) \geq -\frac{\pi}{2}$ for $\omega \geq 0$ by assumption. Therefore by the maximum principle for harmonic functions, $\arg\tilde{a}(z) \geq \arg(z + \alpha)$ for all z with $\Im z \geq 0$, $\Re z \geq -\alpha$. This implies that $\Im\tilde{a}(i\omega) \geq -\alpha\omega\Re\tilde{a}(i\omega)$ for all $\omega \geq 0$. Now let $b(t) = de^{-\alpha t}$ with $d = \min\{\frac{c_1}{2\alpha^2}, \frac{1}{2\alpha}\}$, where c_1 is as in (21). Thus $b(0) = d$, $b' = -\alpha b$, and $\tilde{b}(s) = \frac{d}{s+\alpha}$. Consider the kernel

$$a_1(t) = a(t) + \lambda a * b(t) - \alpha b(t)$$

with Laplace transform

$$\tilde{a}_1(s) = \tilde{a}(s)\left(1 + \frac{\lambda d}{s + \alpha}\right) - \frac{\alpha d}{s + \alpha}$$

for $0 \leq \lambda \leq 1$. Then $\tilde{a}_1$ is analytic for $\Re s > -\alpha$, and for $s = i\omega, \omega \geq 0$ one has

$$\begin{aligned}\Re\tilde{a}_1(i\omega) &= \Re\tilde{a}(i\omega)\left(1 + \frac{\lambda d\alpha}{\omega^2+\alpha^2}\right) + \Im\tilde{a}(i\omega)\frac{\lambda d\omega}{\omega^2+\alpha^2} - \frac{\alpha^2 d}{\omega^2+\alpha^2} \\ &> \frac{c_1}{2(\omega^2+\alpha^2)} + \frac{1}{2}\Re\tilde{a}(i\omega) - \Re\tilde{a}(i\omega)\frac{\lambda d\alpha\omega^2}{\omega^2+\alpha^2} - \frac{\alpha^2 d}{\omega^2+\alpha^2} \geq 0\end{aligned}$$

uniformly in $\lambda \in [0,1]$, by the choice of d. In addition, $\Re\tilde{a}_1(s)$ is bounded uniformly in λ and s on the strip $-\frac{\alpha}{2} \leq \Re s$. One can therefore find a positive $\gamma < \alpha$ such that whenever $\Re s \geq -\gamma$, then $\frac{d}{2} + \Re a_1(s) \geq 0$.

Now the estimate (19) can be proved for small λ, say $\lambda \leq \min\{1, \frac{2\gamma}{3d}\}$. Forming the convolution of (17) with λb and adding the result to (17), one obtains the equation

$$z'(t) + \lambda b * z'(t) + \lambda\,(a + \lambda b * a) * z = 0$$

or equivalently

$$z'(t) + \lambda\,(dz(t) + a_1 * z(t)) = \lambda b(t) = \lambda d e^{-\alpha t}\,,$$

where a_1 is as above (depending also on λ). Set $\delta = \frac{\lambda d}{2}$ and as before $z_\delta(t) = e^{\delta t}z(t)$, $a_{1,\delta}(t) = e^{\delta t}a_1(t)$. The resulting equation for z_δ is

$$z_\delta'(t) + \lambda\left(\frac{d}{2}z_\delta(t) + a_{1,\delta} * z_\delta(t)\right) = \lambda d e^{(\delta-\alpha)t}, \quad z_\delta(0) = 1\,.$$

Since $\Re\tilde{a}_{1,\delta}(i\omega) = \Re\tilde{a}_1(-\delta + i\omega) \geq -\frac{d}{2}$, part a) of this proof implies that

$$|z_\delta(t)| \leq 1 + \int_0^\infty \lambda d e^{-\alpha t + \lambda d t/2} = 1 + \frac{\lambda d}{\alpha - \lambda\frac{d}{2}} \leq 2\,,$$

where the last inequality follows from $\lambda d \leq \frac{2}{3}\gamma$. This implies $|z(t)| \leq 2e^{-\lambda\epsilon t}$ with $\epsilon = \frac{d}{2}$, whenever $\lambda \leq \min\{\frac{2\gamma}{3d}, 1\}$. Reducing ϵ if necessary, inequality (19) is proved for $\lambda \leq 1$.

The proof for $\lambda \geq 1$ is similar. One considers the kernel function

$$a_2(t) = a(t) + a * b(t) - \lambda^{-1}\alpha b(t)$$

with Laplace transform

$$\tilde{a}_2(s) = \tilde{a}(s)\left(1 + \frac{d}{s+\alpha}\right) - \frac{\lambda^{-1}\alpha d}{s+\alpha}\,.$$

Then by the same argument, $\frac{d}{2} + \Re a_2(s) \geq 0$ whenever $\Re s \geq -\kappa$, for some $\kappa > 0$ that does not depend on $\lambda \geq 1$. Then form the convolution of (17) with b and add the result to (17). This yields the equation

$$z'(t) + dz(t) + \lambda\,(a_2 * z(t)) = b(t) = d e^{-\alpha t}\,.$$

with $a_2(t) = a(t) + a * b(t) - \lambda^{-1}\alpha b(t)$. Set $\delta = \min\{\kappa, \frac{d}{2}, \alpha - d\}$ and $z_\delta = e^{\delta t}$, $a_{2,\delta} = e^{\delta t} a_2(t)$. Then

$$z_\delta'(t) + (d-\delta) z_\delta(t) + \lambda\left(a_{2,\delta} * z_\delta(t)\right) = d e^{(\delta-\alpha)t}, \quad z_\delta(0) = 1\,.$$

Since $\Re \tilde{a}_{2,\delta}(i\omega) = \Re \tilde{a}_2(-\delta + i\omega) \geq -\frac{d}{2}$, part a) again implies that

$$|z_\delta(t)| \leq 1 + \int_0^\infty d e^{(\delta-\alpha)t} = 1 + \frac{d}{\alpha-\delta} \leq 2\,,$$

where the last inequality follows from $\delta \leq \alpha - d$. Therefore, $|z(t)| \leq 2e^{-\delta t} \leq 2e^{-\epsilon t}$ if ϵ as chosen earlier or possibly lowered, whenever $\lambda \geq 1$. The proof is now complete. □

Lemma A.2. *Let A_∞, $A_n \in L^1(0, T_0; \mathbb{R})$ for $n \geq 1$ such that $A_n \to A_\infty$ in L^1. For $\rho \geq 0$, let $w_n(\rho, \cdot)$ be the solution of*

$$w_n(\rho, t) + \rho A_n * w_n(\rho, t) = 1$$

for $0 \leq t \leq T_0$. Then $w_n(\rho, \cdot) \to w_\infty(\rho, \cdot)$ uniformly in $t \in [0, T_0]$, locally uniformly in ρ, where $w_\infty(\rho, t) + \rho A_\infty * w_\infty(\rho, t) = 1$. In particular, asume that A is a regularly varying kernel with index $\beta \in (-1, 1]$ and eventually positive, and set $A_T(t) = A(Tt)$. Then the solutions $v_\lambda(\cdot)$ of

$$v_\lambda(t) + \frac{\lambda}{A(T)\Gamma(1+\beta)} A_T * v_\lambda(t) = 1 \tag{22}$$

converge uniformly in t and locally uniformly in λ to $E_{1+\beta}(-\lambda t^{1+\beta})$.

Proof. The first assertion follows from a standard argument for Neumann series. Since $\frac{A(Tt)}{A(T)\Gamma(1+\beta)} \to \frac{t^\beta}{\Gamma(1+\beta)}$, pointwise in t and also in $L^1(0, T_0)$, the second assertion follows as well. □

B. Two examples

Here are two explicit examples of integro-differential equations of the form (5) for which the assumptions in the main results are not satisfied and the conclusions fail as well.

First consider (5) with the kernel $a(t) = \cos(t)$ and $a_0 = 0$. Thus $\tilde{a}(s) = \frac{s}{s^2+1}$, and therefore this kernel is positive definite; $\Re \tilde{a}(s) \geq 0$ for all s in the right half plane. Since $A(t) = \sin(t)$, the kernel is not regularly varying. Taking the Laplace transform with respect to t and the Fourier transform with respect to x, one obtains that the Fourier-Laplace transform solution $\hat{\tilde{u}}$ satisfies

$$s\hat{\tilde{u}} + \frac{|\xi^2| s}{s^2+1} \hat{\tilde{u}} = \hat{u}_0\,.$$

After solving for $\hat{\tilde{u}}$ and inverting the Laplace transform, one obtains

$$\hat{u}(\xi,t) = \left(\frac{1}{|\xi|^2+1} + \frac{|\xi|^2}{|\xi|^2+1}\cos\left(\sqrt{1+|\xi|^2}t\right)\right)\hat{u}_0(\xi)\,.$$

Thus $u(x,t) = u_1(x) + u_2(x,t)$, where $u_1 - \Delta u_1 = u_0$ and u_2 solves the Klein–Gordon equation $u_{2,tt} + u_2 = \Delta u_2$ with initial data $u_2(\cdot,0) = u_0 - u_1$, $u_{2,t}(\cdot,0) = 0$. Locally in x, $u(\cdot,t) \to u_1$ as $t \to \infty$, since the contributions from u_2 are radiated off to infinity. There is a nontrivial time-asymptotic limit (attained e.g. pointwise a.e. for sufficiently smooth initial data) that depends on the initial data.

As a second example, consider (5) with the kernel $a(t) = -e^{-t}$ and $a_0 = 1$. Thus $a_0 + \tilde{a}(s) = \frac{s}{s+1}$, and this kernel is also positive definite. Here $A(t) = e^{-t}$, and the kernel A can be viewed as regularly varying with index $\beta = -\infty$. As before, taking the Laplace transform with respect to t and the Fourier transform with respect to x, one obtains that the Fourier–Laplace transform solution $\hat{\tilde{u}}$ satisfies

$$s\hat{\tilde{u}} + \frac{|\xi^2|s}{s+1}\hat{\tilde{u}} = \hat{u}_0\,.$$

The equation can be solved for $\hat{\tilde{u}}$ and the Laplace transform can be inverted, and the result is

$$\hat{u}(\xi,t) = \left(\frac{1}{|\xi|^2+1} + \frac{|\xi|^2}{|\xi|^2+1}e^{-(1+|\xi|^2)t}\right)\hat{u}_0(\xi)\,.$$

In this case therefore $u(x,t) = u_1(x) + u_2(x,t)$, where as before $u_1 - \Delta u_1 = u_0$ and u_2 now solves the diffusion equation $u_{2,t} + u_2 = \Delta u_2$ with initial data $u_2(\cdot,0) = u_0 - u_1$. Again, there is a nontrivial time-asymptotic limit that depends on the initial data, namely $u(\cdot,t) - u_1 = O(t^{-\frac{n}{2}}e^{-t})$ as $t \to \infty$, uniformly in x.

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