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Conditions for Correct Solvability of a Simplest Singular Boundary Value Problem of General Form. II

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Abstract. We consider the singular boundary value problem

$$-r(x)y'(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R} \tag{1}$$

$$\lim_{|x|\to\infty} y(x) = 0, \tag{2}$$

where $f \in L_p(\mathbb{R})$, $p \in [1,\infty]$ ($L_\infty(\mathbb{R}) := C(\mathbb{R})$), r is a continuous positive function on $\mathbb{R}$, $0 \le q \in L_1^{\text{loc}}(\mathbb{R})$. A solution of this problem is, by definition, any absolutely continuous function y satisfying the limit condition and almost everywhere the differential equation. This problem is called correctly solvable in a given space $L_p(\mathbb{R})$ if for any function $f \in L_p(\mathbb{R})$ it has a unique solution $y \in L_p(\mathbb{R})$ and if the following inequality holds with an absolute constant $c_p \in (0,\infty)$:

$$\|y\|_{L_p(\mathbb{R})} \le c_p \|f\|_{L_p(\mathbb{R})}, \quad \forall\, f \in L_p(\mathbb{R}).$$

We find a relationship between r, q, and the parameter $p \in [1,\infty]$, which guarantees the correctly solvability of the problem (1) and (2) in $L_p(\mathbb{R})$.

Keywords. First order linear differential equation, correct solvability

Mathematics Subject Classification (2000). 34B05

1. Introduction

We consider the singular boundary value problem

$$-r(x)y'(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R} \tag{1.1}$$

$$\lim_{|x|\to\infty} y(x) = 0. \tag{1.2}$$

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Here and throughout the sequel, $f \in L_p(\mathbb{R})$, $p \in [1,\infty]$ ($L_\infty(\mathbb{R}) := C(\mathbb{R})$) and

$$0 < r \in C^{\text{loc}}(\mathbb{R}), \quad 0 \le q \in L_1^{\text{loc}}(\mathbb{R}). \tag{1.3}$$

(In (1.3), we use the symbol $C^{\text{loc}}(\mathbb{R})$ to denote the set of functions defined and continuous in $\mathbb{R}$.) Throughout the paper, we assume that the above conventions are satisfied. We also define a solution of (1.1)–(1.2) as any absolutely continuous function y satisfying (1.2) and (1.1) almost everywhere on $\mathbb{R}$.

Note that the problem (1.1)–(1.2) was already considered in [1]. In particular, in [1], there were obtained general (unconditional) criteria for its correct solvability in $L_p(\mathbb{R})$, $p \in [1,\infty]$ (see §2 below for the definition of correct solvability of problem (1.1)–(1.2).) In the present paper we continue the investigation started in [1]. Our general goal is as follows: under a certain requirement (in addition to (1.3)), find conditions for correct solvability of problem (1.1)–(1.2) which can be expressed solely in terms of the functions r and q. To make this more precise, let us present one of the main results of [1]:

Theorem 1.1 (§4). *Let $p \in (1,\infty)$, $p' = p(p-1)^{-1}$. Problem* (1.1) – (1.2) *is correctly solvable in $L_p(\mathbb{R})$ if and only if the following conditions hold together:*

1) $M_p < \infty$. *Here* $M_p = \sup_{x\in\mathbb{R}} M_p(x)$, *where* (1.4)

$$M_p(x) = \Big[\int_{-\infty}^x \exp\Big(-p\int_t^x \frac{q(\xi)}{r(\xi)}d\xi\Big)dt\Big]^{\frac1p} \cdot \Big[\int_x^\infty \frac{1}{r(t)^{p'}} \exp\Big(-p'\int_x^t \frac{q(\xi)}{r(\xi)}d\xi\Big)dt\Big]^{\frac1{p'}}, \quad x\in\mathbb{R}; \tag{1.5}$$

2) $S_1 = \infty, \quad S_1 \overset{def}{=} \int_{-\infty}^0 \frac{q(t)}{r(t)}dt;$ (1.6)

3) $A_{p'} < \infty$. *Here* $A_{p'} = \sup_{x\in\mathbb{R}} A_{p'}(x)$, *where* (1.7)

$$A_{p'}(x) = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}}, \quad x\in\mathbb{R}; \tag{1.8}$$

$$d(x) = \inf_{d>0}\{d : \Phi(x,d) = 2\}, \quad \Phi(x,d) = \int_{x-d}^{x+d}\frac{q(t)}{r(t)}dt, \quad x\in\mathbb{R}. \tag{1.9}$$

It is easy to see that Theorem 1.1 "does not answer" the posed question. Indeed, in Theorem 1.1, correct solvability (or unsolvability) of problem (1.1)–(1.2) in $L_p(\mathbb{R})$, $p \in (1,\infty)$, is determined by the values of the functionals M_p, S_1 and $A_{p'}$ in the functions r and q, and we have to make a conclusion looking at the functions r and q themselves. At the same time, the values of the functionals M_p, S_1 and $A_{p'}$ are essential for the investigation of problem (1.1)–(1.2) since Theorem 1.1 is a criterion for its correct solvability. Thus we have to find a narrower version of Theorem 1.1 such that the condition 1), 2) and 3) are either equivalent to certain pointwise conditions for the functions r and q themselves or are satisfied automatically. We solve this problem in §3 in Theorem 3.2 which is the main result of the present paper. In addition, we present two additional assertions which complement the paper [1] (see Theorems 3.1 and 3.3).

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2. Preliminaries

Below we present the definition of correct solvability as well as assertions used in the proofs of the main results of this paper.

Definition 2.1 ([1]). We call the problem (1.1)–(1.2) *correctly solvable* in a given space $L_p(\mathbb{R})$ if the following conditions hold:

I) For every function $f \in L_p(\mathbb{R})$, there exists a unique solution $y \in L_p(\mathbb{R})$ of (1.1)–(1.2).

II) The solution $y \in L_p(\mathbb{R})$ of (1.1)–(1.2) satisfies the following inequality with an absolute constant $c_p \in (0, \infty)$:

$$\|y\|_p \le c_p \|f(x)\|_p, \quad \forall\, f \in L_p(\mathbb{R}).$$

Theorem 1.1 contains the criterion for the correct solvability in $L_p(\mathbb{R})$ of the problem (1.1)–(1.2) in the case $p \in (1, \infty)$. For the cases $p = 1$ and $p = \infty$, see Theorems 2.2 and 2.3 below.

Theorem 2.2 ([1]). *Problem* (1.1)–(1.2) *is correctly solvable in* $L_1(\mathbb{R})$ *if and only if the following conditions hold together:*

1) $S_1 = \infty$ *(see* (1.6)*)*; (2.1)

2) $r_0 > 0,\ r_0 = \inf_{x\in\mathbb{R}} r(x) > 0$; (2.2)

3) $M_1 < \infty$. *Here* $M_1 = \sup_{x\in\mathbb{R}} M_1(x) < \infty,$ *where* (2.3)

$$M_1(x) = \frac{1}{r(x)} \int_{-\infty}^{x} \exp\left(- \int_t^x \frac{q(\xi)}{r(\xi)} d\xi \right) dt, \quad x \in \mathbb{R}.$$

Theorem 2.3 ([1]). *Problem* (1.1)–(1.2) *is correctly solvable in* $C(\mathbb{R})$ *if and only if* $A_0 = 0$, *where* $A_0 = \lim_{|x|\to\infty} A(x)$. *Here*

$$A(x) = \int_x^{\infty} \frac{1}{r(t)} \exp\left(- \int_x^t \frac{q(\xi)}{r(\xi)} d\xi \right) dt, \quad x \in \mathbb{R}. \tag{2.4}$$

Moreover, if $A_0 = 0$, *then* $S_1 = \infty$ *(see* (1.6)*)*.

We note that in §§3–6 below we use some technical assertions and their formulations from [1]. We give their formulation in the course of our exposition.

3. Main results

Here and throughout the sequel, the symbols $c, c(\cdot), c_1, c_2, \dots$ denote absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations.

The next statement is a useful complement to Theorem 2.3.

Theorem 3.1 (§4). *Suppose that the functions r and q satisfy the conditions*

1) $S_1 = \infty$ *(see* (1.6)*)*;
2) $q_0 > 0$, *where* $q_0 = \inf_{x\in\mathbb{R}} q(x)$;
3) $q(x) \to \infty$ *as* $|x| \to \infty$.

Then problem (1.1)–(1.2) *is correctly solvable in* $C(\mathbb{R})$.

The main result of the present paper is the following.

Theorem 3.2 (§5). *Suppose that the following conditions hold:*

1) *The functions r and q are positive and continuous on $\mathbb{R}$.*
2) *There exists $a \ge 1$ and $b > 0$ and an interval (α, β) such that*

$$\frac{1}{a} \le \frac{r(t)}{r(x)}, \quad \frac{q(t)}{q(x)} \le a, \quad \text{for} \quad |t - x| \le b\,\frac{r(x)}{q(x)}, \ x \notin (\alpha, \beta); \tag{3.1}$$

and, moreover, $\gamma = \gamma(a, b) \le 1$. Here $\gamma = 3a^2 \exp(-\frac{b}{a^2})$. Then problem (1.1) – (1.2) *is correctly solvable in $L_p(\mathbb{R})$, $p \in [1, \infty]$, if and only if the conditions from the following table are satisfied:*

(3.2)

Space	$L_1(\mathbb{R})$	$L_p(\mathbb{R})$, $1 < p < \infty$	$C(\mathbb{R})$
Conditions for correct solvability	$r_0 > 0$ $q_0 > 0$	$\sigma_{p'} > 0$ $q_0 > 0$	$q(x) \to \infty$ *as* $\lvert x\rvert \to \infty$

Here $p' = \frac{p}{p-1}$ for $p \in (1, \infty)$ and

$$r_0 = \inf_{x\in\mathbb{R}} r(x), \quad q_0 = \inf_{x\in\mathbb{R}} q(x), \quad \sigma_{p'} = \inf_{x\in\mathbb{R}} r(x)^{\frac{1}{p}} q(x)^{\frac{1}{p'}}. \tag{3.3}$$

The next statement is a useful complement to Theorems 1.1, 2.2, and 2.3.

Theorem 3.3 (§6). *Suppose that the following conditions are satisfied:*

$$1)\ S_1 = S_2 = \infty,\ \textit{where}\ S_1 = \int_{-\infty}^{0} \frac{q(t)}{r(t)}dt,\ S_2 = \int_{0}^{\infty} \frac{q(t)}{r(t)}dt; \tag{3.4}$$

2) *there exists* $\delta > 0$ *such that, for* $x \in \mathbb{R}$, *(see* (1.9)*)*

$$d(t) \geq \delta d(x), \quad \textit{for}\ |t-x| \leq d(x). \tag{3.5}$$

Then if

$$\lim_{x\to-\infty} q(x) = 0 \quad \textit{or} \quad \lim_{x\to\infty} q(x) = 0$$

holds, problem (1.1)–(1.2) *cannot be correctly solvable in* $L_p(\mathbb{R})$ *for any* $p \in [1,\infty]$.

This theorem needs an explanation; for this we use the following lemma.

Lemma 3.4 ([1]). *Let* $S_1 = \infty$ *(see* (3.4)*). Then the function* $d(x)$ *is defined for* $x \in \mathbb{R}$. *Moreover,* $d(x)$ *is continuous and positive on* $\mathbb{R}$, *and the following estimates hold:*

$$|d(x+h) - d(x)| \leq |h| \quad \textit{if} \quad |h| \leq d(x),\ x \in \mathbb{R}. \tag{3.6}$$

From (3.6) it follows that for any $\varepsilon \in [0,1]$ and for every $x \in \mathbb{R}$ the following estimates hold:

$$(1-\varepsilon)d(x) \leq d(t) \leq (1+\varepsilon)d(x) \quad \text{for} \quad |t-x| \leq \varepsilon d(x). \tag{3.7}$$

Indeed, let $h = t-x$. Then $|h| \leq \varepsilon d(x) \leq d(x)$, $t = x+h$; and, in view of (3.6), we obtain $|d(t)-d(x)| = |d(x+h)-d(x)| \leq |h| \leq \varepsilon d(x)$, which implies $\left|\frac{d(t)}{d(x)}-1\right| \leq \varepsilon$ and hence (3.7). Thus, we see that inequality (3.5) slightly strengthens the a priori property (3.7), and therefore Theorem 3.3 can be applied to a broad class of problems (1.1)–(1.2).

4. Proof of the theorem on correct solvability in $C(\mathbb{R})$

Proof of Theorem 3.1. Let us check that $A(x) \to 0$ as $|x| \to \infty$ (see (2.4)). Fix $\varepsilon > 0$. Then there is an interval (x_1, x_2) such that $q(x) \geq \frac{3}{\varepsilon}$ for $x \notin (x_1, x_2)$. To estimate $A(x)$ for $x \notin (x_1, x_2)$, we consider the cases a) $x \geq x_2$ and b) $x \leq x_1$ separately. In case a) we have (see (2.4)):

$$A(x) \leq \frac{\varepsilon}{3} \int_x^{\infty} \frac{q(t)}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)}d\xi\right) dt \leq \frac{\varepsilon}{3} < \varepsilon.$$

To estimate $A(x)$ in case b), we write $A(x)$ in the following form:

$$A(x) = \int_x^{x_1} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt + \int_{x_1}^{x_2} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt$$
$$+ \int_{x_2}^{\infty} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt$$

Now we estimate each summand of $A(x)$ separately. We get, for $x \le x_1$,

$$A_1(x) := \int_x^{x_1} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt \le \frac{\varepsilon}{3} \int_x^{x_1} \frac{q(t)}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt \le \frac{\varepsilon}{3}$$

and

$$A_2(x) := \int_{x_1}^{x_2} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt \le \frac{1}{q_0} \exp\left(-\int_x^{x_1} \frac{q(\xi)}{r(\xi)} d\xi\right).$$

Since $S_1 = \infty$, there is $x_0 = x_0(\varepsilon) \ll x_1$ such that $\frac{1}{q_0} \exp\big(-\int_x^{x_1} \frac{q(\xi)}{r(\xi)} d\xi\big) \le \frac{\varepsilon}{3}$ for $x \le x_0$, hence $A_2(x) \le 3^{-1}\varepsilon$ for $x \le x_0$. Finally, from

$$A_3(x) := \int_{x_2}^{\infty} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt \le \frac{\varepsilon}{3} \int_{x_2}^{\infty} \frac{q(t)}{r(t)} \exp\left(-\int_{x_2}^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt$$

it follows $A_3(x) \le \frac{\varepsilon}{3}$. Hence for $x \notin (x_0, x_2)$, we have

$$A(x) = A_1(x) + A_2(x) + A_3(x) \le \varepsilon, \quad x \notin (x_0, x_2),$$

which implies $\lim_{|x|\to\infty} A(x) = 0$. It remains to refer to Theorem 2.3. $\square$

5. Proof of the main result

To prove Theorem 3.2, we need some lemmas. When stating them, we assume that the hypotheses of Theorem 3.2 are satisfied. Below we often use an obvious statement which, for convenience, is formulated as a separate assertion.

Lemma 5.1 ([1]). *Let $\varphi(x)$ and $\psi(x)$ be positive and continuous functions for $x \in \mathbb{R}$. If there exist a constant $c \in [1, \infty)$ and an interval (x_1, x_2) such that*

$$c^{-1}\psi(x) \le \varphi(x) \le c\psi(x) \qquad \textit{for} \quad x \notin (x_1, x_2), \tag{5.1}$$

then equalities (5.1) *remain true for all $x \in \mathbb{R}$, possibly after the replacement of c by a bigger constant.*

Proof. The function $f(x) = \frac{\varphi(x)}{\psi(x)}$ is continuous and positive for $x \in [x_1, x_2]$. Hence its minimum m and maximum M on the segment $[x_1, x_2]$ are finite positive numbers. Let $c_1 = \max\{c, \frac{1}{m}, M\}$. Then $\frac{1}{c_1}\psi(x) \le \varphi(x) \le c_1\psi(x)$ for $x \in \mathbb{R}$. $\square$

Lemma 5.2. *Let $x \in \mathbb{R}$ be given. Let a sequence $\{x_k\}_{k=-\infty}^{\infty}$ be given as follows:*

$$x_0 = x, \quad x_{k+1} = x_k + b\frac{r(x_k)}{q(x_k)} \qquad \text{for} \quad k = 0, 1, 2, \dots \tag{5.2}$$

$$x_0 = x, \quad x_{k-1} = x_k - b\frac{r(x_k)}{q(x_k)} \qquad \text{for} \quad k = 0, -1, -2, \dots. \tag{5.3}$$

Here b is taken from (3.1)*. Then we have*

$$\lim_{k\to-\infty} x_k = -\infty, \qquad \lim_{k\to\infty} x_k = \infty. \tag{5.4}$$

Proof. Both limits in (5.4) are checked in a similar way. Let us prove, for example, the second one. Assume the contrary. The sequence (5.2) is, by construction, monotone increasing. If (5.4) does not hold, then there is $z < \infty$ such that $x_k < z$ for $k \ge 0$. Then the sequence (5.2) has a limit $z_0 \le z$. Moreover, $\infty > z - x \ge \sum_{k=0}^{\infty}(x_{k+1} - x_k) = b\sum_{k=0}^{\infty}\frac{r(x_k)}{q(x_k)}$, which implies $\lim_{k\to\infty}\frac{r(x_k)}{q(x_k)} = 0$, in contradiction to $\lim_{k\to\infty}\frac{r(x_k)}{q(x_k)} = \frac{r(z_0)}{q(z_0)} \ne 0$. $\square$

Lemma 5.3. *Let $\theta \in [0, b]$ (see* (3.1)*). Denote*

$$\omega^{(+)}(x) = \left[x, x + \theta\frac{r(x)}{q(x)}\right], \quad \omega^{(-)}(x) = \left[x - \theta\frac{r(x)}{q(x)}, x\right], \qquad x \in \mathbb{R}. \tag{5.5}$$

Then for $x \notin (\alpha, \beta)$ (see (3.1)*), the following inequalities hold:*

$$\frac{\theta}{a^2} \le \int_{\omega^{(+)}(x)} \frac{q(t)}{r(t)}dt, \quad \int_{\omega^{(-)}(x)} \frac{q(t)}{r(t)}dt \le \theta a^2. \tag{5.6}$$

Proof. The inequalities (5.6) follow from (3.1):

$$\int_{\omega^{(+)}(x)} \frac{q(t)}{r(t)}dt = \int_{\omega^{(+)}(x)} \frac{q(t)}{q(x)} \cdot \frac{q(x)}{r(x)} \cdot \frac{r(x)}{r(t)}dt \ge \frac{1}{a^2} \cdot \frac{q(x)}{r(x)} \cdot \theta\frac{r(x)}{q(x)} = \frac{\theta}{a^2}$$
$$\int_{\omega^{(+)}(x)} \frac{q(t)}{r(t)}dt = \int_{\omega^{(+)}(x)} \frac{q(t)}{q(x)} \cdot \frac{q(x)}{r(x)} \cdot \frac{r(x)}{r(t)}dt \le a^2 \cdot \frac{q(x)}{r(x)} \cdot \theta\frac{r(x)}{q(x)} = \theta a^2. \qquad \square$$

Lemma 5.4. *We have (see* (3.4)*)*

$$S_1 = \infty, \qquad S_2 = \infty. \tag{5.7}$$

Proof. In (5.2), set $x_0 = 0$. By Lemma 5.2, there is $k_0 \gg 1$ such that the points x_k for $k \geq k_0$ are outside the interval (α, β) from condition (3.1). Then by Lemma 5.3, we have

$$\infty \geq S_2 \geq \int_{x_{k_0}}^{\infty} \frac{q(t)}{r(t)} dt = \sum_{k=k_0}^{\infty} \int_{x_k}^{x_{k+1}} \frac{q(t)}{r(t)} dt \geq \sum_{k=k_0}^{\infty} \frac{b}{a^2} = \infty$$

which implies $S_2 = \infty$. The equality $S_1 = \infty$ can be checked in a similar way. □

Lemma 5.5. *Let $a \geq 1$, $b > 0$, and $\gamma \leq 1$ (see Theorem 3.2). Then $b \geq a^2$.*

Proof. If $b < a^2$, then $3 \leq 3a^2 \leq e^{b/a^2} \leq e$, a contradiction. □

Lemma 5.6 ([1]). *Let $S_1 = \infty$ (see (3.4)). Then the function $d(x)$ is defined for $x \in \mathbb{R}$ (see (1.9) and Lemma 3.4). Moreover, the inequality $\eta \geq d(x)$ (resp., $0 \leq \eta \leq d(x)$) holds if and only if*

$$\int_{x-\eta}^{x+\eta} \frac{q(t)}{r(t)} dt \geq 2 \qquad \left(\textit{resp.,} \quad \int_{x-\eta}^{x+\eta} \frac{q(t)}{r(t)} dt \leq 2 \right).$$

Lemma 5.7. *For a given $x \in \mathbb{R}$, the equation in $d \geq 0$*

$$\int_{x-d}^{x+d} \frac{q(\xi)}{r(\xi)} d\xi = 2$$

has a unique positive solution $d = d(x)$. Moreover,

$$\frac{1}{a^2} \frac{r(x)}{q(x)} \leq d(x) \leq a^2 \frac{r(x)}{q(x)} \quad \textit{for} \quad x \notin (\alpha, \beta), \tag{5.8}$$

$$c^{-1} \frac{r(x)}{q(x)} \leq d(x) \leq c \frac{r(x)}{q(x)}, \quad x \in \mathbb{R}. \tag{5.9}$$

Proof. According to (5.7) and by Lemma 5.6, we only have to prove the estimates (5.8) and (5.9). Let $\eta_1(x) = \frac{1}{a^2} \frac{r(x)}{q(x)}$, $x \notin (\alpha, \beta)$. Since $b \geq a^2 \geq a^{-2}$ because of Lemma 5.5, from (5.5) it follows that

$$\int_{x-\eta_1(x)}^{x+\eta_1(x)} \frac{q(t)}{r(t)} dt = \int_{x-\eta_1(x)}^{x} \frac{q(t)}{r(t)} dt + \int_{x}^{x+\eta_1(x)} \frac{q(t)}{r(t)} dt \leq a^2 \frac{1}{a^2} + a^2 \frac{1}{a^2} = 2.$$

Hence $d(x) \geq \eta_1(x)$ by Lemma 5.6. Let now $\eta_2(x) = a^2 \frac{r(x)}{q(x)}$, $x \notin (\alpha, \beta)$. Since $b \geq a^2$ by Lemma 5.5, from (5.5) it follows that

$$\int_{x-\eta_2(x)}^{x+\eta_2(x)} \frac{q(t)}{r(t)} dt = \int_{x-\eta_2(x)}^{x} \frac{q(t)}{r(t)} dt + \int_{x}^{x+\eta_2(x)} \frac{q(t)}{r(t)} dt \geq a^2 \frac{1}{a^2} + a^2 \frac{1}{a^2} = 2.$$

Hence $d(x) \leq \eta_2(x)$ by Lemma 5.6, which implies (5.8). Since the function $d(x)$ is continuous and positive (see Lemma 3.4), the inequalities (5.9) follows from Lemma 5.1. □

Lemma 5.8. *Let $x \notin (\alpha, \beta)$ (see (3.1)), and let $\{x_k\}_{k=-\infty}^{\infty}$ be the sequence from Lemma 5.2. Then the following inequalities hold:*

$$\int_{x_0}^{x_k} \frac{q(t)}{r(t)} dt \geq \frac{bk}{a^2} \quad \text{for} \quad x \geq \beta, \ k = 0, 1, 2, \dots \tag{5.10}$$

$$\int_{x_k}^{x_0} \frac{q(t)}{r(t)} dt \geq \frac{b|k|}{a^2} \quad \text{for} \quad x \leq \alpha, \ k = 0, -1, -2, \dots. \tag{5.11}$$

Proof. If $k = 0$, relation (5.10) is obvious. For $k \geq 1$ it follows from Lemmas 5.2 and 5.3:

$$\int_{x_0}^{x_k} \frac{q(t)}{r(t)} dt = \sum_{\ell=0}^{k-1} \int_{x_\ell}^{x_{\ell+1}} \frac{q(t)}{r(t)} dt \geq \sum_{\ell=0}^{k-1} \frac{b}{a^2} = \frac{b}{a^2} k.$$

Inequality (5.11) can be checked in a similar way. □

Lemma 5.9. *Let $x \notin (\alpha, \beta)$ (see (3.1)), and let $\{x_k\}_{k=-\infty}^{\infty}$ be the sequence from Lemma 5.2. Then the following inequalities hold:*

$$a^{-k} \leq \frac{r(x_k)}{r(x_0)}, \quad \frac{q(x_k)}{q(x_0)} \leq a^k \quad \text{for} \quad x \geq \beta, \ k = 1, 2, \dots \tag{5.12}$$

$$a^{-|k|} \leq \frac{r(x_k)}{r(x_0)}, \quad \frac{q(x_k)}{q(x_0)} \leq a^{|k|} \quad \text{for} \quad x \leq \alpha, \ k = -1, -2, \dots. \tag{5.13}$$

Proof. Let $x \geq \beta$. Then from (3.1) and (5.2), it follows that

$$\frac{1}{a} \leq \frac{r(x_\ell)}{r(x_{\ell-1})}, \quad \frac{q(x_\ell)}{q(x_{\ell-1})} \leq a \quad \text{for} \quad \ell = 1, 2, \dots, k; \ k \geq 1.$$

After multiplying these inequalities, we obtain (5.12). Estimates (5.13) can be checked in a similar way. □

Lemma 5.10. *For $p \in [1, \infty)$ and $x \in \mathbb{R}$, the following inequalities hold:*

$$c^{-1} \frac{r(x)}{q(x)} \leq I_p(x) = \int_{-\infty}^{x} \exp\left(-p \int_t^x \frac{q(\xi)}{r(\xi)} d\xi \right) dt \leq c \frac{r(x)}{q(x)}, \quad c = c(p). \tag{5.14}$$

Proof. The proof of the lower bound in (5.14) is based on Lemma 5.7:

$$\begin{aligned} I_p(x) &\geq \int_{x-d(x)}^{x} \exp\left(-p \int_t^x \frac{q(\xi)}{r(\xi)} d\xi \right) dt \\ &\geq d(x) \exp\left(-p \int_{x-d(x)}^{x+d(x)} \frac{q(\xi)}{r(\xi)} d\xi \right) \\ &\geq c^{-1} \frac{r(x)}{q(x)}. \end{aligned}$$

To prove the upper bound in (5.14), consider two separate cases: 1) $x \le \alpha$; 2) $x \ge \beta$. In case 1), we use below the sequence (5.3), relations (5.11), (5.13) and the inequality $\gamma \le 1$ (see Theorem 3.2 and (5.14):

$$\begin{aligned}
I_p(x) &= \sum_{k=-\infty}^{0} \int_{x_{k-1}}^{x_k} \exp\left(-p\int_t^{x_0} \frac{q(\xi)}{r(\xi)} d\xi\right) dt \\
&\le \sum_{k=-\infty}^{0} (x_k - x_{k-1}) \exp\left(-p\int_{x_k}^{x_0} \frac{q(\xi)}{r(\xi)} d\xi\right) \\
&\le b \sum_{k=-\infty}^{0} \frac{r(x_k)}{q(x_k)} \exp\left(-p\frac{b}{a^2}|k|\right) \\
&= b\frac{r(x_0)}{q(x_0)} \sum_{k=-\infty}^{0} \frac{r(x_k)}{r(x_0)} \frac{q(x_0)}{q(x_k)} \exp\left(-p\frac{b}{a^2}|k|\right) \\
&\le b\frac{r(x)}{q(x)} \sum_{k=-\infty}^{0} a^{2|k|} \exp\left(-p\frac{b}{a^2}|k|\right) \\
&\le c\frac{r(x)}{q(x)}.
\end{aligned}$$

Consider now case 2). Let us write $I_p(x)$ in the form

$$\begin{aligned}
I_p(x) &= \exp\left(-p\int_0^x \frac{q(\xi)}{r(\xi)} d\xi\right) \int_{-\infty}^{x} \exp\left(p\int_0^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt \\
&= \exp\left(-p\int_0^x \frac{q(\xi)}{r(\xi)} d\xi\right) f(x)
\end{aligned}$$

where

$$f(x) \stackrel{\text{def}}{=} \int_{-\infty}^{x} \exp\left(p\int_0^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt, \quad x \ge \beta. \tag{5.15}$$

From (5.15) and case 1) above, it follows that the integral $I_p(x)$ exists for $x \ge \beta$. Moreover, according to (5.7), we have the equality

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \int_{-\infty}^{x} \exp\left(p\int_0^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt = \infty. \tag{5.16}$$

Let us define the integral $I_p(x,\beta)$:

$$I_p(x,\beta) = \int_{\beta}^{x} \exp\left(-p\int_t^x \frac{q(\xi)}{r(\xi)} d\xi\right) dt, \quad x \ge \beta,$$

and write for $I_p(x,\beta)$ an analogue of the representation (5.15):

$$I_p(x,\beta)=\exp\left(-p\int_0^x\frac{q(\xi)}{r(\xi)}d\xi\right)\int_\beta^x\exp\left(p\int_0^t\frac{q(\xi)}{r(\xi)}d\xi\right)dt$$
$$=\exp\left(-p\int_0^x\frac{q(\xi)}{r(\xi)}d\xi\right)f_\beta(x)$$

where

$$f_\beta(x)\overset{\text{def}}{=}\int_\beta^x\exp\left(p\int_0^t\frac{q(\xi)}{r(\xi)}d\xi\right)dt,\quad x\ge\beta. \tag{5.17}$$

Here, according to (5.7), we have

$$\lim_{x\to\infty}f_\beta(x)=\lim_{x\to\infty}\int_\beta^x\exp\left(p\int_0^t\frac{q(\xi)}{r(\xi)}d\xi\right)dt=\infty. \tag{5.18}$$

From (5.15), (5.16), (5.17), (5.18) and L'Hôpital's rule, it follows that

$$\lim_{x\to\infty}\frac{I_p(x)}{I_p(x,\beta)}=\lim_{x\to\infty}\frac{f(x)}{f_\beta(x)}=1.$$

Let $m\ge\beta$ be such that for $x\ge m$, the following inequality holds:

$$I_p(x)\le 2I_p(x,\beta),\qquad x\ge m\ge\beta. \tag{5.19}$$

Consider the sequence (5.3). By Lemma 5.2, for $x\ge m\ge\beta$ there is $\ell\le 0$ such that

$$x_\ell\ge\beta,\qquad x_{\ell-1}\le\beta. \tag{5.20}$$

Let us show that here one can choose m so that for all $x\ge m$ the number ℓ in (5.20) satisfies the inequality $\ell\le-1$. Assume the contrary. Let $\{m_s\}_{s=1}^\infty$ be any monotone sequence increasing to infinity with $m_1>\beta$. By the assumption, for every m_s, $s\ge1$, there is $x^{(s)}\ge m_s$ such that $x^{(s)}-b\frac{r(x^{(s)})}{q(x^{(s)})}\le\beta$. This means that inequalities (3.1) can be extended to the interval $[\beta,x^{(s)}]$ because $[\beta,x^{(s)}]\subseteq\left[x^{(s)}-b\frac{r(x^{(s)})}{q(x^{(s)})},x^{(s)}\right]$, and hence

$$\int_\beta^{x^{(s)}}\frac{q(t)}{r(t)}dt=\int_\beta^{x^{(s)}}\frac{q(t)}{q(x^{(s)})}\cdot\frac{q(x^{(s)})}{r(x^{(s)})}\cdot\frac{r(x^{(s)})}{r(t)}dt\le a^2b.$$

Since here $x^{(s)}\ge m_s\to\infty$ as $s\to\infty$, the integral S_2 converges (see (5.7)), a contradiction. Therefore, in the sequel we choose m big enough so that $m>\beta$, (5.19) holds, and for all $x\ge m$ we always have $\ell\le-1$ in (5.20).

To estimate $I_p(x,\beta)$, we use the sequence (5.3), relations (5.20), (5.11), (5.13) and inequality $\gamma \le 1$ (see Theorem 3.2):

$$\begin{aligned}
I_p(x,\beta) &= \int_\beta^x \exp\left(-p\int_t^x \frac{q(\xi)}{r(\xi)}d\xi\right)dt\\
&= \sum_{k=\ell+1}^{0}\int_{x_{k-1}}^{x_k} \exp\left(\int_t^{x_0}\frac{q(\xi)}{r(\xi)}d\xi\right)dt + \int_\beta^{x_\ell}\exp\left(-p\int_t^{x_0}\frac{q(\xi)}{r(\xi)}d\xi\right)dt\\
&\le \sum_{k=\ell+1}^{0}(x_k - x_{k-1})\exp\left(-p\int_{x_k}^{x_0}\frac{q(\xi)}{r(\xi)}d\xi\right)\\
&\quad + (x_\ell - \beta)\exp\left(-p\int_{x_\ell}^{x_0}\frac{q(\xi)}{r(\xi)}d\xi\right)\\
&\le b\sum_{k=\ell}^{0}\frac{r(x_k)}{q(x_k)}\exp\left(-p\frac{b}{a^2}|k|\right)\\
&= b\frac{r(x_0)}{q(x_0)}\sum_{k=\ell}^{0}\frac{r(x_k)}{r(x_0)}\cdot\frac{q(x_0)}{q(x_k)}\exp\left(-p\frac{b}{a^2}|k|\right)\\
&\le b\frac{r(x)}{q(x)}\sum_{k=0}^{|\ell|}a^{2k}\exp\left(-p\frac{b}{a^2}k\right)\\
&\le b\frac{r(x)}{q(x)}\sum_{k=0}^{\infty}\left[a^2\exp\left(-\frac{b}{a^2}\right)\right]^k = c\frac{r(x)}{q(x)}.
\end{aligned} \tag{5.21}$$

From (5.21) and (5.19), we obtain the estimates (5.14) for $x \ge m$. Thus inequalities (5.14) are proved for $x \notin (\alpha, m)$. To complete the proof of (5.14), it remains to apply Lemma 5.1. □

Proof of Theorem 3.2 *for* $p = 1$. *Necessity.* Suppose that problem (1.1)–(1.2) is correctly solvable in $L_1(R)$. Then $r_0 > 0$ and $M_1 < \infty$ because of Theorem 2.2 (see (2.2)–(2.3)). From (2.3) and (5.14), it follows that

$$M_1 = \sup_{x\in\mathbb{R}}\frac{1}{r(x)}\int_{-\infty}^x \exp\left(-\int_t^x\frac{q(\xi)}{r(\xi)}d\xi\right)dt = \sup_{x\in\mathbb{R}}\frac{1}{r(x)}I_1(x) \ge c^{-1}\sup_{x\in\mathbb{R}}\frac{1}{q(x)}$$

which implies that

$$q_0 = \inf_{x\in\mathbb{R}} q(x) \ge \left(\sup_{x\in\mathbb{R}}\frac{1}{q(x)}\right)^{-1} \ge \frac{c^{-1}}{M_1} > 0.$$

Proof of Theorem 3.2 *for* $p = 1$. *Sufficiency.* Since $S_1 = \infty$ (see (5.7)) and $r_0 > 0$ (see (3.3)), in the space $L_1(R)$ correct solvability of problem (1.1)–(1.2) is guaranteed by the inequality $M_1 < \infty$ (see Theorem 2.2). Below we use Lemma 5.10 and condition $q_0 > 0$ (see (3.3)) to check this requirement (see (2.3), (5.14) and (3.3)):

$$M_1 = \sup_{x\in\mathbb{R}} \frac{1}{r(x)} \int_{-\infty}^{x} \exp\left(- \int_t^x \frac{q(\xi)}{r(\xi)} d\xi \right) dt = \sup_{x\in\mathbb{R}} \frac{I_1(x)}{r(x)} \le c \sup_{x\in\mathbb{R}} \frac{1}{q(x)} < \infty. \qquad \square$$

Proof of Theorem 3.2 *for* $p \in (1,\infty)$. *Necessity.* Suppose that for some $p \in (1,\infty)$, problem (1.1)–(1.2) is correctly solvable in $L_p(R)$. Then $M_p < \infty$ by Theorem 1.1 (see (1.4)–(1.5)). Let $x \in \mathbb{R}$ be arbitrary. In the following relations, we use Lemmas 5.4 and 5.7 (see (1.4)–(1.5)):

$$\begin{aligned}
M_p(x) &= \left[\int_{-\infty}^{x} \exp\left(-p \int_t^x \frac{q(\xi)}{r(\xi)} d\xi \right) dt \right]^{\frac{1}{p}} \\
&\quad \cdot \left[\int_x^{\infty} \frac{1}{r(t)^{p'}} \exp\left(-p' \int_x^t \frac{q(\xi)}{r(\xi)} d\xi \right) dt \right]^{\frac{1}{p'}} \\
&\ge \left[\int_{x-d(x)}^{x} \exp\left(-p \int_t^x \frac{q(\xi)}{r(\xi)} d\xi \right) dt \right]^{\frac{1}{p}} \\
&\quad \cdot \left[\int_x^{x+d(x)} \frac{1}{r(t)^{p'}} \exp\left(-p' \int_x^t \frac{q(\xi)}{r(\xi)} d\xi \right) dt \right]^{\frac{1}{p'}} \\
&= \exp\left(- \int_{x-d(x)}^{x+d(x)} \frac{q(\xi)}{r(\xi)} d\xi \right) d(x)^{\frac{1}{p}} \left[\int_x^{x+d(x)} \frac{dt}{r(t)^{p'}} \right]^{\frac{1}{p'}} \\
&= e^{-2} d(x)^{\frac{1}{p}} \left[\int_x^{x+d(x)} \frac{dt}{r(t)^{p'}} \right]^{\frac{1}{p'}}.
\end{aligned} \tag{5.22}$$

Below we assume that $x \notin (\alpha,\beta)$ (see(3.1)) and continue estimate (5.22) using (3.1) and (5.8):

$$\begin{aligned}
\infty > e^2 M_p &\ge d(x)^{\frac{1}{p}} \left[\int_x^{x+d(x)} \frac{dt}{r(t)^{p'}} \right]^{\frac{1}{p'}} \\
&\ge \left(\frac{1}{a^2} \cdot \frac{r(x)}{q(x)} \right)^{\frac{1}{p}} \left[\int_x^{x+a^{-2}\frac{r(x)}{q(x)}} \left(\frac{r(x)}{r(t)} \frac{1}{r(x)} \right)^{p'} dt \right]^{\frac{1}{p'}} \\
&\ge \frac{1}{a^3 q(x)}
\end{aligned}$$

which implies $q(x) \ge (e^2 a^3 M_p)^{-1}$ for $x \notin (\alpha,\beta)$, hence $q_0 > 0$.

Furthermore, by Theorem 1.1, $A_{p'}$ is also finite (see (1.7)–(1.8)). In the following relations we use (3.1) and Lemma 5.6 for $x \notin (\alpha,\beta)$:

$$\infty > A_{p'} \ge \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}} \ge \int_{x-\frac{r(x)}{a^2 q(x)}}^{x+\frac{r(x)}{a^2 q(x)}} \left(\frac{r(x)}{r(t)} \frac{1}{r(x)} \right)^{p'} dt \ge \frac{2}{a^{p'+2}} \frac{1}{r(x)^{p'-1} q(x)}$$

which implies $r(x)^{\frac{1}{p}}q(x)^{\frac{1}{p'}} \ge A_{p'}^{-\frac{1}{p'}}\frac{1}{a^3}$ and hence $\sigma_{p'} > 0$.

Proof of Theorem 3.2 *for* $p \in (1,\infty)$. *Sufficiency.* Below we need the following assertion.

Lemma 5.11. *Let* $\sigma_{p'} > 0$ *(see* (3.3)*). Then the following inequalities hold:*

$$c^{-1} \le r(x)^{p'-1}q(x)K_p(x) \le c, \quad x \in \mathbb{R}. \tag{5.23}$$

Here $p' \ge 1$*, and*

$$K_p(x) \overset{def}{=} \int_x^\infty \frac{1}{r(x)^{p'}} \exp\left(-p'\int_x^t \frac{q(\xi)}{r(\xi)}d\xi\right)dt, \quad x \in \mathbb{R}. \tag{5.24}$$

Proof. Let us first verify that the integral (5.24) converges for every $x \in \mathbb{R}$. Indeed, from the condition $\sigma_{p'} > 0$ follows $r(t)^{\frac{1}{p}}q(t)^{\frac{1}{p'}} \ge \sigma_{p'} > 0$, $t \in \mathbb{R}$, and hence $r(t)^{p'-1}q(t) \ge \sigma_{p'}^{p'}$ for $t \in \mathbb{R}$ which implies

$$\frac{1}{r(t)^{p'}} \le \frac{1}{\sigma_{p'}^{p'}} \cdot \frac{q(t)}{r(t)}, \qquad t \in \mathbb{R}. \tag{5.25}$$

From (5.25) and (5.7) for $x \in \mathbb{R}$, we now obtain (see (5.24))

$$K_p(x) \le \frac{1}{\sigma_{p'}^{p'}}\int_x^\infty \frac{q(t)}{r(t)} \exp\left(-p'\int_x^t \frac{q(\xi)}{r(\xi)}d\xi\right)dt = \frac{1}{p'\sigma_{p'}^{p'}} < \infty.$$

To check the lower bound from (5.23), we assume $x \notin (\alpha,\beta)$. Then according to (5.8) and (3.1), we have

$$\begin{aligned}
K_p(x) &\ge \int_x^{x+d(x)} \frac{1}{r(t)^{p'}} \exp\left(-p'\int_x^t \frac{q(\xi)}{r(\xi)}d\xi\right)dt \\
&\ge \exp\bigg(-p'\int_{x-d(x)}^{x+d(x)} \frac{q(\xi)}{r(\xi)}d\xi\bigg)\int_x^{x+d(x)} \frac{dt}{r(t)^{p'}} \\
&\ge e^{-2p'}\int_x^{x+\frac{r(x)}{a^2q(x)}} \left(\frac{r(x)}{r(t)}\frac{1}{r(x)}\right)^{p'}dt \\
&\ge \frac{e^{-2p'}}{a^{p'+2}}\frac{1}{r(x)^{p'-1}q(x)}.
\end{aligned}$$

Taking into account Lemma 5.1, the latter inequality gives the lower bound from (5.23) for all $x \in \mathbb{R}$. To prove the upper bound from (5.23), we consider separate cases: 1) $x \geq \beta$ and 2) $x \leq \alpha$. In case 1) we use below the sequence $\{x_k\}_{k=0}^{\infty}$ (see (5.2)) and inequalities (5.10), (5.12), (3.1) and $\gamma \leq 1$:

$$\begin{aligned} K_p(x) &\leq \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} \frac{1}{r(t)^{p'}} \exp\left(-p' \int_{x_0}^{x_k} \frac{q(\xi)}{r(\xi)} d\xi\right) dt \\ &\leq \sum_{k=0}^{\infty} e^{-p' \frac{b}{a^2} k} \int_{x_k}^{x_{k+1}} \left(\frac{r(x_k)}{r(t)} \frac{1}{r(x_k)}\right)^{p'} dt \\ &\leq a^{p'} b \sum_{k=0}^{\infty} \frac{e^{-p' \frac{b}{a^2} k}}{r(x_k)^{p'-1} q(x_k)} \\ &= \frac{c_1}{r(x_0)^{p'-1} q(x_0)} \sum_{k=0}^{\infty} \left(\frac{r(x_0)}{r(x_k)}\right)^{p'-1} \left(\frac{q(x_0)}{q(x_k)}\right) e^{-p' \frac{b}{a^2} k} \\ &\leq \frac{c_2}{r(x)^{p'-1} q(x)} \sum_{k=0}^{\infty} \left(a e^{-\frac{b}{a^2}}\right)^{kp'} \\ &\leq \frac{c}{r(x)^{p'-1} q(x)}. \end{aligned}$$

Thus estimate (5.23) holds for $x \geq \beta$.

Consider case 2). Let us introduce the function

$$K_p(x, \alpha) = \int_x^{\alpha} \frac{1}{r(t)^{p'}} \exp\left(-p' \int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt, \quad x \leq \alpha.$$

Let $x \leq m < \alpha$ (we shall choose m later). Then

$$\begin{aligned} K_p(x) &= K_p(x, \alpha) + \exp\left(-p' \int_x^{\alpha} \frac{q(\xi)}{r(\xi)} d\xi\right) K_p(\alpha) \\ &\leq K_p(x, \alpha) + \frac{1}{p' \sigma_{p'}^{p'}} \exp\left(-p' \int_x^{\alpha} \frac{q(\xi)}{r(\xi)} d\xi\right) \\ &= K_p(x, \alpha) \left\{1 + \frac{c}{K_p(x, \alpha)} \exp\left(-p' \int_x^{\alpha} \frac{q(\xi)}{r(\xi)} d\xi\right)\right\}. \end{aligned} \tag{5.26}$$

Here we have

$$\begin{aligned} \exp\left(p' \int_x^{\alpha} \frac{q(\xi)}{r(\xi)} d\xi\right) K_p(x, \alpha) &= \int_x^{\alpha} \frac{1}{r(t)^{p'}} \exp\left(p' \int_t^{\alpha} \frac{q(\xi)}{r(\xi)} d\xi\right) dt \\ &\geq \int_m^{\alpha} \frac{dt}{r(t)^{p'}}. \end{aligned} \tag{5.27}$$

Denote

$$C(\alpha) = \int_{-\infty}^{\alpha} \frac{dt}{r(t)^{p'}}, \quad \delta(m) = \int_m^{\alpha} \frac{dt}{r(t)^{p'}}$$

and choose m as follows:

$$m = \begin{cases} \theta_1 & \text{if } C(\alpha) = \infty \text{ and } \int_{\theta_1}^{\alpha} \frac{dt}{r(t)^{p'}} = 1 \\ \theta_2 & \text{if } C(\alpha) < \infty \text{ and } \int_{\theta_2}^{\alpha} \frac{dt}{r(t)^{p'}} = \frac{C(\alpha)}{2}. \end{cases}$$

With such a choice of m, from (5.26) and (5.27) it follows that

$$K_p(x) \le cK_p(x,\alpha), \quad x \le m, \quad c = 1 + \frac{\delta(m)^{-1}}{p'\sigma_{p'}^{p'}}. \tag{5.28}$$

Let now $x \le m$. For $x_0 = x$ consider the sequence (5.2). By Lemma 5.2, for $x \le m \le \alpha$, there is $\ell \ge 0$ such that

$$x_\ell \le \alpha, \quad x_{\ell+1} > \alpha. \tag{5.29}$$

Let us show that m can be chosen so small that for all $x \le m$ the number ℓ in the inequalities (5.29) satisfies the inequality $\ell \ge 1$. Assume the contrary. Let $\{m_s\}_{s=1}^{\infty}$ be any monotone sequence decreasing to $-\infty$ with $m_1 < \alpha$. By the assumption, for every m_s, $s \ge 1$, there is $x_s \le m_s$ such that $x_s + b\frac{r(x_s)}{q(x_s)} \ge \alpha$. This means that the inequalities (3.1) can be extended to the interval $[x_s, \alpha]$ because $[x_s, \alpha] \subseteq \left[x_s, x_s + b\frac{r(x_s)}{q(x_s)}\right]$ and hence

$$\int_{x_s}^{\alpha} \frac{q(t)}{r(t)} dt = \int_{x_s}^{\alpha} \frac{q(t)}{q(x_s)} \cdot \frac{q(x_s)}{r(x_s)} \cdot \frac{r(x_s)}{r(t)} dt \le a^2 b < \infty;$$

and since here $x_s \le m_s \to -\infty$ as $s \to \infty$, the integral S_1 converges (see (5.7)), a contradiction. Therefore, below m is chosen so small that $m < \alpha$, (5.28) holds, and in the inequalities (5.29) we always have $\ell \ge 1$. When estimating $K_p(x,\alpha)$, we use sequences (5.2) and relations (5.29) with $\ell \ge 1$, (3.1) with $\gamma \ge 1$ and (5.13):

$$\begin{aligned}
&K_p(x,\alpha) \\
&\le \sum_{k=0}^{\ell-1} \int_{x_k}^{x_{k+1}} \frac{1}{r(t)^{p'}} \exp\left(-p' \int_{x_0}^{x_k} \frac{q(\xi)}{r(\xi)} d\xi\right) dt + \int_{x_\ell}^{\alpha} \frac{1}{r(t)^{p'}} \exp\left(-p' \int_{x_0}^{x_\ell} \frac{q(\xi)}{r(\xi)} d\xi\right) dt \\
&\le \sum_{k=0}^{\ell-1} e^{-p'\frac{b}{a^2}k} \int_{x_k}^{x_{k+1}} \left(\frac{r(x_k)}{r(t)} \frac{1}{r(x_k)}\right)^{p'} dt + e^{-p'\frac{b}{a^2}\ell} \int_{x_\ell}^{\alpha} \left(\frac{r(x_\ell)}{r(t)} \frac{1}{r(x_\ell)}\right)^{p'} dt \\
&\le a^{p'} b \left[\sum_{k=0}^{\ell-1} \frac{1}{r(x_k)^{p-1} q(x_k)} e^{-p'\frac{b}{a^2}k}\right] + a^{p'} \frac{(\alpha - x_\ell)}{r(x_\ell)^{p'}} e^{-p'\ell\frac{b}{a^2}} \\
&\le a^{p'} b \left[\sum_{k=0}^{\ell-1} \frac{1}{r(x_k)^{p-1} q(x_k)} e^{-p'\frac{b}{a^2}k}\right] + a^{p'} \frac{x_{\ell+1} - x_\ell}{r(x_\ell)^{p'}} e^{-p'\frac{b}{a^2}k} \\
&= \frac{c_1}{r(x_0)^{p'-1} q(x_0)} \sum_{k=0}^{\ell} \left(\frac{r(x_0)}{r(x_k)}\right)^{p'-1} \left(\frac{q(x_0)}{q(x_k)}\right) e^{-p'\frac{b}{a^2}k}.
\end{aligned}$$

We get

$$K_p(x,\alpha) \le c_1 \frac{1}{r(x)^{p'-1}q(x)} \sum_{k=0}^{\ell} \frac{a^{kp'}}{e^{p'\frac{b}{a^2}k}} \le \frac{c_2}{r(x)^{p'-1}q(x)}.$$

Thus the upper bound from (5.23) holds for $x \notin (m,\beta)$. To finish the proof of (5.23), it remains to apply Lemma 5.1. □

Let us now go to the proof of the theorem. Here we use Theorem 1.1. Since $S_1 = \infty$ (see (5.7) and (1.6)), to apply Theorem 1.1 it is enough to prove that $M_p < \infty$ and $A_{p'} < \infty$ (see (1.4) and (1.7)). In the next estimate for M_p we use Lemmas 5.10 and 5.11 and condition (3.2):

$$M_p = \sup_{x\in\mathbb{R}} (I_p(x))^{\frac{1}{p}} (K_p(x))^{\frac{1}{p'}} \le c \sup_{x\in\mathbb{R}} \left(\frac{r(x)}{q(x)}\right)^{\frac{1}{p}} \frac{1}{r(x)^{\frac{1}{p}} q(x)^{\frac{1}{p'}}} \le \frac{c}{q_0} < \infty.$$

To check the inequality $A_{p'} < \infty$, let us first estimate the function $A_{p'}(x)$ for $x \notin (\alpha,\beta)$ (see (1.7) and (3.1)). Below we use Lemmas 5.7 and 5.5, (3.2) and (3.1):

$$A_{p'}(x) = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}} \le \int_{x-a^2\frac{r(x)}{q(x)}}^{x+a^2\frac{r(x)}{q(x)}} \left(\frac{r(x)}{r(t)}\frac{1}{r(x)}\right)^{p'} dt \le 2a^{p'+2} \frac{1}{r(x)^{p'-1}q(x)} \le \frac{c}{\sigma_{p'}^{p'}}$$

and hence

$$\sup_{x\notin(\alpha,\beta)} A_{p'}(x) < \infty. \tag{5.30}$$

Note that the function $A_{p'}(x)$ is continuous for $x \in \mathbb{R}$ because so is $d(x)$ (see Lemma 3.4). Therefore, $A_{p'}(x)$ is bounded on $[\alpha,\beta]$. Together with (5.30), this leads to the inequality $A_{p'} < \infty$. Thus problem (1.1)–(1.2) is correctly solvable in $L_p(R)$ for $p \in (1,\infty)$ by Theorem 1.1. □

Proof of Theorem 3.2 *for* $p = \infty$. *Necessity.* Suppose that problem (1.1)–(1.2) is correctly solvable in $C(\mathbb{R})$. Then equality $A_0 = 0$ holds, and $S_1 = \infty$ (see Theorem 2.3). For $x \in \mathbb{R}$, Lemma 5.7 implies (see (2.4))

$$A(x) \ge \int_x^{x+d(x)} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r\xi)} d\xi\right) dt > 0. \tag{5.31}$$

From the equality $A_0 = 0$ and (5.31), it follows that

$$\lim_{|x|\to\infty} \int_x^{x+d(x)} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt = 0. \tag{5.32}$$

Furthermore, for $x \notin (\alpha, \beta)$, using (3.1) and (5.8) we obtain

$$\begin{aligned}\int_x^{x+d(x)} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt &\geq \exp\left(-\int_{x-d(x)}^{x+d(x)} \frac{q(\xi)}{r(\xi)} d\xi\right) \int_x^{x+d(x)} \frac{dt}{r(t)} \\ &= e^{-2} \int_x^{x+d(x)} \frac{dt}{r(t)} \\ &\geq e^{-2} \int_x^{x+\frac{r(x)}{a^2 q(x)}} \frac{r(x)}{r(t)} \frac{dt}{r(x)} \\ &\geq \frac{e^{-2}}{a^3} \frac{1}{q(x)} > 0. \end{aligned} \tag{5.33}$$

From (5.32) and (5.33) we get $\lim\limits_{|x|\to\infty} \frac{1}{q(x)} = 0$. Hence $q(x) \to \infty$ as $|x| \to \infty$.

Proof of Theorem 3.2 *for* $p = \infty$. *Sufficiency.* In this case the statement of the theorem is an obvious consequence of Lemma 5.4 and Theorem 3.1. □

6. Proof of the theorem on correct unsolvability in L_p

Proof of Theorem 3.3. We consider the cases 1) $p = 1$; 2) $p \in (1, \infty)$, and 3) $p = \infty$, separately. Since the cases $x \to -\infty$ and $x \to +\infty$ are treated similarly, let us, for example, consider $q(x) \to 0$ as $x \to \infty$.

`Case 1):` Let $p = 1$. Assume the contrary: problem (1.1)–(1.2) is correctly solvable in $L_1(\mathbb{R})$. Then $M_1 < \infty$ (see (2.3)), and for any $t \in \mathbb{R}$, we get

$$M_1 \geq M_1(t) \geq \frac{1}{r(t)} \int_{t-d(t)}^t \exp\left(-\int_\xi^t \frac{q(s)}{r(s)} ds\right) d\xi \geq \frac{d(t)}{r(t)} \exp\left(-\int_{t-d(t)}^{t+d(t)} \frac{q(s)}{r(s)} ds\right)$$

which implies

$$\frac{d(t)}{r(t)} \leq e^2 M_1, \quad t \in \mathbb{R}. \tag{6.1}$$

Let us integrate inequality (6.1) along the interval $[x - d(x), x + d(x)]$, $x \in \mathbb{R}$ and use (3.5) to get

$$2e^2 M_1 d(x) \geq \int_{x-d(x)}^{x+d(x)} \frac{d(t)}{r(t)} dt \geq \delta d(x) \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)}, \quad x \in \mathbb{R}$$

so that

$$\sup_{x\in\mathbb{R}} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \leq 2e^2 \delta^{-1} M_1 < \infty. \tag{6.2}$$

Let ε be a given positive number. Then $q(x) \le \varepsilon$ for $x \ge c(\varepsilon) \gg 1$. Since $S_2 = \infty$, then $x - d(x) \to \infty$ as $x \to \infty$ (see [1, (3.18)]). Therefore, there is $x_0(\varepsilon) \ge c(\varepsilon) \gg 1$ such that $x - d(x) \ge c(\varepsilon)$ for $x \ge x_0(\varepsilon)$. For $x \ge x_0(\varepsilon)$, we get from $2 = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{r(t)} dt \le \varepsilon \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)}$ that

$$\lim_{x\to\infty} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} = \infty. \tag{6.3}$$

This leads to a contradiction between (6.2) and (6.3).

`Case 2):` Let $p \in (1, \infty)$. Assume the contrary: problem (1.1)–(1.2) is correctly solvable in $L_p(\mathbb{R})$ and $M_p < \infty$ (see (1.4)). Denote $z_1(x) = x - d(x)$, $z_2(x) = x + d(x)$. Below we use (1.4), (1.5), (1.9) and (3.5):

$$\begin{aligned} M_p &\ge \left[\int_{-\infty}^{z_1(x)} \exp\left(-p \int_t^{z_1(x)} \frac{q(\xi)}{r(\xi)} d\xi \right) dt \right]^{\frac{1}{p}} \\ &\quad \cdot \left[\int_{z_1(x)}^{\infty} \frac{1}{r(t)^{p'}} \exp\left(-p' \int_{z_1(x)}^{t} \frac{q(\xi)}{r(\xi)} d\xi \right) dt \right]^{\frac{1}{p'}} \\ &\ge \left[\int_{z_1(x)-d(z_1(x))}^{z_1(x)} \exp\left(-p \int_t^{z_1(x)} \frac{q(\xi)}{r(\xi)} d\xi \right) dt \right]^{\frac{1}{p}} \\ &\quad \cdot \left[\int_{z_1(x)}^{z_2(x)} \frac{1}{r(t)^{p'}} \exp\left(-p' \int_{z_1(x)}^{t} \frac{q(\xi)}{r(\xi)} d\xi \right) dt \right]^{\frac{1}{p'}} \\ &\ge \exp\left(-\int_{z_1(x)-d(z_1(x))}^{z_1(x)+d(z_1(x))} \frac{q(\xi)}{r(\xi)} d\xi \right) \\ &\quad \cdot \exp\left(-\int_{z_1(x)}^{z_2(x)} \frac{q(\xi)}{r(\xi)} d\xi \right) \cdot d(z_1(x))^{\frac{1}{p}} \left[\int_{z_1(x)}^{z_2(x)} \frac{dt}{rt)^{p'}} \right]^{\frac{1}{p'}} \\ &\ge e^{-4} \delta^{\frac{1}{p}} d(x)^{\frac{1}{p}} \left[\int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}} \right]^{\frac{1}{p'}}. \end{aligned}$$

This implies

$$\sup_{x\in\mathbb{R}} d(x)^{\frac{1}{p}} \left[\int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}} \right]^{\frac{1}{p'}} \le \delta^{-\frac{1}{p}} e^4 M_p < \infty, \quad x \in \mathbb{R}. \tag{6.4}$$

On the other hand, let $\varepsilon > 0$ be given. Below $x \ge x_0(\varepsilon)$ (see the proof of (6.3)) it holds

$$2 = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{r(t)} dt \le \varepsilon \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \le 2^{\frac{1}{p}} \varepsilon d(x)^{\frac{1}{p}} \left[\int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}} \right]^{\frac{1}{p'}}$$

so that

$$\lim_{x\to\infty} d(x)^{\frac{1}{p}} \left[\int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}}\right]^{\frac{1}{p'}} = \infty. \tag{6.5}$$

This leads to a contradiction between (6.4) and (6.5).

`Case 3)`: Let $p = \infty$. Assume the contrary: problem (1.1)–(1.2) is correctly solvable in $C(\mathbb{R})$. We have $x - d(x) \to \infty$ as $x \to \infty$ (see case 1) above). Since

$$\begin{aligned}\int_{x-d(x)}^{\infty} \frac{1}{r(t)} \exp\left(-\int_{x-d(x)}^{t} \frac{q(\xi)}{r(\xi)}d\xi\right)dt \\ &\ge \int_{x-d(x)}^{x+d(x)} \frac{1}{r(t)} \exp\left(-\int_{x-d(x)}^{t} \frac{q(\xi)}{r(\xi)}d\xi\right) dt \\ &\ge \exp\left(-\int_{x-d(x)}^{x+d(x)} \frac{q(\xi)}{r(\xi)}d\xi\right) \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \\ &= e^{-2} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} > 0. \end{aligned} \tag{6.6}$$

Then from (6.6) and Theorem 2.3 it follows that

$$\lim_{x\to\infty} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} = 0. \tag{6.7}$$

On the other hand, let $\varepsilon > 0$ be given. Below $x \ge x_0(\varepsilon)$ (see the proof of (6.3)):

$$2 = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{r(t)}dt \le \varepsilon \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \quad \Rightarrow \quad \lim_{x\to\infty} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} = \infty. \tag{6.8}$$

This leads to a contradiction between (6.7) and (6.8). □

References

[1] Chernyavskaya, N. and Shuster, L., Conditions for correct solvability of a simplest singular boundary value of general form. I. *Z. Anal. Anwendungen* 25 (2006), 205 – 235.

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