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Stochastic Comparison of Solutions of Stochastic Functional Differential Equations

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Abstract. A stochastic comparison of solutions of nonlinear stochastic functional differential equations with different drift and diffusion coefficients is obtained. Some known results are generalized.

Keywords. Stochastic functional differential equation, stochastic comparison, Itô formula

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1. Introduction

It is well-known that comparison theorems represent a useful tool in the theory of differential equations both in the deterministic and in the stochastic case. A first comparison theorem for ordinary stochastic differential equations was published by Skorohod [11] in 1961. As a consequence of the enormous progress in stochastic analysis his result was generalized and extended by Yamada [12], Malliavin [8], Ikeda [6], and many others. For example, Ikeda [6] studied the temporally homogeneous stochastic differential equations with the identical diffusion terms; Ferreyra [2], Galchuk [3], Huang [5] and O'Brien [10] considered the temporally homogeneous stochastic differential equations with the different diffusion terms; Abundo [1] and Geiss [4] investigated nonhomogeneous stochastic differential equations with different drift and diffusion terms.

Considering the effect of the time delay, Maizenberg [9] recently made the first attempt to study the comparison of solutions of stochastic functional differential equations. However, Maizenberg [9] only focused on the case when the diffusion terms are identical. In the present paper, we will consider the nonlinear nonhomogeneous stochastic functional differential equations with different drift and diffusion terms. The conditions in terms of the drift and diffusion coefficients for comparison of the solutions of these equations are formulated. Our result generalizes the results of [1] and [9].

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2. Main results

Throughout this paper, unless otherwise specified, we let $h > 0$ and $C([-h,0];\mathbb{R})$ denote the family of continuous functions φ from $[-h,0]$ to $\mathbb{R}$ with the norm $\|\varphi\| = \sup_{-h\le\theta\le 0}|\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}$.

Consider the following two stochastic delay differential equations:

$$\begin{cases} dX(t) = b(t,X(t),X_t)dt + \sigma(t,X(t))\,dw(t) \\ X(s) = f,\ s\in[-h,0] \end{cases} \tag{1}$$

$$\begin{cases} dY(t) = c(t,Y(t),Y_t)dt + \mu(t,Y(t))\,dw(t) \\ Y(s) = g,\ s\in[-h,0], \end{cases} \tag{2}$$

where $X_t = \{X(t+\theta) : -h \le \theta \le 0\}, Y_t = \{Y(t+\theta) : -h\le\theta\le 0\}$, which are regarded as two $C([-h,0];R)$-valued stochastic processes, $w(t)$ is a standard Wiener process, f, g are continuous functions on $[-h,0]$. We make the following assumptions on the coefficients:

(A1) The functions σ, μ : $\mathbb{R}^+\times\mathbb{R}\to R$, and functionals b, c : $\mathbb{R}^+\times\mathbb{R}\times C([-h,0];\mathbb{R}) \to R$ are jointly continuous, and there exists a constant $K > 0$ such that for every $t\ge 0$, $x,y\in\mathbb{R}$, $\psi,\varphi\in C([-h,0];\mathbb{R})$,

$$\big|b(t,x,\psi) - b(t,y,\varphi)\big| + \big|c(t,x,\psi) - c(t,y,\varphi)\big| \le K\big(|x-y| + \|\psi-\varphi\|\big)$$
$$b^2(t,x,\psi) + c^2(t,x,\psi) + \sigma^2(t,x) + \mu^2(t,x) \le K\big(|x^2| + \|\psi^2\|\big).$$

(A2) σ, μ are bounded functions, and there exists a strictly increasing function $\rho : \mathbb{R}^+ \to \mathbb{R}$ such that $\rho(0) = 0$, $\int_{0^+}\rho^{-2}(u)du = +\infty$ and

$$\big|\sigma(t,x) - \sigma(t,y)\big| \le \rho(|x-y|)$$
$$\big|\mu(t,x) - \mu(t,y)\big| \le \rho(|x-y|)$$

for every $t\ge 0$, $x,y\in R$.

Remark 1. Conditions (A1) and (A2) assure the uniqueness and non-explosion of the solutions (see [6] or [7]); (A2) holds, for instance, if σ and μ are Lipschitiz-continuous, or Hölder-continuous of order greater than $\frac{1}{2}$.

Now we state our main result.

Theorem 1. *Assume that* (A1), (A2) *hold, the functions* $\sigma(t,x), \mu(t,x)$ *are differentiable with respect to* x *and* t *respectively, and there exists a continuous function* $\alpha(t,x) : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *such that* $\alpha(t,x)$ *is also differentiable with respect to* x *and* t*, respectively, and for* $\psi\in C([-h,0];\mathbb{R})$

$$\tilde{b}(t,x,\psi) := \frac{b(t,x,\psi)\alpha(t,x)}{\sigma(t,x)} + \frac{\sigma(t,x)\partial_x\alpha(t,x) - \alpha(t,x)\partial_x\sigma(t,x)}{2} + \int_{f(0)}^{x}\frac{\sigma(t,u)\partial_t\alpha(t,u) - \alpha(t,x)\partial_t\sigma(t,u)}{\sigma^2(t,u)}du$$

$$\tilde{c}(t,x,\psi) := \frac{c(t,x,\psi)\alpha(t,x)}{\mu(t,x)} + \frac{\mu(t,x)\partial_x\alpha(t,x) - \alpha(t,x)\partial_x\mu(t,x)}{2} + \int_{g(0)}^{x} \frac{\mu(t,u)\partial_t\alpha(t,u) - \alpha(t,u)\partial_t\mu(t,u)}{\mu^2(t,u)}du$$

satisfy the conditions

(i) $\forall\, t \geq 0$, $\forall x, y \in \mathbb{R}$, $\forall \psi, \varphi \in C([-h,0];\mathbb{R})$,

$$|\tilde{b}(t,x,\psi) - \tilde{b}(t,y,\varphi)| + |\alpha(t,x) - \alpha(t,y)| \leq L_0(|x-y| + \|\psi - \varphi\|)$$
$$\tilde{c}(t,x,\psi) - \tilde{c}(t,y,\varphi)| + |\alpha(t,x) - \alpha(t,y)| \leq L_0(|x-y| + \|\psi - \varphi\|),$$

where $L_0 \geq 0$ *is a constant;*

(ii) $\forall\, t \geq 0$, $\forall\, x \in \mathbb{R}$, $\forall \psi \in C([-h,0];\mathbb{R})$,

$$\tilde{b}(t,x,\psi) \leq \tilde{c}(t,x,\psi);$$

(iii) $\forall\, t \geq 0, \forall\, x \in \mathbb{R}, \forall \psi, \varphi \in C([-h,0];\mathbb{R})$ *with* $\psi(\theta) \leq \varphi(\theta)$ *for any* $\theta \in [-h,0]$*, then*

$$\tilde{b}(t,x,\psi(\theta)) \leq \tilde{b}(t,x,\varphi(\theta)).$$

In addition, suppose that for all $t \geq 0$ *and* $x \in \mathbb{R}$,

$$\frac{\alpha(t,x)}{\sigma(t,x)} > 0, \qquad \frac{\alpha(t,x)}{\mu(t,x)} > 0 \tag{3}$$

$$\int_{f(0)}^{x} \frac{\alpha(t,u)}{\sigma(t,u)}du \geq \int_{g(0)}^{x} \frac{\alpha(t,u)}{\mu(t,u)}du. \tag{4}$$

Then, if initial conditions $f, g \in C([-h,0];\mathbb{R})$ *satisfy that* $f \leq g$*, for the solutions of equations* (1) *and* (2)*, the following holds:*

$$\mathbf{P}\{X(t) \leq Y(t), \forall\, t \geq 0\} = 1.$$

Proof. We assume that $f(0) = g(0)$; if $f(0) < g(0)$, the assertion follows from the case $f(0) = g(0)$. Let us consider the functions

$$F(t,x) = \begin{cases} \int_{f(0)}^{x} \frac{\alpha(t,u)}{\sigma(t,u)}du, & t > 0 \\ f(t), & t \in [-h,0] \end{cases}$$

$$G(t,x) = \begin{cases} \int_{f(0)}^{x} \frac{\alpha(t,u)}{\mu(t,u)}du, & t > 0 \\ g(t), & t \in [-h,0]. \end{cases}$$

From condition (3), we get that $\frac{\partial F}{\partial x} > 0, \frac{\partial G}{\partial x} > 0$. Now fixed $t > 0$, for any $z \in J_t := \{z : z = F(t,x), \text{ for some } x \in \mathbb{R}\}$ there exists a unique $x = x(z)$ such that

$F(t,x) = z$. Analogously for any $z' \in J'_t := \{z' : z' = G(t,x),\ \text{for some } x \in \mathbb{R}\}$ there exists a unique $x = x(z')$ such that $G(t,x) = z'$.

Define the processes $X_1(t)$ and $Y_1(t)$ by

$$X_1(t) = F(t, X(t))$$
$$Y_1(t) = G(t, Y(t)).$$

By Itô formula, we get for fixed $t > 0$

$$dX_1(t) = \tilde{b}(t, X(t), X_t)|_1\, dt + \alpha(t, X(t))\, dw(t)$$
$$dY_1(t) = \tilde{c}(t, Y(t), Y_t)|_1\, dt + \alpha(t, X(t))\, dw(t),$$

where $\tilde{b}(t, X(t), X_t)|_1$ means that $\tilde{b}(t, X(t), X_t)$ is calculated in $(t, X(t), X_t)$ such that $F(t, X(t)) = X_1(t)$, and $\tilde{c}(t, Y(t), Y_t)|_1$ means that $\tilde{c}(t, Y(t), Y_t)$ is calculated in $(t, Y(t), Y_t)$ such that $G(t, Y(t)) = Y_1(t)$.

Now, $X_1(t)$ and $Y_1(t)$ are the solutions of stochastic functional differential equations with the identical diffusion terms and initial conditions $f(t)$ and $g(t), t \in [-h, 0]$, respectively. By the recent results of [9], we have

$$\mathbf{P}\{X_1(t) \leq Y_1(t), \forall\ t \geq 0\} = 1. \tag{5}$$

On the other hand, we claim that for fixed t,

$$F(t,x) \leq G(t,y) \ \Rightarrow\ x \leq y. \tag{6}$$

In fact, by the condition (4), it follows that $F(t,y) \geq G(t,y)$, so if $F(t,x) \leq G(t,y)$ then it follows $F(t,x) \leq G(t,y) \leq F(t,y)$. Moreover, since $F(t,x)$ is increasing with respect to the second variable, this implies $x \leq y$. Combing with (5) and (6), we have $X(t) \leq Y(t)$ with probability 1 for all $t \geq 0$. □

Remark 2. In the case when $h = 0$, that is, the stochastic functional equations (1) and (2) reduce to the stochastic ordinary differential equations, our result Theorem 1 with $\alpha(t,x) \equiv \sqrt{2}$ is reduced to Theorem 2.1 in [1]. So our result generalizes the main result in [1].

Remark 3. For the special case when $\sigma(t,x) \equiv \mu(t,x)$, that is to say, stochastic functional equations (1) and (2) with different diffusion terms are reduced to stochastic functional differential equations with the identical diffusion term, if we choose $\alpha(t,x) = \sigma(t,x)$, then our result Theorem 1 is the same one as the theorem in [9]. So our result generalizes the main result in [9].

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References

[1] Abundo, M., Some comparison results of solutions of diffusion and jump-diffusion equations. *Adv. Appl. Stat.* 2 (2002), 51 – 78.

[2] Ferreyra, G. and Sunda, P., Comparison of solutions of stochastic equations and applications. *Stochastic Anal. Appl.* 18 (2000), 211 – 229.

[3] Galchuk, L. I. and Davis, M. H. A., A note on a comparison theorem for equations with different diffusions. *Stochastics* 6 (1981/82), 147 – 149.

[4] Geiss, C. and Manthey, R., Comparison theorems for stochastic differential equations in finite and infinite dimensions. *Stoch. Processes Appl.* 53 (1994), 23 – 55.

[5] Huang, Z. Y., A comparison theorem for solutions of stochastic differential equations and its applications. *Proc. Amer. Math. Soc.* 91 (1984), 611 – 617.

[6] Ikeda, N. and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes.* Amsterdam: North-Holland 1981.

[7] Kolmanovskii, V. B. and Myshkis, A. D., *Introduction to the Theory and Applications of Functional Differential Equations*. Dordrecht: Kluwer 1999.

[8] Malliavin, P., *Geometrie Differentielle Stochastique.* Seminar on Higher Mathematics 64. Montreal: Univ. Press Montreal 1978.

[9] Maizenberg, T. L., On the comparison of solutions of stochastic equations with aftereffect (in Russian). *Avtomat. i Telemekh.* (2001), 89 – 93; transl. in: *Autom. Remote Control* 62 (2001), 758 – 761.

[10] O'Brien, G. L., A new comparison theorem for solutions of stochastic differential equations. *Stochastics* 3 (1980), 245 – 249.

[11] Skorokhod, A. V., *Investigations in the theory of random processes* (in Russian). Kiev: Univ. Press Kiev 1961.

[12] Yamada, T., On a comparison theorem for solutions of stochastic differential equations and its applications. *J. Math. Kyoto Univ.* 11 (1971), 155 – 167.

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