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On Ren-Kähler's Paper ”Hardy-Littlewood Inequalities and Q_p-Spaces”

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Abstract. In this note we prove that a harmonic function u on the unit ball $B \subset \mathbb{R}^n$ belongs to the harmonic mixed norm space $\mathcal{A}^{p,q}_s(B)$, when $p, q \in (0, \infty]$ and $s > 0$, if and only if all weighted tangential derivatives of order k (with positive orders of derivatives) belong to the related weighted Lebesgue mixed norm space $\mathcal{L}^{p,q}_s(B)$. Our proof of the result for the case $q \in (0, 1)$ and k is odd, corrects the corresponding one in the paper: G. Ren and U. Kähler, Hardy-Littlewood inequalities and Q_p-spaces, *Z. Anal. Anwendungen* 24 (2005), 375 – 388.

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1. Introduction

Throughout this paper $B(a, r) = \{x \in \mathbb{R}^n \mid |x - a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \mathbb{R}^n$, B the open unit ball in $\mathbb{R}^n$, $rB = B(0, r)$, $S = \partial B = \{ x \in \mathbb{R}^n \mid |x| = 1\}$ is the boundary of B and $S_r = \{x \in \mathbb{R}^n \mid |x| = r\}$. Let further dV denote the Lebesgue measure on $\mathbb{R}^n$, $d\sigma$ the surface measure on S, σ_n the surface area of S, dV_N the normalized Lebesgue measure on B, $d\sigma_N$ the normalized surface measure on S.

For a given multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with each α_i, $i \in \{1, \dots, n\}$, a nonnegative integer, we use notations $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where ∂_j denotes the differentiation with respect to the jth variable.

Let $\mathcal{H}(B)$ denote the set of all harmonic functions on B. Some basic facts on harmonic functions can be found, for example, in [1].

For $u \in \mathcal{H}(B)$ and $p \in (0, \infty)$, we denote the integral mean of u by

$$M_p^p(u, r) = \int_S |u(r\zeta)|^p d\sigma_N(\zeta), \quad r \in [0, 1)$$

while $M_\infty(u, r) = \sup_{|x|<r} |u(x)|$, $r \in [0, 1)$.

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The *harmonic Hardy space* $\mathcal{H}^p(B)$, $p \in (0,\infty)$ consists of all $u \in \mathcal{H}(B)$ such that $\sup_{0<r<1} M_p(u,r) < \infty$.

The *mixed norm space* $\mathcal{A}^{p,q}_s(B)$, $p,q \in (0,\infty)$, $s \in \mathbb{R}$, consists of all $u \in \mathcal{H}(B)$ such that

$$\|u\|_{p,q,s} = \left(\int_0^1 M_q^p(u,r)(1-r)^{ps-1}dr\right)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, then

$$\mathcal{A}^{\infty,q}_s(B) = \Big\{u \in \mathcal{H}(B) \;\Big|\; \sup_{0<r<1}(1-r^2)^s M_q(u,r) < \infty\Big\}.$$

Let $\mathcal{R}_a = aI + \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$ where I denotes the identity operator. For $a = 0$, $\mathcal{R}_a$ is the standard radial differential operator $\mathcal{R}$. By $T_{ij}u = x_i\frac{\partial u}{\partial x_j} - x_j\frac{\partial u}{\partial x_i}$ we denote the tangential derivatives, where $1 \le i < j \le n$. Note that tangential derivatives of harmonic functions are again harmonic. Given a nontrivial multi-index α we use the notation $T^\alpha = T^{\alpha_1}_{i_1j_1}\cdots T^{\alpha_n}_{i_nj_n}$ for any choice of $i_1,\dots,i_n$ and $j_1,\dots,j_n$.

In [16] Ren and Kähler nicely note that the following result can be obtained by some modification of known ones:

Theorem A. *Let $0 < p,q \le \infty$ $0 < s < \infty$ and $k \in \mathbb{N}$. If $u \in \mathcal{H}(B)$, then the following quantities are equivalent:*

a) $\|u\|_{p,q,s}$;

b) $|u(0)| + \|\mathcal{R}^k u\|_{p,q,s+k}$;

c) $\sum_{|\alpha|<k}|\partial^\alpha u(0)| + \sum_{|\alpha|=k}\|\partial^\alpha u\|_{p,q,s+k}$.

They also claim that in [4] Choe–Koo–Yi have proved the following asymptotic relation:

$$\|u\|_{p,q,s} \asymp |u(0)| + \|T^k u\|_{p,q,s+k},$$

when $p = q \in [1,\infty]$, $s = 0$ and $k \in \mathbb{N}$, for every $u \in \mathcal{H}(B)$ (see [16, p. 380]). The above means that there are finite positive constants C and C' independent of u such that the left and right hand sides $L(u)$ and $R(u)$ satisfy

$$CR(u) \le L(u) \le C'R(u)$$

for all harmonic u.

However, there is a number of somewhat vague points in their paper [16]. First, they defined the norm $\|u\|_{p,q,s}$ wrongly, by $\int_B M_q^p(u,r)(1-r)^{ps-1}dr$ (see [16, p. 378]) which is probably a misprint. Second, the tangential derivative T is defined by $\{T_{ij}\}_{i<j}$ which looks like a set, and then it is not clear what $\|T^k u\|_{p,q,s+k}$ means. In our opinion the operator T should have been defined

by $T = \sum_{i<j} T_{ij}$. Third, in order to obtain Choe–Koo–Yi's result s should be equal to $\frac{1}{p}$, that is, Ren and Kähler have probably meant that

$$\|u\|_{p,p,\frac{1}{p}} \asymp |u(0)| + \sum_{|\alpha|=k} \|T^\alpha u\|_{p,p,\frac{1}{p}+k}$$

bearing in mind the fact that

$$\|u\|_{b^p} = \left(\int_B |u(x)|^p dV(x)\right)^{\frac{1}{p}} \asymp \|u\|_{p,p,\frac{1}{p}},$$

when $p \geq 1$. Note that it is not immediately clear why these two norms are equivalent in the case $p \in (0,1)$. As we know the monotonicity of the integral means plays a crucial role in proving the asymptotic relation for the case $p \geq 1$.

Motivated by Choe–Koo–Yi's result they tried to prove the following nice result:

Theorem 1. *Let* $0 < p, q \leq \infty$ $0 < s < \infty$ *and* $k \in \mathbb{N}$. *If* $u \in \mathcal{H}(B)$, *then*

$$\|u\|_{p,q,s} \asymp |u(0)| + \sum_{|\alpha|=k} \|T^\alpha u\|_{p,q,s+k}. \tag{1}$$

However, their proof of Theorem 1 is not true when $q \in (0,1)$ and k is odd. They assert that Propositions 3.8 and 3.9 in [16] are true for all $q \in (0,\infty]$ and that their proofs are exactly the same as the proofs of [15, Propositions 3.1 and 3.2]. Unfortunately, this is also not true. More specifically, in the proof of Proposition 3.1 on page 64, they use Bocher's Theorem 1 in [3], which was proved for analytic functions on the unit ball in $\mathbb{C}^n$ (moreover, on circular domains). Further, Hardy–Littlwood's maximal theorem (see [15, p. 66]) for the case of harmonic functions on the unit ball holds only when $q > 1$, unlike the case of analytic functions in the unit ball of $\mathbb{C}^n$. Beside that the standard trick of using the slice functions $g_\zeta(\lambda) = g(\lambda\zeta)$, where $\lambda \in \mathbb{C}$ and $\zeta \in \partial B$, cannot be applied for the case of harmonic functions.

Probably the biggest mistake in [16] appears in the proof of Theorem 3.10 where Ren and Kähler use the fact that the integral means $M_q(u,r)$ of harmonic functions are nondecreasing. Unfortunately, if $q \in (0,1)$ this is also not true causing several steps in the proof of Theorem 3.10 to be incorrect. For example, it is well known that the Poisson kernel $u(x) = P(x,\zeta) = \frac{1-|x|^2}{|x-\zeta|^n}$ where $\zeta \in S$ is fixed, is a harmonic function on B satisfying the condition $\lim_{r\to 1} \int_S |u(r\zeta)|^p d\sigma_N(\zeta) = 0$.

Another property which is specific for the case $p \in (0,1)$ is a kind of subharmonicity. We say that a locally integrable function f on B possesses *HL-property* with a constant $c > 0$ if

$$f(a) \leq \frac{c}{r^n} \int_{B(a,r)} f(x) dV(x) \quad \text{whenever } \overline{B}(a,r) \subset B.$$

For example, every subharmonic function ([11]) possesses *HL*-property with $c = \frac{1}{v_n}$. In [10] Hardy and Littlewood proved that $|u|^p$, when $p > 0$ and $n = 2$, also possesses *HL*-property whenever u is a harmonic function in B. In the case $n \geq 3$ a generalization was made by Fefferman and Stein [6].

Using Fefferman–Stein's result it follows that the estimate in [4, Lemma 2.2] also holds when $p \in (0,1)$. Namely, the following result holds true:

Lemma 1. *Let $p \in (0,\infty)$ and α be a multi-index. Suppose u is harmonic on a proper open subset G of $\mathbb{R}^n$. Then, we have*

$$|\partial^\alpha u(x)|^p \leq \frac{C}{d^{n+p|\alpha|}(x,\partial G)} \int_G |u(y)|^p dV(y) \quad (x \in G),$$

where $d(x,\partial G)$ denotes the distance from x to the boundary ∂G. The constant C depends only on n, p and α.

Mixed norm spaces and weighted Bergman spaces of analytic or harmonic functions of one or several variables have been studied extensively. For closely related results to Theorem A and Theorem 1, see, for example, [2, 4, 5], [7]–[10], [14], [17]–[25] and the references therein.

The organization of the paper is as follows: In Section 2 we formulate and prove two auxiliary results, which we use in the proof of Theorem 1. The main result of this paper (Theorem 1) is proved in Section 3.

We have to say that throughout the rest of the paper C will denote a constant not necessarily the same at each occurrence.

2. Auxiliary results

In this section we give two auxiliary results which we use in the proof of Theorem 1 in the next section. The following lemma was proved in [4, Lemma 5.1]:

Lemma 2. *Given an integer $m \geq 1$, there is a smooth differential operator E_m of order $2m-1$ with bounded coefficients such that*

$$\mathcal{R}^{2m}u = \Big(-\sum_{i<j} T_{ij}^2\Big)^m u + E_m u$$

for functions u harmonic on B.

The next lemma is a generalization of Proposition 5.1 in [4].

Lemma 3. *Let $0 < p, q \leq \infty$, $\varepsilon \in (0,1)$ and m be a positive integer. Then there is a positive constant $C = C(p,q,\varepsilon,m)$ such that*

$$\sup_{|x|\leq r} |u(x) - u(0)| \leq C \sum_{|\alpha|=m} \|T^\alpha u\|_{p,q,s+m} \tag{2}$$

whenever $0 < r < 1-\varepsilon$ and u is harmonic on B.

Proof. From the proof of Proposition 5.1 in [4] we have that there is a positive constant C independent of u such that

$$|u(x) - u(0)| \leq C \sup_{\eta \in S_r} \sum_{i<j} |T_{ij} u(\eta)| = C \sup_{\eta \in S_r} \sum_{|\alpha|=1} |T^\alpha u(\eta)| \tag{3}$$

for every $x \in S_r$. Applying (3) to the harmonic function $T_{i_1 j_1} u$ and using the fact $T_{i_1 j_1} u(0) = 0$ we obtain

$$|T_{i_1 j_1} u(x)| \leq C \sup_{\eta \in S_r} \sum_{|\alpha|=1} |T^\alpha T_{i_1 j_1} u(\eta)| \leq \sup_{\eta \in S_r} \sum_{|\alpha|=2} |T^\alpha u(\eta)|,$$

for every $x \in S_r$ and every $1 \leq i_1 < j_1 \leq n$. Continuing this process it follows that, for every $m \in \mathbb{N}$, there is a positive constant C independent of u such that $|u(x) - u(0)| \leq C \sup_{\eta \in S_r} \sum_{|\alpha|=m} |T^\alpha u(\eta)|$ for every $x \in S_r$. Using the maximum principle and Lemma 3.3 in [16] (with $k = 1$, $\alpha \to s+m$ and $f = T^\alpha u$), it follows that

$$\begin{aligned} \sup_{rB} |u(x) - u(0)| &\leq C \sup_{\eta \in S_r} \sum_{|\alpha|=m} |T^\alpha u(\eta)| \\ &\leq C \sup_{w \in rB} \sum_{|\alpha|=m} |T^\alpha u(w)| \\ &\leq C(p, q, \varepsilon, m) \sum_{|\alpha|=m} \|T^\alpha u\|_{p,q,s+m}. \end{aligned}$$

□

3. Proof of the main result

In this section we prove the main result in this paper.

Proof of Theorem 1. Let α be a multi-index of order k. Then by Theorem A and the definition of tangential derivatives we have that

$$\|T^\alpha u\|_{p,q,s+k} \leq C \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha u\|_{p,q,s+k} \leq C \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha u\|_{p,q,s+|\alpha|} \leq C \|u\|_{p,q,s} .$$

From this and [16, Lemma 3.3], it follows that

$$|u(0)| + \sum_{|\alpha|=k} \|T^\alpha u\|_{p,q,s+k} \leq C \|u\|_{p,q,s}. \tag{4}$$

Now we prove that there is a constant C such that

$$\|u\|_{p,q,s} \leq C \biggl(|u(0)| + \sum_{|\alpha|=k} \|T^\alpha u\|_{p,q,s+k} \biggr).$$

Let T_{ij} be any tangential differential operator and $|\alpha| = 2m-1$. Then, by Theorem A applied to the function $T^\alpha u$, we have

$$\begin{aligned}\|(1-|x|)^{2m} T_{ij} T^\alpha u\|_{p,q,s} &\le C\|(1-|x|)\nabla T^\alpha u (1-|x|)^{2m-1}\|_{p,q,s}\\ &\le C\|(1-|x|)^{2m-1} T^\alpha u\|_{p,q,s}\end{aligned}$$

which implies that

$$\sum_{|\alpha|=2m} \|T^\alpha u\|_{p,q,s+2m} \le C \sum_{|\alpha|=2m-1} \|T^\alpha u\|_{p,q,s+2m-1}. \tag{5}$$

Let E_m be the differential operator in Lemma 2. Then, we have

$$\|\mathcal{R}^{2m} u\|_{p,q,s+2m} \le C\Big(\sum_{|\alpha|=2m} \|T^\alpha u\|_{p,q,s+2m} + \|(1-|x|)^{2m} E_m u\|_{p,q,s}\Big). \tag{6}$$

Let $\delta \in (0,1)$. Then by Theorem A we have

$$\begin{aligned}\Big(\int_{1-\delta}^1 M_q^p(E_m u, r)(1-r^2)^{p(s+2m)-1} dr\Big)^{\frac{1}{p}} &\le C\delta\|(1-|x|)^{2m-1} E_m u\|_{p,q,s}\\ &\le C_1\delta\|u\|_{p,q,s}.\end{aligned} \tag{7}$$

On the other hand, by Lemma 1 and Lemma 3, we have

$$\begin{aligned}\Big(\int_0^{1-\delta} M_q^p(E_m u, r)(1-r^2)^{p(s+2m)-1} dr\Big)^{\frac{1}{p}} &\le C \sup_{|x|<1-\delta} |(1-|x|)^{2m-1} E_m u(x)|\\ &\le C \sup_{|x|<1-\delta/2} |u(x)|\\ &\le C\Big(|u(0)| + \sum_{|\alpha|=2m} \|T^\alpha u\|_{p,q,s+2m}\Big).\end{aligned} \tag{8}$$

From (7) and (8), it follows that

$$\|(1-|x|)^{2m} E_m u\|_{p,q,s} \le C\Big(|u(0)| + \sum_{|\alpha|=2m} \|T^\alpha u\|_{p,q,s+2m}\Big) + C_1\delta\|u\|_{p,q,s}, \tag{9}$$

for some C_1 independent of δ. Since $|u(0)| + \|\mathcal{R}^{2m} u\|_{p,q,s+2m} \asymp \|u\|_{p,q,s}$ and from (6) and (9), it follows that

$$\|u\|_{p,q,s} \le C\Big(|u(0)| + \sum_{|\alpha|=2m} \|T^\alpha u\|_{p,q,s+2m}\Big) + C\delta\|u\|_{p,q,s}. \tag{10}$$

Taking in (10) δ sufficiently small we obtain the result, for k even. If k is odd then the result follows from (4), (5) and the asymptotics (1) for k even, finishing the proof of the theorem for the case $p \in (0,\infty)$. The proof in the case $p = \infty$ is simpler and is omitted. □

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