Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 26 (2007), 495–506

Estimates of Approximate Solutions and Well-Posedness in Vector Optimization

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Abstract. In this paper we estimate the sizes of approximate solution sets for vector optimization from outside and inside, respectively. In terms of an important scalarization function, we obtain some estimates of approximate solutions for well-posed vector optimization, and some estimates of approximate solutions for well-posed vector optimization under perturbations.

Keywords. Vector optimization, well-posedness, extended well-posedness, approximate solution, estimate

Mathematics Subject Classification (2000). Primary 90C31, secondary 65K05, 49K40

1. Introduction

The study of approximate solutions is very important in the theory of optimization. From numerical and practical points of view, the task is always to find the approximate solution instead of the exact solution of the problem because the finding of exact solution costs much, and sometimes even is impossible. Given an approximate solution, it is necessary to estimate its distance to the exact solution. Such estimate is an important and interesting subject of stability analysis of optimization problems, see, e.g., [1]–[3], [22] and the references therein. Another important issue related to stability analysis is on the well-posedness of optimization which deals with the continuity property of the solutions with respect to data's perturbations. An initial notion of well-posedness was first introduced by Tykhonov [19], already known as Tykhonov well-posedness, which means the existence and uniqueness of solution, and the convergence of every "minimizing sequence" to the unique solution of the problem. In the following years, various notions of well-posedness were introduced and studied. Concerning well-posedness in scalar optimization, we can refer the readers to [8],

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[19]–[23] and the references therein. In [22], Zolezzi studied the sets of approximate solutions for well-posed scalar optimization. Some estimates of the size of approximate solution sets were obtained for Tykhonov well-posed optimization, and further for extended well-posed (called well-posed under perturbations in [22]) optimization. In recent years, the concepts of well-posedness have been generalized to vector optimization; see, e.g., [4, 5, 7], [9]–[12], [14, 15]. A natural problem is: whether or not some analogous estimates can be established for well-posed vector optimization? The paper is denoted to the study of this topic. In terms of an important function ξ (defined in the sequel, see also [9, 13, 17]) we establish some estimates of approximate solutions for well-posed vector optimization, and some estimates of approximate solutions for well-posed vector optimization under perturbations. Our results generalize and improve the corresponding results by Zolezzi [22].

2. Preliminaries and notations

Let X and Y be normed vector spaces. Denote by $B(x,\delta)$ ($\bar{B}(x,\delta)$) the open (closed) ball, centered at x with radius δ. The space Y is endowed with a partial order induced by a pointed, closed and convex cone C with $\mathrm{int\,C} \neq \emptyset$ in the following way:

$$\begin{aligned} x \leq_C y \quad &\Leftrightarrow y - x \in C \\ x \leq_{\mathrm{int\,C}} y \ &\Leftrightarrow y - x \in \mathrm{int\,C}, \end{aligned}$$

where $\mathrm{int}\,C$ denotes the interior of C. Assume that P is a parametric metric space, p^* is a fixed point in P, and that L is a closed ball in P centered at p^* with a positive radius. Let $J : X \mapsto Y$ and $I : X \times L \mapsto Y$ be vector-valued functions. Consider the following global vector optimization problems:

$$\begin{aligned} &(X, J): && \min_{x\in X} J(x) \\ &(X, I(\cdot, p)): && \min_{x\in X} I(x, p). \end{aligned}$$

Assume always that

$$J(x) = I(x, p^*), \quad \forall x \in X.$$

(X, J) is called the original problem and $(X, I(\cdot, p))$ is called the perturbed problem of the original problem corresponding to the parameter $p \in L$. A point $y^* \in I(X, L)$ is called a minimal point of $(X, I(\cdot, p))$ (for short (p) if no confusion arises) iff

$$I(X, p) \cap (y^* - C) = \{y^*\}.$$

A point $x^* \in X$ is called an *efficient solution* of (p) iff $I(x^*, p)$ is a minimal point of (p). Denote by $\mathrm{Min(X, I(\cdot, p))}$ and $\mathrm{Eff(X, I(\cdot, p))}$ (or $\mathrm{Min(p)}$ and $\mathrm{Eff(p)}$, respectively, if no confusion arises) the sets of minimal points and efficient solutions of (p), respectively. Fix an $e \in \mathrm{intC}$. In the sequel, we use frequently the following function $\xi : Y \to \mathbb{R}$ (see [9, 13, 17]) defined by

$$\xi(y) = \min\{t \in \mathbb{R} : y \in te - C\}, \quad \forall y \in Y.$$

Lemma 2.1 ([9, 11, 13, 17]). *The following conclusions hold:*

(i) *The function ξ is continuous, monotone (with respect to C) and sublinear;*

(ii) *For any $y \in Y$, $\xi(y) = \max_{\lambda \in C^* \setminus \{0\}} \frac{\lambda(y)}{\lambda(e)}$, where C^* is the usual dual cone of C.*

The following concepts of well-posedness for vector optimization were introduced and studied by Dentcheva and Helbig [7] and Huang [10]–[12].

Definition 2.2 ([7]). (X, J) is said to be *well-posed of type* 1 at a point $v \in \mathrm{Eff(p^*)}$ iff

$$\inf_{\alpha > 0} \mathrm{diam}\, L(\mathrm{v}, \mathrm{q}, \alpha) = 0, \quad \forall \mathrm{q} \in \mathrm{C},$$

where $L(v, q, \alpha) = \{x \in X : J(x) \leq_C J(v) + \alpha q\}$ and diam denotes the diameter of a set.

Note that if (X, J) is well-posed of type 1 at a point $v \in \mathrm{Eff(p^*)}$, then $J^{-1}(J(v)) = \{v\}$.

Definition 2.3 ([10]). (X, J) is said to be *well-posed of type* 2 at a point $v \in \mathrm{Eff(p^*)}$ iff $x_n \to v$, for any sequence $\{x_n\} \subset X$ with $J(x_n) \to J(v)$.

Remark 2.4. As pointed out by Huang [10], well-posedness of type 1 implies well-posedness of type 2, but the converse is not true in general.

Definition 2.5 ([11, 12]). (X, J) is said to be *extended well-posed* iff

(i) $\mathrm{Eff(p)} \neq \emptyset$ for all $p \in L$;

(ii) $\forall (p_n)$ with $p_n \to p^*$, $\forall (a_n)$ with $0 \leq \alpha_n \to 0$, $\forall (y_n)$ with $y_n \in \mathrm{Min(p_n)}$ (for every n), $\forall (x_n)$ with $x_n \in X$ and $I(x_n, p_n) \leq_C y_n + \alpha_n e$ (for every n), $\exists x^* \in \mathrm{Eff(p^*)} : x_n \to x^*$.

Note that if (X, J) verifies conditions (i) and (ii) of Definition 2.5, then $\mathrm{Eff(p^*)}$ is a singleton (see the statements of Propositions 4.7 and 4.9 below).

Definition 2.6 ([16]). A set-valued mapping $F : X \mapsto 2^Y$ is said to be

(i) *upper Hausdorff continuous* at $x_0 \in X$ iff for any neighborhood V of 0 in Y there exists a neighborhood W of x_0 in X such that $F(x) \subset F(x_0) + V$ for all $x \in W$;

(ii) *lower Hausdorff continuous* at $x_0 \in X$ iff for any neighborhood V of 0 in Y there exists a neighborhood W of x_0 in X such that $F(x_0) \subset F(x) + V$ for all $x \in W$.

Definition 2.7. A set-valued mapping $F : X \mapsto 2^Y$ is said to be

(i) *C-upper semicontinuous at $x_0 \in X$* iff for any $d \in \operatorname{int} \mathrm{C}$ there exists a neighborhood W of x_0 in X such that $F(x) \subset F(x_0) + d - \operatorname{int} \mathrm{C}$ for all $x \in W$;

(ii) *C-lower semicontinuous at $x_0 \in X$* iff for any $d \in \operatorname{int} \mathrm{C}$ there exists a neighborhood W of x_0 in X such that $F(x_0) \subset F(x) + d - \operatorname{int} \mathrm{C}$ for all $x \in W$.

F is called *C-upper* (*C-lower*) *semicontinuous* iff F is C-upper (C-lower) semicontinuous at every $x_0 \in X$.

Remark 2.8. (i) When F is single-valued, Definition 2.7 reduces to the definition of cone semicontinuity in the sense of Luc [13], Bianchi, Hadjisavvas and Schaible [6], and Tanaka [18].

(ii) Upper (lower) Hausdorff continuity implies C-upper(lower) semicontinuity.

3. Estimates of approximate solutions and well-posedness

In this section we give some estimates of approximate solutions for well-posed vector optimization. In what follows we fix $u^* \in \mathrm{Eff}(\mathrm{p}^*)$ and $e \in \operatorname{int} \mathrm{C}$. Set $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. Consider the set $L(u^*, e, \epsilon)$, which is defined as in Definition 2.2. To estimate the size of $L(u^*, e, \epsilon)$ we need the following important lemma and functions:

Lemma 3.1 (Theorem 2.3.1 of [9]). *Given $y \in Y$ and $t \in R$. Then $\xi(y) \le t$ if and only if $y \le_C te$.*

Set

$$
\begin{aligned}
\alpha(\epsilon) &= \sup\{\|u - u^*\| : u \in X, \xi(J(u) - J(u^*)) \le \epsilon\} \\
\Delta(\epsilon) &= \sup\{\delta > 0 : \|u - u^*\| \le \delta \Rightarrow \xi(J(u) - J(u^*)) \le \epsilon\} \\
c^*(t) &= \inf\{\xi(J(u) - J(u^*)) : u \in X, \|u - u^*\| = t\} \\
q_1(s) &= \sup\{t \ge 0 : c^*(t) \le s\} \\
q_2(s) &= \sup\{t \ge 0 : c^*(t) < s\} \\
k(t) &= \sup\{\xi(J(u) - J(u^*)) : \|u - u^*\| \le t\} \\
k_1(s) &= \sup\{t \ge 0 : k(t) \le s\}.
\end{aligned}
$$

By Lemma 3.1, it is easy to see that $\alpha(\epsilon)$ and $\Delta(\epsilon)$ are the radii of the largest open ball centered at u^* contained in $L(u^*, e, \epsilon)$, and the smallest closed ball centered at u^* containing $L(u^*, e, \epsilon)$, respectively; in particular, $\Delta(\epsilon) \le \alpha(\epsilon)$. By the definitions, it is clear that α, Δ, k and k_1 are nondecreasing on $[0, \infty)$, $k(0) = 0, k_1(0) \ge 0$, and $\alpha(0) \ge 0$, but $\Delta(\epsilon)$ might be $-\infty$ for some $\epsilon > 0$.

Proposition 3.2. *For all positive number ϵ, the following relations hold:*

$$q_2(\epsilon) \le \alpha(\epsilon) \le q_1(\epsilon) \tag{1}$$

$$k_1(\epsilon) = \max\{0, \Delta(\epsilon)\}. \tag{2}$$

Proof. We first prove (1). By the definition of c^*, we have

$$\xi(J(u) - J(u^*)) \ge c^*(\|u - u^*\|), \quad \forall u \in X.$$

For any $u \in L(u^*, e, \epsilon)$, it follows from Lemma 3.1 that $c^*(\|u - u^*\|) \le \epsilon$. From the definition of q_1, we get $\|u - u^*\| \le q_1(\epsilon)$, and so $\alpha(\epsilon) \le q_1(\epsilon)$. To prove the left side of (1), we only consider the case when $\alpha(\epsilon) < +\infty$ (in fact, the left side of (1) holds trivially when $\alpha(\epsilon) = +\infty$). Suppose by contradiction that $q_2(\epsilon) > \alpha(\epsilon)$ for some ϵ. By the definition of q_2, there exists some $T \ge 0$ such that

$$c^*(T) < \epsilon, \quad T > \alpha(\epsilon). \tag{3}$$

Again from the definition of c^*, there exists $u \in X$ with $\|u - u^*\| = T$ such that $\xi(I(u) - I(u^*)) < \epsilon$. From the definition of α, we obtain $T \le \alpha(\epsilon)$, contradicting (3). Thus (1) is proved.

Next, we prove (2). Fix $\epsilon > 0$. If $k_1(\epsilon) = 0$, then $k_1(\epsilon) \le \max\{0, \Delta(\epsilon)\}$. Else take $0 < \gamma < k_1(\epsilon)$. Then there exists $t > \gamma$ with $k(t) \le \epsilon$. Hence $\xi(J(u) - J(u^*)) \le \epsilon$ for $\|u - u^*\| \le t$, and so $\Delta(\epsilon) \ge t > \gamma$. Therefore, $k_1(\epsilon) \le \Delta(\epsilon)$. Thus $k_1(\epsilon) \le \max\{0, \Delta(\epsilon)\}$ for all $\epsilon > 0$.

Take now $\gamma > k_1(\epsilon)\,(\ge 0)$. Then $k(\gamma) > \epsilon$, and so there exists $u \in X$ such that $\|u - u^*\| \le \gamma$ and $\xi(J(u) - J(u^*)) > \epsilon$. It follows that $\Delta(\epsilon) \le \gamma$. Hence $\Delta(\epsilon) \le k_1(\epsilon)$. Since k_1 is nondecreasing on $[0, \infty)$ and $k_1(0) \ge 0$, $\max\{0, \Delta(\epsilon)\} \le k_1(\epsilon)$. □

Remark 3.3. Proposition 3.2 generalizes Proposition 1 of Zolezzi in [22].

Proposition 3.4. *Assume that X is a finite-dimensional space and J is C-lower semicontinuous. Then $\alpha(\epsilon) = q_1(\epsilon)$ for all $\epsilon > 0$.*

Proof. By Proposition 3.2, it is sufficient to show that $\alpha(\epsilon) \ge q_1(\epsilon)$. When $q_1(\epsilon) = +\infty$, from the definitions of q_1 and α, one has $\alpha(\epsilon) = +\infty$. Suppose that $q_1(\epsilon) < +\infty$. By the definition of q_1, there exists a sequence $\{t_n\} \subset R_+$ such that $t_n \to q_1(\epsilon)$, $c^*(t_n) < \epsilon + \frac{1}{n}$. By the definition of c^*, there exists a sequence $\{u_n\} \subset X$ such that $\|u_n - u^*\| = t_n$, $\xi(J(u_n) - J(u^*)) \le \epsilon + \frac{1}{n}$. Since X is finite-dimensional, without loss of generality, we can suppose that $u_n \to u$. It follows that

$$\|u - u^*\| = q_1(\epsilon), \quad J(u_n) \in J(u^*) + \Big(\epsilon + \frac{1}{n}\Big)e - C. \tag{4}$$

Since J is C-lower semicontinuous, for any $d \in \operatorname{int} \mathrm{C}$ there exists n_0 such that $J(u_n) \in J(u) - d + \operatorname{int} \mathrm{C}$, for all $n > n_0$. This together with (4) implies that

$J(u) \in J(u^*)+d+(\epsilon+\frac{1}{n})e-\operatorname{int} \mathrm{C}$, for all $n > n_0$. Since $d \in \operatorname{int} \mathrm{C}$ is arbitrary, it follows that $J(u) \leq_C J(u^*)+\epsilon e$. Therefore, $\|u-u^*\| = q_1(\epsilon)$, $\xi(J(u)-J(u^*)) \leq \epsilon$, whence $\alpha(\epsilon) \geq q_1(\epsilon)$. □

Proposition 3.5. *If* (X, J) *is well-posed of type* 1 *at* $u^* \in \mathrm{Eff}(\mathrm{p}^*)$, *then* $q_1(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Proof. If $q_1(\epsilon) \not\to 0$ as $\epsilon \to 0$, then there exist $\delta > 0$ and $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that $q_1(\epsilon_n) \geq \delta > 0$. By the definition of q_1, there exists t_n with $t_n \geq \delta$ such that $c^*(t_n) \leq \epsilon_n$. Then there exists $u_n \in X$ with $\|u_n - u^*\| = t_n$ such that $\xi(J(u_n) - J(u^*)) \leq \epsilon_n$. From Lemma 3.1, we have $u_n \in L(u^*, e, \epsilon_n)$, and so $t_n = \|u_n - u^*\| \leq \operatorname{diam} L(\mathrm{u}^*, \mathrm{e}, \epsilon_\mathrm{n}) \to 0$ since (X, J) is well-posed of type 1 at u^*. This gives a contradiction. □

Remark 3.6. By Propositions 3.2, 3.4 and 3.5, $\alpha(\epsilon) \to 0$ and $q_2(\epsilon) \to 0$ as $\epsilon \to 0$ whenever (X, J) is well-posed of type 1 at $u^* \in \mathrm{Eff}(\mathrm{p}^*)$.

4. Estimates of approximate solutions and extended well-posedness

In this section we give some estimates of approximate solutions of extended well-posed vector optimization taking into account perturbations.

Define

$$T_1(\epsilon, \delta) = \bigcup_{p \in \bar{B}(p^*, \delta)} \{u \in X : I(u, p) \in \mathrm{Min(p)} + \epsilon \mathrm{e} - \mathrm{C}\}$$

$$T_2(\epsilon, \delta) = \bigcup_{p \in \bar{B}(p^*, \delta)} \{u \in X : I(u, p) \in \mathrm{Min(p^*)} + \epsilon \mathrm{e} - \mathrm{C}\}.$$

To estimate the sizes of $T_1(\epsilon, \delta)$ and $T_2(\epsilon, \delta)$ from outside, we need the following functions:

$$c_1^*(t, s) = \inf_{\|u-u^*\|=t, p \in \bar{B}(p^*, s)} \inf_{y \in \mathrm{Min(p)}} \xi(I(u, p) - y)$$

$$c_2^*(t, s) = \inf_{\|u-u^*\|=t, p \in \bar{B}(p^*, s)} \inf_{y \in \mathrm{Min(p^*)}} \xi(I(u, p) - y)$$

$$\alpha_i(\epsilon, \delta) = \sup\{\|u - u^*\| : u \in T_i(\epsilon, \delta)\}, \quad i = 1, 2$$

$$q_1^i(s, \delta) = \sup\{t \geq 0 : c_i^*(t, \delta) \leq s)\}, \qquad i = 1, 2$$

$$q_2^i(s, \delta) = \sup\{t \geq 0 : c_i^*(t, \delta) < s)\}, \qquad i = 1, 2$$

$$\omega_1(\delta) = \sup_{p \in \bar{B}(p^*, \delta)} \sup\{\xi(y) : y \in \mathrm{Min(p^*)} - \mathrm{Min(p)}\}$$

$$\omega_2(\delta) = \sup_{p \in \bar{B}(p^*, \delta)} \sup\{\xi(y) : y \in \mathrm{Min(p)} - \mathrm{Min(p^*)}\}.$$

Proposition 4.1. *For all positive numbers ϵ and δ, the following inequalities hold:*

$$q_2^1(\epsilon,\delta) \leq \alpha_1(\epsilon,\delta) \leq q_1^1(\epsilon,\delta) \tag{5}$$

$$q_2^2(\epsilon,\delta) \leq \alpha_2(\epsilon,\delta) \leq q_1^2(\epsilon,\delta). \tag{6}$$

Proof. To prove (5), let $u \in T_1(\epsilon,\delta)$. Then there exists $p \in \bar{B}(p^*,\delta)$ such that $I(u,p) \in \mathrm{Min(p)} + \epsilon \mathrm{e} - \mathrm{C}$. From Lemma 3.1, we have $\xi(I(u,p)-y) \leq \epsilon$ for some $y \in \mathrm{Min(p)}$. This together with the definition of c_1^* implies that $c_1^*(\|u-u^*\|,\delta) \leq \epsilon$. By the definition of q_1^1, we obtain $\|u-u^*\| \leq q_1^1(\epsilon,\delta)$, and so $\alpha_1(\epsilon,\delta) \leq q_1^1(\epsilon,\delta)$.

To prove the left side of (5), suppose by contradiction that $q_2^1(\epsilon,\delta) > \alpha_1(\epsilon,\delta)$ for some ϵ and δ. Then there exists some t such that $t > \alpha_1(\epsilon,\delta)$, $c_1^*(t,\delta) < \epsilon$. By the definition of c_1^*, there exist u with $\|u-u^*\| = t$ and $p \in \bar{B}(p^*,\delta)$ such that

$$\inf_{y \in \mathrm{Min(p)}} \xi(I(u,p)-y) < \epsilon.$$

This together with Lemma 3.1 implies that $I(u,p) \in y + \epsilon e - C$ for some $y \in \mathrm{Min(p)}$. By the definition of α_1, we get $t \leq \alpha_1(\epsilon,\delta)$, a contradiction. Thus (5) is proved.

Similarly, we can prove (6). $\square$

Proposition 4.2. *For all positive numbers ϵ and δ, the following inequalities hold:*

$$q_2^1(\epsilon-\omega_2(\delta),\delta) \leq \alpha_2(\epsilon,\delta) \leq q_1^1(\epsilon+\omega_1(\delta),\delta) \tag{7}$$

$$q_2^2(\epsilon-\omega_1(\delta),\delta) \leq \alpha_1(\epsilon,\delta) \leq q_1^2(\epsilon+\omega_2(\delta),\delta). \tag{8}$$

Proof. To prove (7), let $u \in T_2(\epsilon,\delta)$. Then there exists $p \in \bar{B}(p^*,\delta)$ such that $I(u,p) \in y^* + \epsilon e - C$ for some $y^* \in \mathrm{Min(p^*)}$. By Lemmas 2.1 and 3.1, for given $y \in \mathrm{Min(p)}$ we have

$$\xi(I(u,p)-y) \leq \xi(y^*-y+\epsilon e) \leq \xi(y^*-y)+\epsilon \leq \omega_1(\delta)+\epsilon.$$

It follows that $c_1^*(\|u-u^*\|,\delta) \leq \omega_1(\delta)+\epsilon$. By the definition of q_1^1, we obtain $\|u-u^*\| \leq q_1^1(\epsilon+\omega_1(\delta),\delta)$ and so $\alpha_2(\epsilon,\delta) \leq q_1^1(\epsilon+\omega_1(\delta),\delta)$.

To prove the left side of (7), only the case when $\alpha_2(\epsilon,\delta) < +\infty$ and $q_2^1(\epsilon - \omega_2(\delta),\delta) > -\infty$ is of interest. Suppose by contradiction that $\alpha_2(\epsilon,\delta) < q_2^1(\epsilon - \omega_2(\delta),\delta)$ for some ϵ and δ. Then there exists some t such that $t > \alpha_2(\epsilon,\delta)$, $c_1^*(t,\delta) < \epsilon - \omega_2(\delta)$. By the definition of c_1^*, there exist u with $\|u-u^*\| = t$ and $p \in \bar{B}(p^*,\delta)$ such that

$$\inf_{y \in \mathrm{Min(p)}} \xi(I(u,p)-y) < \epsilon - \omega_2(\delta).$$

This together with Lemma 3.1 yields that $I(u,p) \in y + (\epsilon-\omega_2(\delta))e - C$ for some $y \in \mathrm{Min(p)}$. For given $y^* \in \mathrm{Min(p^*)}$, it follows that

$$\xi(I(u,p)-y^*) \leq \xi(y-y^*+(\epsilon-\omega_2(\delta))e) \leq \xi(y-y^*)+(\epsilon-\omega_2(\delta)) \leq \epsilon.$$

Thus $t \le \alpha_2(\epsilon, \delta)$, a contradiction. Thus (7) is proved.

Similarly, we can prove (8). $\square$

Remark 4.3. Proposition 4.2 generalizes Proposition 2 of Zolezzi in [22].

Next we estimate the sizes of $T_i(\epsilon, \delta)$ from inside in terms of the following functions:

$$\Delta_i(\epsilon, \delta) = \sup\{t \ge 0 : B(u^*, t) \subset T_i(\epsilon, \delta)\}, \quad i = 1, 2$$

$$k^1(t, \delta) = \inf_{p \in \bar{B}(p^*, \delta)} \sup_{u \in \bar{B}(u^*, t)} \inf_{y \in \mathrm{Min(p)}} \xi(I(u, p) - y)$$

$$k^2(t, \delta) = \inf_{p \in \bar{B}(p^*, \delta)} \sup_{u \in \bar{B}(u^*, t)} \inf_{y \in \mathrm{Min(p^*)}} \xi(I(u, p) - y)$$

$$k_i^*(\epsilon, \delta) = \sup\{t \ge 0 : k^i(t, \delta) \le \epsilon\}, \quad i = 1, 2.$$

Proposition 4.4. *For all positive numbers ϵ and δ, the following inequalities hold:*

$$k_1^*(\epsilon, \delta) \le \Delta_1(\epsilon, \delta) \le q_1^1(\epsilon, \delta) \tag{9}$$

$$k_2^*(\epsilon, \delta) \le \Delta_2(\epsilon, \delta) \le q_2^1(\epsilon, \delta). \tag{10}$$

Proof. To prove the left side of (9), suppose by contradiction that $k_1^*(\epsilon, \delta) > \Delta_1(\epsilon, \delta)$ for some ϵ and δ. Then there exists some $t \ge 0$ such that $t > \Delta_1(\epsilon, \delta)$, $k^1(t, \delta) \le \epsilon$. By the definition of k^1, there exists some $p \in \bar{B}(p^*, \delta)$ such that

$$\sup_{u \in \bar{B}(u^*, t)} \inf_{y \in \mathrm{Min(p)}} \xi(I(u, p) - y) \le \epsilon.$$

By Lemma 3.1, for every $u \in \bar{B}(u^*, t)$ there exists $y \in \mathrm{Min(p)}$ such that $I(u, p) \in y + \epsilon e - C$. This proves that $\bar{B}(u^*, t) \subset T_1(\epsilon, \delta)$ and so $\Delta_1(\epsilon, \delta) \ge t$, a contradiction. Thus the left side of (9) is proved.

Similarly, we can prove the left side of (10). The right sides of (9) and (10) follow immediately from Proposition 4.1 and the fact $\Delta_i(\epsilon, \delta) \le \alpha_i(\epsilon, \delta)$, $i = 1, 2$. $\square$

Proposition 4.5. *For all positive numbers ϵ and δ, the following inequalities hold:*

$$k_2^*(\epsilon - \omega_1(\delta), \delta) \le \Delta_1(\epsilon, \delta) \le q_1^2(\epsilon + \omega_2(\delta), \delta) \tag{11}$$

$$k_1^*(\epsilon - \omega_2(\delta), \delta) \le \Delta_2(\epsilon, \delta) \le q_1^1(\epsilon + \omega_1(\delta), \delta). \tag{12}$$

Proof. To prove the left side of (11), assume by contradiction that $k_2^*(\epsilon - \omega_1(\delta), \delta) > \Delta_1(\epsilon, \delta)$ for some ϵ and δ. Then there exists some $t \ge 0$ such that $t > \Delta_1(\epsilon, \delta)$, $k^2(t, \delta) \le \epsilon - \omega_1(\delta)$. By the definition of k^2, there exists $p \in \bar{B}(p^*, \delta)$ such that

$$\sup_{u \in \bar{B}(u^*, t)} \inf_{y \in \mathrm{Min(p^*)}} \xi(I(u, p) - y) \le \epsilon - \omega_1(\delta).$$

This means that for every $u \in \bar{B}(u^*, t)$, $\xi(I(u,p) - y^*) \leq \epsilon - \omega_1(\delta)$ for some $y^* \in \mathrm{Min(p^*)}$. Let $y \in \mathrm{Min(p)}$. From Lemma 3.1, we have $I(u,p) - y \in y^* - y + (\epsilon - \omega_1(\delta))e - C$. It follows from Lemmas 2.1 and 3.1 that

$$\xi(I(u,p) - y) \leq \xi(y^* - y + (\epsilon - \omega_1(\delta))e) \leq \xi(y^* - y) + (\epsilon - \omega_1(\delta)) \leq \epsilon.$$

Again from Lemma 3.1, we obtain $I(u,p) \in y + \epsilon e - C$. Summarizing, we get $\bar{B}(u^*, t) \subset T_1(\epsilon, \delta)$ and so $t \leq \Delta_1(\epsilon, \delta)$, a contradiction. Thus the left side of (11) is proved.

Similarly we can prove the left side of (12). The right sides of (11) and (12) follow immediately from Proposition 4.2 and the fact $\Delta_i(\epsilon, \delta) \leq \alpha_i(\epsilon, \delta)$, $i = 1, 2$. □

Remark 4.6. Propositions 4.4 and 4.5 generalize Proposition 3 of Zolezzi in [22].

To end this paper, we give characterizations of extended well-posedness of (X, J) in terms of the behavior of diam $T_i(\epsilon, \delta)$ when $\mathrm{Eff(p^*)}$ is a singleton.

Proposition 4.7. *Assume that* $\mathrm{Eff(p)} \neq \emptyset$ *for all* $p \in L$. *Then condition*

$$\mathrm{diam}\, \mathrm{T}_2(\epsilon, \delta) \to 0 \qquad \textit{as } (\epsilon, \delta) \to (0, 0) \tag{13}$$

is necessary for extended well-posedness of (X, J) *and* $\mathrm{Eff(p^*)}$ *to be a singleton provided that the mapping* $p \mapsto \mathrm{Min(p)}$ *is* C*-lower semicontinuous at* p^**, and sufficient provided that the mapping* $p \mapsto \mathrm{Min(p)}$ *is* C*-upper semicontinuous at* p^* *and* I *is* C*-lower semicontinuous at* (x, p^*) *for all* $x \in X$.

Proof. Suppose that (13) holds, the mapping $p \mapsto \mathrm{Min(p)}$ is C-upper semicontinuous at p^* and I is C-lower semicontinuous at (x, p^*) for all $x \in X$. Then $\mathrm{Eff(p^*)}$ is a singleton since $\mathrm{Eff(p^*)} = \mathrm{T}_2(0,0)$. Set $\mathrm{Eff(p^*)} = \{\mathrm{u}^*\}$. Let $\{p_n\} \subset L$,$\{x_n\} \subset X$, $\{\epsilon_n\} \subset R_+$, $y_n \in \mathrm{Min(p_n)}$ be such that

$$p_n \to p^*, \quad \epsilon_n \to 0, \quad I(x_n, p_n) \in y_n + \epsilon_n e - C.$$

By (13), for given $a > 0$ there exists $b > 0$ such that

$$\mathrm{diam}\ T_2(\epsilon, \delta) < \mathrm{a}, \qquad \text{whenever } 0 < \epsilon \leq \mathrm{b},\ 0 < \delta \leq \mathrm{b}. \tag{14}$$

Then there exists n_0 such that $\|p_n - p^*\| \leq b$ and $\epsilon_n \leq \frac{b}{2}$ for all $n > n_0$. Since $p \mapsto \mathrm{Min(p)}$ is C-upper semicontinuous, there exists n_1 such that $\mathrm{Min(p_n)} \subset \mathrm{Min(p^*)} + \frac{\mathrm{b}}{2}\mathrm{e} - \mathrm{int\, C}$, for all $n > n_1$, and so

$$I(x_n, p_n) \in y_n + \frac{b}{2}e - C \subset \mathrm{Min(p^*)} + \mathrm{be} - \mathrm{C}, \quad \forall \mathrm{n} > \mathrm{n}_2, \tag{15}$$

where $n_2 = \max\{n_0, n_1\}$. Summarizing, for any $a > 0$ we have from (14) and (15) that $\|x_n - x_k\| \le \operatorname{diam} \mathrm{T}_2(\epsilon, \delta) \le a$ for all sufficiently large n, k, and so $x_n \to x$ for some $x \in X$. Since I is C-lower semicontinuous at (x, p^*) and $p \mapsto \mathrm{Min(p)}$ is C-upper semicontinuous at p^*, for any $d \in \operatorname{int} \mathrm{C}$ there exists n_3 such that

$$\begin{aligned} I(x, p^*) &\in I(x_n, p_n) + d - \operatorname{int} C \\ &\subset y_n + \epsilon_n e - C + d - \operatorname{int} C \\ &\subset (\mathrm{Min(p^*)} + \mathrm{d} - \operatorname{int} \mathrm{C}) + \epsilon_\mathrm{n} \mathrm{e} + \mathrm{d} - \operatorname{int} \mathrm{C} \\ &\subset \mathrm{Min(p^*)} + 2\mathrm{d} + \epsilon_\mathrm{n} \mathrm{e} - \operatorname{int} \mathrm{C} \end{aligned}$$

for all $n > n_3$. It follows that $I(x, p^*) \in \mathrm{Min(p^*)} - \mathrm{C}$. This yields $x \in \mathrm{Eff(p^*)} = \{\mathrm{u}^*\}$, and so (X, J) is extended well-posed.

Conversely, suppose that (X, J) is extended well-posed and $\mathrm{Eff(p^*)} = \{\mathrm{u}^*\}$. If (13) does not hold, then there exist $\epsilon_n \to 0$, $\delta_n \to 0$, u_n and v_n in $T_2(\epsilon_n, \delta_n)$, and $a > 0$ such that

$$\|u_n - v_n\| \ge a. \tag{16}$$

By the definition of T_2, there exist $p_n \to p^*$ and $q_n \to p^*$, such that

$$I(u_n, p_n) \in \mathrm{Min(p^*)} + \epsilon_\mathrm{n} \mathrm{e} - \mathrm{C}, \quad \mathrm{I(v_n, q_n)} \in \mathrm{Min(p^*)} + \epsilon_\mathrm{n} \mathrm{e} - \mathrm{C}.$$

Since the mapping $p \mapsto \mathrm{Min(p)}$ is C-lower semicontinuous at p^*, $\mathrm{Min(p^*)} \subset \mathrm{Min(p_n)} + \epsilon_\mathrm{n} \mathrm{e} - \operatorname{int} \mathrm{C}$ and $\mathrm{Min(p^*)} \subset \mathrm{Min(q_n)} + \epsilon_\mathrm{n} \mathrm{e} - \operatorname{int} \mathrm{C}$ for all sufficiently large n. It follows that

$$\begin{aligned} I(u_n, p_n) &\in \mathrm{Min(p_n)} + \epsilon_\mathrm{n} \mathrm{e} - \operatorname{int} \mathrm{C} + \epsilon_\mathrm{n} \mathrm{e} - \mathrm{C} \subset \mathrm{Min(p_n)} + 2\epsilon_\mathrm{n} \mathrm{e} - \mathrm{C} \\ I(v_n, q_n) &\in \mathrm{Min(q_n)} + \epsilon_\mathrm{n} \mathrm{e} - \operatorname{int} \mathrm{C} + \epsilon_\mathrm{n} \mathrm{e} - \mathrm{C} \subset \mathrm{Min(q_n)} + 2\epsilon_\mathrm{n} \mathrm{e} - \mathrm{C}. \end{aligned}$$

Since (X, J) is extended well-posed, $u_n \to u^*$ and $v_n \to u^*$. This contradicts (16). □

Remark 4.8. Proposition 4.7 generalizes Theorem 1 of Zolezzi in [22].

Following the proof of Proposition 4.7, we have

Proposition 4.9. *Assume that* $\mathrm{Eff(p)} \neq \emptyset$ *for all* $p \in L$. *Then condition*

$$\operatorname{diam} \mathrm{T}_1(\epsilon, \delta) \to 0 \qquad \text{as } (\epsilon, \delta) \to (0, 0)$$

holds if and only if (X, J) *is extended well-posed and* $\mathrm{Eff(p^*)}$ *is a singleton.*

Acknowledgement. The authors thank two anonymous referees for their helpful comments and suggestions. This work was partially supported by the National Natural Science Foundation of China, the Applied Research Project of Sichuan Province (05JY029-009-1) and Specialized Research Fund for the Doctoral Program of Higher Education (20060610005).

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Received September 6, 2005; May 15, 2006