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A Sharp Stability Criterion for Soliton-Type Propagating Phase Boundaries in Korteweg's Model

Kevin Zumbrun

Abstract. Recently, Benzoni–Gavage, Danchin, Descombes, and Jamet have given a sufficient condition for linear and nonlinear stability of solitary wave solutions of Korteweg's model for phase-transitional isentropic gas dynamics in terms of convexity of a certain "moment of instability" with respect to wave speed, which is equivalent to variational stability with respect to the associated Hamiltonian energy under a partial subset of the constraints of motion; they conjecture that this condition is also necessary. Here, we compute a sharp criterion for spectral stability in terms of the second derivative of the Evans function at the origin, and show that it is equivalent to the variational condition obtained by Benzoni–Gavage et al., answering their conjecture in the positive.

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1. Introduction

Motivated by recent work of Benzoni–Gavage et al [2], we investigate in this paper stability of "soliton-type" (i.e., homoclinic) traveling wave solutions

$$U(x,t) = \bar{U}(x-st), \quad \lim_{z\to\pm\infty} \bar{U}(z) = U_\infty, \tag{1.1}$$

$U=(v,u)$, $\bar{U}=(\bar{v},\bar{u})$ of the Korteweg model

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= -\kappa v_{xxx}, \end{aligned} \tag{1.2}$$

for isentropic phase-transitional gas dynamics, written in Lagrangian coordinates, with v denoting specific volume, u particle velocity, p pressure, and $\kappa > 0$ a coefficient of capillarity, taken for simplicity to be constant. The

K. Zumbrun: Dept. Mathematics, Indiana University, USA; kzumbrun@indiana.edu

extension of our results to general $\kappa(v)$ as considered in [2] is straightforward; see Remarks 1.6 and 1.8. As pointed out in [5], system (1.2) is also formally equivalent to the "good" Boussinesq equation modeling shallow-water flow.

Equations (1.1), accounting for the effects of compressibility and capillarity, but neglecting viscosity, are of dispersive, type, in contrast to the dissipative type of the usual compressible Navier–Stokes equations. Indeed, as discussed in greater generality in [2], (1.1) has the Hamiltonian structure

$$U_t = \mathcal{J}\delta\mathcal{H}, \tag{1.3}$$

where

$$\mathcal{J} = \partial_x J := \partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.4}$$

is a constant-coefficient skew-symmetric first-order differential operator and $\delta\mathcal{H}$ is a second-order differential operator corresponding to the variational derivative of the Hamiltonian functional

$$\mathcal{H} = \int H, \quad H = \frac{1}{2}(u - u_\infty)^2 - \int_{v_\infty}^{v} (p(z) - p(v_\infty))dz + \frac{1}{2}\kappa(v)(v_x)^2, \tag{1.5}$$

of the (relative) total energy H of the system, with $\kappa(v) \equiv \kappa$ constant.

Formally,

$$(d/dt)\mathcal{H}(U) = \langle \delta\mathcal{H}, U_t\rangle = \langle \delta\mathcal{H}, \mathcal{J}\delta\mathcal{H}\rangle = 0,$$

so that the Hamiltonian is one conserved quantity of motion. A second (formally) conserved quantity, arising as a consequence of group invariance under translation (see [14]) is the relative generalized momentum

$$\mathcal{Q}(U) = \frac{1}{2}\langle J(U - U_\infty), \quad (U - U_\infty)\rangle = \int (u - u_\infty)(v - v_\infty)(x)\, dx \tag{1.6}$$

(formally, $J = (\partial_x J)^{-1}\partial_x$, as prescribed in [14]). Two additional conserved quantities are the relative masses

$$\mathcal{P}_1(U) = \int (v - v_\infty)(x)\, dx, \quad \mathcal{P}_2(U) = \int (u - u_\infty)(x)\, dx.$$

The existence of these further quantities is associated with the fact that operator $\mathcal{J}$ is not onto, a circumstance that turns out to be significant.

With this framework, it was shown by Benzoni–Gavage et al [2] that stability of solitons may be investigated by variational methods, following the formalism of [6, 14]. Specifically, one may compute that solitary wave solutions are critical points of the Hamiltonian $\mathcal{H}$ under constraint $\mathcal{Q}$, satisfying the Euler–Lagrange equation (itself Hamiltonian)

$$(\delta\mathcal{H} + s\delta\mathcal{Q})(\bar{U}) = 0, \tag{1.7}$$

where speed s plays the role of a Lagrange multiplier. Such solutions occur in a one-parameter family $\bar U^s$. Formally, *strict variational stability* of $\bar U$ with respect to constraint $\mathcal{Q}$ is thus sufficient for *time-evolutionary orbital stability* of the family $\{\bar U^s\}$, since, then, (i) a minimum $\bar U^{s(\mathcal{Q})}$ should therefore persist under small changes in $\mathcal{Q}$, and (ii) within each level surface of $\mathcal{Q}$, the conserved quantity $\mathcal{H}(U)-\mathcal{H}(\bar U^{s(\mathcal{Q})})$ should control $\|U-\bar U^{s(\mathcal{Q})}\|^2$ in the underlying Hilbert norm.

Denote by

$$\mathcal{L}^s := (\delta^2\mathcal{H} + s\delta^2\mathcal{Q})|_{\bar U^s} \tag{1.8}$$

the self-adjoint operator given by the constrained Hessian about $\bar U^s$. Then, strict variational stability corresponds to positivity of $\mathcal{L}^s$ on the kernel of $\delta\mathcal{Q}$. On the other hand, changing to moving coordinates $\tilde x = x - st$ for which $\bar U^s$ is a stationary solution, and linearizing about $U = \bar U^s(\tilde x)$, we obtain linearized time-evolution equations $U_t = LU$, where

$$L := \mathcal{J}\mathcal{L}^s$$

denotes the (non-self-adjoint) linearized operator about the wave; see (2.5)–(2.6). Thus, time-evolutionary stability is related to spectral stability of L, i.e., nonexistence of spectra with positive real part; see Definition 2.1.

In [14], Grillakis et al made rigorous the intuitive argument (i)–(ii) above, showing that variational stability indeed implies nonlinear orbital stability in a quite general framework (which applies here). Moreover, under the assumption that $\mathcal{L}^s$ *have at most one negative eigenvalue*, they gave a necessary and sufficient condition for strict variational stability of $\bar U^s$ at a prescribed speed $s = \bar s$, in terms of strict convexity with respect to s of the "moment of instability"

$$m(s) := (\mathcal{H} + s\mathcal{Q})(\bar U^s)$$

(in the terminology of [6]), i.e.,

$$(d^2m/ds^2)(\bar s) = (d/ds)\mathcal{Q}(\bar U^s) = \langle(\bar U - U_\infty), J(\partial \bar U^{\bar s}/\partial s)\rangle > 0. \tag{1.9}$$

Finally, for $\mathcal{J}$ *onto*, they showed by construction of a suitable Lyapunov function that strict failure of (1.9) implies time-evolutionary instability, completely deciding the issue of stability.

Remark 1.1. Relation (1.9) shows that convexity of the moment of instability is equivalent to monotone increase with respect to s of the generalized momentum $\mathcal{Q}$ along $\{\bar U^s\}$.[1] Local monotonicity of $\mathcal{Q}$ along $\{\bar U^s\}$ is necessary for the picture of stability described above (in particular, for strict variational

[1] In the notation of [14], monotone decrease with respect to $\omega = -s$.

stability), since the map from s to $\mathcal{Q}(\bar{U}^s)$ must be locally invertible if all sufficiently small perturbations (corresponding to small variations in $\mathcal{Q}$) are to lie near some $\bar{U}^s$. Through the key relation

$$\mathcal{L}^s(\partial \bar{U}^s/\partial s) = -\delta Q \tag{1.10}$$

(obtained by differentiating (1.7) with respect to s; see [2, Proposition 4]), we find that (1.9) is equivalent to

$$d^2m/ds^2 = -\langle \partial \bar{U}^s/\partial s, \mathcal{L}^s \partial \bar{U}^s/\partial s\rangle.$$

Remark 1.2. The condition that $\mathcal{L}^s$ have at most one unstable eigenvalue is important for the conclusion of equivalence (of variational and time-evolutionary stability). Even for finite-dimensional ODE, it is easy to construct examples for which $\mathcal{L}$, self-adjoint, has two unstable eigenvalues but $L = \mathcal{J}\mathcal{L}$ is stable, with $\mathcal{J}$ bijective. For example, consider the system generated by $\mathcal{H}(x,y) = |x|^2 - |y|^2$, $x, y \in \mathbb{R}^2$, $\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$, with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In general, the numbers of unstable eigenvalues are linked by parity considerations. See [16, 17] for a more detailed discussion of the relation between the number of unstable eigenvalues of $\mathcal{L}$ and $\mathcal{J}\mathcal{L}$.

In [2], a simple formula is given for $m(s)$ and evaluated numerically to show that regions of both convexity and nonconvexity of $m(\cdot)$ may arise, depending on physical parameters. The first case corresponds with orbital stability, as discussed above. However, *since $\mathcal{J}$ is not onto*, the second case is inconclusive by the theory of [14]. The authors conjecture nonetheless that convexity is necessary as well as sufficient for stability, so that the second case in fact corresponds to instability.

Here, we investigate time-evolutionary stability directly, using alternative, Evans function methods introduced in [1, 3, 13, 25, 26] to obtain a sharp criterion for spectral time-evolutionary stability of a prescribed wave $\bar{U}^{\bar{s}}$ in terms of the sign of the second derivative at the origin of the Evans function $D(\lambda)$, an analytic function whose zeroes correspond in location and multiplicity with eigenvalues of the linearized operator $L = \mathcal{J}\mathcal{L}^{\bar{s}}$ about the wave. Specifically, we obtain the following results deciding in the positive the conjecture of [2].

Theorem 1.3. *$D''(0) = c(d^2m/ds^2)(\bar{s})$ for a nonzero constant c.*

Corollary 1.4. *Traveling waves* (1.1) *of* (1.2) *are time-evolutionarily (linearly and nonlinearly) stable if $d^2m/ds^2 > 0$ and only if $d^2m/ds^2 \geq 0$, i.e., if they are strictly variationally stable with respect to constraint $\mathcal{Q}$, and only if they are nonstrictly variationally stable.*

Theorem 1.3 generalizes a number of similar results obtained in [25] for various related scalar models, and indeed holds in far greater generality for systems of abstract form (1.3). To see why, and to better understand in general the relation between the moment condition and variational and time-evolutionary stability, notice that, in this generality, (1.6) and(1.9) become, formally (see [14]),

$$\mathcal{Q} = \frac{1}{2}\big\langle \mathcal{J}^{-1}\partial_x(U - U_\infty), (U - U_\infty)\big\rangle$$

and

$$(d^2m/ds^2)(\bar s) = \big\langle \mathcal{J}^{-1}\partial_x(\bar U - U_\infty), (\partial \bar U^{\bar s}/\partial s)\big\rangle = \big\langle \mathcal{J}^{-1}\partial_x \bar U, (\partial \bar U^{\bar s}/\partial s)\big\rangle.$$

On the other hand, differentiation of the traveling-wave equation with respect to x and s, respectively, reveals that $f_1 = \partial_x \bar U$ is a right zero-eigenfunction of the linearized operator L and $f_2 = -\partial_s \bar U^{\bar s}$ a generalized zero-eigenfunction of height two; for further discussion, see Section 2. Noting that left and right zero eigenfunctions $\tilde f_1$ and f_1 of $L = \mathcal{J}\mathcal{L}$ are related, formally, by $\tilde f_1 = \mathcal{J}^{-1} f_1$, we find the general relation

$$(d^2m/ds^2)(\bar s) = -\langle \tilde f_1, f_2\rangle. \tag{1.11}$$

But, vanishing of the inner product (1.11) of the genuine left eigenfunction $\tilde f_1$ against the generalized right eigenfunction f_1 of height two precisely detects existence of a generalized eigenfunction of height three (by Jordan form), i.e., algebraic multiplicity of order two or more of the eigenvalue $\lambda = 0$. Thus, $(d^2m/ds^2)(\bar s)$ must be a nonzero multiple of $D''(0)$, since vanishing of $D''(0)$ detects the same phenomenon.

Formula (1.11) reveals a direct (formal) link between the moment condition and time-evolutionary stability that, moreover, does not require invertibility of $\mathcal{J}$ on the whole space, but only on the range of ∂_x, through the distinguished roles of $(\partial \bar U^{\bar s}/\partial x)$ and $-(\partial \bar U^{\bar s}/\partial s)$ as zero eigenfunction and generalized eigenfunction of L. Recall that these same functions played critical roles in the argument of [14] linking the moment condition and variational stability: $(\partial \bar U^{\bar s}/\partial x)$ as zero eigenfuction of $\mathcal{L}$ and $-(\partial \bar U^{\bar s}/\partial s)$ through relation (1.10).

This formal argument can be made rigorous using the extended spectral theory of [29, Section 6], valid essentially wherever an Evans function can be analytically defined,[2] concerning Jordan structure at eigenvalues embedded in essential spectrum, along with additional assumptions assuring that there exist no additional (extended) genuine zero eigenfunctions other than f_0; see Remark 1.6. We shall instead follow the more concrete approach of direct Evans function calculations using the specific structure at hand, which provide at the same time sign information. However, we note that these are quite similar to those on which the abstract development of [29] is based (see in particular the proof of [29, Proposition 6.3]).

[2]See also [23, 27] for extensions to operators L of the degenerate type considered here.

Remark 1.5. Along similar lines, the general relation

$$\partial_\lambda^k D(0) = c\langle \tilde{f}_1, f_k\rangle \tag{1.12}$$

has been established by Kapitula [15] for general (not necessarily Hamiltonian) systems for which $\lambda = 0$ is an isolated eigenvalue of geometric multiplicity one of the linearized operator L under consideration. In the present (Hamiltonian) case, $\lambda = 0$ is embedded in the essential spectrum of L, and so this result does not apply. Indeed, Benzoni–Gavage, Serre, and Zumbrun [3] have shown in the context of viscous conservation laws that (1.12) does not hold in general for embedded eigenvalue $\lambda = 0$, but rather must be corrected by the addition of appropriate boundary terms at plus and minus spatial infinity; see also related results in [18, 19, 20] for perturbed NLS equations. Thus, the argument above reflects partly the special features of the Hamiltonian case.

Remark 1.6. When $\mathcal{J}$ is a differential operator of the general form $\partial_x J$ considered here, with J an invertible (not necessarily local) operator onto L^2 (the most general case falling under the sufficient but not necessary theory of [14]), we may take $\mathcal{J}^{-1}\partial_x = J^{-1}$ in the discussion above, and

$$\tilde{f}_1 = \mathcal{J}^{-1} f_1 = \mathcal{J}^{-1}\partial_x(\bar{U} - U_\infty) = J^{-1}(\bar{U} - U_\infty).$$

Noting that $\tilde{f}_1$ decays exponentially as $x \to \pm\infty$, we find that it is indeed a left genuine zero-eigenfunction of L, making rigorous sense of relation (1.11). To complete the rest of the formal argument sketched above, note that bounded solutions of $Lf = 0$ correspond to nearby homoclinic connections with different endstates, hence have equal limits as $x \to \pm\infty$, and this eliminates them as possible extended eigenfunctions (which might in general be merely bounded [29]) unless they in fact vanish at $\pm\infty$. Thus, the standard assumption that $\bar{U}$ be a transverse connection, ensuring that the L^2 kernel of L is one-dimensional, is sufficient to ensure that the extended kernel of L is also one-dimensional, i.e., there is a unique (up to constant factor) extended genuine eigenfunction f_1 with dual eigendirection $\tilde{f}_1$.

Moreover, since $\tilde{f}_1$, f_1, and f_2 are all exponentially decaying, there exists an extended generalized eigenfunction f_3, bounded but not necessarily decaying at infinity, if and only if $\langle \tilde{f}_1, f_2\rangle = 0$. For, $Lf_3 = f_2$ implies

$$\langle \tilde{f}_1, f_2\rangle = \langle \tilde{f}_1, Lf_3\rangle = \langle L^* f_1, f_3\rangle = 0.$$

On the other hand, if the extended Jordan chain is order two, then (see [29, Proposition 5.3 (iii)]) the extended spectral projection

$$\mathcal{P} = f_1\langle \tilde{f}_2, \cdot\rangle + f_2\langle \tilde{f}_1, \cdot\rangle$$

preserves exponentially decaying elements of the extended eigenspace; in particular $\mathcal{P} f_2 = f_2$, whence $\langle \tilde{f}_1, f_2\rangle \neq 0$. Finally, recalling ([29, Theorem 6.3 (ii)]) that the Evans function vanishes with multiplicity equal to the dimension of the corresponding extended eigenspace, we obtain the result

$$D''(0) = c(d^2m/ds^2)(\bar{s}), \quad c \neq 0 \tag{1.13}$$

for this general class: in particular, for the general isentropic Korteweg equations discussed in [2] consisting of (1.3)–(1.5) with arbitrary $\kappa(v) > 0$.

Remark 1.7. The same extended spectrum argument used in Remark 1.6, applied to general (not necessarily Hamiltonian) PDE, yields the remarkable fact that the dual version

$$\partial_\lambda^k D(0) = \tilde{c}\langle \tilde{f}_k, f_1 \rangle, \tag{1.14}$$

$\tilde{c} \neq 0$, of (1.12) remains valid in complete generality, *without additional boundary terms*, for embedded eigenvalues with (extended) geometric multiplicity one for which the associated eigenfunction f_1 is exponentially decaying, a condition that is in the traveling wave context essentially always satisfied.[3] For, then $\mathcal{P}f_1 = f_1$, where $\mathcal{P} = \sum_{j=1}^K f_j \langle \tilde{f}_{K-j+1}, \cdot \rangle$ and $K \geq k$ is the order of vanishing of the Evans function, whence $\langle \tilde{f}_k, f_1 \rangle = 0$ if and only if $K \geq k+1$, or, equivalently, $\partial_\lambda^k D(0) = 0$. Boundary terms arise in the forward formula through the integration by parts converting $\langle \tilde{f}_k, f_1 \rangle = \langle \tilde{f}_k, L^{k-1} f_k \rangle$ to $-1^{k-1}\langle (L^*)^{k-1} \tilde{f}_k, f_k \rangle = -1^{k-1}\langle \tilde{f}_1, f_k \rangle$, except in the case, as in Remark 1.6 above, that $f_1, \dots, f_k$ or $\tilde{f}_1, \dots, \tilde{f}_k$ decay exponentially. (Recall that extended eigenfunctions f_j, $\tilde{f}_j$ do not necessarily decay at infinity, except, by assumption, f_1, but rather grow at most algebraically [3, 29].)

Remark 1.8. The relation (1.13) by itself does not give give the complete stability information of Corollary 1.4, but requires further information on the *sign of* c (with D suitably normalized: more precisely, the relation between $\mathrm{sgn}(c)$ and $\mathrm{sgn}D(+\infty)$). In practice, this is often not restrictive, since we may calibrate the sign of c by continuation (homotopy) to a case where stability is decided by the theory of [14], i.e., either $d^2m/ds^2 > 0$ (variational stability) or $\mathcal{J}$ onto. For example, in the case considered here, the collection of all soliton solutions of all isentropic Korteweg models comprises an open set in parameter space, so we may conclude by the (numerically observed) existence of stable waves. Alternatively, the regular perturbation $\mathcal{J}^\theta := \partial_x J + \theta K$, $K := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, θ sufficiently small, preserves sign information while converting $\mathcal{J}$ to an operator $\mathcal{J}^\theta$ that is onto for all $\theta \neq 0$.

[3] Likewise, the obvious extensions to higher geometric multiplicity remain valid so long as all genuine extended eigenfunctions are exponentially decaying.

A still more general approach, not limited to the Hamiltonian setting, is to work directly from (1.14), computing the sign by a direct calculation like that of Section 3. In practice, one may sometimes determine the normalization without doing the full calculation, in which case it is easier to determine $\tilde{f}_k$, f_1 numerically; see, for example, the analysis of stability of undercompressive traveling waves of thin-film models in [4, Proposition 2.11 and Footnote 6 on duality]. We point out that the righthand side of (1.14) may by itself be considered as a generalized Melnikov integral, like d^2m/ds^2 giving geometric information about the dynamics of the traveling-wave ODE; see [23, Section 4.2] (especially eqns. (4.9)–(4.13)) for a general duality principle linking dual eigenfunctions to solutions of the adjoint ODE.

Remark 1.9. There appears in [2] the statement, apparently contradicting Corollary 1.4, that there exist stable solitons that are variationally unstable. However, the time-evolutionarily stable waves considered in [2] are in the standard sense variationally stable; indeed, this is the property that is used to prove time-evolutionary stability. The instability that is referred to, rather, is of the constrained Hessian, (1.8), with respect to *unconstrained* perturbations (i.e., perturbations outside the kernel of $\delta\mathcal{Q}$), a notion that is necessary but not sufficient for instability with respect to constraint $\mathcal{Q}$.

Note. After the completion of this paper, we have learned of results of Bridges and Derks [7, 8] establishing the relation $D''(0) = c(d^2m/ds^2)$ for the good Boussinesq system, which is equivalent to the main example (1.2) studied here. More generally, they derive formulae generalizing those of Pego and Weinstein [25] for scalar equations for the first nonvanishing derivative of a "symplectic Evans function" at the origin, for systems that can be put in the multi-symplectic form $MZ_t + KZ_x = \nabla S(Z)$, where $Z \in \mathbb{R}^{2n}$, M, K are constant skew-symmetric $2n \times 2n$ matrices, and S is a smooth function on $\mathbb{R}^{2n}$. These include in principle a rather large class of Hamiltonian PDE, overlapping with but apparently distinct from the class (described in Remark 1.6) to which our methods apply.

The multi-symplectic approach has the advantage that it is equation independent once a change of coordinates to multi-symplectic form has been found, yielding automatically a characterization of the sign of c in (1.13) in terms of the geometry of the phase space of the traveling-wave ODE (cf. Remark 1.8, par. two). On the other hand, our methods are somewhat more straightforward, being carried out in the original coordinates and motivated by the simple relation (1.11). In addition (see Remarks 1.7–1.8), they yield useful partial information also in the general, non-Hamiltonian case.

2. Preliminaries

Substituting (1.1) into (1.2), we obtain the traveling-wave equation

$$\begin{aligned} -sv' - u' &= 0 \\ -su' + p(v)' &= -\kappa v''', \end{aligned} \tag{2.1}$$

or, substituting the first equation into the second, and integrating from $-\infty$, the Hamiltonian ODE (nonlinear oscillator)

$$v'' = \kappa^{-1}\big(s^2 v + p(v) - s^2 v_\infty - p(v_\infty)\big). \tag{2.2}$$

Alternatively, we may write (2.1) formally as $-sU' = \mathcal{J}\mathcal{H}(U)$, or

$$\mathcal{J}(\delta\mathcal{H} + sJ)(U) = \mathcal{J}\delta(\mathcal{H} + s\mathcal{Q})(U) = 0. \tag{2.3}$$

By the Hamiltonian structure of (2.2), it follows that homoclinic orbits persist under changes in speed s and endstate U_∞, forming for fixed U_∞ a one-parameter family $\bar{U}^s$, $s \geq 0$, as described in the introduction. The equations are invariant under shifts in velocity u, so that any value of u_∞ is possible. However, the requirement that U_∞ be a saddle-point of (2.1) enforces on v_∞ the condition

$$s^2 + p'(v_\infty) < 0. \tag{2.4}$$

Making the standard change of coordinates $x \to x - st$ to a rest frame for the traveling wave, we may investigate its stability as an equilibrium solution $U(x,t) = \bar{U}(x)$ of $U_t - sU_x = \mathcal{J}\delta\mathcal{H}(U)$, or

$$U_t = \mathcal{J}\delta(\mathcal{H} + s\mathcal{Q})(U). \tag{2.5}$$

Linearizing (2.5) about $\bar{U}$, we obtain

$$U_t = LU := \mathcal{J}\mathcal{L}^{\bar{s}}U, \tag{2.6}$$

where $\mathcal{L}^s$ is defined as in (1.8) and $\bar{s}$ denotes the speed of the wave $\bar{U}$ under investigation.

Definition 2.1. The wave $\bar{U}$ is *spectrally stable* if the spectrum $\sigma(L)$ of the linearized operator L about the wave is contained in the nonpositive complex half-plane $\{\lambda : \Re\lambda \leq 0\}$.

Routine calculation (see, e.g., [2, 14, 25], or Section 3 below) shows that the essential spectrum of L consists of the entire imaginary axis, so that stability is at best of neutral, or bounded type, rather than asymptotic stability. This may be seen, likewise, by the fact that the equations are time-space reversible. Spectral stability is therefore determined by the point spectrum of L: specifically, whether there lie eigenvalues off of the imaginary axis. The following general result of Pego and Weinstein, a quantitative (linear) version of the previously-remarked relation between variational and time-evolutionary stability, gives a way to bound the number of such eigenvalues.

Lemma 2.2 ([25]). *For a linear operator L factoring as $L = \mathcal{J}\mathcal{L}$ with $\mathcal{J}$ skew-symmetric and $\mathcal{L}$ self-adjoint, the number of eigenvalues of L in the positive complex half-plane $\{\lambda : \Re\lambda > 0\}$ is less than or equal to the number of negative eigenvalues of $\mathcal{L}$.*

Proof. See [25, Theorem 3.1]. □

Lemma 2.2 was used in [2] to establish the following upper bound. This is not needed in the present context, in which we seek to establish *instability*, but gives useful additional information; see Remark 3.3. (It is necessary for stability; see Remark 2.4.)

Corollary 2.3 ([2]). *The number of unstable (i.e., positive real part) eigenvalues of $L = \mathcal{J}\mathcal{L}^{\bar{s}}$ is less than or equal to one.*

Proof. Eigenvalues of $\mathcal{L}^{\bar{s}}$ may be shown to correspond to eigenvalues of a second-order scalar Sturm–Liouville operator $Mv := v'' - \kappa^{-1}(s^2 + \alpha)v$, $\alpha := p'(\bar{v}(x))$ in the variable v. Since $\bar{v}'$ by translation invariance is a zero eigenfunction of this operator (see further discussion below; or, just differentiate (2.1)), the number of unstable eigenvalues by standard Sturm–Liouville theory is equal to the number of nodes (zeroes) of $\bar{v}'$, which, by (2.1) may be seen to be one. See [2] for details. □

A general approach to resolving the issue of variational vs. evolutionary stability when variational methods fail to decide the question, introduced in [25], is via the Evans function $D(\lambda)$, an analytic function taking real values to real values, whose zeroes correspond to eigenvalues of L; for origins of the Evans function, see [9, 10, 11, 12, 1]. By translational invariance, $D(0)$ necessarily vanishes. For, differentiating (2.3) with respect to x, we obtain $L\bar{U}' = 0$, hence $\bar{U}'$ is a zero eigenfunction of L. Likewise, existence of a one-parameter family $\bar{U}^s$ of solutions with the same endstate implies that $D'(0) = 0$. For, differentiating (2.3) with respect to s yields $L(\partial\bar{U}^s/\partial s)|_{s=\bar{s}} = -\bar{U}'$, hence $-(\partial\bar{U}^s/\partial s)|_{s=\bar{s}}$ is a generalized zero eigenfunction of L and there is a nontrivial Jordan block for L at $\lambda = 0$.

Since $D(+\infty)$ by standard considerations (see [1, 25, 13]) has a constant, nonzero sign as parameters are varied, this means that, in the generic case that $D''(0) \neq 0$, the number of unstable (i.e., positive) real roots of D (eigenvalues of L) is odd or even depending on the sign of $D''(0)$. Used in conjunction with Lemma 2.2, this observation can yield complete information. Namely, when the total number is at most one, spectral stability is determined by the sign of the second derivative of the Evans function.

The second derivative $D''(0)$ was evaluated in [25] for scalar KdV-type equations and shown in several related cases to be exactly the second derivative with respect to s of the moment of instability, thus establishing necessity of

$d^2m/ds^2 \geq 0$ along with sufficiency of $d^2m/ds^2 > 0$ for linearized and nonlinear stability in those cases. Their method of computing D and its derivatives does not apply in the system case considered here. However, it was shown in [13] and [3], respectively, that quite similar formulae may be obtained for $D'(0)$ and higher derivatives $(d/d\lambda)^k D(0)$ in the system case; see also [7, 8, 18, 19, 20, 22].

In the remainder of the paper, we calculate $D''(0)$ by the method of [3] and show by explicit computation that it is a nonzero multiple of $(d^2m/ds^2)(\bar{s})$, thus establishing Theorem 1.3 and Corollary 1.4.

Remark 2.4. The relation $(d^2m/ds^2)(\bar{s}) = cD''(0)$, $c \neq 0$, shows that the assumption that $\mathcal{L}^s$ have at most one negative eigenvalue is necessary for the result of [14] that $(d^2m/ds^2)(\bar{s}) > 0$ implies stability. For, by the discussion above, the sign of $D''(0)$, hence of (d^2m/ds^2), counts the parity of the number of unstable eigenvalues, which might be even in general despite instability of L.

3. Evans function calculations

3.1. Construction of the Evans function. Writing out the eigenvalue equation $(L-\lambda)U = 0$, L as in (2.6), we obtain

$$\begin{aligned} \lambda v - sv' - u' &= 0 \\ \lambda u - su' + (\alpha v)' &= -\kappa v''', \end{aligned} \tag{3.1}$$

where $\alpha := p'(\bar{v}(x))$ as in (2.1), which may be written in phase variables $W = (u, v, v', v'')^{\mathrm{tr}}$ as a first order system of ODE $W' = A(x,\lambda)W$, or

$$\begin{pmatrix} u \\ v \\ v' \\ v'' \end{pmatrix}' = \begin{pmatrix} 0 & \lambda & -s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\kappa^{-1}\lambda & \kappa^{-1}s\lambda & \kappa^{-1}(-\alpha - s^2) & -\kappa^{-1}\alpha'\bar{v}_x \end{pmatrix} \begin{pmatrix} u \\ v \\ v' \\ v'' \end{pmatrix}. \tag{3.2}$$

Using the fact (easily verified for (2.1)) that $\bar{U}(x)$ converges exponentially to its limit U_∞ as $x \to \pm\infty$, we may construct an Evans function for (3.2) by the general method described in [13]; see [27] for a more recent exposition.

Examining first the constant-coefficient limit

$$\begin{aligned} \lambda v - sv' - u' &= 0 \\ \lambda u - su' + \alpha_\infty v' &= -\kappa v''' \end{aligned}$$

of (3.1), $\alpha_\infty < 0$ by (2.4), we find, taking the Fourier transform $\partial_x \to i\xi$, that the spectrum of limiting, constant coefficient operator L_∞ of L satisfies dispersion relation

$$\lambda(\xi) = si\xi \pm \sqrt{\alpha_\infty - \kappa\xi^4}, \quad \xi \in \mathbb{R}, \tag{3.3}$$

hence consists of the imaginary axis. By a standard result of Henry [21] on asymptotically constant-coefficient operators, we thus find that the essential spectrum of L likewise consists of the imaginary axis, while spectra lying in the strictly unstable half-plane $\Lambda := \{\lambda : \Re\lambda > 0\}$ consists entirely of isolated eigenvalues.

The same calculation shows that the dimensions of the stable and unstable subspaces of the limiting coefficient matrix $A_\infty(\lambda) := \lim_{z\to\pm\infty} A(z,\lambda)$ is *constant* in Λ. For, substituting $\mu = i\xi$ into the characteristic equation

$$(\lambda - s\mu)^2 + \alpha_{-\infty}\mu^2 + \kappa\mu^4 = 0 \tag{3.4}$$

determining eigenvalues $\mu(\lambda)$, we obtain the dispersion relation (3.3); thus, eigenvalues $\mu(\lambda)$ may cross the imaginary axis only along the dispersion curve, i.e., for λ on the imaginary axis. Taking $\lambda \to +\infty$ along the real axis in (3.4), we find that $\lambda^2 \sim -\kappa\mu^4$, so that μ lie along the fourth roots of -1. Thus, both stable and unstable subspaces of $A_\infty(\lambda)$ have dimension two in Λ. An elementary matrix perturbation calculation at $\lambda = 0$ shows that these extend analytically to a neighborhood of the origin, $\lambda = 0$, with limiting values at $\lambda = 0$ given by the direct sum of stable (resp. unstable) subspace of $A(\lambda)$ and the vector

$$W = (\sqrt{-\alpha_\infty}, 1, 0, 0)^{\mathrm{tr}} \quad (\text{resp. } (-\sqrt{-\alpha_\infty}, 1, 0, 0)^{\mathrm{tr}}),$$

with $U = (v, u)$ coordinate equal to the unstable (resp. stable) subspace

$$U = (1, \sqrt{-\alpha_\infty})^{\mathrm{tr}} \quad (\text{resp. } (1, -\sqrt{-\alpha_\infty})^{\mathrm{tr}})$$

of the convection matrix $a := \left(\begin{smallmatrix} -s & -1 \\ \alpha_\infty & -s \end{smallmatrix}\right)$. (Recall, (2.4), that $\det a < 0$, so that a has one real positive and one real negative eigenvalue.)

Applying the framework of [13], we find that, on Λ, the subspaces of solutions decaying at $+\infty$ and $-\infty$ of the variable-coefficient equations (3.1) likewise have dimension two. Moreover, there exist choices of bases W_1^+, W_2^+ and W_3^-, W_4^- for these subspaces that are analytic in λ on Λ and extend analytically to $\Lambda \cup B(0,r)$ for $r > 0$ sufficiently small. At $\lambda = 0$, we may choose $W_1^+(\cdot,0) = W_4^-(\cdot,0) = \partial_x \bar W^{\bar s}$ asymptotically decaying, $\bar W^{\bar s} := (\bar u^{\bar s}, \bar v^{\bar s}, \bar v_x^{\bar s}, \bar v_{xx}^{\bar s})^{\mathrm{tr}}$, and $W_2^+(\cdot,0)$, $W_3^-(\cdot,0)$ asymptotically constant, with

$$\begin{aligned} \lim_{x\to+\infty} W_2^+(x,0) &= (c,1,0,0)^{\mathrm{tr}} \\ \lim_{x\to-\infty} W_3^-(x,0) &= (-c,1,0,0)^{\mathrm{tr}}, \end{aligned} \tag{3.5}$$

$c := \sqrt{-\alpha_\infty} > 0$. The Evans function is defined as the Wronskian

$$D(\lambda) := \det \begin{pmatrix} W_1^+ & W_2^+ & W_3^- & W_4^- \end{pmatrix}|_{x=0},$$

zeroes of which detect nontrivial intersection between decaying manifolds of solutions at plus/minus spatial infinity, i.e., decaying solutions of eigenvalue equation (3.1).

3.2. Calculation of $D''(0)$. We now compute $D''(0)$ by a simplified version (taking advantage of special structure) of the approach of [3].

Proof of Theorem 1.3. Following [3] (see also related calculations of [29, Section 6]), we choose a convenient basis for the calculation of derivatives at the origin, based on the Jordan chain of L at $\lambda = 0$. Namely, observing that $Z := \partial_\lambda U$ satisfies at $\lambda = 0$ the first-order generalized eigenvalue equation

$$(L - \lambda)Z = U \tag{3.6}$$

if $U(\lambda, \cdot)$ satisfies for all λ the eigenvalue equation $(L-\lambda)U = 0$, we may arrange that not only $W_1^+(0) = W_4^-(0) = \partial_x \bar{W}^{\bar{s}}$ as above, but also

$$\partial_\lambda W_1^+(0) = \partial_\lambda W_4^-(0) = -(\partial \bar{W}^s/\partial s)(\bar{s}).$$

For, W_1^+ and W_4^- may be chosen as "fast modes", decaying uniformly exponentially as $x \to \pm\infty$ for $|\lambda|$ sufficiently small, whence $\partial_\lambda W_1^+(0)$ and $\partial_\lambda W_4^-(0)$ are uniformly exponentially decaying at their associated spatial infinities as well, and (their U components) satisfy the generalized eigenvalue equation (3.6), which uniquely specifies them up to exponentially decaying solutions of the eigenvalue equation: in this case, multiples $c_1 W_1^+$ and $c_2 W_4^-$, respectively, which may be removed by the analytic change of coordinates $W_j \to \lambda c_j W_j$.

With this normalization, we find immediately that

$$D(0) = \det\begin{pmatrix}\partial_x \bar{W}^{\bar{s}} & W_2^+ & W_3^- & \partial_x \bar{W}^{\bar{s}}\end{pmatrix} = 0$$

and

$$\begin{aligned} D'(0) &= \det\begin{pmatrix}\partial_\lambda W_1^+ & W_2^+ & W_3^- & W_4^-\end{pmatrix} + \cdots \\ &\quad + \det\begin{pmatrix}W_1^+ & W_2^+ & W_3^- & \partial_\lambda W_4^-\end{pmatrix} \\ &= \det\begin{pmatrix}\partial_x \bar{W}^{\bar{s}} & W_2^+ & W_3^- & \partial_s \bar{W}^{\bar{s}}\end{pmatrix} \\ &\quad + \det\begin{pmatrix}\partial_s \bar{W}^{\bar{s}} & W_2^+ & W_3^- & \partial_x \bar{W}^{\bar{s}}\end{pmatrix} = 0. \end{aligned}$$

By a similar computation, we find that

$$\begin{aligned} D''(0) &= \det\begin{pmatrix}\partial_\lambda^2 W_1^+ & W_2^+ & W_3^- & W_4^-\end{pmatrix} \\ &\quad + \det\begin{pmatrix}W_1^+ & W_2^+ & W_3^- & \partial_\lambda^2 W_4^-\end{pmatrix} \\ &= \det\begin{pmatrix}W_1^+ & W_2^+ & W_3^- & (\partial_\lambda^2 W_4^- - \partial_\lambda^2 W_1^+)\end{pmatrix}, \end{aligned} \tag{3.7}$$

$W_1^+ = \partial_x \bar{W}^{\bar{s}}$, where $Y_j := \partial_\lambda^2 U_j$ satisfy at $\lambda = 0$ the second-order generalized eigenvalue equation

$$(L - \lambda)Y = Z. \tag{3.8}$$

(Recall that $Z := \partial_\lambda U$ satisfies the first-order generalized eigenvalue equation (3.6).)

Now, setting $\lambda = 0$ and integrating (3.1) from $x = \pm\infty$ to $x = 0$, we obtain

$$\begin{aligned} sv + u &= (sv + u)(\pm\infty), \\ -su + \alpha v + \kappa v'' &= (-su + \alpha v)(\pm\infty), \end{aligned} \tag{3.9}$$

for each $(u, v) = (u, v)^{\pm}_j$ associated with $W^{\pm}_j$. In particular, the righhand sides of (3.9) vanish for $(u, v) = (u, v)^{+}_1$. Likewise, setting $(\tilde u, \tilde v) = (\tilde u, \tilde v)^{\pm}_j := \partial^2_\lambda (u, v)^{\pm}_j$, $j = 1, 4$, using (3.8), and recalling that we chose $(\tilde u, \tilde v)$ to vanish at spatial infinity, we find that

$$\begin{aligned} s\tilde v + \tilde u &= -\int_{\pm\infty}^{x} (\partial \bar v^{\bar s}/\partial s)\, dx, \\ -s\tilde u + \alpha \tilde v + \kappa \tilde v'' &= \int_{\pm\infty}^{x} (\partial \bar u^{\bar s}/\partial s)\, dx. \end{aligned} \tag{3.10}$$

Performing the row operations corresponding the the lefthand side of (3.9), i.e., adding $\kappa^{-1}(-s\tilde u + \alpha \tilde v)$ to v'', then adding sv to u, and using (3.9), (3.5), and (3.10), reduces (3.7) to

$$D''(0) = \det \begin{pmatrix} 0 & s + c & s - c & C_v \\ v^+_1 & v^+_2 & v^-_3 & (\partial^2_\lambda v^-_4 - \partial^2_\lambda v^+_1) \\ (v^+_1)' & (v^+_2)' & (v^-_3)' & (\partial^2_\lambda v^-_4 - \partial^2_\lambda v^+_1)' \\ 0 & -sc - c^2 & sc - c^2 & C_u \end{pmatrix},$$

where $C_v := -\int_{-\infty}^{+\infty} (\partial \bar v^{\bar s}/\partial s)\, dx$ and $C_u := \kappa^{-1} \int_{-\infty}^{+\infty} (\partial \bar u^{\bar s}/\partial s)\, dx$. Since the matrix $\begin{pmatrix} s+c & s-c \\ -sc - c^2 & sc - c^2 \end{pmatrix}$ is invertible, by (2.4), there exist constants d_2, d_3 such that

$$d_3 \begin{pmatrix} s - c \\ sc - c^2 \end{pmatrix} - d_2 \begin{pmatrix} s + c \\ -sc - c^2 \end{pmatrix} = \begin{pmatrix} C_v \\ C_u \end{pmatrix}.$$

Performing the corresponding column operation to eliminate C_v, C_u, we obtain, finally,

$$\begin{aligned} D''(0) &= \det \begin{pmatrix} 0 & s + c & s - c & 0 \\ v^+_1 & v^+_2 & v^-_3 & (\partial^2_\lambda v^-_4 - \partial^2_\lambda v^+_1 + d_2 v^+_2 - d_3 v^-_3) \\ (v^+_1)' & (v^+_2)' & (v^-_3)' & (\partial^2_\lambda v^-_4 - \partial^2_\lambda v^+_1 + d_2 v^+_2 - d_3 v^-_3)' \\ 0 & -sc - c^2 & sc - c^2 & 0 \end{pmatrix} \\ &= C\gamma, \end{aligned} \tag{3.11}$$

where

$$C = \det \begin{pmatrix} s + c & s - c \\ -sc - c^2 & sc - c^2 \end{pmatrix} = 2c(p'(v_\infty) + s^2) < 0.$$

and

$$\gamma = \det \begin{pmatrix} v^+_1 & (\partial^2_\lambda v^-_4 - \partial^2_\lambda v^+_1 + d_2 v^+_2 - d_3 v^-_3) \\ (v^+_1)' & (\partial^2_\lambda v^-_4 - \partial^2_\lambda v^+_1 + d_2 v^+_2 - d_3 v^-_3)' \end{pmatrix}.$$

Expand now $\gamma = \gamma_-(0) - \gamma_+(0)$, where

$$\gamma_-(x) := \det \begin{pmatrix} v_1^+ & \hat{v}_4 \\ (v_1^+)' & (\hat{v}_4)' \end{pmatrix}, \qquad \gamma_+(x) := \det \begin{pmatrix} v_1^+ & \hat{v}_1 \\ (v_1^+)' & (\hat{v}_1)' \end{pmatrix}$$

and

$$\hat{v}_4 := \partial_\lambda^2 v_4^- - d_3 v_3^-, \qquad \hat{v}_1 := \partial_\lambda^2 v_1^+ - d_2 v_2^+.$$

Since $\hat{v}_4$ is bounded as $x \to -\infty$, while $v_1^+ = v_4^-$ decays exponentially,

$$\gamma_-(-\infty) = 0. \tag{3.12}$$

Similarly,

$$\gamma_+(+\infty) = 0. \tag{3.13}$$

By (3.9), (3.10), $v_1^+ = \bar{v}_x$ satisfies $v'' + \kappa^{-1}(s^2 + \alpha)v = 0$, or

$$\begin{pmatrix} v \\ v' \end{pmatrix}' - \begin{pmatrix} 0 & 1 \\ -\kappa^{-1}(s^2+\alpha) & 0 \end{pmatrix} \begin{pmatrix} v \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

while $\hat{v}_\pm$ satisfy $\hat{v}'' + \kappa^{-1}(s^2+\alpha)\hat{v} = \kappa^{-1}\int_{-\infty}^x [-s(\partial \bar{v}^{\bar{s}}/\partial s) + (\partial \bar{u}^{\bar{s}}/\partial s)]dx + \hat{C}$, or

$$\begin{pmatrix} \hat{v} \\ \hat{v}' \end{pmatrix}' - \begin{pmatrix} 0 & 1 \\ -\kappa^{-1}(s^2+\alpha) & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{v}' \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix},$$

with $\hat{C}$ constant,

$$F := \kappa^{-1} \int_{-\infty}^x [-s(\partial \bar{v}^{\bar{s}}/\partial s) + (\partial \bar{u}^{\bar{s}}/\partial s)]\, dx + \hat{C}. \tag{3.14}$$

Thus, using inhomogeneous Abel's formula and (3.12)-(3.13), we may evaluate the difference γ between Wronskians γ_+ and γ_- at $x = 0$ as the Melnikov-type integral

$$\begin{aligned}
\gamma &= \gamma_-(0) - \gamma_+(0) \\
&= \int_{-\infty}^{+\infty} \det \begin{pmatrix} v_1^+ & 0 \\ 0 & F \end{pmatrix} (x)\, dx \\
&= \int_{-\infty}^{+\infty} \bar{v}_x F(x)\, dx \\
&= \kappa^{-1} \int_{-\infty}^{+\infty} \bar{v}_x(x) \int_{-\infty}^x [-s(\partial \bar{v}^{\bar{s}}/\partial s) + (\partial \bar{u}^{\bar{s}}/\partial s)](y)\, dy\, dx \\
&= \kappa^{-1} \int_{-\infty}^{+\infty} (\bar{v} - v_\infty)[-s(\partial \bar{v}^{\bar{s}}/\partial s) + (\partial \bar{u}^{\bar{s}}/\partial s)](x)\, dx,
\end{aligned}$$

where in going from the third to the fourth line we are using the fact that term $\int_{-\infty}^{+\infty} \hat{C} v_x(x)\, dx$ coming from (3.14) integrates to zero and in going from the fourth to the final line we are using integration by parts.

Substituting the relation $-s(\bar v - v_\infty) = (\bar u - u_\infty)$ coming from the first equation in (2.1) and recalling (1.9), Remark 1.1, we obtain, finally,

$$\begin{aligned}\gamma &= \kappa^{-1}\int_{-\infty}^{+\infty}[(\bar u - u_\infty)(\partial \bar v^{\bar s}/\partial s) + (\bar v - v_\infty)(\partial \bar u^{\bar s}/\partial s)](x)\,dx\\ &= \kappa^{-1}\int_{-\infty}^{+\infty}(\bar U - U_\infty)J(\partial \bar U^{\bar s}/\partial s)(x)\,dx\\ &= \kappa^{-1}(d^2m/ds^2)(\bar s).\end{aligned} \tag{3.15}$$

Combining (3.11) and (3.15), we obtain

$$D''(0) = (-C/\kappa)(d^2m/ds^2)(\bar s), \tag{3.16}$$

$-C/\kappa > 0$, completing the proof. □

3.3. Proof of Corollary 1.4. By standard considerations [3, 13, 25, 27] we have also the following.

Lemma 3.1. *As* $\lambda \to +\infty$ *along the real axis,* $\mathrm{sgn} D(\lambda)$ *has limit* $+1$.

Proof. First, note that $D(\lambda)$ does not vanish for $\Re\lambda$ sufficiently large, a standard fact associated with well-posedness of the linearized time-evolutionary system; this may be established either by asymptotic ODE theory as in [13, 27] or by elementary energy estimates as in [3]. Thus, $D(\lambda)$ has a (nonzero) limiting sign as claimed.

To determine the value of this sign, we may consider a homotopy from system (3.1) to the constant-coefficient equation

$$\begin{aligned}\lambda v - u' &= 0\\ \lambda u + \kappa v''' &= 0,\end{aligned} \tag{3.17}$$

capturing high-frequency behavior, for which bases $V_j = \left(\frac{\lambda}{\mu_j}, 1, \mu_j, \mu_j^2\right)$, $\mu_j = \theta_j\sqrt{\lambda}$ of exponential solutions may be explicitly calculated for all λ, where θ_j are the fourth roots of -1.

Choosing bases V_1, $\frac{1}{\sqrt{\lambda}}(V_2 - V_1)$ and V_3, $\frac{1}{\sqrt{\lambda}}(V_4 - V_3)$ of decaying subspaces at $\pm\infty$, we find that these extend continuously to $\lambda = 0$, with the projection onto (u, v) coordinates of the limiting bases equal to the standard Euclidean basis $(1, 0)$, $(0, 1)$; in particular, the projection of the limiting subspaces is nonsingular. Likewise, it is nonsingular for the bases at $\lambda = 0$ for the original system, and (by identical calculation) for all systems strictly between the two. Thus, choosing continuously initializing bases at $\lambda = 0$ for the family of systems, we find that the signs of $D(+\infty)$ and of the determinants $d_\pm$ of the projections of decaying subspaces at $\pm\infty$ onto (u, v) coordinates are all constant under homotopy, with, moreover $\mathrm{sgn} D(+\infty) d_- d_+$ explicitly calculable from system (3.17). We may therefore determine $\mathrm{sgn} D(+\infty)$ for the original system by the straightforward calculation of d_- and d_+, yielding the result. □

Remark 3.2. An alternative, somewhat simpler approach, using the homoclinic structure of the wave, is to note that the limiting subspaces at plus and minus infinity of decaying solutions of the eigenvalue equation are complementary subspaces of the limiting coefficient matrix A, hence transverse. Thus, their determinant (real-valued, by construction) is of constant sign independent of λ, which may be related to $\text{sgn}D(+\infty)$ by explicit calculation. Calculating the value at $\lambda = 0$ of this determinant then gives the result. This is similar in spirit to the original argument of [25]. However, this approach does not extend to the heteroclinic case, and so we have chosen to give the more general argument above. For related arguments, see [27, 28].

We may now easily obtain the main result.

Proof of Corollary 1.4. It has been shown in [2] by variational considerations that $(d^2m/ds^2)(\bar{s}) > 0$ implies linearized and nonlinear stability, using Corollary 2.3 and the argument of [14]. Thus, we need only establish that $(d^2m/ds^2)(\bar{s}) < 0$ implies instability, for which it is sufficient to show that there is an L^2 eigenfunction of L with positive real eigenvalue λ. Defining the *stability index*

$$\Gamma := \text{sgn}D''(0)D(+\infty),$$

we have, provided $\Gamma \neq 0$, that the number of positive real eigenvalues of L (zeroes of D) is odd or even, according as Γ is negative or positive, by standard degree theory on the line (recall, D is real-valued, by construction, for λ real). Thus, $\Gamma < 0$ implies existence of at least one positive real eigenvalue, for which (see Section 3) the associated eigenfunction is necessarily exponentially decaying as $x \to \pm\infty$. Noting that $\text{sgn}\Gamma = \text{sgn}(d^2m/ds^2)(\bar{s})$ by (3.16) combined with Lemma 3.1, we are done. $\square$

Remark 3.3. By Lemma 2.2, there is in fact precisely one unstable eigenvalue, necessarily real, in case $(d^2m/ds^2)(\bar{s}) < 0$.

4. Remarks on the viscous case

Finally, we comment briefly on the viscous case, as modeled by

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= \epsilon u_{xx} - \kappa v_{xxx}, \end{aligned} \tag{4.1}$$

$\epsilon > 0$. Stability of traveling waves of this model has been studied in detail in [24, 26] for a parameter range of (κ, ϵ) for which (4.1) may be converted by a change of dependent variables to a strictly parabolic problem.

Including viscosity of arbitrarily small strength $\epsilon > 0$, one finds by the energy argument of [24] that all homoclinic connections of the $\epsilon = 0$ equations (2) break, save for zero-speed solitary waves identical to those of (1.2). Moreover, by a calculation similar to but much simpler than that of the previous section, we find that all of these waves are unstable, with $D(0) = 0$, $D'(0) = c\epsilon \int (\bar{v}_x)^2 dx$, $c \neq 0$. That is, even infinitesimal viscosity will either break or destabilize any solitary wave, whether stable or unstable for the $\epsilon = 0$ model, leaving unclear the physical implications of stability or instability with respect to (1.2). However, depending whether the $\epsilon = 0$ version is stable or unstable, the unstable root will be near or far from the origin, corresponding perhaps to some type of metastable phenomenon.

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