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On Two-Dimensional Immersions of Prescribed Mean Curvature in $\mathbb{R}^n$

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Abstract. We consider two-dimensional immersions of disc-type in $\mathbb{R}^n$. We focus on well known classical concepts and study the nonlinear elliptic systems of such mappings. Using an Osserman-type condition we give a-priori estimates of the principle curvatures for graphs with prescribed mean curvature fields and derive a theorem of Bernstein type for minimal graphs.

Keywords. Two-dimensional immersions, prescribed mean curvature, higher codimension, curvature estimates

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Introduction

The main result of our paper is an estimate of the principal curvatures of two-dimensional graphs with prescribed mean curvature in Euclidean space $\mathbb{R}^n$ in terms of certain a-priori data.

The notations which we need to formulate this result are introduced in the first chapter: In Section 1.1. a definition of differential geometric immersions with smooth sections of the normal bundle; in Section 1.2 an introduction to conformally parametrized immersions with prescribed mean curvature fields; and in Section 1.3 our main theorem with a brief discussion. In particular, we will infer a theorem of Bernstein type for minimal graphs.

Before we give a detailed proof in Chapter 3 we recall important concepts of the differential geometry of two-dimensional immersions in $\mathbb{R}^n$ using the classical Ricci calculus. Among these are the differential equations of Weingarten and Gauss, as well as the integrability conditions of Ricci which lead us to the notion of the normal sectional curvature.

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While in the first chapter we introduce the non-linear elliptic mean curvature systems in a more abstract way, in Chapter 2 these systems arise naturally from the differential geometric identities. A variational problem which illustrates the appearance of mean curvature systems as minimal surfaces in certain Riemannian spaces, and an example for our main theorem complete this second part.

Chapter 3 presents a detailed proof of our curvature estimate. Essentially we use ideas from [12] (see also [13] and [14]) where the author applies fundamental results on non-linear elliptic systems with quadratic gradient growth which were developed in [6].

1. Basic notations and the main theorem

1.1. Basic notations. Denote by $B := \{(u,v) \in \mathbb{R}^2 : u^2+v^2<1\}$ the open unit disc in $\mathbb{R}^2$ and by $\overline{B} \subset \mathbb{R}^2$ its topological closure. For positive integers $n \geq 3$ we consider two-dimensional immersions

$$X(u,v) = (x^1(u,v), \ldots, x^n(u,v)), \quad (u,v) \in \overline{B}, \tag{1.1}$$

of the regularity class $X \in C^{3+\alpha}(B,\mathbb{R}^n) \cap C^0(\overline{B},\mathbb{R}^n)$, $\alpha \in (0,1)$, such that

$$\operatorname{rank} \partial X(u,v) \equiv \operatorname{rank} \begin{pmatrix} x_u^1(u,v) & x_v^1(u,v) \\ \vdots & \vdots \\ x_u^n(u,v) & x_v^n(u,v) \end{pmatrix} = 2 \quad \forall\,(u,v) \in B, \tag{1.2}$$

where the indices u and v denote the partial derivatives w.r.t. u and v.

Definition 1.1. We define $\mathfrak{C}(B,\mathbb{R}^n)$ to be the *set of all immersions* $X = X(u,v)$ with the properties (1.1) to (1.2).

We infer that the tangent vectors $X_u = X_u(u,v)$ and $X_v = X_v(u,v)$ are linearly independent at any $(u,v) \in B$ and span the two-dimensional tangent plane at that point, namely

$$\mathcal{T}_X(w) := \operatorname{Span}\big\{X_u(w), X_v(w)\big\}, \quad w = u+iv \in B. \tag{1.3}$$

The normal space $\mathcal{N}_X(w) := \mathcal{T}_X(w)^\perp$ at $w \in B$ is a $(n-2)$-dimensional vector space spanned by vectors $N_1(w), \ldots, N_{n-2}(w)$ such that there hold the orthonormality relations

$$N_\Sigma(w) \cdot N_\Theta(w)^t = \delta_{\Sigma\Theta} := \begin{cases} 1 & \text{if } \Sigma = \Theta \\ 0 & \text{if } \Sigma \neq \Theta \end{cases} \qquad \forall\,\Sigma,\Theta \in \{1,\ldots,n-2\} \tag{1.4}$$

with the Kronecker symbol $\delta_{\Sigma\Theta}$. Here, the upper t means the transposed vector. Let $X \in \mathfrak{C}(B,\mathbb{R}^n)$. Then there exists an orthonormal set $\{N_1(w), \ldots, N_{n-2}(w)\}$

such that $N_\Sigma \in C^{2+\alpha}(B, S^{n-1})$ for all $\Sigma = 1, \ldots, n-2$, where $S^{n-1} := \{Z \in \mathbb{R}^n : |Z|^2 = 1\}$, and

$$N_\Sigma(w) \cdot X_{u^\ell}(w)^t = 0 \quad \forall\, \Sigma = 1, \ldots, n-2,\ \ell = 1, 2,\ \forall\, w \in B. \tag{1.5}$$

Definition 1.2. We call a set $\{N_1, \ldots, N_{n-2}\}$ from the assumption above an *orthonormal normal (ON-) section* of the immersion $X = X(u, v)$.

Example 1.3. Given functions $\varphi_1, \ldots, \varphi_{n-2} \in C^{3+\alpha}(\Omega, \mathbb{R})$ on a bounded domain $\Omega \subset \mathbb{R}^2$, we define unit vectors

$$\begin{aligned} \widetilde{N}_1 &:= \frac{1}{\sqrt{1 + |\nabla \varphi_1|^2}}\,(-\varphi_{1,x}, -\varphi_{1,y}, 1, 0, \ldots, 0), \\ \widetilde{N}_2 &:= \frac{1}{\sqrt{1 + |\nabla \varphi_2|^2}}\,(-\varphi_{2,x}, -\varphi_{2,y}, 0, 1, \ldots, 0),\ \ldots \end{aligned} \tag{1.6}$$

which are normal to the graph $(x, y, \varphi_1(x, y), \ldots, \varphi_n(x, y))$. Here, ∇ denotes the Euclidean gradient. Using Gram–Schmidt orthonormalization these vectors can be transformed into an ON-normal section $\{N_1, \ldots, N_{n-2}\}$.

1.2. Immersions with prescribed mean curvature fields. Following [15] we introduce conformal parameters $(u, v) \in B$ into the immersion $X = X(u, v)$ such that there hold in B the conformality relations

$$|X_u(u, v)|^2 = W(u, v) = |X_v(u, v)|^2\,, \quad X_u(u, v) \cdot X_v(u, v)^t = 0. \tag{1.7}$$

Here, $W = W(u, v)$ means the area element

$$\begin{aligned} W(u, v) &:= \sqrt{g_{11}(u, v) g_{22}(u, v) - g_{12}(u, v)^2} \\ g_{ij}(u, v) &:= X_{u^i}(u, v) \cdot X_{u^j}(u, v)^t\,, \end{aligned} \tag{1.8}$$

of the surface. Note that $W(u, v) > 0$ for all $(u, v) \in B$.

Definition 1.4. The functions $g_{ij} = g_{ij}(u, v)$, $i, j = 1, 2$, from (1.8) are called the *coefficients of the first fundamental form* of the immersion $X = X(u, v)$.

Given a vector valued function $\mathcal{H}\colon \mathbb{R}^n \to \mathbb{R}^n$ we define a scalar field $H\colon \mathbb{R}^n \times S^{n-1} \to \mathbb{R}$ by

$$H(X, Z) := \mathcal{H}(X) \cdot Z^t \quad \text{for } X \in \mathbb{R}^n,\ Z \in S^{n-1}\,. \tag{1.9}$$

Now, we introduce the notion of prescribed mean curvature fields $H(X, Z)$.

Definition 1.5. Let $X \in \mathfrak{C}(B, \mathbb{R}^n)$ with an ON-normal section $\{N_1, \ldots, N_{n-2}\}$ be given. Then we call $X = X(u,v)$ a *conformally parametrized immersion* of prescribed mean curvature field $H \in C^0(\mathbb{R}^n \times S^{n-1}, \mathbb{R})$ iff there hold (1.7) and

$$\triangle X = 2H(X, N_1)WN_1 + \cdots + 2H(X, N_{n-2})WN_{n-2} \quad \text{in } B \tag{1.10}$$

with the Euclidean Laplace operator $\triangle$. The immersion is called a *minimal surface* iff $H(X,Z) \equiv 0$, that is $\triangle X = 0$ in B.

Remark 1.6. The differential system (1.10) is invariant w.r.t. orthogonal changes of the ON-normal section $\{N_1, \ldots, N_{n-2}\}$. For the proof let $\mathbf{O} = (o_{\Sigma\Omega})_{\Sigma,\Omega=1,\ldots,n}$ be an orthogonal matrix which in $\overline{B} \subset \mathbb{R}^2$ satisfies the relations

$$\begin{aligned} 2\sum_{\Sigma=1}^{n-2} o_{\Sigma\Omega}(w)^2 &= \sum_{\Sigma=1}^{n-2} o_{\Omega\Sigma}(w)^2 = 1 && \text{for } \Omega = 1, \ldots, n-2 \\ \sum_{\Sigma=1}^{n-2} o_{\Sigma\Omega}(w) o_{\Sigma\Omega'}(w) &= \sum_{\Sigma=1}^{n-2} o_{\Omega\Sigma}(w) o_{\Omega'\Sigma}(w) = 0 && \text{for } \Omega \neq \Omega' \,. \end{aligned}$$

We introduce a new ON-normal section $\{\widetilde{N}_1, \ldots, \widetilde{N}_{n-2}\}$ via $\widetilde{N}_\Sigma := \sum_{\Omega=1}^{n-2} o_{\Sigma\Omega} N_\Omega$, $\Sigma = 1, \ldots, n-2$. The stated invariance follows from

$$\begin{aligned} \sum_{\Sigma=1}^{n-2} H(X, \widetilde{N}_\Sigma)\widetilde{N}_\Sigma &= \sum_{\Sigma=1}^{n-2} \Big\{ \mathcal{H}(X) \cdot \sum_{\Omega=1}^{n-2} o_{\Sigma\Omega} N_\Omega^t \Big\} \sum_{\Omega'=1}^{n-2} o_{\Sigma\Omega'} N_{\Omega'} \\ &= \sum_{\Omega=1}^{n-2} \Big(\sum_{\Sigma=1}^{n-2} o_{\Sigma\Omega}^2 \Big) \Big\{ \mathcal{H}(X) \cdot N_\Omega^t \Big\} N_\Omega \\ &\quad + \sum_{\substack{\Omega,\Omega'=1 \\ \Omega\neq\Omega'}}^{n-2} \Big(\sum_{\Sigma=1}^{n-2} o_{\Sigma\Omega} o_{\Sigma\Omega'} \Big) \Big\{ \mathcal{H}(X) \cdot N_\Omega^t \Big\} N_{\Omega'} \\ &= \sum_{\Omega=1}^{n-2} \Big\{ \mathcal{H}(X) \cdot N_\Omega^t \Big\} N_\Omega \\ &= \sum_{\Omega=1}^{n-2} H(X, N_\Omega) N_\Omega \,. \end{aligned}$$

1.3. The main theorem. We admit mean curvature fields $H \in C^0(\mathbb{R}^n \times S^{n-1}, \mathbb{R})$ which additionally satisfy the following Hölder and Lipschitz assumptions:

$$\begin{gathered} |H(X,Z)| \le h_0 \quad \forall\, X \in \mathbb{R}^n \,,\ Z \in S^{n-1}, \quad \text{and} \\ |H(X_1, Z_1) - H(X_2, Z_2)| \le h_1 |X_1 - X_2|^\alpha + h_2 |Z_1 - Z_2| \\ \forall\, X_1, X_2 \in \mathbb{R}^n \,,\ Z_1, Z_2 \in S^{n-1} \,. \end{gathered} \tag{1.11}$$

Theorem 1.7. *Let an immersion $X \in \mathfrak{C}(B, \mathbb{R}^n)$ of prescribed mean curvature field $H = H(X, Z)$ be given such that* (1.11) *holds. Furthermore, we assume:*

(A1) *the immersion $X = X(u, v)$ is a conformal reparametrization of a graph*

$$\begin{gathered}(x, y, \varphi_1(x, y), \ldots, \varphi_{n-2}(x, y)) \\ \varphi_\Sigma \in C^{3+\alpha}(\Omega, \mathbb{R}) \quad \text{for } \Sigma = 1, \ldots, n-2,\end{gathered} \tag{1.12}$$

over a bounded and simply connected domain $\Omega \subset \mathbb{R}^2$;

(A2) *the surface represents a geodesic disc $\mathfrak{B}_r(X_0)$ of geodesic radius $r > 0$ and with center $X_0 := (0, \ldots, 0)$ such that, with a real constant $d_0 > 0$, it holds*

$$\operatorname{Area}[\mathfrak{B}_r(X_0)] := \iint\limits_B W(u, v)\, du\, dv \le d_0 r^2 \tag{1.13}$$

for the area of the geodesic disc, where $d_0 \in (0, +\infty)$ does not depend on r;

(A3) *at every point $w \in B$, each normal vector of the immersion makes an angle of at least $\omega > 0$ with the x_1-axis.*

Then, for any orthonormal basis $\{\overline{N}_1, \ldots, \overline{N}_{n-2}\}$ of the normal space at X_0 there exists a constant

$$\Theta = \Theta\big(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, n, \alpha\big) \in (0, +\infty) \tag{1.14}$$

such that it holds

$$\kappa_{\Sigma,1}(0,0)^2 + \kappa_{\Sigma,2}(0,0)^2 \le \frac{1}{r^2}\big\{(h_0 r)^2 + \Theta\big\} \tag{1.15}$$

for the principal curvatures $\kappa_{\Sigma,1}$ and $\kappa_{\Sigma,2}$ w.r.t. $\overline{N}_\Sigma$ for all $\Sigma = 1, \ldots, n-2$.

Remark 1.8. 1. The principal curvatures $\kappa_{\Sigma,i} = \kappa_{\Sigma,i}(u, v)$ are defined as the eigenvalues of a Weingarten form w.r.t. a unit normal vector N_Σ (see Section 2.1 for details).

2. An estimate of the principal curvatures can be proved under the assumption $\operatorname{Area}[X] \le M_0$ with any constant $M_0 \in (0, +\infty)$ instead of (A2), where Θ from (1.14) will depend on M_0. However, (A2) leads to the following Bernstein type result.

Consider minimal graphs $(x, y, \varphi_1, (x, y), \ldots, \varphi_{n-2}(x, y))$, $(x, y) \in \mathbb{R}^2$. Because its Gauss curvature is non-positive (see Section 2.1), by a theorem of Hadamard (see [9, Theorem 3.4.16]) we can introduce geodesic discs $\mathfrak{B}_r(X_0)$ for all $X_0 = (x_0, y_0, \varphi_1(x_0, y_0), \ldots, \varphi_{n-2}(x_0, y_0))$ and all $r > 0$. Then the limit $r \to \infty$ yields the

Corollary 1.9. *Let $X = X(x, y)$, $(x, y) \in \mathbb{R}^2$, be a complete minimal graph with the properties:*

(i) *there exists* $X_0 = (x_0, y_0, \varphi_1(x_0, y_0), \ldots, \varphi_{n-2}(x_0, y_0))$ *and a radius* $r_0 > 0$ *such that all geodesic discs* $\mathfrak{B}_r(X_0)$ *with center* X_0 *and radius* $r \geq r_0$ *satisfy*

$$\text{Area}[\mathfrak{B}_r(X_0)] \leq d_0 r^2 \quad \forall\, r \geq r_0 \tag{1.16}$$

with a constant $d_0 \in (0, +\infty)$ *which does not depend on* r;

(ii) *each normal vector of the graph makes an angle of at least* $\omega > 0$ *with the* x_1*-axis.*

Then $X = X(x, y)$ *is a linear mapping.*

Proof. For any point $X_1 = (x_1, y_1, \varphi_1(x_1, y_1), \ldots, \varphi_{n-2}(x_1, y_1))$ on the graph we have

$$\text{Area}[\mathfrak{B}_r(X_1)] \leq 4 d_0 r^2 \quad \forall\, r \geq \max\{r_0, d(X_0, X_1)\} \tag{1.17}$$

where $d(X_0, X_1) \geq 0$ is the inner distance between X_0 and X_1 on the surface. This holds because of the inclusion

$$\mathfrak{B}_r(X_1) \subset \mathfrak{B}_{2r}(X_0) \quad \forall\, r \geq \max\{r_0, d(X_0, X_1)\} \tag{1.18}$$

and assumption (i).

Since $K \leq 0$ for the Gaussian curvature we can consider geodesic discs $\mathfrak{B}_r(X_1)$ for all $r \in (0, +\infty)$ on account of Hadamard's theorem. With the aid of [15] we introduce conformal parameters into such a geodesic disc.

Using the curvature estimate (1.15) and letting $r \to \infty$ shows that all principal curvatures at X_1 vanish which proves the Corollary (note that Θ does not depend on r since $h_0, h_1, h_2 = 0$). □

Remark 1.10. 1. In [11] Osserman proved that a complete two-dimensional minimal surface in $\mathbb{R}^n$ is a plane if all of its normal vectors make a certain positive angle with a fixed axis in space (compare with assumption (A3)). The method of his proof is based essentially on results of complex analysis and it does not need a growth condition of the form (1.13).

2. In [7] a Bernstein type result for minimal submanifolds is proved. The methods established there were generalized in [8] to prove curvature estimates for submanifolds with parallel mean curvature fields. Due to the higher dimension of the manifolds itself the authors assume a-priori bounds for the gradients.

3. Curvature estimates and related Bernstein type result for minimal submanifolds can also be found in [19] where the authors extend methods from [17] for minimal immersions with vanishing normal sectional curvature (see also [20], and (2.13) below).

4. Our method of proof uses essentially results from [6], and follows [12] where curvature estimates for two-dimensional immersions of mean curvature type in $\mathbb{R}^3$ where established.

2. Differential geometry of surfaces in $\mathbb{R}^n$

2.1. Mean and Gaussian curvature fields and principal curvatures. Let $X \in \mathfrak{C}(B,\mathbb{R}^n)$ be given with an ON-normal section $\{N_1,\ldots,N_{n-2}\}$. Consider the forms

$$(L^k_{\Sigma,i})_{i,k=1,2} := (L_{\Sigma,ij}g^{jk})_{i,k=1,2} \in \mathbb{R}^{2\times 2}\,, \quad \Sigma = 1,\ldots,n-2, \tag{2.1}$$

with the coefficients $g^{ij} = g^{ij}(u,v)$ of the inverse of the first fundamental form, i.e., $g_{ij}g^{jk} = \delta^k_i$ with the Kronecker symbol δ^k_i using the summation convention.

Definition 2.1. The *mean curvature* $H_\Sigma = H_\Sigma(u,v)$ and the *Gaussian curvature* $K_\Sigma = K_\Sigma(u,v)$ *in direction* N_Σ, $\Sigma = 1,\ldots,n-2$, are defined as

$$H_\Sigma := \frac{1}{2}\,\mathrm{trace}\,(L^k_{\Sigma,i})_{i,k=1,2} = \frac{L_{\Sigma,11}g_{11} - 2L_{\Sigma,12}g_{12} + L_{\Sigma,22}g_{22}}{2(g_{11}g_{22} - g_{12}^2)} \tag{2.2}$$

and

$$K_\Sigma := \det\,(L^k_{\Sigma,i})_{i,k=1,2} = \frac{L_{\Sigma,11}L_{\Sigma,22} - L^2_{\Sigma,12}}{g_{11}g_{22} - g_{12}^2}\,. \tag{2.3}$$

The *principal curvatures* $\kappa_{\Sigma,1}$, $\kappa_{\Sigma,2}$ w.r.t. N_Σ are the eigenvalues of $(L^k_{\Sigma,i})_{i,k=1,2}$, that is

$$H_\Sigma = \frac{\kappa_{\Sigma,1} + \kappa_{\Sigma,2}}{2}\,, \quad K_\Sigma = \kappa_{\Sigma,1}\kappa_{\Sigma,2}\,, \quad \Sigma = 1,\ldots,n-2. \tag{2.4}$$

Definition 2.2. Let the immersion $X \in \mathfrak{C}(B,\mathbb{R}^n)$ be given with an ON-normal section $\{N_1,\ldots,N_{n-2}\}$. The *Gaussian curvature* of $X = X(u,v)$ is defined by

$$K(u,v) := \sum_{\Sigma=1}^{n-2} K_\Sigma(u,v), \quad (u,v) \in B. \tag{2.5}$$

Remark 2.3. 1. Similarly to the proof of the invariance of the mean curvature system w.r.t. changes of the ON-normal section in 1.2, one can show the invariance of the Gauss curvature $K = K(u,v)$.

2. For minimal surfaces we have $K_\Sigma \le 0$ for all $\Sigma = 1,\ldots,n-2$, therefore $K \le 0$.

3. Up to sign, $K = K(u,v)$ is the non-trivial component of the Riemannian curvature tensor

$$R_{nijk} = R^\ell_{ijk}g_{\ell n} = \big(\Gamma^\ell_{ij,u^k} - \Gamma^\ell_{ik,u^j} + \Gamma^m_{ij}\Gamma^\ell_{mk} - \Gamma^m_{ik}\Gamma^\ell_{mj}\big)g_{\ell n} \tag{2.6}$$

with the Christoffel symbold Γ^k_{ij} defined in (2.11). In particular, evaluating the tangent components of $X_{u^iuv} - X_{u^ivu} = 0$ yields

$$R_{2112} = \sum_{\Sigma=1}^{n-2} K_\Sigma W^2\,. \tag{2.7}$$

This is the so-called theorema egregium.

2.2. The differential equations. We want to express N_{Σ,u^i} and $X_{u^iu^j}$ in terms of the moving frame $\{X_u, X_v, N_1, \ldots, N_{n-2}\}$.

Proposition 2.4. *Let $X \in \mathfrak{C}(B, \mathbb{R}^n)$ be given with an ON-normal section $\{N_1, \ldots, N_{n-2}\}$. Then there hold the Weingarten equations*

$$N_{\Sigma,u^i} = -L_{\Sigma,ij} g^{jk} X^t_{u^k} + \sigma^{\Theta}_{\Sigma,i} N_{\Theta}\,, \quad i = 1, 2,\ \Sigma = 1, \ldots, n-2, \tag{2.8}$$

with the torsion coefficients

$$\sigma^{\Theta}_{\Sigma,i} := \begin{cases} N_{\Sigma,u^i} \cdot N^t_{\Theta} & \text{if } \Sigma \neq \Theta \\ 0 & \text{if } \Sigma = \Theta\,, \end{cases} \tag{2.9}$$

as well as the Gauss equations

$$X_{u^iu^j} = \Gamma^k_{ij} X_{u^k} + \sum_{\Sigma=1}^{n-2} L_{\Sigma,ij} N_{\Sigma}\,, \quad i, j = 1, 2, \tag{2.10}$$

with the Christoffel symbols

$$\Gamma^k_{ij} := \frac{1}{2}\, g^{k\ell}(g_{j\ell,i} + g_{\ell i,j} - g_{ij,\ell}), \quad g_{ij,k} := g_{ij,u^k}\,. \tag{2.11}$$

For the proofs of these equations we refer to [3].

The $\sigma^{\Omega}_{\Sigma,i} = \sigma^{\Omega}_{\Sigma,i}(u, v)$ are also called the *coefficients of the normal connection.* This notation becomes clear from the next result.

Corollary 2.5. *Let the immersion $X \in \mathfrak{C}(B, \mathbb{R}^n)$ be given with an ON-normal section $\{N_1, \ldots, N_{n-2}\}$. Then there hold the Ricci equations*

$$\sigma^{\Omega}_{\Sigma,2,u} - \sigma^{\Omega}_{\Sigma,1,v} + \sigma^{\Theta}_{\Sigma,2}\sigma^{\Omega}_{\Theta,1} - \sigma^{\Theta}_{\Sigma,1}\sigma^{\Omega}_{\Theta,2} = (L_{\Sigma,2j}L_{\Omega,k1} - L_{\Sigma,1j}L_{\Omega,k2})g^{jk} \tag{2.12}$$

for $\Sigma, \Omega = 1, \ldots, n-2$.

These identities follow by evaluating $N_{\Sigma,uv} - N_{\Sigma,vu} = 0$ for $\Sigma = 1, \ldots, n-2$ (see, e.g., [3]). Note the similarity of the left hand side in (2.12) with the Riemannian curvature tensor in (2.6).

Definition 2.6. The normal curvature tensor of $\{N_1, \ldots, N_{n-2}\}$ is given by

$$S^{\Omega}_{\Sigma,ij} := \sigma^{\Omega}_{\Sigma,i,u^j} - \sigma^{\Omega}_{\Sigma,j,u^i} + \sigma^{\Theta}_{\Sigma,i}\sigma^{\Omega}_{\Theta,j} - \sigma^{\Theta}_{\Sigma,j}\sigma^{\Omega}_{\Theta,i}\,. \tag{2.13}$$

Remark 2.7. Consider the two-dimensional plane $\sigma := \mathrm{Span}\,\{N_{\Sigma}, N_{\Omega}\} \subset \mathbb{R}^n$. Then $S_{ij}(\sigma) := S^{\Omega}_{\Sigma,ij}$ is invariant w.r.t. changes of the orthonormal basis of σ. Thus $S_{ij}(\sigma)$ represents a sectional curvature in the normal bundle.

2.3. The mean curvature system. Using conformal parameters $(u, v) \in B$, the Christoffel symbols satisfy $\Gamma_{11}^1 + \Gamma_{22}^1 = 0$, $\Gamma_{11}^2 + \Gamma_{22}^2 = 0$. Together with (2.2) and (2.10) we calculate

$$\begin{aligned}\triangle X &= (\Gamma_{11}^1 + \Gamma_{22}^1) X_u + (\Gamma_{11}^2 + \Gamma_{22}^2) X_v + \sum_{\Sigma=1}^{n-2} (L_{\Sigma,11} + L_{\Sigma,22}) N_\Sigma \\ &= 2 \sum_{\Sigma=1}^{n-2} H_\Sigma W N_\Sigma \,. \end{aligned} \tag{2.14}$$

This is exactly the mean curvature system from (1.10).

Corollary 2.8. *Under the assumptions* $|H_1|, \ldots, |H_{n-2}| \leq h_0$ *with a real constant* $h_0 \in [0, +\infty)$ *there holds the estimate*

$$|\triangle X| \leq 2(n-2) h_0 |X_u||X_v| \leq (n-2) h_0 |\nabla X|^2 \quad \text{in } B. \tag{2.15}$$

The quadratic growth in the gradient allows, e.g., the following enclosure principle (see [5]).

Corollary 2.9. *Let the conformally parametrized immersion* $X \in \mathfrak{C}(B, \mathbb{R}^n)$ *be given with an ON-normal section* $\{N_1, \ldots, N_{n-2}\}$. *Let* $|H_1|, \ldots, |H_{n-2}| \leq h_0$ *for the associated mean curvature field of* $X = X(u, v)$ *with a real constant* h_0 *such that*

$$0 \leq h_0 \sup_{(u,v) \in B} |X(u, v)| \leq \frac{1}{n-2} \,. \tag{2.16}$$

Then it holds

$$\sup_{(u,v) \in B} |X(u, v)|^2 = \sup_{(u,v) \in \partial B} |X(u, v)|^2 \tag{2.17}$$

where we set $\partial B := \{(u, v) \in \mathbb{R}^2 \,:\, u^2 + v^2 = 1\}$.

Proof. Using (2.15) and (2.16) we estimate

$$\triangle |X|^2 = 2|\nabla X|^2 + 2\triangle X \cdot X^t \geq 2 \left\{1 - (n-2)|X| h_0\right\} |\nabla X|^2 \geq 0. \tag{2.18}$$

The maximum principle yields the statement. □

2.4. An example from the calculus of variations. We want to discuss a variational problem (see [2] for $n = 3$) whose critical points $X = X(u, v)$ satisfy the above mean curvature system together with the continuity assumptions (1.11).

Proposition 2.10. *Let the conformally parametrized immersion* $X \in \mathfrak{C}(B, \mathbb{R}^n)$ *be critical for*

$$\mathcal{F}[X] := \iint_B \Gamma(X) W \, du \, dv \tag{2.19}$$

with a positive weight function $\Gamma \in C^{1+\alpha}(\mathbb{R}^n, \mathbb{R})$*. Let* $\{N_1, \ldots, N_{n-2}\}$ *be an ON-normal section. Then* $X = X(u, v)$ *satisfies the mean curvature system*

$$\triangle X = 2 \sum_{\Sigma=1}^{n-2} H_\Sigma W N_\Sigma \quad \text{in } B \tag{2.20}$$

with the mean curvatures

$$H_\Sigma = H(X, N_\Sigma) = \frac{\Gamma_X(X) \cdot N_\Sigma^t}{2\Gamma(X)}, \quad \Sigma = 1, \ldots, n-2. \tag{2.21}$$

Proof. (i) We introduce the unit normal field

$$\mathfrak{N}(u, v) := \sum_{\Sigma=1}^{n-2} \gamma^\Sigma(u, v) N_\Sigma(u, v), \quad \sum_{\Sigma=1}^{n-2} (\gamma^\Sigma)^2 = 1, \tag{2.22}$$

with coefficients $\gamma^\Sigma \in C^{2+\alpha}(\overline{B}, \mathbb{R})$ and consider the variation $\widetilde{X}(u, v) = X(u, v) + \varepsilon\varphi(u, v)\mathfrak{N}(u, v)$ with a test function $\varphi \in C_0^\infty(B, \mathbb{R})$ and $\varepsilon \in (-\varepsilon_0, +\varepsilon_0)$. We calculate

$$\widetilde{X}_u = X_u + \varepsilon\varphi_u\mathfrak{N} + \varepsilon\varphi\mathfrak{N}_u\,, \quad \widetilde{X}_v = X_v + \varepsilon\varphi_v\mathfrak{N} + \varepsilon\varphi\mathfrak{N}_v\,, \tag{2.23}$$

and therefore

$$\begin{aligned} \widetilde{X}_u^2 &= W + 2\varepsilon\varphi\, X_u \cdot \mathfrak{N}_u^t + o(\varepsilon) \\ \widetilde{X}_v^2 &= W + 2\varepsilon\varphi\, X_v \cdot \mathfrak{N}_v^t + o(\varepsilon) \\ \widetilde{X}_u \cdot \widetilde{X}_v^t &= \varepsilon\big\{X_u \cdot \mathfrak{N}_v^t + X_v \cdot \mathfrak{N}_u^t\big\}\, \varphi + o(\varepsilon). \end{aligned} \tag{2.24}$$

(ii) We define the forms

$$\mathfrak{L}_{ij} := X_{u^iu^j} \cdot \mathfrak{N}^t = -X_{u^i} \cdot \mathfrak{N}_{u^j}^t = -X_{u^j} \cdot \mathfrak{N}_{u^i}^t\,, \quad i, j = 1, 2. \tag{2.25}$$

Note that $\mathfrak{N}_{u^i} \cdot \mathfrak{N}^t = 0$ due to $\mathfrak{N}^2 = 1$, and $X_{u^i} \cdot \mathfrak{N}_{u^j}^t = -X_{u^iu^j} \cdot \mathfrak{N}^t$ in view of $X_{u^i} \cdot \mathfrak{N}^t = 0$. Furhermore, it holds $X_u \cdot \mathfrak{N}_v^t = X_v \cdot \mathfrak{N}_u^t$. Then, (2.24) can be written in the form

$$\begin{aligned} \widetilde{X}_u^2 &= W - 2\varepsilon\varphi\mathfrak{L}_{11} + o(\varepsilon) \\ \widetilde{X}_v^2 &= W - 2\varepsilon\varphi\mathfrak{L}_{22} + o(\varepsilon) \\ \widetilde{X}_u \cdot \widetilde{X}_v^t &= -\, 2\varepsilon\varphi\mathfrak{L}_{12} + o(\varepsilon) \end{aligned} \tag{2.26}$$

which yields the variation formula

$$\delta g_{11} = -2\varphi\mathfrak{L}_{11}\,, \quad \delta g_{12} = -2\varphi\mathfrak{L}_{12}\,, \quad \delta g_{22} = -2\varphi\mathfrak{L}_{22}\,. \tag{2.27}$$

From $2W\delta W = g_{22}\,\delta g_{11} - 2g_{12}\,\delta g_{12} + g_{11}\,\delta g_{22}$ we obtain

$$\delta W = \frac{1}{2}\,\delta g_{11} + \frac{1}{2}\,\delta g_{22} = -\left\{\mathfrak{L}_{11} + \mathfrak{L}_{22}\right\}\varphi = -\,2\mathfrak{H}W\varphi \tag{2.28}$$

with the mean curvature field $\mathfrak{H} := \frac{1}{2}\,\mathfrak{L}_{ij}g^{ij}$.

(iii) Together with $\delta\Gamma(X) = \Gamma_X(X)\cdot\mathfrak{N}^t\,\varphi$ we infer

$$\delta\,\mathcal{F}[X] = \iint\limits_B \left\{\Gamma_X(X)\cdot\mathfrak{N}^t - 2\Gamma(X)\mathfrak{H}(X,\mathfrak{N})\right\}W\varphi\,du\,dv \tag{2.29}$$

for all $\varphi \in C_0^\infty(B,\mathbb{R})$. Then $\delta\mathcal{F}[X] = 0$ gives

$$\mathfrak{H}(X,\mathfrak{N}) = \frac{\Gamma_X(X)\cdot\mathfrak{N}^t}{2\Gamma(X)}, \tag{2.30}$$

where $\gamma^\Sigma = \gamma^\Sigma(u,v)$ is chosen arbitrarily.

(iv) Let $\gamma^\Sigma \equiv 1$ for any $\Sigma \in \{1,\ldots,n-2\}$ and $\gamma^\Omega \equiv 0$ for all $\Omega \neq \Sigma$. Then $\mathfrak{N} = N_\Sigma$, $\mathfrak{H} = H_\Sigma$, such that (2.20) follows. □

Remark 2.11. If we endow $\mathbb{R}^n$ with the Riemannian metric

$$ds^2 := \Gamma(x_1,\ldots,x_n)\left\{(dx^1)^2 + \cdots + (dx^n)^2\right\}, \tag{2.31}$$

then $\mathcal{F}[X]$ measures the area of an immersion $X = X(u,v)$ in the Riemannian space $(\mathbb{R}^n, ds^2)$. Thus, minimal surfaces in this Riemannian space are surfaces with mean curvature field $\mathfrak{H}(X,\mathfrak{N})$ from (2.30) in $\mathbb{R}^n$. An example of such a space is obtained by stereographic projection of the sphere S^n into $\mathbb{R}^n$, where

$$\Gamma(X) = \frac{4}{(1+|X|^2)^2}\,, \quad X = (x^1,\ldots,x^n). \tag{2.32}$$

2.5. An example: The holomorphic graph (w, w^2). Let us consider the graph

$$\begin{gathered} X(u,v) := (u, v, u^2 - v^2, 2uv), \\ (u,v) \in \overline{B}_R := \left\{(u,v)\in\mathbb{R}^2 \,:\, u^2+v^2 \le R^2\right\} \end{gathered} \tag{2.33}$$

in $\mathbb{R}^4$ which is generated by the holomorphic function $\Phi(w) := w^2$, $w = u + iv$.

Note that this graph can be extended to a graph over the whole plane $\mathbb{R}^2$ but it is *not* a plane. We show that (A1) and (A2) of our Theorem are satisfied, while (A3) holds with an angle $\omega = \omega(R)$ such that $\omega(R) \to 0$ for $R \to \infty$. Therefore, we cannot apply Corollary 1.

Statement 1. *The graph is a conformally parametrized minimal graph over $\overline{B}_R$. Furthermore, it can be extended to a complete and non-linear minimal graph over $\mathbb{R}^2$. In particular, it holds $H(X,Z) \equiv 0$ for the mean curvature field.*

Statement 2. *The unit vectors*

$$\begin{aligned} N_1 &:= \frac{1}{\sqrt{1+4u^2+4v^2}}\,(-2u, 2v, 1, 0) \\ N_2 &:= \frac{1}{\sqrt{1+4u^2+4v^2}}\,(-2v, -2u, 0, 1) \end{aligned} \tag{2.34}$$

form an ON-normal section $\{N_1, N_2\}$ *over* $\overline{B}_R$.

Let $c(t) = (u(t), v(t)) \subset \overline{B}_R$, $t \in [0, T]$, denote a continuously differentiable curve such that

(i) $c(0) = (0, 0)$ and $c(T) \in \partial B_R$;

(ii) $|\dot{c}(t)|^2 = 1$ for all $t \in [0, T]$ where the dot denotes differentiation w.r.t. the variable t.

Denote the class of all these curves by $\mathfrak{D}$.

Statement 3. *It holds*

$$\text{Area}[\mathfrak{B}_r(X_0)] \le 192\pi r^2 \quad \textit{for large } r > 0,\ X_0 = (0, \dots, 0). \tag{2.35}$$

Proof. We give a lower bound for the length $\mathcal{L}[c]$ of the image curve $X \circ c(t)$ for any $c \in \mathfrak{D}$ on the surface. Assume w.l.o.g. that $|u(T)| \ge |v(T)|$, in particular $|u(T)| \ge \frac{R}{2}$. Define $t^* := \sup\left\{t \in (0, T) \,:\, |u(t)| \le \frac{R}{4}\right\}$. Note that $T - t^* \ge \frac{R}{4}$ because of the arc length parametrization. Then we estimate as follows:

$$\mathcal{L}[c] = \int\limits_0^T \sqrt{1+4u^2+4v^2}\,dt \ge \int\limits_{t^*}^T \sqrt{1+4u^2}\,dt \ge \frac{R}{2}\,(T - t^*) \ge \frac{R^2}{8}\,. \tag{2.36}$$

In the same way we treat the case $|v(T)| \ge |u(T)|$.

Now if we define $r := \min_{c\in\mathfrak{D}} \mathcal{L}[c]$, then the geodesic disc $\mathfrak{B}_r(X_0)$, $X_0 := (0, \dots, 0)$, projects into $\overline{B}_R$. Using (2.36) we estimate the area of this geodesic disc:

$$\text{Area}[\mathfrak{B}_r(X_0)] \le \iint\limits_{B_R} (1 + 4u^2 + 4v^2)\,du\,dv \le 192\pi r^2, \quad \forall R \ge 1 \tag{2.37}$$

since $\iint_{B_R}(1+4u^2+4v^2)\,du\,dv = \int_0^{2\pi}\int_0^R (1+4\varrho^2)\varrho\,d\varrho\,d\varphi = 2\pi(\frac{R^2}{2} + R^4) \le 3\pi R^4$ for all $R \ge 1$. $\square$

Thus, Assumption (A2) is satisfied at least for large $r > 0$.

Statement 4. (A3) *does not hold for* $R \to \infty$.

Proof. The condition that each normal vector N makes an angle of at least $\omega > 0$ with the x_1-axis means $|N \cdot (1, 0, \ldots, 0)^t| \le \cos\omega < 1$. But consider N_1 from (2.34) for $v = 0$, then

$$|N_1 \cdot (1, 0, \ldots, 0)^t| = \frac{2|u|}{\sqrt{1 + 4u^2}} \longrightarrow 1 \quad \text{for } |u| \to \infty \tag{2.38}$$

which proves the statement. □

3. Proof of the main theorem

We set $H_\Sigma(X) \equiv H(X, N_\Sigma)$. At first it holds

$$\begin{aligned} \kappa_{\Sigma,1}(0,0)^2 + \kappa_{\Sigma,2}(0,0)^2 &= 4H_\Sigma(0,0)^2 - 2K_\Sigma(0,0) \\ &\le 4h_0^2 + 2|K_\Sigma(0,0)| \\ &= \frac{1}{r^2}\left\{(2h_0 r)^2 + r^2|K_\Sigma(0,0)|\right\} \end{aligned} \tag{3.1}$$

for $\Sigma = 1, \ldots, n-2$. The desired curvature bound follows from an estimate of

$$K_\Sigma(0,0) = \frac{(X_{uu} \cdot N_\Sigma)(X_{vv} \cdot N_\Sigma) - (X_{uv} \cdot N_\Sigma)^2}{W^2}\Big|_{(0,0)}. \tag{3.2}$$

This means that (i) we have to find a lower bound for the area element, and (ii) we have to establish an upper bound for the second derivatives of the immersion.

1. In the first part we will prove the estimate

$$\frac{W(w)}{r^2} \ge C_1 \quad \text{for } w \in B_{\frac{1}{2}}(0,0) \tag{3.3}$$

with a constant $C_1 = C_1(h_0 r, d_0, \sin\omega, n) > 0$.

1.1. Due to the graph property (A1) it is not difficult to find a global ON-normal section $\{N_1, \ldots, N_{n-2}\}$ on the surface: Note that the vectors $e_3 := (0, 0, 1, 0, \ldots, 0), \ldots, e_n := (0, \ldots, 0, 1)$ are not in any tangent plane of the surface beause

$$\begin{gathered} \frac{1}{\sqrt{1 + |\nabla\varphi_1|^2}}(-\varphi_{1,x}, -\varphi_{1,y}, 1, 0, \ldots, 0), \\ \vdots \qquad \vdots \qquad \vdots \\ \frac{1}{\sqrt{1 + |\nabla\varphi_{n-2}|^2}}(-\varphi_{n-2,x}, -\varphi_{n-2,y}, 0, \ldots, 0, 1) \end{gathered} \tag{3.4}$$

are normal to the surface and their inner products with e_Σ do not vanish. Therefore, the projections

$$\begin{aligned} N_1^* &:= e_3 - \frac{e_3 \cdot X_u^t}{|X_u|^2}\, X_u - \frac{e_3 \cdot X_v^t}{|X_v|^2}\, X_v \\ N_2^* &:= e_4 - \frac{e_4 \cdot X_u^t}{|X_u|^2}\, X_u - \frac{e_4 \cdot X_v^t}{|X_v|^2}\, X_v\,, \quad \ldots \end{aligned} \tag{3.5}$$

can be transformed into an ON-normal section $\{N_1, \ldots, N_{n-2}\}$. In the first part of the proof we will work with this section.

1.2. Using conformal parameters $(u, v) \in B$ it holds

$$\begin{aligned} \triangle X &= 2H(X, N_1)WN_1 + 2H(X, N_2)WN_2 + \ldots \\ &\quad + 2H(X, N_{n-2})WN_{n-2} \quad \text{in } B. \end{aligned} \tag{3.6}$$

From (1.11) we infer the estimate

$$|\triangle X(u, v)| \leq (n-2)h_0 |\nabla X(u, v)|^2 \quad \text{in } B. \tag{3.7}$$

The special structure of this differential inequality – the quadratic growth in the gradient – enables us to apply the methods of [6].

We cite two important consequences of our assumptions.

1.3. Assumption (A2) yields: Let $\Gamma(B)$ be the set of all continuous and piecewise differentiable curves $\gamma : [0, 1] \to \overline{B}$, such that $\gamma(0) = (0, 0)$ and $\gamma(1) \in \partial B$. Then (see [12])

$$\inf_{\gamma \in \Gamma(B)} \int_0^1 \left| \frac{d}{dt} X \circ \gamma(t) \right| dt \geq r. \tag{3.8}$$

1.4. Assumption (A3) gives $|\nabla x^1|^2 \geq W \sin^2 \omega$ in B. The proof can be taken from [11, Lemma 1.1], where the author makes essential use of the conformal representation of the surface.

For the estimate of the area element we define several auxiliary functions and apply Heinz' results on elliptic systems in $\mathbb{R}^2$ from [6].

1.5. We denote by $F^*(u, v) := (x^1(u, v), x^2(u, v)) : \overline{B} \to \mathbb{R}^2$ the plane mapping w.r.t. $X = X(u, v)$. Then we have

(i) $|\triangle F^*(w)| \leq \dfrac{4h_0}{\sin^2 \omega}\, |\nabla F^*(w)|^2$ for all $w \in B$

because we estimate $|\triangle F^*| \leq |\triangle X| \leq (n-2)h_0 |\nabla X|^2 = 2(n-2)h_0 W \leq \frac{4h_0}{\sin^2 \omega}\, |\nabla x^1|^2 \leq \frac{4h_0}{\sin^2 \omega}\, |\nabla F^*|^2$;

(ii) $|\nabla X(w)|^2 \le \dfrac{2}{\sin^2 \omega} |\nabla F^*(w)|^2$ for all $w \in B$

which follows from $|\nabla X|^2 = 2W \le \frac{2}{\sin^2 \omega} |\nabla x^1|^2 \le \frac{2}{\sin^2 \omega} |\nabla F^*|^2$.

1.6. Let $w_0 \in B$ and $\nu \in (0,1)$ be given such that $B_{2\nu}(w_0) := \{w \in B : |w - w_0| < 2\nu\} \subset B$. We consider the mapping $Y(w) := \frac{1}{r}\{X(w_0 + 2\nu w) - X(w_0)\}$, $w \in \overline{B}$, and the corresponding plane mapping $F(w) := (y^1(w), y^2(w)) : \overline{B} \to \mathbb{R}^2$. The immersion $Y = Y(w)$ satisfies

$$|Y_u(w)|^2 = \frac{4\nu^2}{r^2} W(w_0 + 2\nu w) = |Y_v(w)|^2, \quad Y_u(w) \cdot Y_v(w)^t = 0 \tag{3.9}$$

and due to (3.7)

$$|\triangle Y(w)| \le (n-2)(h_0 r)|\nabla Y(w)|^2 \quad \text{in } B. \tag{3.10}$$

1.7. Together with 1.5 (ii) we infer

$$\begin{aligned} |\triangle F(w)| &\le |\triangle Y(w)| \le (n-2)(h_0 r)|\nabla Y(w)|^2 \\ &= \frac{8(n-2)\nu^2(h_0 r)}{r^2} W(w_0 + 2\nu w) \\ &= \frac{4(n-2)\nu^2(h_0 r)}{r^2} |\nabla X(w_0 + 2\nu w)|^2 \\ &\le \frac{8(n-2)\nu^2(h_0 r)}{r^2 \sin^2 \omega} |\nabla F^*(w_0 + 2\nu w)|^2 \\ &= \frac{8(n-2)\nu^2(h_0 r)}{r^2 \sin^2 \omega} \frac{r^2}{4\nu^2} |\nabla F(w)|^2 \\ &\le \frac{2(n-2)(h_0 r)}{\sin^2 \omega} |\nabla F(w)|^2 \end{aligned} \tag{3.11}$$

for all $w \in B$. Furthermore, from (A1) we infer that $F = F(u,v)$ is one-to-one and has positive Jacobian $J_F(w) > 0$ for all $w \in B$. Additionally, assumption (A2) gives

$$\mathcal{D}[F] \le \mathcal{D}[Y] \le \frac{1}{r^2} \mathcal{D}[X] \le 2d_0\,. \tag{3.12}$$

with the Dirichlet energy

$$\mathcal{D}[Z] := \iint\limits_B \left\{|Z_u|^2 + |Z_v|^2\right\} du\, dv \tag{3.13}$$

We apply [6, Theorem 6, p. 254] to obtain the following inner gradient estimate: There is a constant $c_1 = c_1(h_0 r, d_0, \sin \omega, n) \in (0, +\infty)$ such that

$$|\nabla F(u,v)| \le c_1(h_0 r, d_0, \sin \omega, n) \quad \forall\, (u,v) \in B_{\frac{1}{2}}(0,0). \tag{3.14}$$

1.8. From 1.7, (3.11) we get

$$\begin{aligned} \frac{1}{r^2}\, W(w_0+2\nu w) \leq\ & \frac{1}{4\nu^2\sin^2\omega}\, c_1(h_0 r, d_0, \sin\omega, n) \\ =:\ & c_2(h_0 r, d_0, \sin\omega, \nu, n) \end{aligned} \tag{3.15}$$

for all $w \in B$. In particular, we arrive at

$$\frac{1}{r^2}\, W(w) \leq c_2(h_0 r, d_0, \sin\omega, \nu, n) \quad \forall\, w \in B_{\frac{1}{2}}(0,0). \tag{3.16}$$

This estimate will be used in the second part of the proof.

1.9. Because $J_F(w) > 0$ and $\mathcal{D}[F] \leq 2d_0$ we can apply [6, Lemma 17, p. 255]: There exists a constant $c_3 = c_3(h_0 r, d_0, \sin\omega, n) \in (0, +\infty)$ such that

$$|\nabla F(w)|^2 \leq c_3(h_0 r, d_0, \sin\omega, n) |\nabla F(0,0)|^{\frac{2}{5}} \tag{3.17}$$

for all $w \in B_{\frac{1}{2}}(0,0)$. It follows

$$\begin{aligned} \frac{4\nu^2}{r^2}\, W(w_0+2\nu w) &\leq \frac{1}{\sin^2\omega}\, |\nabla F(w)|^2 \\ &\leq \frac{c_3(h_0 r, d_0, \sin\omega, n)}{\sin^2\omega}\, |\nabla F(0,0)|^{\frac{2}{5}} \\ &\leq \frac{c_3(h_0 r, d_0, \sin\omega, n)}{\sin^2\omega}\, |\nabla Y(0,0)|^{\frac{2}{5}} \\ &= \frac{c_3(h_0 r, d_0, \sin\omega, n)}{\sin^2\omega} \left[\frac{8\nu^2}{r^2}\, W(w_0)\right]^{\frac{1}{5}}. \end{aligned} \tag{3.18}$$

Rearranging yields an inequality of Harnack type

$$\left[\frac{W(w_0)}{r^2}\right]^{\frac{1}{5}} \geq \frac{4 \cdot 8^{-\frac{1}{5}} \nu^{\frac{8}{5}} \sin^2\omega}{c_3(h_0 r, d_0, \sin\omega)}\, \frac{W(w_0+2\nu w)}{r^2} \tag{3.19}$$

for all $w \in B_{\frac{1}{2}}(0,0)$, or equivalently

$$c_4(h_0 r, d_0, \sin\omega, \nu, n) \left[\frac{W(w)}{r^2}\right]^5 \leq \frac{W(w_0)}{r^2} \tag{3.20}$$

for all $w \in B_\nu(w_0)$ with the constant $c_4(h_0 r, d_0, \sin\omega, \nu, n) := \frac{2^7 \nu^8 \sin^{10}\omega}{c_3(h_0 r, d_0, \sin\omega, n)^5} \in (0, +\infty)$.

1.10. (A2) also ensures that we can estimate the area element in at least one point: There is a point $w^* \in B_{1-\nu_0}(0,0)$, $\nu_0 := \min(e^{-4\pi d_0}, \frac{1}{2})$ such that

$$\frac{W(w^*)}{r^2} \geq \frac{1}{4(1-e^{-4\pi d_0})} =: c_5(d_0) > 0\,. \tag{3.21}$$

The constant ν_0 comes from an application of the Courant–Lebesgue lemma.

1.11. We now establish an estimate of the area element: Set $\nu := \frac{1}{2}\nu_0 \in (0, \frac{1}{4}]$ and choose an integer $m = m(\nu) \in \mathbb{N}$ such that $1 - 2\nu \leq \frac{m}{2}\,\nu \leq 1 - \nu$. For an arbitrary $w_0 \in B_{1-\nu_0}(0,0)$ we define the points

$$w_j := \frac{j}{m}\,w^* + \frac{m-j}{m}\,w_0 \quad \text{for } j = 0, \ldots, m \tag{3.22}$$

with $w^* \in B_{1-\nu_0}(0,0)$ from 1.10. We have

$$|w_j| \leq \frac{j}{m}\,|w^*| + \frac{m-j}{m}\,|w_0| < 1 - \nu_0 \tag{3.23}$$

and therefore $B_{2\nu}(w_j) = B_{\nu_0}(w_j) \subset B$. Furthermore, it holds

$$|w_{j+1} - w_j| = \Big| \frac{1}{m}\,w^* - \frac{1}{m}\,w_0 \Big| \leq \frac{1}{m}\,|w^* - w_0| \leq \frac{2(1-\nu_0)}{m} \leq \nu. \tag{3.24}$$

This implies $w_{j+1} \in B_\nu(w_j)$ for $j = 0, \ldots, m-1$.

We apply the Harnack inequality from 1.9, (3.20) and obtain

$$\begin{aligned} l\frac{W(w_0)}{r^2} &\geq c_4 \left[\frac{W(w_1)}{r^2}\right]^5 \geq c_4^{1+5} \left[\frac{W(w_2)}{r^2}\right]^{5^2} \geq \cdots \\ &\cdots \geq c_4^{1+5+5^2+\cdots+5^{m-1}} \left[\frac{W(w_m)}{r^2}\right]^{5^m}. \end{aligned} \tag{3.25}$$

Recall that $w_m = w^*$, and 1.10, (3.21) gives

$$\frac{W(w_0)}{r^2} \geq c_4^{1+5+5^2+\cdots+5^{m-1}} c_5(d_0)^{5^m} =: C_1(h_0 r, d_0, \sin\omega, n) > 0 \tag{3.26}$$

for all $w_0 \in B_{1-\nu_0}(0,0)$. From $\nu_0 \leq \frac{1}{2}$ we conclude

$$\frac{W(w)}{r^2} \geq C_1(h_0 r, d_0, \sin\omega, n) \quad \forall\ w \in B_{\frac{1}{2}}(0,0)\,. \tag{3.27}$$

This completes the first part of the proof.

2. In the second part we estimate the second derivatives of $X = X(u,v)$ using $\triangle X = 2H(X,N_1)WN_1 + \cdots + 2H(X,N_{n-2})WN_{n-2}$. In particular, we have to give Hölder estimates of the right hand side of this equation. We will construct an adequate orthonormal section $\{N_1, \ldots, N_{n-2}\}$ of the normal space.

2.1. Define the auxiliary function

$$Z(u,v) = \frac{1}{r}\,\{X(u,v) - X(0,0)\} = \frac{1}{r}\,X(u,v), \quad (u,v) \in \overline{B}. \tag{3.28}$$

Denoting by W_Z the area element of Z, we have $|Z_u|^2 = W_Z = |Z_v|^2$ and $Z_u \cdot Z_v^t = 0$ in B. It holds $r^2 W_Z = W_X$ with $W_X := |X_u|^2 = |X_v|^2$. We calculate

$$\begin{aligned} \triangle Z &= \frac{2}{r} H(X, N_1) W_X N_1 + \cdots + \frac{2}{r} H(X, N_{n-2}) W_X N_{n-2} \\ &= 2rH(rZ, N_1) W_Z N_1 + \cdots + 2rH(rZ, N_{n-2}) W_Z N_{n-2} \,. \end{aligned} \tag{3.29}$$

2.2. Due to 1.8, (3.16) we have the estimate

$$|\triangle Z(w)| \leq 2(n-2)(rh_0) c_2(h_0 r, d_0, \sin\omega, n) \quad \forall w \in B_{\frac{1}{2}}(0,0). \tag{3.30}$$

Furthermore, we get

$$\begin{aligned} |Z(u,v)| = |Z(u,v) - Z(0,0)| &\leq 2 \max_{w \in B_{\frac{1}{2}}(0,0)} |\nabla Z(w)| \\ &\leq 2\sqrt{2c_2(h_0 r, d_0, \sin\omega)} \quad \text{in } B_{\frac{1}{2}}(0,0). \end{aligned} \tag{3.31}$$

Potential theoretic estimates yield a constant $c_6(h_0 r, d_0, \sin\omega, n, \alpha)$ such that

$$\begin{gathered} |Z_{u^i}(w_1) - Z_{u^i}(w_2)| \leq c_6(h_0 r, d_0, \sin\omega, n, \alpha) |w_1 - w_2|^\alpha \\ \forall\, w_1, w_2 \in B_{\frac{1}{4}}(0,0)\,,\ \forall\ \alpha \in (0,1), \end{gathered} \tag{3.32}$$

where $u^1 = u$, $u^2 = v$ (see, e.g., [16, Chapter XII, §2]). Therefore

$$\begin{gathered} |W_Z(w_1) - W_Z(w_2)| \leq c_7(h_0 r, d_0, \sin\omega, n, \alpha) |w_1 - w_2|^\alpha \\ \forall\, w_1, w_2 \in B_{\frac{1}{4}}(0,0) \end{gathered} \tag{3.33}$$

with the constant $c_7 := 4\sqrt{c_2}\, c_6$.

2.3. Using the mean value theorem we have the Lipschitz estimate

$$\begin{gathered} |Z(w_1) - Z(w_2)| \leq 4\sqrt{2c_2(h_0 r, d_0, \sin\alpha)}\, |w_1 - w_2| \\ \forall\, w_1, w_2 \in B_{\frac{1}{2}}(0,0). \end{gathered} \tag{3.34}$$

In a certain neighborhood of the origin we construct an ON-normal section $\{N_1, \ldots, N_{n-2}\}$ whose Hölder norm can be estimated.

2.4. We choose unit vectors $\overline{N}_1, \ldots, \overline{N}_{n-2} \in \mathbb{R}^n$ such that

$$\overline{N}_\Sigma \cdot Z_{u^j}(0,0)^t = 0, \quad \overline{N}_\Sigma \cdot \overline{N}_\Omega^t = \delta_{\Sigma\Omega}\,, \quad j = 1,2,\ \Sigma, \Omega = 1, \ldots, n-2, \tag{3.35}$$

and define vectors

$$N_\Sigma^*(w) := \overline{N}_\Sigma - \frac{\overline{N}_\Sigma \cdot Z_u(w)^t}{|Z_u(w)|^2} Z_u(w) - \frac{\overline{N}_\Sigma \cdot Z_v(w)^t}{|Z_v(w)|^2} Z_v(w) \quad \text{in } B. \tag{3.36}$$

2.5. These vectors belong to the normal space at $Z(w)$ but they may not be linearly independent. We now determine a $\nu_1 = \nu_1(h_0 r, d_0, \sin\omega, n, \alpha) > 0$ such that

$$|N_\Sigma^*(w)|^2 = 1 - \frac{[\overline{N}_\Sigma \cdot Z_u(w)^t]^2}{W_Z(w)} - \frac{[\overline{N}_\Sigma \cdot Z_v(w)^t]^2}{W_Z(w)} \geq \frac{1}{2} \quad \text{in } B_{\nu_1}(0,0). \tag{3.37}$$

Namely, using (3.35) and 2.2, (3.32) we calculate

$$\begin{aligned} |\overline{N}_\Sigma \cdot Z_{u^\ell}(w)^t|^2 &= |\overline{N}_\Sigma \cdot \{Z_{u^\ell}(w) - Z_{u^\ell}(0,0)\}^t|^2 \\ &\leq |Z_{u^\ell}(w) - Z_{u^\ell}(0,0)|^2 \\ &\leq c_6(h_0 r, d_0, \sin\omega, n, \alpha)^2 |w|^{2\alpha}, \end{aligned} \tag{3.38}$$

for $\ell = 1, 2$, $\Sigma = 1, \ldots, n-2$, and from 1.11, (3.26) we know the lower bound

$$W_Z(w) \geq C_1(h_0 r, d_0, \sin\omega, n) \quad \text{in } B_{\frac{1}{2}}(0,0). \tag{3.39}$$

Thus, (3.37) holds if $\nu_1^{2\alpha} \leq \frac{C_1}{4c_6}$.

2.6. We remark that the vectors $N_\Sigma^*(w)$, $\Sigma = 1, \ldots, n-2$, satisfy the Hölder estimate

$$\begin{gathered} |N_\Sigma^*(w_1) - N_\Sigma^*(w_2)| \leq c_8(h_0 r, d_0, \sin\omega, n, \alpha)|w_1 - w_2|^\alpha \\ \forall\, w_1, w_2 \in B_{\nu_1}(0,0) \end{gathered} \tag{3.40}$$

with a constant $c_8(h_0 r, d_0, \sin\omega, n, \alpha)$. This estimate arises from the Hölder estimate for Z_{u_j} and the lower bound of W_Z.

2.7. For $\Sigma = 1, \ldots, n-2$ we define

$$\widetilde{N}_\Sigma(w) := \frac{N_\Sigma^*(w)}{|N_\Sigma^*(w)|} \quad \text{in } B_{\nu_1}(0,0). \tag{3.41}$$

These vectors are well defined because it holds $|N_k^*(w)|^2 \geq \frac{1}{2}$ in $B_{\nu_1}(0,0)$, but they are not orthogonal. Note that

$$N_\Sigma^* \cdot N_\Omega^* = \frac{(\overline{N}_\Sigma \cdot Z_u^t)(\overline{N}_\Omega \cdot Z_u^t)}{W_Z} + \frac{(\overline{N}_\Sigma \cdot Z_v^t)(\overline{N}_\Omega \cdot Z_v^t)}{W_Z} \tag{3.42}$$

for $\Sigma \neq \Omega$ and therefore, with (3.38),

$$\begin{aligned} |\widetilde{N}_\Sigma \cdot \widetilde{N}_\Omega^t| = \frac{|N_\Sigma^* \cdot N_\Omega^*|}{|N_\Sigma^*||N_\Omega^*|} &\leq \frac{2}{C_1}\left\{|\overline{N}_\Sigma \cdot Z_u^t||\overline{N}_\Omega \cdot Z_u^t| + |\overline{N}_\Sigma \cdot Z_v^t||\overline{N}_\Omega \cdot Z_v^t|\right\} \\ &\leq \frac{4c_6^2}{C_1}|w|^{2\alpha}. \end{aligned} \tag{3.43}$$

2.8. Thus we can find a $\nu_2 = \nu_2(h_0 r, d_0, \sin\omega, n, \alpha)$ with $0 < \nu_2 \le \nu_1$ such that the following vectors (write N_Σ for $N_\Sigma(w)$ and $\widetilde{N}_\Omega$ for $\widetilde{N}_\Omega(w)$) are well-defined in $B_{\nu_2}(0,0)$:

$$\begin{aligned} N_1 &:= \widetilde{N}_1 \\ N_2 &:= \frac{\widetilde{N}_2 - \{N_1 \cdot \widetilde{N}_2^t\} N_1}{\sqrt{1 - \{N_1 \cdot \widetilde{N}_2^t\}^2}} \\ &\vdots \\ N_{n-2} &:= \frac{\widetilde{N}_{n-2} - \{N_1 \cdot \widetilde{N}_{n-2}^t\} N_1 - \ldots - \{N_{n-3} \cdot \widetilde{N}_{n-2}^t\} N_{n-3}}{\sqrt{1 - \{N_1 \cdot \widetilde{N}_{n-2}^t\}^2 - \ldots - \{N_{n-3} \cdot \widetilde{N}_{n-2}^t\}^2}} \,. \end{aligned} \tag{3.44}$$

Namely, we choose $\nu_2 \in (0,1)$ sufficiently small with the property that all denominators in (3.44) are greater than or equal to $\frac{1}{2}$. These vectors form an ON-normal section in $B_{\nu_2}(0,0)$. Furthermore, the Hölder estimates

$$\begin{gathered} |N_\Sigma(w_1) - N_\Sigma(w_2)| \le c_9(h_0 r, d_0, \sin\omega, n, \alpha)|w_1 - w_2|^\alpha \\ \forall\, w_1, w_2 \in B_{\nu_2}(0,0)\,,\ \Sigma = 1, \ldots, n-2 \end{gathered} \tag{3.45}$$

hold with a constant $c_9(h_0 r, d_0, \sin\omega, n, \alpha)$ which can be calculated from the Hölder estimates for the N_Σ^*.

2.9. Now we make use of the differential system

$$\triangle Z = 2rH(rZ, N_1) W_Z N_1 + 2rH(rZ, N_2) W_Z N_2 \tag{3.46}$$

in $B_{\nu_2}(0,0)$. We already established $|\triangle Z(w)| \le 2(n-2)(h_0 r)c_2$ in $B_{\nu_2}(0,0)$ (see 2.2). Using (1.11) we obtain the Hölder estimate

$$\begin{aligned} |H(rZ(w_1), N_\Sigma(w_1)) &- H(rZ(w_2), N_\Sigma(w_2))| \\ &\le h_1 r^\alpha |Z(w_1) - Z(w_2)|^\alpha + h_2 |N_\Sigma(w_1) - N_\Sigma(w_2)| \\ &\le h_1 4^\alpha r^\alpha (2c_2)^{\frac{\alpha}{2}} |w_1 - w_2|^\alpha + h_2 c_9 |w_1 - w_2|^\alpha \,. \end{aligned} \tag{3.47}$$

Thus we can find a constant $c_{10} = c_{10}(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, n, \alpha)$ such that

$$|\triangle Z(w_1) - \triangle Z(w_2)| \le c_{10}|w_1 - w_2|^\alpha \quad \forall\, w_1, w_2 \in B_{\nu_2}(0,0). \tag{3.48}$$

2.10. We set $\nu_3 := \frac{1}{2}\nu_2$. From interior Schauder estimates we infer a constant $C_2(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, n, \alpha) \in (0, +\infty)$ such that there holds

$$|Z_{uu}(w)|, |Z_{uv}(w)|, |Z_{vv}(w)| \le C_2 \quad \text{in } B_{\nu_3}(0,0). \tag{3.49}$$

2.11. From the beginning of the proof we recall

$$\kappa_{1,\Sigma}(0,0)^2 + \kappa_{2,\Sigma}(0,0)^2 \leq \frac{1}{r^2}\left\{(h_0 r)^2 + \frac{|Z_{uu}(0,0)||Z_{vv}(0,0)| + |Z_{uv}(0,0)|^2}{W_Z(0,0)^2}\right\}. \tag{3.50}$$

Setting $\Theta(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, n, \alpha) := \frac{2C_2^2}{C_1^2}$ we arrive at

$$\kappa_{1,\Sigma}(0,0)^2 + \kappa_{2,\Sigma}(0,0)^2 \leq \frac{1}{r^2}\big\{(h_0 r)^2 + \Theta\big\}. \tag{3.51}$$

This completes the proof. □

Remark 3.1. 1. The graph property (A1) is essentially needed in 1.9 where we derived the Harnack-type inequality for the area element. For certain immersions of prescribed mean curvature in $\mathbb{R}^3$ one can establish a modulus of continuity for the spherical mapping which ensures the graph property at least locally.

2. Assumption (A2) is needed e.g. in 1.9 for the gradient estimate in terms of $|\nabla F(0,0)|$, and in 1.10 to ensure a point w^* with the property $W(w^*) \geq r^2 c_5$. For certain stable or generalized stable immersions in $\mathbb{R}^3$ one can realize the constant d_0 in (1.13) (see e.g. [4]).

3. Assumption (A3) is needed in 1.5. to establish the inequality

$$|\triangle F^*(w)| \leq \frac{4h_0}{\sin^2\omega}\,|\nabla F^*(w)|^2$$

for the plane mapping F^*. For immersions in $\mathbb{R}^3$, such an inequality follows already from the conformal parametrization and (A3) is not needed.

4. The assumptions (1.11) on the mean curvature field are needed in 2.9 and 2.10 where we applied Schauder theory to establish upper bounds for the second derivatives of the immersion.

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