Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 27 (2008), 79–94

On a Singular Perturbation Problem for a Class of Variational Inequalities

J. A. Álvarez-Dios, M. Chipot, J. C. García and M. C. Muñiz

Abstract. The goal of this paper is to study the asymptotic behavior of a degenerate singular perturbation problem for a class of variational inequalities depending on a positive parameter ε. We also give an existence and uniqueness result.

Keywords. Variational inequalities, singular perturbation problem, asymptotic behavior

Mathematics Subject Classification (2000). 35J85,47J20

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n, n \geq 1$. We denote by $L^2(\Omega)$ the space of square integrable functions normed by

$$|v|_{2,\Omega} = \left\{ \int_\Omega v^2 \, dx \right\}^{\frac{1}{2}}$$

and by $H^1(\Omega)$ the usual Sobolev space built on $L^2(\Omega)$, which we will suppose normed by

$$\|v\|_{1,2} = \left\{ |v|^2_{2,\Omega} + |\,|\nabla v|\,|^2_{2,\Omega} \right\}^{\frac{1}{2}}. \tag{1}$$

($|\nabla v|$ denotes the Euclidean norm of the gradient. We refer the reader to [1, 6, 7] for details on Sobolev spaces.) We denote by $a \in L^\infty(\Omega)$ a function satisfying

$$0 \leq a \leq \Lambda \quad \text{a.e. } x \in \Omega, \, a \not\equiv 0. \tag{2}$$

J.A. Álvarez-Dios, J. C. García and M.C. Muñiz: Department of Applied Mathematics, University of Santiago de Compostela, 15706 Santiago, Spain; jaaldios@usc.es, carlosgb@usc.es, mcarmen@usc.es
M. Chipot: Institut für Angewandte Mathematik. Universität Zürich. 8057 Zürich; m.m.chipot@math.unizh.ch
The first author has been supported by PGIDIT02PXIA20701AF. The second author has been supported by the Swiss National Science Foundation under the contracts # 20-67618.02 and 20-105155/1. The fourth author has been supported by Xunta de Galicia under research project PGIDIT06PXIB207052PR.

Let K be a nonempty closed convex set of $H^1(\Omega)$ and $f \in (H^1(\Omega))^*$ the dual space of $H^1(\Omega)$. We would like to study in this note problems of the type

$$\begin{cases} u_\varepsilon \in K \\ \displaystyle\int_\Omega \big[\varepsilon A(x, \varepsilon u_\varepsilon)\nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon) + a u_\varepsilon(v - u_\varepsilon)\big]\, dx \geq \langle f, v - u_\varepsilon\rangle \quad \forall v \in K. \end{cases} \tag{3}$$

More precisely, we would like to investigate the behaviour of u_ε when $\varepsilon \to 0$ ($\varepsilon > 0$). Note that if A is the identity matrix and $a(x) \geq \lambda > 0$ a.e. in Ω, then (3) is the archetype of singular perturbation problems, see [9] for instance.

In the above variational inequality $A = A(x, u)$ is a $n \times n$-matrix of the Caratheodory type – i.e. such that

$$x \longmapsto A(x, u) \text{ is measurable } \forall u \in \mathbb{R}, \tag{4}$$

$$u \longmapsto A(x, u) \text{ is continuous a.e. } x \in \Omega. \tag{5}$$

(Here A is considered to be a $\mathbb{R}^{n^2}$-valued mapping.) Moreover we suppose that A is uniformly elliptic with uniformly bounded entries. This can be expressed by the existence of λ, $\Lambda > 0$ such that

$$|A(x, u)| \leq \Lambda \qquad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R} \tag{6}$$

$$\lambda |\xi|^2 \leq A\xi \cdot \xi \qquad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^n. \tag{7}$$

(In (6), $|A|$ denotes the operator norm of matrices subordinated to the Euclidean norm; in (7), $|\xi|$ is the Euclidean norm of ξ, $A\xi$ is the vector obtained by applying the matrix A to ξ and "$\cdot$" denotes the usual scalar product.)

Singular perturbations problems were studied in details in the book [9]. However very little is devoted there to perturbation of variational inequalities or to nonlinearity issues. Allowing function a to degenerate also leads to new interesting behaviours who are beyond the scope of [9]. This is what we would like to investigate here.

From a physical point of view (3) models for instance a slow steady diffusion of a colony of bacteria (see [2]), u_ϵ being the density of the population and $\{x \in \Omega;\, a(x) \neq 0\}$ a domain where some death occurs due for instance to a hostile environment. Function f is the outside supply. The set K helps in imposing some further constraints on the species at stake.

2. Existence and uniqueness of a solution

We have

Theorem 2.1. *Under the assumptions of the introduction, for any $\varepsilon > 0$ there exists a solution to* (3).

Proof. We use the Schauder fixed point theorem in the spirit of [3]. Let

$$\mathcal{K} = \bar{K} \cap B(0, R),$$

where $\bar{K}$ denotes the closure of K in $L^2(\Omega)$, $B(0, R)$ the ball of centre 0 and radius R in $L^2(\Omega)$. For $w \in \mathcal{K}$ there exists a unique $u = T(w)$ solution to

$$\begin{cases} u \in K \\ \displaystyle\int_\Omega \big[\varepsilon A(x, \varepsilon w)\nabla u \cdot \nabla(v-u) + au(v-u)\big]\, dx \geq \langle f, v-u \rangle \quad \forall v \in K. \end{cases} \tag{8}$$

This follows from the theory of variational inequalities. Indeed by (7) we have

$$\begin{aligned} (\lambda\varepsilon \wedge 1) \int_\Omega \big(|\nabla u|^2 + au^2\big)\, dx &\leq \lambda\varepsilon \int_\Omega |\nabla u|^2\, dx + \int_\Omega au^2\, dx \\ &\leq \int_\Omega \big(\varepsilon A(x, \varepsilon w)\nabla u \cdot \nabla u + au^2\big)\, dx \end{aligned} \tag{9}$$

($\wedge$ denotes the minimum of two numbers). Since

$$\|u\|_a = \left\{ \int_\Omega \big(|\nabla u|^2 + au^2\big)\, dx \right\}^{\frac{1}{2}} \tag{10}$$

is a norm equivalent to the norm (1) –(see [5])– we see that

$$a(u, v) = \int_\Omega \big(\varepsilon A(x, \varepsilon w)\nabla u \cdot \nabla v + auv\big)\, dx$$

is a continuous, coercive, bilinear form on $H^1(\Omega)$. Thus (8) admits a unique solution.

Let us fix $v_0 \in K$. Using (9), (8) we derive

$$\begin{aligned} (\lambda\varepsilon \wedge 1)\|u\|_a^2 &\leq \int_\Omega \big(\varepsilon A(x, \varepsilon w)\nabla u \cdot \nabla u + au^2\big)\, dx \\ &\leq \int_\Omega \big(\varepsilon A(x, \varepsilon w)\nabla u \cdot \nabla v_0 + auv_0\big)\, dx - \langle f, v_0 - u\rangle \\ &\leq (\varepsilon \vee 1)\Lambda \int_\Omega (|\nabla u||\nabla v_0| + |u||v_0|)\, dx + |f|_*(\|v_0\|_{1,2} + \|u\|_{1,2}), \end{aligned} \tag{11}$$

(see (2), (6); $|f|_*$ denotes the strong dual norm of f and $\vee$ the maximum of two numbers). From (11) we easily derive

$$(\lambda\varepsilon \wedge 1)\|u\|_a^2 \leq \|u\|_{1,2}\big\{(\varepsilon \vee 1)\,\Lambda\|v_0\|_{1,2} + |f|_*\big\} + |f|_*\|v_0\|_{1,2}.$$

By the equivalence of norms $\|\cdot\|_a$, $\|\cdot\|_{1,2}$ we obtain

$$|u|_{2,\Omega} \leq \|u\|_{1,2} \leq C, \tag{12}$$

where $C = C(\varepsilon, \lambda, \Lambda, v_0, f)$ is independent of w. Taking $R > C$, it follows that T maps $\mathcal{K}$ onto $\mathcal{K}$. Moreover, it is easy to prove that T is compact and continuous (see (12)). This completes the existence result by the Schauder fixed point theorem. □

We now turn to the issue of uniqueness. For that we assume A to be uniformly Lipschitz continuous in u, that is to say

$$|A(x,u) - A(x,v)| \le \gamma |u - v| \quad \text{a.e. } x \in \Omega,\ \forall u, v \in \mathbb{R}, \tag{13}$$

(see (6) for the definition of the matrix norm used here). Moreover, we suppose that K is such that for every nonnegative Lipschitz function F with Lipschitz modulus less than 1 and vanishing on $(-\infty, 0)$, it holds

$$u_1 + F(u_2 - u_1),\ \ u_2 - F(u_2 - u_1) \in K, \quad \forall u_1, u_2 \in K. \tag{14}$$

Then we can show

Theorem 2.2. *Under the above assumptions, in particular if* (13), (14) *hold, the solution of* (3) *is unique.*

Proof. Let $u_1 = u_{\varepsilon,1}$ and $u_2 = u_{\varepsilon,2}$ be two solutions of problem (3). For simplicity we will drop the index ε. Using the test functions defined by (14) in (3) written for u_1 and u_2 respectively, we get

$$\int_\Omega \big[\varepsilon A(x,\varepsilon u_1)\nabla u_1 \cdot \nabla F(u_2 - u_1) + a\, u_1 F(u_2 - u_1)\big]\, dx \ge \langle f, F(u_2 - u_1)\rangle$$
$$-\int_\Omega \big[\varepsilon A(x,\varepsilon u_2)\nabla u_2 \cdot \nabla F(u_2 - u_1) + a\, u_2 F(u_2 - u_1)\big]\, dx \ge -\langle f, F(u_2 - u_1)\rangle.$$

By adding we obtain

$$\varepsilon \int_\Omega \big(A(x,\varepsilon u_1)\nabla u_1 - A(x,\varepsilon u_2)\nabla u_2\big) \cdot \nabla F(u_2 - u_1)\, dx$$
$$+ \int_\Omega a\,(u_1 - u_2) F(u_2 - u_1)\, dx \ge 0,$$

which can also be written as

$$\begin{aligned}\int_\Omega \big\{\varepsilon A(x,\varepsilon u_2)\nabla(u_2 - u_1) \cdot \nabla F(u_2 - u_1)\, dx + a\,(u_2 - u_1) F(u_2 - u_1)\big\}\, dx \\ \le \varepsilon \int_\Omega \big(A(x,\varepsilon u_1) - A(x,\varepsilon u_2)\big)\nabla u_1 \cdot \nabla F(u_2 - u_1)\, dx.\end{aligned} \tag{15}$$

We particularize F by choosing

$$F = F_\delta(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le \delta \\ \delta & \text{if } x > \delta. \end{cases}$$

Noticing that

$$(u_2 - u_1)F_\delta(u_2 - u_1) \geq F_\delta(u_2 - u_1)^2, \;\; \nabla(u_2 - u_1) = \nabla F_\delta(u_2 - u_1) \text{ on } \Omega_\delta$$

where $\Omega_\delta = \{x \in \Omega;\ 0 < (u_2 - u_1)(x) < \delta\}$, we derive from (15)

$$\int_\Omega \big\{\varepsilon A(x, \varepsilon u_2)\nabla F_\delta(u_2 - u_1) \cdot \nabla F_\delta(u_2 - u_1) + a\, F_\delta(u_2 - u_1)^2\big\}\, dx$$
$$\leq \int_\Omega \varepsilon\big(A(x, \varepsilon u_1) - A(x, \varepsilon u_2)\big)\nabla u_1 \cdot \nabla F_\delta(u_2 - u_1)\, dx.$$

By arguing like in (9), it follows that we have

$$(\lambda\varepsilon \wedge 1)\|F_\delta(u_2 - u_1)\|_a^2 \leq \int_\Omega \varepsilon\big(A(x, \varepsilon u_1) - A(x, \varepsilon u_2)\big)\nabla u_1 \cdot \nabla F_\delta(u_2 - u_1) dx.$$

Using (13) we get

$$(\lambda\varepsilon \wedge 1)\|F_\delta(u_2 - u_1)\|_a^2 \leq \varepsilon^2\gamma \int_{\Omega_\delta} |u_1 - u_2|\; |\nabla u_1|\, |\nabla F_\delta(u_2 - u_1)|\, dx$$
$$\leq \varepsilon^2\gamma \left\{\int_{\Omega_\delta} |u_1 - u_2|^2 |\nabla u_1|^2 dx\right\}^{\frac{1}{2}} \|F_\delta(u_2 - u_1)\|_a,$$

by the Cauchy–Schwarz inequality. Using again the equivalence of the norms given by (1), (10), we derive that

$$\|F_\delta(u_2 - u_1)\|_{1,2}^2 \leq C \int_{\Omega_\delta} |u_1 - u_2|^2 |\nabla u_1|^2\, dx$$

where C is independent of δ. It implies

$$\int_\Omega F_\delta(u_2 - u_1)^2\, dx \leq C \int_{\Omega_\delta} |u_1 - u_2|^2 |\nabla u_1|^2\, dx$$

and thus

$$\int_\Omega \chi_{\{u_2 - u_1 > \delta\}}\, \delta^2\, dx \leq C \int_\Omega \chi_{\Omega_\delta}\, \delta^2 |\nabla u_1|^2\, dx.$$

χ denotes the characteristic function of sets, $\{u_2 - u_1 > \delta\} = \{x \in \Omega;\ (u_2 - u_1)(x) > \delta\}$. Dividing by δ^2 it comes

$$\int_\Omega \chi_{\{u_2 - u_1 > \delta\}}\, dx \leq C \int_\Omega \chi_{\Omega_\delta}\, |\nabla u_1|^2\, dx.$$

Letting $\delta \to 0$, since

$$\chi_{\Omega_\delta} \longrightarrow 0,\ \chi_{\{u_2 - u_1 > \delta\}} \longrightarrow \chi_{\{u_2 - u_1 > 0\}} \text{ a.e.,}$$

we obtain by the Lebesgue theorem $\int_\Omega \chi_{\{u_2 - u_1 > 0\}}\, dx = 0$ and thus $u_2 \leq u_1$. Exchanging the roles of u_1 and u_2, the result follows. □

3. Asymptotic behaviour of u_ε

3.1. The convergence of $\varepsilon u_\varepsilon$. Before investigating the behavior of u_ε, it is useful (see [4]) to consider $\varepsilon u_\varepsilon$. Some notation is in order. Let k_0 be an arbitrary element in K. We define

$$K_\varepsilon(k_0) = \varepsilon(K - k_0) = \{\varepsilon(k - k_0),\ k \in K\}, \quad K_0 = \bigcap_{\varepsilon>0} K_\varepsilon(k_0).$$

Then we have

Lemma 3.1. *Let k_0 be an arbitrary element of K.*

(i) *$\{K_\varepsilon(k_0)\}_{\varepsilon>0}$ is a nondecreasing sequence of closed convex sets, i.e., $\varepsilon < \varepsilon'$ implies $K_\varepsilon(k_0) \subset K_{\varepsilon'}(k_0)$.*

(ii) *K_0 is a closed convex set containing 0 independent of $k_0 \in K$.*

Proof. (i) $K - k_0$ is closed, convex, containing 0 and so is $K_\varepsilon(k_0) = \varepsilon(K - k_0)$. Next, assuming $\varepsilon < \varepsilon'$ and considering $\varepsilon(k - k_0) \in K_\varepsilon(k_0)$, we have

$$\varepsilon(k - k_0) = \frac{\varepsilon}{\varepsilon'}\varepsilon'(k - k_0) = \frac{\varepsilon}{\varepsilon'}\varepsilon'(k - k_0) + \Big(1 - \frac{\varepsilon}{\varepsilon'}\Big)0 \in K_{\varepsilon'}(k_0).$$

(ii) K_0 is a closed convex set as an intersection of closed convex sets. It contains 0 since $0 \in K_\varepsilon(k_0)$, for all $\varepsilon > 0$. Let us show that K_0 is independent of the element $k_0 \in K$. For that consider $v \in \bigcap_{\varepsilon>0} K_\varepsilon(k_0)$. Then, for every $\varepsilon > 0$ there exists $k \in K$ such that $v = \varepsilon(k - k_0)$. Taking $k_0' \in K$ and $\varepsilon' > \varepsilon$ we have $v = \varepsilon(k - k_0) = \varepsilon(k - k_0') + \varepsilon(k_0' - k_0)$, and then by (i) $v - \varepsilon(k_0' - k_0) = \varepsilon(k - k_0') \in K_\varepsilon(k_0') \subset K_{\varepsilon'}(k_0')$. Letting $\varepsilon \to 0$, since v is a fixed element, we get $v \in K_{\varepsilon'}(k_0')$, for all $\varepsilon' > 0$. This shows that $\bigcap_{\varepsilon>0} K_\varepsilon(k_0) \subset \bigcap_{\varepsilon>0} K_\varepsilon(k_0')$, and the result follows by exchanging k_0 and k_0'. □

We now introduce

$$W_a = \{v \in K_0,\ av = 0 \text{ a.e. in } \Omega\}. \tag{16}$$

Since W_a is clearly a closed convex set of $H^1(\Omega)$, fixing $\varepsilon = 1$ in Theorem 2.1 it follows that there exists a w_0 solution to

$$\begin{cases} w_0 \in W_a \\ \displaystyle\int_\Omega A(x, w_0)\nabla w_0 \cdot \nabla(v - w_0)\, dx \geq \langle f, v - w_0\rangle \quad \forall v \in W_a. \end{cases} \tag{17}$$

Remark 3.2. The above bilinear form seems not to be coercive on $H^1(\Omega)$, however on W_a one has

$$\int_\Omega A(x, w)\nabla u \cdot \nabla v\, dx = \int_\Omega \big(A(x, w)\nabla u \cdot \nabla v + auv\big)\, dx \quad \forall u, v \in W_a.$$

If in addition we suppose that W_a satisfies (14), then the solution to (17) is unique. The proof follows from Theorem 2.2 where we take $\varepsilon = 1$.

Then we have

Theorem 3.3. *Suppose that* u_ε *is solution to* (3)*. If* W_a *satisfies* (14) *and if* (2)*,* (4)–(7)*,* (13) *hold, we have*

$$\lim_{\varepsilon\to 0} \varepsilon u_\varepsilon = w_0 \quad \text{in } H^1(\Omega) \text{ strong},$$

where w_0 *is the unique solution to* (17)*.*

Remark 3.4. Note at this point that we do not assume the solution to (3) to be unique. Only (17) is supposed to have a unique solution.

We will need the following lemma (see [2, 8]),

Lemma 3.5 (Minty). *The problem* (3) *is equivalent to*

$$\begin{cases} u_\varepsilon \in K \\ \displaystyle\int_\Omega \big[\varepsilon A(x,\varepsilon u_\varepsilon)\nabla v \cdot \nabla(v-u_\varepsilon) + a\, v(v-u_\varepsilon)\big]\, dx \geq \langle f, v-u_\varepsilon\rangle \ \ \forall v \in K. \end{cases} \tag{18}$$

Proof. We reproduce the proof for the reader's convenience. First if (3) holds then

$$\begin{aligned} &\int_\Omega \big[\varepsilon A(x,\varepsilon u_\varepsilon)\nabla v \cdot \nabla(v-u_\varepsilon) + a\, v(v-u_\varepsilon)\big]\, dx \\ &\qquad = \int_\Omega \big[\varepsilon A(x,\varepsilon u_\varepsilon)\nabla(v-u_\varepsilon) \cdot \nabla(v-u_\varepsilon) + a(v-u_\varepsilon)^2\big]\, dx \\ &\qquad\quad + \int_\Omega \big[\varepsilon A(x,\varepsilon u_\varepsilon)\nabla u_\varepsilon \cdot \nabla(v-u_\varepsilon) + a u_\varepsilon(v-u_\varepsilon)\big]\, dx \\ &\qquad \geq \langle f, v-u_\varepsilon\rangle \quad \forall v \in K \end{aligned}$$

(by (2), (3), (7)). Next if (18) holds, replacing v by $u_\varepsilon + t(v-u_\varepsilon)$ which is in K for any $t\in(0,1)$, $v\in K$, we get

$$\begin{aligned} \int_\Omega \big[\varepsilon A(x,\varepsilon u_\varepsilon)&\nabla\{u_\varepsilon + t(v-u_\varepsilon)\}\cdot \nabla t(v-u_\varepsilon) \\ &+ a\,\{u_\varepsilon + t(v-u_\varepsilon)\} t(v-u_\varepsilon)\big]\, dx \geq t\langle f, v-u_\varepsilon\rangle. \end{aligned}$$

Dividing by t and letting $t\to 0$ we get (3). □

We now turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let us take a fixed element u^* in K. Considering $v = (1-\varepsilon)u_\varepsilon + \varepsilon u^* \in K$ in (3) we get

$$\varepsilon\int_\Omega A(x,\varepsilon u_\varepsilon)\nabla u_\varepsilon\cdot\nabla(-\varepsilon u_\varepsilon+\varepsilon u^*)\, dx + \int_\Omega a\, u_\varepsilon(-\varepsilon u_\varepsilon + \varepsilon u^*)\, dx \geq \langle f, -\varepsilon u_\varepsilon + \varepsilon u^*\rangle.$$

This implies setting $v_\varepsilon = \varepsilon u_\varepsilon$

$$\int_\Omega A(x,v_\varepsilon)\nabla v_\varepsilon\cdot\nabla v_\varepsilon\,dx + \frac{1}{\varepsilon}\int_\Omega a\,v_\varepsilon^2\,dx \\ \le \varepsilon\int_\Omega A(x,v_\varepsilon)\nabla v_\varepsilon\cdot\nabla u^*\,dx + \int_\Omega a\,v_\varepsilon u^*\,dx + \langle f, v_\varepsilon - \varepsilon u^*\rangle.$$

Using (6), (7) we derive

$$\lambda\int_\Omega |\nabla v_\varepsilon|^2\,dx + \frac{1}{\varepsilon}\int_\Omega a v_\varepsilon^2\,dx \\ \le \varepsilon\int_\Omega (\Lambda\,|\nabla v_\varepsilon||\nabla u^*| + a|v_\varepsilon||u^*|)\,dx + |f|_*\|v_\varepsilon\|_{1,2} + \varepsilon|f|_*\|u^*\|_{1,2}. \tag{19}$$

Assuming $\varepsilon \le 1$, $\varepsilon\Lambda < 1$ –recall that $\varepsilon \to 0$– we get

$$(\lambda\wedge 1)\|v_\varepsilon\|_a^2 \le \|v_\varepsilon\|_a\|u^*\|_a + |f|_*\|v_\varepsilon\|_{1,2} + |f|_*\|u^*\|_{1,2}.$$

Due to the equivalence of norms $\|\cdot\|_a$, $\|\cdot\|_{1,2}$ we obtain, for some constants independent of ε, $\|v_\varepsilon\|_{1,2}^2 \le C\|v_\varepsilon\|_{1,2} + C'$. It follows that

$$\|v_\varepsilon\|_{1,2} \le C''. \tag{20}$$

and –up to a sequence– there exists $v_0 \in K$ such that when $\varepsilon \to 0$,

$$v_\varepsilon \rightharpoonup v_0 \quad \text{in } H^1(\Omega) \tag{21}$$

$$v_\varepsilon \to v_0 \quad \text{in } L^2(\Omega) \tag{22}$$

$$v_\varepsilon \to v_0 \quad \text{a.e. in } \Omega. \tag{23}$$

From (19), (20) we derive $\int_\Omega a v_\varepsilon^2\,dx \le \varepsilon\, C$, where C is independent of ε. Using Fatou's lemma we infer

$$\int_\Omega a v_0^2\,dx = 0, \quad \text{i.e. } a v_0 = 0 \text{ a.e. in } \Omega. \tag{24}$$

Next we would like to show that $v_0 \in K_0$. Consider $k_0 \in K$. We have $v_\varepsilon = \varepsilon u_\varepsilon = \varepsilon(u_\varepsilon - k_0) + \varepsilon k_0$, and thus for $\varepsilon' > \varepsilon$, by Lemma 3.1, $v_\varepsilon - \varepsilon k_0 = \varepsilon(u_\varepsilon - k_0) \in K_{\varepsilon'}(k_0)$. Letting $\varepsilon \to 0$ we get $v_0 \in K_{\varepsilon'}(k_0)$ for all ε'. It follows that $v_0 \in K_0$ and by (24) $v_0 \in W_a$. Next, considering (18) and multiplying the inequality by ε we have

$$\int_\Omega \big(A(x,v_\varepsilon)\nabla(\varepsilon v)\cdot\nabla(\varepsilon v - v_\varepsilon) + a\,v(\varepsilon v - v_\varepsilon)\big)\,dx \ge \langle f, \varepsilon v - v_\varepsilon\rangle\ \forall v \in K. \tag{25}$$

Consider $w \in W_a$ an arbitrary element. Since $w \in K_0$, for every ε there exists $w_\varepsilon \in K$ such that $w = \varepsilon(w_\varepsilon - k_0)$. Taking

$$v = w_\varepsilon = \frac{w}{\varepsilon} + k_0 \tag{26}$$

in (25) we obtain

$$\begin{aligned}\int_\Omega \Big(& A(x,v_\varepsilon)\nabla(w+\varepsilon k_0)\cdot\nabla(w-v_\varepsilon+\varepsilon k_0) \\ & +a\Big(\frac{w}{\varepsilon}+k_0\Big)(w-v_\varepsilon+\varepsilon k_0)\Big)dx \ge \langle f, w-v_\varepsilon+\varepsilon k_0\rangle.\end{aligned} \tag{27}$$

Since $w \in W_a$, then $aw = 0$ a.e. $x \in \Omega$ and (27) leads to

$$\begin{aligned}&\int_\Omega A(x,v_\varepsilon)\nabla w\cdot\nabla(w-v_\varepsilon)\,dx + \varepsilon\int_\Omega A(x,v_\varepsilon)\nabla w\cdot\nabla k_0\,dx \\ &+\varepsilon\int_\Omega A(x,v_\varepsilon)\nabla k_0\cdot\nabla(w-v_\varepsilon+\varepsilon k_0)\,dx + \int_\Omega a\,k_0\,(-v_\varepsilon+\varepsilon k_0)\,dx \\ &\ge \langle f, w-v_\varepsilon\rangle + \varepsilon\,\langle f,k_0\rangle.\end{aligned} \tag{28}$$

It follows from (23) that $A(x,v_\varepsilon)\nabla w \to A(x,v_0)\nabla w$ in $L^2(\Omega)$, and passing to the limit in (28) we obtain

$$\int_\Omega A(x,v_0)\nabla w\cdot\nabla(w-v_0)\,dx \ge \langle f, w-v_0\rangle \quad \forall w\in W_a.$$

Using Lemma 3.5 with $\varepsilon = 1$, we see that v_0 also satisfies

$$\begin{cases} v_0 \in W_a \\ \displaystyle\int_\Omega A(x,v_0)\nabla v_0\cdot\nabla(w-v_0)\,dx \ge \langle f, w-v_0\rangle \quad \forall w\in W_a, \end{cases}$$

i.e., $v_0 = w_0$ the unique solution to (17). Since the possible limit of $v_\varepsilon = \varepsilon u_\varepsilon$ is unique, it is the whole sequence v_ε that satisfies (21)–(23). Let us now show that the convergence is in fact strong. For that we multiply (3) by ε and take $v = w_\varepsilon$ given by (26). We obtain

$$\int_\Omega \big(A(x,v_\varepsilon)\nabla v_\varepsilon\cdot\nabla(w+\varepsilon k_0-v_\varepsilon) + a\,u_\varepsilon(w+\varepsilon k_0-v_\varepsilon)\big)\,dx \ge \langle f, w+\varepsilon k_0-v_\varepsilon\rangle.$$

Thus rearranging this inequality and taking into account that $\frac{1}{\varepsilon} > 1$, we get

$$\begin{aligned}\int_\Omega \big(A(x,v_\varepsilon)\nabla v_\varepsilon\cdot\nabla v_\varepsilon + av_\varepsilon^2\big)\,dx \le & \int_\Omega A(x,\ v_\varepsilon)\nabla v_\varepsilon\cdot\nabla(w+\varepsilon k_0)\,dx \\ & + \int_\Omega a\,u_\varepsilon\,\varepsilon\,k_0\,dx - \langle f, w+\varepsilon k_0-v_\varepsilon\rangle.\end{aligned} \tag{29}$$

Thus we derive taking $w = w_0$ in (29)

$$
\begin{aligned}
&(\lambda \wedge 1)\|v_\varepsilon - w_0\|_a^2 \\
&\quad \le \int_\Omega \big(A(x, v_\varepsilon)\nabla(v_\varepsilon - w_0)\cdot\nabla(v_\varepsilon - w_0) + a\,(v_\varepsilon - w_0)^2\big)\,dx \\
&\quad = \int_\Omega \big(A(x, v_\varepsilon)\nabla v_\varepsilon\cdot\nabla v_\varepsilon + a\,(v_\varepsilon)^2\big)\,dx \\
&\qquad -\int_\Omega \big\{A(x, v_\varepsilon)\nabla w_0\cdot\nabla v_\varepsilon + A(x, v_\varepsilon)\nabla v_\varepsilon\cdot\nabla w_0\big\}dx + \int_\Omega A(x, v_\varepsilon)\nabla w_0\cdot\nabla w_0\,dx \\
&\quad \le \int_\Omega A(x, v_\varepsilon)\nabla v_\varepsilon\cdot\nabla(w + \varepsilon k_0)dx + \int_\Omega a\,u_\varepsilon\,\varepsilon k_0\,dx - \langle f, w + \varepsilon k_0 - v_\varepsilon\rangle \\
&\qquad -\int_\Omega \big\{A(x, v_\varepsilon)\nabla w_0\cdot\nabla v_\varepsilon + A(x, v_\varepsilon)\nabla v_\varepsilon\cdot\nabla w_0\big\}dx + \int_\Omega A(x, v_\varepsilon)\nabla w_0\cdot\nabla w_0\,dx,
\end{aligned}
$$

which converges towards zero when $\varepsilon \to 0$. This completes the proof of the theorem. □

3.2. Convergence of u_ε. Suppose that we are in dimension 1. Then – due to the embedding $H^1(\Omega) \subset \mathcal{C}(\bar\Omega)$ – we derive from Theorem 3.3 that $v_\varepsilon \to w_0$ in $\mathcal{C}(\bar\Omega)$. In particular

$$u_\varepsilon = \frac{v_\varepsilon}{\varepsilon} \to \operatorname{sign} w_0 \cdot \infty \qquad \text{on } [w_0 \ne 0],$$

and we can expect convergence of u_ε only on the set $[w_0 = 0]$. Due to (16) and since $w_0 \in W_a$ we have $w_0 = 0$ on $\Omega' = \{x \in \Omega;\ a(x) > 0\}$. Now we would like to investigate the behavior of u_ε on this set. For this we will suppose

$$f \in L^2(a\,dx)^*, \tag{30}$$

where we have set

$$
\begin{aligned}
L^2(a\,dx) &= \Big\{v \text{ measurable on } \Omega \text{ such that } \int_\Omega av^2\,dx < +\infty\Big\} \\
L^2(a\,dx)^* &= \text{the dual of } L^2(a\,dx).
\end{aligned}
$$

It is clear that $L^2(a\,dx)$ is a Hilbert space for the scalar product $(u, v)_a = \int_\Omega auv\,dx$, and its dual can be identified to $L^2(a\,dx)$ via the Riesz representation theorem. If f satisfies (30) we have

$$\langle f, v\rangle \le C|v|_a = C\left\{\int_\Omega av^2\,dx\right\}^{\frac{1}{2}} \le C\|v\|_a, \tag{31}$$

and thus $f \in H^1(\Omega)^*$. So, there exists u_ε solution to (3). Moreover, we have

Theorem 3.6. *Let $f \in L^2(a\,dx)^*$ and let u_ε be a solution to* (3). *Then it holds that*

$$u_\varepsilon \to u_0 \ \text{in}\ L^2(a\,dx),$$

where u_0 is the solution to

$$\begin{cases} u_0 \in \overline{K} \ (\text{the closure of } K \ \text{in } L^2(a\,dx)) \\ \displaystyle\int_\Omega a\,u_0(v-u_0)\,dx \geq \langle f, v-u_0\rangle \quad \forall v \in \overline{K}. \end{cases} \tag{32}$$

Proof. Let v_0 be a fixed element in K. Taking $v = v_0$ in (3) and setting $A = A(x, \varepsilon u_\varepsilon)$ we obtain

$$\varepsilon \int_\Omega (A\nabla u_\varepsilon \cdot \nabla(v_0 - u_\varepsilon) + a\,u_\varepsilon(v_0 - u_\varepsilon))\,dx \geq \langle f, v_0 - u_\varepsilon\rangle.$$

Using (7) we deduce

$$\varepsilon\lambda \int_\Omega |\nabla u_\varepsilon|^2\,dx + \int_\Omega a\,u_\varepsilon^2\,dx \leq \varepsilon \int_\Omega A\nabla u_\varepsilon \cdot \nabla v_0\,dx + \int_\Omega a\,u_\varepsilon v_0\,dx + \langle f, u_\varepsilon - v_0\rangle.$$

Recalling (6) we get –see also (31)–

$$\varepsilon\lambda|\,|\nabla u_\varepsilon|\,|_{2,\Omega}^2 + |u_\varepsilon|_a^2 \leq \varepsilon\Lambda|\,|\nabla u_\varepsilon|\,|_{2,\Omega}\,|\,|\nabla v_0|\,|_{2,\Omega} + |u_\varepsilon|_a|v_0|_a + |f|_a^*\{|u_\varepsilon|_a + |v_0|_a\},$$

where $|f|_a^*$ denotes the strong dual norm of f. Setting $N(u_\varepsilon) = \{\varepsilon\lambda|\,|\nabla u_\varepsilon|\,|_{2,\Omega}^2 + |u_\varepsilon|_a^2\}^{\frac{1}{2}}$ one easily deduces that the following holds:

$$N(u_\varepsilon)^2 \leq \left\{\sqrt{\varepsilon}\frac{\Lambda}{\sqrt{\lambda}}|\,|\nabla v_0|\,|_{2,\Omega} + |v_0|_a + |f|_a^*\right\} N(u_\varepsilon) + |f|_a^*|v_0|_a$$

and thus for some constant C independent of ε ($\varepsilon < 1$) we obtain

$$N(u_\varepsilon)^2 \leq C. \tag{33}$$

So, up to a subsequence we have $u_\varepsilon \rightharpoonup u$ in $L^2(a\,dx)$. From (3) we derive

$$\begin{aligned} \int_\Omega a u_\varepsilon v\,dx &\geq \langle f, v-u_\varepsilon\rangle - \varepsilon\int_\Omega A\nabla u_\varepsilon \cdot \nabla v\,dx + \int_\Omega a u_\varepsilon^2\,dx \\ &\geq \langle f, v-u_\varepsilon\rangle - \varepsilon\Lambda|\,|\nabla u_\varepsilon|\,|_{2,\Omega}\,|\,|\nabla v|\,|_{2,\Omega} + \int_\Omega a u_\varepsilon^2\,dx \\ &\geq \langle f, v-u_\varepsilon\rangle - \sqrt{\varepsilon}C' + \int_\Omega a u_\varepsilon^2\,dx \end{aligned} \tag{34}$$

by (33). Passing to the limit inf in ε we get

$$\int_\Omega a\,u\,v\,dx \geq \langle f, v-u\rangle + \int_\Omega a u^2\,dx, \quad \forall v \in K.$$

By density the above inequality holds for every $v \in \overline{K}$ and $u = u_0$ solution to (32). By uniqueness of the limit it follows that the whole sequence u_ε converges to u_0 in $L^2(a\,dx)$ weakly. Taking $v = u_0$ in (34) and passing to the lim sup in ε we obtain $\limsup \int_\Omega a u_\varepsilon^2\,dx \le \int_\Omega a u_0^2\,dx \le \liminf \int_\Omega a u_\varepsilon^2\,dx$. Thus it holds $\lim_{\varepsilon\to 0} \int_\Omega a u_\varepsilon^2\,dx = \int_\Omega a u_0^2\,dx$. This establishes the strong convergence of u_ε and completes the proof. □

In the case where $u_0 \in K$ we can estimate more precisely the rate of convergence of u_ε toward u_0 and show

Theorem 3.7. *Suppose that $u_0 \in K$. Then we have*

$$||u_\varepsilon||_{1,2} \le C_1, \quad |u_\varepsilon - u_0|_a \le \sqrt{\varepsilon}\, C_2,$$

where C_1 and C_2 are two constants independent of ε.

Proof. Since $u_0 \in K$, we can choose $v = u_0$ in (3) and $v = u_\varepsilon$ in (32). Adding up we obtain $\varepsilon \int_\Omega A\nabla u_\varepsilon \cdot \nabla(u_0 - u_\varepsilon)\,dx - \int_\Omega a\,(u_\varepsilon - u_0)^2\,dx \ge 0$. This can also be written as $\varepsilon \int_\Omega A\nabla(u_\varepsilon - u_0 + u_0) \cdot \nabla(u_0 - u_\varepsilon)\,dx - \int_\Omega a\,(u_\varepsilon - u_0)^2\,dx \ge 0$. Therefore

$$\varepsilon \int_\Omega A\nabla(u_0 - u_\varepsilon) \cdot \nabla(u_0 - u_\varepsilon)\,dx + \int_\Omega a\,(u_\varepsilon - u_0)^2 dx \le \varepsilon \int_\Omega A\nabla u_0 \cdot \nabla(u_0 - u_\varepsilon)\,dx.$$

Thus

$$\varepsilon\lambda|\,|\nabla(u_\varepsilon - u_0)|\,|_{2,\Omega}^2 + |u_\varepsilon - u_0|_a^2 \le \varepsilon\Lambda|\,|\nabla u_0|\,|_{2,\Omega}|\,|\nabla(u_\varepsilon - u_0)|\,|_{2,\Omega}$$

(see (6), (7)). It follows that $|\,|\nabla(u_\varepsilon - u_0)|\,|_{2,\Omega} \le \frac{\Lambda}{\lambda}|\,|\nabla u_0|\,|_{2,\Omega}$ and $|u_\varepsilon - u_0|_a^2 \le \varepsilon\frac{\Lambda^2}{\lambda}|\,|\nabla u_0|\,|_{2,\Omega}$. This completes the proof since $\|\cdot\|_a$ is equivalent to $\|\cdot\|_{1,2}$. □

4. Some examples

4.1. The case where K is bounded. In this case $K_0 = W_a = \{0\}$ but one can also see directly –since u_ε is bounded– that $v_\varepsilon = \varepsilon u_\varepsilon \to 0$ in $H^1(\Omega)$.

4.2. The case of a vector space. If $K = V$ is a closed subspace of $H^1(\Omega)$, then $K_0 = V$, $W_a = \{v \in V;\ av = 0 \text{ a.e. in } \Omega\}$, and w_0 is the weak solution to

$$w_0 \in W_a, \qquad \int_\Omega A(x, w_0)\nabla w_0 \cdot \nabla v\,dx = \langle f, w\rangle \quad \forall w \in W_a$$

(see also [4]). Note that $w_0 = 0$ when $a > 0$ a.e. in Ω.

Now if $f \in (L^2(a\,dx))^*$ by the Riesz representation theorem there exists a unique $u \in L^2(a\,dx)$ such that

$$\langle f, v\rangle = (u, v)_a, \quad \forall v \in L^2(a\,dx) \tag{35}$$

and u_0 is such that –see (32)–

$$u_0 \in \overline{V}, \qquad (u_0, v)_a = (u, v)_a, \quad \forall v \in \overline{V},$$

where $\overline{V}$ denotes the closure of V in $L^2(a\,dx)$ (to see that replace v by $u_0 \pm v$ in (32)). In the case where V is dense in $L^2(a\,dx)$ one has

$$u_0 = u \tag{36}$$

This is the case in particular when $V = H^1(\Omega)$, $H_0^1(\Omega)$.

4.3. The case of the obstacle problem. Consider for instance

$$K = \{v \in H_0^1(\Omega);\ v \geq \varphi \text{ a.e. in } \Omega\}$$

where φ is a function satisfying $\varphi \in H^1(\Omega)$, $\varphi \leq 0$ on Γ. Then clearly $\varphi^+ \in K$ and

$$\begin{aligned} K_\varepsilon(\varphi^+) &= \{\varepsilon(v - \varphi^+);\ v \in K\} \\ &= \Big\{w \in H_0^1(\Omega);\ \frac{w}{\varepsilon} + \varphi^+ \geq \varphi \text{ a.e. in } \Omega\Big\} \\ &= \{w \in H_0^1(\Omega);\ w \geq -\varepsilon\varphi^- \text{ a.e. in } \Omega\}. \end{aligned}$$

It follows that $K_0 = \{w \in H_0^1(\Omega);\, w \geq 0 \text{ a.e. in } \Omega\}$, $W_a = \{w \in K_0;\, aw = 0 \text{ a.e. in } \Omega\}$. This determines the solution w_0 in this case.

Suppose now to simplify that $a = a_0\,\chi_{\Omega'}$, where $\Omega' \subset \Omega$ is a measurable subset and a_0 a function satisfying $0 < \lambda \leq a_0 \leq \Lambda$ a.e. in Ω'. It is easy to see in this case that $L^2(a\,dx) = L^2(\Omega')$. Thus

$$\overline{K} = \{v \in L^2(a\,dx);\ v \geq \varphi \text{ a.e. in } \Omega'\}. \tag{37}$$

Indeed, one has $K \subset \overline{K}$. Moreover if $v \in L^2(a\,dx)$ satisfies $v \geq \varphi$ a.e. in Ω', consider $v_n \in H_0^1(\Omega)$ such that $v_n \to v\chi_{\Omega'}$ in $L^2(\Omega)$ (recall that $H_0^1(\Omega)$ is dense in $L^2(\Omega)$). Then $v_n \vee \varphi \in K$, $v_n \vee \varphi \to v$ in $L^2(\Omega')$. This shows (37). If we introduce u such that (35) holds, then problem (32) can be written

$$u_0 \in \overline{K}, \qquad (u_0, v - u_0)_a \geq (u, v - u_0)_a, \quad \forall v \in \overline{K}. \tag{38}$$

We claim that it holds that

$$u_0 = u \vee \varphi. \tag{39}$$

Indeed, first $u \vee \varphi \in \overline{K}$. Moreover for $v \in \overline{K}$ –i.e. $v \geq \varphi$ a.e. in Ω'– it holds that

$$\int_{\Omega'} a\,\{(u \vee \varphi) - u\}\,\{v - (u \vee \varphi)\}\,dx = \int_{\Omega' \cap \{u < \varphi\}} a\,\{\varphi - u\}\,\{v - \varphi\}\,dx \geq 0,$$

i.e., $u \vee \varphi$ satisfies (38) and (39) is proved.

4.4. An example in one dimension. Taking $\Omega = (0,1)$, $\Omega' = (0, \frac{1}{2})$, $a = \chi_{\Omega'}$ and $\eta \in \mathbb{R}$, let us choose K as $K = \{v \in H^1(\Omega),\ v - \eta \in H_0^1(\Omega)\}$. It is easy to see that K is a closed, convex and nonempty subset from $H^1(\Omega)$. In order to linearize our problem, we take $A(x,u)$ and f equal to one; thus problem (3) reads

$$u_\varepsilon \in K; \quad \varepsilon \int_\Omega u'_\varepsilon(v' - u'_\varepsilon)\, dx + \int_{\Omega'} u_\varepsilon(v - u_\varepsilon)\, dx \geq \int_\Omega (v - u_\varepsilon)\, dx, \quad \forall v \in K.$$

Taking $v = u_\varepsilon \pm w$ where $w \in H_0^1(\Omega)$ we see after an integration by parts that u_ε is solution to

$$u_\varepsilon \in K; \qquad \int_\Omega (-\varepsilon u''_\varepsilon + u_\varepsilon\, \chi_{\Omega'} - 1) w\, dx = 0, \quad \forall w \in H_0^1(\Omega),$$

and this implies that u_ε solves the following ordinary differential equation:

$$-\varepsilon u''_\varepsilon + u_\varepsilon = 1 \quad \text{in } (0, \tfrac{1}{2}), \qquad -\varepsilon u''_\varepsilon = 1 \quad \text{in } (\tfrac{1}{2}, 1)$$
$$u_\varepsilon(0) = \eta = u_\varepsilon(1)$$

with the continuity conditions $u_\varepsilon^-(\frac{1}{2}) = u_\varepsilon^+(\frac{1}{2})$, $u'^-_\varepsilon(\frac{1}{2}) = u'^+_\varepsilon(\frac{1}{2})$. Using $u_\varepsilon(0) = \eta$ it is straightforward to obtain

$$u_\varepsilon(x) = 1 + (\eta - 1) e^{\frac{-x}{\sqrt{\varepsilon}}} + 2\, A \sinh\Big(\frac{x}{\sqrt{\varepsilon}}\Big), \quad x \in (0, \frac{1}{2}),$$

where A is given in terms of $u^\star = u_\varepsilon^-(\frac{1}{2}) = u_\varepsilon^+(\frac{1}{2})$ by

$$A = \frac{u^\star - 1 + (1 - \eta) e^{\frac{-1}{2\sqrt{\varepsilon}}}}{2 \sinh\big(\frac{1}{2\sqrt{\varepsilon}}\big)}.$$

Moreover, in the interval $(\frac{1}{2}, 1)$ the solution reads

$$u_\varepsilon(x) = 2\, u^\star(1 - x) + \eta\, (2x - 1) + \frac{1}{4\, \varepsilon}(-2x^2 + 3x - 1).$$

Finally, using the continuity condition for the derivatives at $x = \frac{1}{2}$ we obtain

$$u^\star = \frac{2\, \eta \sqrt{\varepsilon} + \frac{1}{4\sqrt{\varepsilon}} + (\eta - 1) e^{\frac{-1}{2\sqrt{\varepsilon}}} + \big[(\eta - 1) e^{\frac{-1}{2\sqrt{\varepsilon}}} + 1\big] \coth\big(\frac{1}{2\sqrt{\varepsilon}}\big)}{2\sqrt{\varepsilon} + \coth\big(\frac{1}{2\sqrt{\varepsilon}}\big)}.$$

Applying theorem 3.3 it yields that $\varepsilon u_\varepsilon \to w_0$ where w_0 solves the problem

$$w_0 \in W_a, \qquad \int_0^1 w'_0(v' - w'_0)\, dx \geq \int_0^1 (v - w_0)\, dx, \quad \forall v \in W_a$$

with $W_a = \{v \in H^1_0(\Omega),\ v = 0 \text{ a.e. in } \Omega'\}$. Taking $v = w_0 \pm w$ we get after integration by parts

$$w_0 \in W_a, \qquad \int_{\frac{1}{2}}^{1} (-w_0'' - 1) w \, dx = 0, \quad \forall w \in W_a,$$

and then we deduce that w_0 solves

$$w_0 = 0 \quad \text{in } (0, \tfrac{1}{2}), \qquad -w_0'' = 1 \quad \text{in } (\tfrac{1}{2}, 1)$$
$$w_0(\tfrac{1}{2}) = 0 = w_0(1).$$

Therefore we have that

$$w_0 = \begin{cases} 0, & \text{in } (0, \frac{1}{2}) \\ \frac{1}{4}(-2x^2 + 3x - 1) & \text{in } (\frac{1}{2}, 1). \end{cases}$$

Figure 1 shows $\varepsilon u_\varepsilon$ for several values of ε and its limit w_0 taking η equal to one.

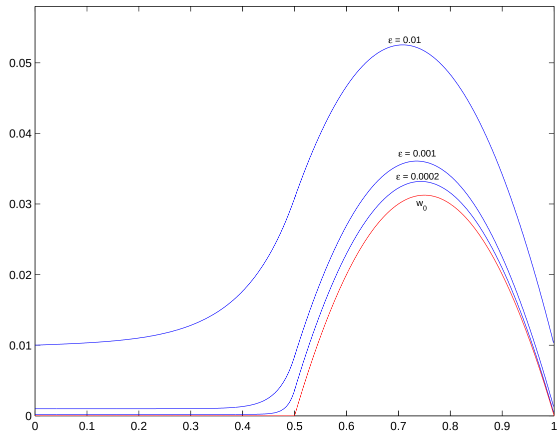

Figure 1: $\varepsilon u_\varepsilon$ and w_0.

Let us choose to simplify $\eta = 1$. We obtain by straightforward computations in the interval $(0, \frac{1}{2})$

$$u_\varepsilon(x) = 1 + \frac{(u^\star - 1)}{\sinh\left(\frac{1}{2\sqrt{\varepsilon}}\right)} \sinh\left(\frac{x}{\sqrt{\varepsilon}}\right)$$

with

$$u^\star - 1 = \frac{1}{4\sqrt{\varepsilon}\left(2\sqrt{\varepsilon} + \coth\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right)}.$$

Thus we deduce that $u_\varepsilon(x) \to 1$ in $(0, \frac{1}{2})$ but the convergence is not strong in $L^2(0, \frac{1}{2})$; indeed, we have

$$\int_0^{\frac{1}{2}} (u_\varepsilon(x) - 1)^2 dx = \frac{1}{16\,\varepsilon\left(2\sqrt{\varepsilon} + \coth\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right)^2} \frac{1}{\left(\sinh\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right)^2} \int_0^{\frac{1}{2}} \sinh\left(\frac{x}{\sqrt{\varepsilon}}\right)^2 dx.$$

Then

$$\int_0^{\frac{1}{2}} (u_\varepsilon(x) - 1)^2 dx = \frac{\sqrt{\varepsilon}\, \sinh\left(\frac{1}{\sqrt{\varepsilon}}\right) - 1}{64\,\varepsilon\left(2\sqrt{\varepsilon} + \coth\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right)^2 \left(\sinh\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right)^2},$$

and this integral goes to infinity when $\varepsilon \to 0$. This is due to the fact that f defined by $\langle f, v\rangle = \int_\Omega v(x)dx$ does not belong to $L^2(a\,dx)^*$. On the other hand for this value of η, if we choose f such that $\langle f, v\rangle = \int_{\Omega'} v(x)dx$, f belongs to $L^2(a\,dx)^*$ and u_ε solution to

$$u_\varepsilon \in K; \quad \varepsilon \int_\Omega u_\varepsilon'(v' - u_\varepsilon')\,dx + \int_{\Omega'} u_\varepsilon(v - u_\varepsilon)\,dx \ge \int_{\Omega'} (v - u_\varepsilon)\,dx, \quad \forall v \in K,$$

is the solution of (see above)

$$-\varepsilon u_\varepsilon'' + u_\varepsilon = 1 \quad \text{in } (0, \tfrac{1}{2}), \qquad -\varepsilon u_\varepsilon'' = 0 \quad \text{in } (\tfrac{1}{2}, 1)$$
$$u_\varepsilon(0) = 1 = u_\varepsilon(1)$$

with the continuity conditions $u_\varepsilon^-(\frac{1}{2}) = u_\varepsilon^+(\frac{1}{2})$ and $u'^-_\varepsilon(\frac{1}{2}) = u'^+_\varepsilon(\frac{1}{2})$. It is straightforward to deduce that u_ε equals 1 for all $\varepsilon > 0$ and the convergence towards f is then here strong.

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Received January 16, 2006; revised April 19, 2006