# Artificial Boundary Conditions
# for the Stokes and Navier-Stokes Equations
# in Domains that are Layer-Like at Infinity

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**Abstract.** Artificial boundary conditions are presented to approximate solutions to Stokes- and Navier-Stokes problems in domains that are layer-like at infinity. Based on results about existence and asymptotics of the solutions $v^\infty$, $p^\infty$ to the problems in the unbounded domain $\Omega$ the error $v^\infty - v^R$, $p^\infty - p^R$ is estimated in $H^1(\Omega_R)$ and $L^2(\Omega_R)$, respectively. Here $v^R$, $p^R$ are the approximating solutions on the truncated domain $\Omega_R$, the parameter $R$ controls the exhausting of $\Omega$. The artificial boundary conditions involve the Steklov-Poincaré operator on a circle together with its inverse and thus turn out to be a combination of local and nonlocal boundary operators. Depending on the asymptotic decay of the data of the problems, in the linear case the error vanishes of order $O(R^{-N})$, where $N$ can be arbitrarily large.

**Keywords.** Stokes Problem in layers, Navier-Stokes system, artificial boundary conditions, exact boundary conditions, Steklov-Poincaré operator

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## 1. Introduction

Layer-like domains appear in many topics of mathematical physics, related to film flows, lubrication patterns, plates etc. In the present paper a layer like domain is a domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$, and $\Omega$ coincides with the layer

$$\Lambda = \left\{ x = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, |z| < \tfrac{1}{2} \right\} \tag{1.1}$$

outside the ball $\mathbb{B}_{R_0} = \{x \in \mathbb{R}^3 : |x| < R_0\}$ of radius $R_0 > 1$. We consider the Stokes equations – and further Navier-Stokes equations – with Dirichlet

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boundary conditions

$$
\begin{aligned}
-\nu\Delta v^\infty + \nabla p^\infty &= f &&\text{in } \Omega \\
\nabla \cdot v^\infty &= 0 &&\text{in } \Omega \\
v^\infty &= 0 &&\text{on } \partial\Omega \, .
\end{aligned}
\tag{1.2}
$$

The vector $v^\infty = (v_1^\infty, v_2^\infty, v_3^\infty)$ stands for the velocity and the scalar $p^\infty$ for the pressure in a fluid with constant viscosity $\nu > 0$. In domains of type (1.1) besides the question of uniqueness and existence of solutions also the asymptotic behavior of $v, p$ at infinity is important in dependance of the decay properties of $f$ for various reasons. One context is the following:

Computational schemes for boundary value problems in unbounded domains require the reduction to a problem in a bounded region. A very common practice is to cut the unbounded domain by taking the intersection with a bounded one and prescribe an *artificial boundary condition* (ABC) on the truncation surface. The choice of the truncation surfaces is usually governed by the geometry of the domains, the choice of the ABCs by the structure of differential operators. An opportune ABC should lead to a well posed problem which is accessible for numerics and leaves a minimal truncation error. The latter feature leads to non reflecting (absorbing, exact) ABC, they produce the restriction of the original solution to the truncated domain. However, with the exception of trivial examples they are nonlocal and require information like the structure of a Fourier expansion for the solution, e.g., information which usually exists only for homogeneous linear systems and simple geometries (see [3, 6, 8, 31, 35], e.g.).

Local ABC normally leave a truncation error but can mostly be handled with finite element methods and are available for inhomogeneous systems as well as for nonlinear problems, e.g., the Navier-Stokes system. Their choice is based on the asymptotic behavior of solutions at infinity. In particular, for elliptic boundary value problems in exterior domains and domains with cylindrical or conical outlets to infinity, ABCs in differential form were systematically developed during the last decades (see, e.g., [1, 2, 4, 5, 7, 9, 10, 22, 23, 26, 32, 34] and the papers quoted there). The common feature of local ABCs are estimates for the truncation error of the form $\|u^\infty - u^R\| = O(R^{-\gamma})$ as $R$ tends to infinity, with some $\gamma > 0$. Here $R$ is a parameter which controls the size of the truncated domain (usually the radius of a ball), $u^\infty$ is the solution to the original problem, and $u^R$ the approximating solution. The order $\gamma$ of the error is limited by the asymptotic decay of the problem's data *and* the choice of the boundary operator. This means even if the right hand sides of the boundary value problem have compact support, the choice of a an ABC in differential form fixes a $\gamma_{max}$, and of course the aim is then to obtain $\gamma_{max}$ as large as possible. Usually the estimates of the truncation error require a careful analysis for various boundary value problems in weighted Sobolev spaces.

These questions were barely investigated up to now in layer like domains, although they represent a class of domains with noncompact boundaries that are important for applications. However, to the best of our knowledge, there exists only one paper [24] where ABC were constructed in a layer-like domain for the Neumann problem for the Poisson equation without assuming axial symmetry which turns the three-dimensional problem into a two-dimensional one.

Our results are based on asymptotic expansions at infinity of solutions to the Stokes problem (1.2) and to the Navier-Stokes problem

$$
\begin{aligned}
-\nu\Delta v^\infty + \left(v^\infty \cdot \nabla\right)v^\infty + \nabla p^\infty &= f && \text{in } \Omega \\
\nabla \cdot v^\infty &= 0 && \text{in } \Omega \\
v^\infty &= 0 && \text{on } \partial\Omega\,.
\end{aligned}
\tag{1.3}
$$

These asymptotic expansions (see formulae (2.3)–(2.5)) were obtained in [19] with the help of a method developed in [14–17], they contain the plane harmonics $P_N$.

Other than in exterior domains, in a layer-like domain it is useful to define $\Omega_R$ as the intersection of $\Omega$ with an infinite cylinder of radius $R$, whose axis coincides with the $z$-axis. Then the boundary $\partial\Omega_R$ consists of two parts, the
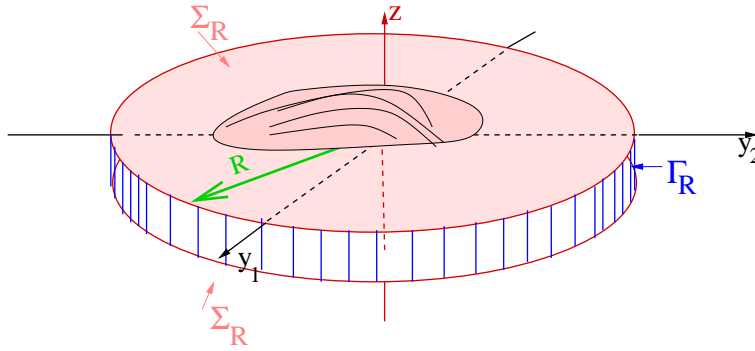


Figure 1: The domain $\Omega_R$.

common part of $\partial\Omega$ and $\partial\Omega_R$ denoted by $\Sigma_R$, and the truncation surface $\Gamma_R$. Using the notation of cylindrical coordinates $x = (y,z) \leftrightarrow (r,\varphi,z)$ with $r = (y_1^2 + y_2^2)^{\frac{1}{2}}$, this means

$$
\begin{aligned}
\Omega_R &= \{x = (r,\varphi,z) \in \Omega : r < R\} \\
\Sigma_R &= \{x \in \partial\Omega : r < R\} \\
\Gamma_R &= \left\{x \in \Omega : r = R, |z| < \tfrac{1}{2}\right\}.
\end{aligned}
$$

Note that for $R > R_0$, $\Gamma_R$ coincides with the lateral boundary of the cylinder (or better: truncated layer) $\Lambda_R = \left\{(r,\varphi,z) \in \mathbb{R}^3 : r < R, |z| < \tfrac{1}{2}\right\}$. The

approximation problem in the bounded domain $\Omega_R$ is composed from the Stokes (or Navier-Stokes) equations, the Dirichlet conditions restricted to $\Sigma_R$, and the ABC on $\Gamma_R$, in the linear case this means

$$
\begin{aligned}
-\nu\Delta v^R + \nabla p^R &= f && \text{in } \Omega_R \\
\nabla \cdot v^R &= 0 && \text{in } \Omega_R \\
v^R &= 0 && \text{on } \Sigma_R \\
M_R\big(v^R, p^R\big) &= 0 && \text{on } \Gamma_R,
\end{aligned}
\tag{1.4}
$$

where the operator $M_R$ has to be chosen properly. "Properly" means here that the problem (1.4) is well posed and the operator $M_R$ vanishes on the main asymptotic terms of $(v^\infty, p^\infty)$ – the latter feature arises from the experiences with ABC in other situations.

We describe the boundary operator $M_R$ briefly: Let $v_r$, $v_\varphi$ and $v_z$ denote the components of a vector field $v$ related to cylindrical coordinates $(r, \varphi, z)$. Any smooth function $F(y, z)$ on $\Gamma_R$ can be written as

$$
F(y, z) =: \left(z^2 - \frac{1}{4}\right)\overline{F}(y) + F^\#(y, z) \quad \text{with}
$$
$$
\overline{F}(y) = 30 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(z^2 - \frac{1}{4}\right) F(y, z)\,dz.
$$

Further let $\Pi_R$ denote the external Steklov-Poincaré operator (or Dirichlet-to-Neumann operator, see formulae (3.3)–(3.8) for more details) on the circle $\mathbb{S}_R = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 = R^2\}$, and finally $\overline{F}_\bullet = \overline{F} - (2\pi R)^{-1} \int_{\mathbb{S}_R} \overline{F}$ the projection of $\overline{F}$ onto the mean value free functions. Then the operator $M_R$ is defined by

$$
M_R(v, p) = \begin{pmatrix}
v_r^\# \\
v_\varphi^\# \\
v_z \\
\nu\dfrac{\partial}{\partial r}\overline{v_r} - \overline{p} + \nu\left\{\Pi_R\overline{v_r} + \dfrac{1}{R}\overline{v_r} + 10\,\Pi_R^{-1}\big(\overline{v_r}\big)_\bullet\right\} \\
\nu\dfrac{\partial}{\partial r}\overline{v_\varphi} + \nu\left\{\Pi_R\overline{v_\varphi} + \dfrac{1}{R}\overline{v_\varphi}\right\}
\end{pmatrix}
\tag{1.5}
$$

on $\Gamma_R$. *Why* it should have this particular form, this is explained in Section 3.

The boundary operator here is a combination of local and nonlocal operators. In Section 4 we prove existence of a unique solution to problem (1.4) with $M_R$ as in (1.5) (Theorem 4.6) and an error estimate of the form (see formula (4.30) in Theorem 4.8)

$$
\|v^\infty|_{\Omega_R} - v^R; H^1(\Omega_R)^3\| + R^{-1}\|p^\infty|_{\Omega_R} - p^R; L_2(\Omega_R)\| \le C_N\, R^{-N}\, \|f\|_{(N)}, \tag{1.6}
$$

where the constant $C_N$ does not depend on the radius $R \geq R_0$ and an appropriate weighted norm $\|f\|_{(N)}$ of the right-hand side in the original problem. We emphasize that, for the linear problem, the exponent $N$ can be made arbitrarily large provided that the right-hand side $f$ decays quickly enough. This is due to the fact that here the features of asymptotic ABC and non-reflecting ABC are combined; moreover, this result cannot be achieved without knowing the asymptotic form of the solution.

Let us also give a short guide through the other sections of the paper. The results on existence, uniqueness and the asymptotics of the solutions to (1.2) are outlined in Sections 2. As already mentioned, the ABC for the linear problem are derived in Section 3. The well-posedness of the approximation problem and error estimates are proved in Section 4. The most tricky point is here to find a solution to the continuity equation together with an estimate that controls the behavior of $H^1(\Omega_R)$-norm with respect to $R$ (Lemma 4.3).

The last two sections are devoted to the Navier-Stokes problem (1.3). Under suitable restrictions for the data it is possible to obtain solutions to the nonlinear problem with the same ABC as for the linear problem together with error estimates of type (1.6) (see Theorem 6.4). However, by using existence results of [20] and the results on the asymptotic behavior of the solutions to (1.2) (see [19, 27]) and (1.3) it becomes clear how the nonlinearity influences the asymptotics at infinity of suitable strong solutions to (1.3) – these results are explained in Section 5. Thus for the nonlinear problem the order of convergence is limited by $N \leq 3$ in (1.6), even if the right hand side $f$ is infinitely smooth with compact support.

## 2. General notations, basic function spaces and asymptotics of solutions to the Stokes problem

In view of the particular geometry of a layer-like domain $\Omega$ it is convenient to fix the following conventions: We always have $y \in \mathbb{R}^2$, $x \in \mathbb{R}^3$ with $x = (y, z)$, with corresponding Euclidean norms $|y|$, $|x|$, and we use $(r, \varphi)$ to denote polar coordinates related to $y$ as well as $(r, \varphi, z)$ to denote cylindrical coordinates related to $x$.

We recall some standard notations for function spaces: For an arbitrary domain $G \subset \mathbb{R}^n$ (here only $n = 2, 3$) with closure $\overline{G}$ and boundary $\partial G$, the notation $C_0^\infty(G)$ indicates the set of all smooth functions with compact support in $G$, the symbol $H^m(G)$, $m \in \mathbb{N}$, stands for the Sobolev space containing all functions $w \in L^2(G)$ such that all derivatives $\partial^\alpha w \in L^2(G)$ up to $|\alpha| = m$ (using the common multi-index terminology), by $\mathring{H}^m(G)$ we indicate the closure of $C_0^\infty(G)$ in $H^m(G)$. We use the lower index $y$ in $\nabla_y$, $\partial_y^\alpha$ to indicate derivatives with respect to the plane variables.

We indicate the scalar-product in $L^2(G)$ by $(\,\cdot\,,\cdot\,)_G$ – without distinguishing between scalar functions and vector fields, similarly we use $(\,\cdot\,,\cdot\,)_\Xi$ for suitable manifolds $\Xi$ (mostly $\Xi \subset \partial\Omega_R$).

As shown in [15, 17–19], the following anisotropic weighted Sobolev norms (2.1) are especially adapted to a wide class of elliptic boundary value problems in layer-like domains. We recall that $x = (y,z)$ and $r = |y|$, thus derivatives $\partial^\beta = \partial_x^\beta$ can be split into $\partial_x^\beta = \partial_y^\alpha\partial_z^j$, with $|\alpha| + j = |\beta|$. By $L_\beta^2(\Omega)$, we understand the space of all locally square summable functions with finite norm

$$\|w; L_\beta^2(\Omega)\| = \|(1 + r)^\beta w; L^2(\Omega)\|.$$

We also introduce the space $\mathcal{W}_\beta^l(\Omega)$ as the completion of $C_0^\infty(\overline{\Omega})$ with respect to the anisotropic weighted norm

$$\|w; \mathcal{W}_\beta^l(\Omega)\| = \left\{ \sum_{|\alpha|+j\leq l} \|\partial_y^\alpha\partial_z^j w; L_{\beta-l+|\alpha|}^2(\Omega)\|^2 \right\}^{\frac{1}{2}}. \tag{2.1}$$

In contrast to the usual "isotropic" Kondratiev norm (see, e.g., [11, 21]) where derivatives of a fixed order in any direction are provided with the same exponent in the weight function, the weighted norm (2.1) is called "anisotropic" [17]. We emphasize that for each differentiation in $y_1$ and $y_2$ the weight exponent in (2.1) is increased by 1, while for derivatives in $z$ the weight exponent is kept.

The first part of the following lemma on the weak solution of problem (1.2) is a special part of [20, Theorems 3.1, 3.2], while the second part follows from [18, Theorem 4.1 (i)] .

**Lemma 2.1.**

(i) *Let $f \in L^2(\Omega)^3$ and $\beta < -1$. There exist $v^\infty \in \mathring{H}^1(\Omega)^3$ and $p^\infty \in L_\beta^2(\Omega)$ which satisfy relations (1.2)$_{2,3}$ and the integral identity*

$$\nu\big(\nabla v^\infty, \nabla w\big)_\Omega = \big(p^\infty, \nabla \cdot w\big)_\Omega + \big(f, w\big)_\Omega \qquad \forall w \in C_0^\infty(\Omega)^3.$$

*The solution $(v^\infty, p^\infty)$ is determined up to an additive constant in its pressure component.*

(ii) *If $-1 < \beta < 0$ and additionally $f \in L_{\beta+2}^2(\Omega)$, then there exists a unique weak solution $v^\infty \in \mathring{H}^1(\Omega)^3$ and $p^\infty \in L_\beta^2(\Omega)$ to problem (1.2). In this case the estimate*

$$\|v^\infty; H^1(\Omega)\| + \|p^\infty; L_\beta^2(\Omega)\| \leq c_\beta \|f; L_{\beta+2}^2(\Omega)\|,$$

*is valid where the constant $c_\beta$ depends on $\nu$, $\beta$, and $\Omega$, but is independent of $f$.*

Note that the assumption on $f$ used in Lemma 2.1 can be weakened (cf. [20]). The additive constant in pressure appears in the first part of the lemma because a constant function $p$ belongs to the space $L^2_\beta(\Omega)$ if $\beta < -1$.

If the right-hand side $f$ of problem (1.2) decays sufficiently fast, the solution $(v^\infty, p^\infty)$ gets a special asymptotic form, as it was shown in [19], here we present simplified results which are sufficient for the further use in this paper. To this end, we introduce a cut-off function $\chi$ specified as follows:

$$\chi \in C_0^\infty(\mathbb{R}), \qquad \chi(t) = \begin{cases} 1 & \text{for } t > 1 \\ 0 & \text{for } t \geq 2. \end{cases} \tag{2.2}$$

Further we distinguish between the vector of longitudinal components, $v_y^\infty$, and the transversal component $v_z^\infty$ of the vector $v^\infty$.

**Theorem 2.2.** *Let $l \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, $-1 < \beta < 0$, $N \in \mathbb{N} := \{1, 2, 3, \dots\}$ and*

$$f \in \mathcal{W}_\gamma^{l+2}(\Omega)^3, \quad N + l + 4 > \gamma > N + l + 3\,.$$

*Then for the solution $v^\infty \in \overset{o}{H}{}^1(\Omega)^3$, $p \in L^2_\beta(\Omega)$ to the Stokes problem (1.2) the following asymptotic representation is valid:*

$$p^\infty(y, z) = P^\infty(y, z) + \widetilde{p}^\infty(y, z), \quad v^\infty(y, z) = V^\infty(y, z) + \widetilde{v}^\infty(y, z) \tag{2.3}$$

*with*

$$\begin{aligned} P^\infty(y, z) &= \left(1 - \chi(R_0^{-1}|y|)\right) P_N(y) \\ V_y^\infty(y, z) &= \left(1 - \chi(R_0^{-1}|y|)\right) \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)\nabla_y P_N(y) \\ V_z^\infty(y, z) &= 0. \end{aligned} \tag{2.4}$$

*The function $P_N$ is a plane harmonic, namely*

$$P_N(y) = \sum_{j=1}^N r^{-j}\left(a_j \cos(j\varphi) + b_j \sin(j\varphi)\right), \quad y = (r\cos\varphi, r\sin\varphi) \tag{2.5}$$

*with suitable constants $a_j$, $b_j$. The remainders satisfy the inclusions*

$$\begin{aligned} \widetilde{v}_y^\infty &\in \mathcal{W}_{\gamma-1}^{l+2}(\Omega)^2, & \widetilde{v}_z^\infty &\in \mathcal{W}_\gamma^{l+2}(\Omega) \\ \widetilde{p}^\infty &\in \mathcal{W}_{\gamma-1}^{l+3}(\Omega), & \partial_z\widetilde{p}^\infty &\in \mathcal{W}_\gamma^{l+2}(\Omega). \end{aligned} \tag{2.6}$$

*Moreover, these remainders and the coefficients $a_j$ and $b_j$ fulfil the estimate*

$$\begin{aligned} &\|\widetilde{v}_y^\infty; \mathcal{W}_{\gamma-1}^{l+2}(\Omega)\| + \|\widetilde{v}_z^\infty; \mathcal{W}_\gamma^{l+2}(\Omega)\| + \|\widetilde{p}^\infty; \mathcal{W}_{\gamma-1}^{l+3}(\Omega)\| \\ &+ \|\partial_z\widetilde{p}^\infty; \mathcal{W}_\gamma^{l+2}(\Omega)\| + \sum_{j=1}^N \left(|a_j| + |b_j|\right) \leq c_{l,\gamma}\,\|f; \mathcal{W}_\gamma^{l+2}(\Omega)\| \end{aligned} \tag{2.7}$$

*with a constant $c_{l,\gamma}$, independent of the right-hand side $f$.*

The results presented above follow from [19, Theorem 5.3 (ii)] and [18, Theorem 4.1]. The estimates of the remainders in (2.4) are not optimal with respect to the smoothness properties of the data and the solutions, in particular, the assumptions on the right-hand side $f$ are too restrictive.

We emphasize that different weight indices in (2.6) and (2.7) reflect the different asymptotic behavior at infinity of $v_y^\infty$, $p^\infty$ and $v_z^\infty$. If the right hand side $f \in H_{loc}^{l+2}(\Omega)$ has a compact support, then $f \in \mathcal{W}_\gamma^{l+2}(\Omega)^3$ for *any* $\gamma \in \mathbb{R}$ and Theorem 2.2 provides an explicit information on the power-law asymptotic behavior of the solution. Especially in the case $f \in C_0^\infty(\overline{\Omega})^3$ the indices $l$ and $N$ can be taken arbitrarily large. Sending $N$ to infinity then *formally* there appear series for $V^\infty$, $P^\infty$, we emphasize that the series do not converge in general. It can be easily verified that the detached terms $V^\infty$, $P^\infty$ in (2.4) do not belong to the spaces indicated in (2.6). However, in the case $\gamma \in (N+l+3, N+l+4)$, the next asymptotic terms, which appear if we replace the term $P_N$ in (2.4) by

$$P_{N+1}(y) - P_N(y) = r^{-N-1}\big(a_{N+1}\cos((N+1)\varphi) + b_{N+1}\sin((N+1)\varphi)\big),$$

belong to those spaces.

## 3. The choice of the artificial boundary conditions

In this section we motivate the choice of the operator $M_R$ and the weak formulation of the approximation problem (1.4). The main idea can be explained as follows: To any vector field $v$ we associate the components $(v_r, v_\varphi, v_z)$ related to the cylindrical coordinates, further we denote by $e_r$ the unit vector in the radial direction. Suppose $(v^R, p^R)$ is a sufficiently smooth solution to problem $(1.4)_{1,2,3}$, and $w \in H^1(\Omega_R)$ is a divergence free vector field with $w = 0$ on $\Sigma_R$ (the "top" and the "bottom" faces of $\Omega_R$). By partial integration we obtain

$$\big(f, w\big)_{\Omega_R} = \nu\big(\nabla v^R, \nabla w\big)_{\Omega_R} - \big(\nu\partial_r v^R - e_r p^R, w\big)_{\Gamma_R}. \tag{3.1}$$

For sufficiently smooth vector-fields $w$, we choose now a suitable decomposition $w = w^\flat + w^{\flat\flat}$ near the truncation surface $\Gamma_R$ together with a linear operator $B$ which is well defined for $w^{\flat\flat}|_{\Gamma_R}$ and define the artificial boundary condition on $\Gamma_R$ as combination of Dirichlet conditions for $v^R$ and mixed conditions:

$$\left.\begin{aligned} (v^R)^\flat &= 0 \\ \partial_r(v^R)^{\flat\flat} - (e_r p^R)^{\flat\flat} &= -B(v^R)^{\flat\flat} \end{aligned}\right\} \text{ on } \Gamma_R. \tag{3.2}$$

The decomposition and the operator $B$ are fixed in such a way, that the following two properties hold true:

I. For *any* $N \in \mathbb{N}$, the main asymptotic terms $(V^\infty, P^\infty)$ of the solution $(v^\infty, p^\infty)$ described in Theorem 2.2 fulfill (3.2) on $\Gamma_R$.

II. The quadratic form $\mathbf{q}_R(w, w) := (Bw^{\flat\flat}, w^{\flat\flat})_{\Gamma_R}$ is continuous on $H^1(\Omega_R)^3$ and nonnegative for all $w \in H^1(\Omega_R)$.

The second condition will ensure the unique solvability of the approximation problem (1.4) while using $(3.1)_1$, the first one will lead to good error estimates, as we will see in the next section. Since the number of summands in $V_y^\infty$ and $P^\infty$ in the representation (2.3)–(2.4) increases with $N$ we cannot expect to satisfy Condition I with a local operator $B$. Then again for $|y| > 2R_0$ and any fixed $z \in (-\frac{1}{2}, \frac{1}{2})$, the components of $V_y^\infty$ and $P^\infty$ consist of plane harmonics. Thus we introduce the notation $\mathbb{S}_R = \{y \in \mathbb{R}^2 : |y| = R\}$; let further $H^s(\mathbb{S}_R)$ denote the usual Sobolev-Slobodetskiĭ space. It is well known that for $R > 0$, $s \geq \frac{1}{2}$, $h \in H^s(\mathbb{S}_R)$, there exists a unique bounded harmonic extension $H$ to the domain $\{y \in \mathbb{R}^2 : |y| > R\}$. The external Steklov-Poincaré operator $\Pi_R$ on $\mathbb{S}_R$ is defined then by $\Pi_R h = -\partial_r H|_{\mathbb{S}_R} \in H^{s-1}(\mathbb{S}_R)$ (see [30], e.g.). By means of Fourier series, $\Pi_R$ as well as its inverse can be calculated elementarily. We recall that for any $s \in \mathbb{R}$, $R > 0$, the space $H^s(\mathbb{S}_R)$ can be identified with the set of (weighted) Fourier series

$$h(y) = \alpha_0 + \sum_{j=1}^\infty R^{-j}\{\alpha_j \cos(j\varphi) + \beta_j \sin(j\varphi)\}, \tag{3.3}$$

$(y = (R\cos\varphi, R\sin\varphi))$ such that $|\alpha_0|^2 + \sum j^{2s}R^{-2j}(|\alpha_j|^2 + |\beta_j|^2) < \infty$ *, and the corresponding bounded harmonic extension of $h$ to the exterior of the circle $\mathbb{S}_R$ is

$$H(y) = \alpha_0 + \sum_{j=1}^\infty r^{-j}\{\alpha_j \cos(j\varphi) + \beta_j \sin(j\varphi)\}. \tag{3.4}$$

We collect the properties of $\Pi_R$ we need for the following, each of which can be derived easily with (3.3) and (3.4).

**Proposition 3.1.** *For $s \geq \frac{1}{2}$, $h \in H^s(\mathbb{S}_R)$ with Fourier representation (3.3), it holds*

$$\Pi_R h(R\cos\varphi, R\sin\varphi) = \sum_{j=1}^\infty jR^{-1-j}\{\alpha_j \cos(j\varphi) + \beta_j \sin(j\varphi)\}. \tag{3.5}$$

*The kernel of $\Pi_R$ consists of the constant functions, while the range of $\Pi_R$ is given by $H_\bullet^{s-1}(\mathbb{S}_R) =: \{h \in H^{s-1}(\mathbb{S}_R) : a_0 = 0 \text{ in } (3.3)\}$. Thus for any $s \geq \frac{1}{2}$, the operator $\Pi_R$ defines an isomorphism from $H_\bullet^s(\mathbb{S}_R)$ onto $H_\bullet^{s-1}(\mathbb{S}_R)$. For $h_\bullet \in H_\bullet^{s-1}(\mathbb{S}_R)$, i.e. $a_0 = 0$ in (3.3), we have*

$$\Pi_R^{-1}h_\bullet(R\cos\varphi, R\sin\varphi) = \sum_{j=1}^\infty \frac{R^{1-j}}{j}\{\alpha_j \cos(j\varphi) + \beta_j \sin(j\varphi)\}. \tag{3.6}$$

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*Note that the transformation $\mathbf{y} = R^{-1}y$ relates to $h$ the function $\mathbf{h}(\mathbf{y}) = h(R\mathbf{y})$, defined on the unit circle, with Fourier coefficients $\tilde\alpha_j = R^{-j}\alpha_j$, $\tilde\beta_j = R^{-j}\beta_j$

If $g \in H^1(\mathbb{S}_R)$ and $h \in L^2(\mathbb{S}_R)$ with Fourier coefficients $\alpha_{j1}, \beta_{j1}$ and $\alpha_{j2}, \beta_{j2}$, respectively, then

$$(\Pi_R g, h)_{\mathbb{S}_R} = \pi \sum_{j=1}^{\infty} R^{-2j} j (\alpha_{j1} \alpha_{j2} + \beta_{j1} \beta_{j2}).$$

In particular, for $g, h \in H^1(\mathbb{S}_R)$ and $g_\bullet, h_\bullet \in L^2_\bullet(\mathbb{S}_R)(= \{h : \int_{\mathbb{S}_R} h = 0\})$ it follows

$$(\Pi_R g, h)_{\mathbb{S}_R} = (g, \Pi_R h)_{\mathbb{S}_R}, \qquad (\Pi_R^{-1} g_\bullet, h_\bullet)_{\mathbb{S}_R} = (g_\bullet, \Pi_R^{-1} h_\bullet)_{\mathbb{S}_R}. \qquad (3.7)$$

Moreover, the following scaling properties are valid:

$$\Pi_R h = R^{-1} \Pi h(R \cdot), \qquad \Pi_R^{-1} h = R \Pi^{-1} h(R \cdot) \qquad (3.8)$$

with the notations $\mathbb{S}$ and $\Pi$ for the unit circle and its external Steklov-Poincaré operator. □

Taking into account the special form of $V^\infty$, $P^\infty$ in the representation (2.4), we introduce a convenient decomposition of functions defined on a neighborhood of $\Gamma_R$. Namely we put $\psi(z) = z^2 - \frac{1}{4}$, then an easy calculation shows $\int_{-\frac{1}{2}}^{\frac{1}{2}} |\psi|^2 = \frac{1}{30}$. For $F \in L^2(\Gamma_R)$, we obtain

$$F(y, z) = \overline{F}(y) \psi(z) + F^\#(y, z) \quad \text{with}$$

$$\overline{F}(y) = 30 \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(z) F(y, z) dz, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(z) F^\#(y, z) dz = 0, \qquad (3.9)$$

valid for almost all $y \in \mathbb{S}_R := \{y \in \mathbb{R}^2 : |y| = R\}$. It is also clear that this construction works componentwise for vector fields and also on any subset of the layer $\Lambda$ of the form $A \times (-\frac{1}{2}, \frac{1}{2})$ where, e.g., $A$ is an open subset of $\mathbb{R}^2$. We mention two simple, but useful identities for sufficiently regular functions $F$ defined in a neighborhood of $\Gamma_R$:

$$(\overline{F}\psi, \overline{G}\psi)_{\Gamma_R} = \frac{1}{30}(\overline{F}, \overline{G})_{\mathbb{S}_R}, \qquad \overline{\partial_{y_j} F} = \partial_{y_j} \overline{F}, j = 1, 2. \qquad (3.10)$$

**Lemma 3.2.** *For any $N \in \mathbb{N}$, $R > 2R_0$, the main asymptotic terms $(V_y^\infty, V_z^\infty)$ and $P^\infty$ defined as in (2.4) fulfill the following boundary conditions on $\Gamma_R$:*

$$V_z^\infty(y, z) = 0, \qquad (V_y^\infty)^\#(y, z) = 0 \qquad (see\ (3.9))$$

$$\nu \frac{\partial}{\partial r} \overline{V_r^\infty}(y) - \overline{P^\infty}(y) = -\nu \left\{ \Pi_R \overline{V_r^\infty}(y) + \frac{1}{R} \overline{V_r^\infty}(y) + 10 \Pi_R^{-1} (\overline{V_r^\infty})_\bullet(y) \right\}$$

$$\nu \frac{\partial}{\partial r} \overline{V_\varphi^\infty} = -\nu \left\{ \Pi_R \overline{V_\varphi^\infty} + \frac{1}{R} \overline{V_\varphi^\infty} \right\}, \qquad (3.11)$$

*where "$\bullet$" stands for the projection on the space of mean-value free functions on $\mathbb{S}_R$,*

$$w_\bullet(y) = w(y) - \frac{1}{2\pi R} \int\limits_{\mathbb{S}_R} w(y)\, ds_y,$$

$\Pi_R$, $\Pi_R^{-1}$ *are defined as in Proposition 3.1 and* $V_y^\infty = (V_r^\infty, V_\phi^\infty)$.

*Proof.* The Dirichlet conditions $(3.11)_1$ follow immediately from the asymptotic representation $(2.4)_{2,3}$ together with (3.10), if we observe that $1 - \chi(R_0^{-1}|y|) = 1$ for $|y| > 2R_0$. From (3.10) we obtain in particular $\partial_r V_r^\infty = \psi \partial_r \overline{V_r^\infty}$, $\partial_r V_\varphi^\infty = \psi \partial_r \overline{V_\varphi^\infty}$, $-5P^\infty = \overline{P^\infty}$ on $\Gamma_R$, while from (2.4) and (2.5) we get $\overline{V_r^\infty}(y) = -\frac{1}{2\nu} R^{-1} \sum_{j=1}^N j R^{-j}\big(a_j \cos(j\varphi)) + b_j \sin(j\varphi)\big)$, hence

$$\nu \frac{\partial}{\partial r} \overline{V_r^\infty}(y) - \overline{P^\infty(y)} = \sum_{j=1}^N \left\{ \frac{1}{2} j(j+1) R^{-2-j} + 5 R^{-j} \right\} \big( a_j \cos(j\varphi) + b_j \sin(j\varphi) \big).$$

From here it follows $(3.11)_2$ with (3.5) and (3.6), we point out that $\big(\overline{V_r^\infty}\big)_\bullet$ belongs to the domain of the inverse operator $\Pi_R^{-1}$. Analogous calculations lead from (2.3) with (3.5) to $(3.11)_3$. $\qquad\square$

Bearing the remarks at the beginning of this section in mind, this leads to the operator operator $M_R$ defined by (1.5) in the approximation problem (1.4). We point out that (3.11) is just (3.2), with

$$v^\flat = (v_y - \overline{v_y}\psi, v_z), \quad v^{\flat\flat} = (\overline{v_y}\psi, 0), \quad Bv^{\flat\flat} = (\psi \overline{B}\,\overline{v_y}, 0), \tag{3.12}$$

where $\overline{B}$ is given by the right hand sides of $(3.11)_{2,3}$, and $(v^\flat, w^{\flat\flat})_{\Gamma_R} = 0$ for all $v, w \in L^2(\Gamma_R)$.

Finally we observe the following: if $w = (w_y, w_z) \in H^1(\Omega_R)^3$ fulfils the Dirichlet conditions

$$w = 0 \ \text{ on } \Sigma_R, \qquad w_z = 0, \quad w_y^\# = 0 \ \text{ on } \Gamma_R \tag{3.13}$$

and $v^R$, $p^R$ is a solution to (1.5), (1.4), then again using identities (3.10), the integral over $\Gamma_R$ in (3.1) reduces to

$$
\begin{aligned}
-\big(\nu\partial_r v^R &- e_r p^R, w\big)_{\Gamma_R} \\
&= -\frac{1}{30}\left\{ \big(\nu\partial_r \overline{v_r^R} - \overline{p^R}, \overline{w_r}\big)_{\mathbb{S}_R} + \big(\nu\partial_r \overline{v_\varphi}, \overline{w_\varphi}\big)_{\mathbb{S}_R} \right\} \\
&= \frac{\nu}{30}\left\{ \big(\Pi_R \overline{v_r^R}, \overline{w_r}\big)_{\mathbb{S}_R} + \big(\Pi_R \overline{v_\varphi^R}, \overline{w_\varphi}\big)_{\mathbb{S}_R} + \frac{1}{R}\big(\overline{v_r^R}, \overline{w_r}\big)_{\mathbb{S}_R} \right. \\
&\quad \left. + \frac{1}{R}\big(\overline{v_\varphi^R}, \overline{w_\varphi}\big)_{\mathbb{S}_R} + 10\big(\Pi_R^{-1}\big(\overline{v_r^R}\big)_\bullet, \big(\overline{w_r}\big)_\bullet\big)_{\mathbb{S}_R} \right\} := \mathbf{q}_R(v^R, w).
\end{aligned}
\tag{3.14}
$$

Clearly, $\mathbf{q}_R(w, w)$ defines a nonnegative quadratic form on $H^1(\Omega_R)^3$.

## 4. Solution of the linear approximation problem

After the formal derivation of the ABC we establish the weak approximation problem and show the existence of weak solutions. In the following we always assume $R > 2R_0$ at least (with $R_0$ as in Section 1) and introduce the domains

$$\Lambda_R = \{(y, z) \in \Lambda : r = |y| < R\} \subset \mathbb{R}^3$$
$$A_R = \left\{y \in \mathbb{R}^2 : \frac{R}{2} < r < R\right\} \subset \mathbb{R}^2 \tag{4.1}$$
$$\Xi_R = A_R \times \left(-\frac{1}{2}, \frac{1}{2}\right) \subset \mathbb{R}^3.$$

Obviously we have $\Xi_R \subset \Omega_R \cap \Lambda_R$. For $v \in H^1(\Omega_R)^3$ and $x \in \Xi_R$, we can use the decomposition (3.9) and obtain $v_y(x) = \overline{v_y}(y)\psi(z) + (v_y)^\#(x)$, and clearly

$$\|\overline{v_y}; H^1(A_R)\| + \|v_y^\#; H^1(\Xi_R)\| \leq C\|v; H^1(\Omega_R)\|$$

with a constant independent on $R$. Thus, the space

$$\mathcal{H}(\Omega_R) = \{w \in H^1(\Omega_R)^3 : w \text{ fulfils } (3.13)\}$$

is a closed subspace of $H^1(\Omega_R)^3$.

**Definition 4.1** (Weak solution of the approximation problem)**.** We put

$$\mathcal{H}_\sigma(\Omega_R) = \{w \in \mathcal{H}(\Omega_R) : \nabla \cdot w = 0\}.$$

If $\Phi$ is a linear functional on $\mathcal{H}(\Omega_R)$, continuous with respect to the $H^1(\Omega_R)$-norm, we call a pair $V \in \mathcal{H}_\sigma(\Omega_R), P \in L^2(\Omega_R)$ as above a weak solution to the general approximation problem, provided

$$\Phi(w) = \nu(\nabla V, \nabla w)_{\Omega_R} - (P, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(V, w), \tag{4.2}$$

where $\Phi(w)$ indicates the value of the functional $\Phi$ at the test function $w$ and $\mathbf{q}_R$ is defined as in (3.14).

A weak solution to the approximation problem (1.4) is a pair $(v^R, p^R)$ which satisfies the definition above with $\Phi(w) = (f, w)_{\Omega_R}$.

As usual, the existence of a weak solution to Problem (4.2) is reduced to prove the existence of $v^R \in \mathcal{H}_\sigma(\Omega_R)$ by means of the Lax-Milgram lemma, and then recover the pressure while treating the problem $\nabla \cdot w = g$. We start with the auxiliary result on the solution of the divergence equation. To this end we recall a well known result on this problem.

**Proposition 4.2** ( [12], see also [33, Lemma 2.3.1])**.** *Let $\omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, and $G \in L^2_\bullet(\omega)$ (i.e., $\int_\omega G dx = 0$). Then there exists $W \in \overset{\circ}{H}{}^1(\omega)^3$ with $\nabla \cdot W = G$ and*

$$\|W; H^1(\omega)\| \leq C(\omega)\|G; L^2(\omega)\|. \tag{4.3}$$

**Lemma 4.3** (Solution of the continuity equation). *For any $g \in L^2(\Omega_R)$ there exists a $w \in \mathcal{H}(\Omega_R)$ with $\nabla \cdot w = g$ and*

$$\|w; H^1(\Omega_R)\| + R^{\frac{3}{2}}\|w_y; L^2(\Gamma_R)\| \leq CR\|g; L^2(\Omega_R)\|, \tag{4.4}$$

*where $C$ is independent of $g$ and $R$.*

*Proof.* The basic idea is the following: We split the problem on $\Omega_R$ into a problem on the fixed domain, which contains the perturbed part of the boundary, and a problem on the domain $\Omega_R \setminus \Omega_{R_0}$, which can be considered as part of the cylinder $\Lambda_R$. On $\Lambda_R$ the dependence on $R$ of the norms is controlled by a scaling argument. In both parts we use Proposition 4.2, thus we have to juggle a bit with mean values.

To fill in the details, let us first assume that $R > 3R_0$, and recall the notation $x = (y, z)$, $|y| = r$. We define the vector field $\mathscr{W}^T(x)$ by

$$\mathscr{W}_y^T(x) = \left(z^2 - \frac{1}{4}\right)\nabla_y\left((1 - \chi(T^{-1}r))\ln r\right), \quad \mathscr{W}_z^T(x) = 0,$$

where $\chi$ is the same cut-off function as in (2.4). For $\frac{R}{2} \geq T \geq R_0$, it is obvious that $\mathscr{W}^T|_{\Omega_R} \in \mathcal{H}(\Omega_R)$, and since $\Delta_y \ln|y| = 0$ for $|y| \neq 0$, we have

$$\nabla \cdot \mathscr{W}^T(x) = 0 \quad \text{for } r < T \text{ and } r > 2T. \tag{4.5}$$

Integration by parts gives, for any $R \geq 2T$,

$$\int_{\Omega_R} \nabla \cdot \mathscr{W}^T(x)dx = \int_{\Gamma_R} \mathscr{W}_r^T(x)ds = \int_{\Gamma_R}\left(z^2 - \frac{1}{4}\right)\partial_r \ln r dx = -\frac{\pi}{3}.$$

We put $G_{3R_0} = \int_{\Omega_{3R_0}} g(x)\, dx$, then clearly

$$|G_{3R_0}| \leq C\|g; L^2(\Omega_R)\|. \tag{4.6}$$

Now we look for the solution $w$ to the continuity equation as $w(x) = w^*(x) - \frac{3}{\pi}G_{3R_0}\mathscr{W}^{R_0}(x)$. Then $w^*$ has to solve

$$\nabla \cdot w^* = g + 3\pi^{-1}G_{3R_0}\nabla \cdot \mathscr{W}^{R_0} =: g^1 + g^2$$
$$g^1 = \mathcal{X}_{\Omega_{3R_0}}\left(g + 3\pi^{-1}G_{3R_0}\nabla \cdot \mathscr{W}^{R_0}\right)$$
$$g^2 = \mathcal{X}_{\Omega_R \setminus \Omega_{3R_0}}\left(g + 3\pi^{-1}G_{3R_0}\nabla \cdot \mathscr{W}^{R_0}\right) = \mathcal{X}_{\Omega_R \setminus \Omega_{3R_0}}g,$$

where $\mathcal{X}_\omega$ is the indicator function of the set $\omega$, for the representation of $g^2$ we used (4.5). By construction, we have $\int_{\Omega_{3R_0}} g^1 = 0$, hence by Proposition 4.2, we find $w^1 \in \mathring{H}^1(\Omega_{3R_0})$ which, after extension with zero, fulfils $\nabla \cdot w^1 = g^1$ on $\Omega_R$, and by (4.3) and (4.6),

$$\|w^1; H^1(\Omega_R)\| = \|w^1; H^1(\Omega_{3R_0})\| \leq C\|g^1; L^2(\Omega_{3R_0})\| \leq C\|g; L^2(\Omega_R)\|. \tag{4.7}$$

It remains to find $w^2 \in H^1(\Omega_R \backslash \Omega_{R_0})$ with $\nabla \cdot w^2 = g^2$ on $\Omega_R \backslash \Omega_{R_0}$, $w^2 = 0$ on $\Sigma_R \cup \Gamma_{R_0}$, and $w^2$ fulfils (3.13). Then the extension with zero on $\Omega_{R_0}$ leads to an element in $\mathcal{H}(\Omega_R)$ which solves $\nabla \cdot w^2 = g^2$.

To construct $w^2$ together with the desired estimates we first extend $g^2$ with zero to the whole cylinder $\Lambda_R$ and use a scaling argument. With

$$\mathbf{y} = \frac{y}{R}, \quad \mathbf{x} = (\mathbf{y}, z), \quad \mathbf{g}^2(\mathbf{x}) = g^2(R\mathbf{y}, z) \text{ and}$$
$$\mathbf{v}(\mathbf{x}) = \left( \frac{1}{R} v_y(R\mathbf{y}, z), v_z(R\mathbf{y}, z) \right) \tag{4.8}$$

we get: The problem $\nabla \cdot v = g^2$ in $\Lambda_R$ is equivalent to $\nabla_{\mathbf{x}} \cdot \mathbf{v} = \mathbf{g}^2$ in $\Lambda_1$. Moreover, we have

$$\|\mathbf{g}^2; L^2(\Lambda_1)\| = CR^{-1}\|g^2; L^2(\Lambda_R)\|. \tag{4.9}$$

Now we use a similar trick as above. We put $\mathbf{G} = \int_{\Lambda_1} \mathbf{g}^2(\mathbf{x}) d\mathbf{x}$, then

$$|\mathbf{G}| \leq C\|\mathbf{g}^2; L^2(\Lambda_1)\|, \tag{4.10}$$

and we look for $\mathbf{v}$ as

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^*(\mathbf{x}) - \frac{3}{\pi} \mathbf{G} \mathscr{W}^{\frac{1}{2}}(\mathbf{x}) \quad \text{with } \nabla_{\mathbf{x}} \cdot \mathbf{v}^*(\mathbf{x}) = \mathbf{g}^2(\mathbf{x}) + \frac{3}{\pi} \mathbf{G} \nabla_{\mathbf{x}} \cdot \mathscr{W}^{\frac{1}{2}}(\mathbf{x}).$$

Since the right-hand side of the divergence equation is now mean value free, Proposition 4.2 gives $\mathbf{v}^* \in \overset{\circ}{H}{}^1(\Lambda_1)$ and the estimate

$$\|\nabla_{\mathbf{x}} \mathbf{v}^*; L^2(\Lambda_1)\| \leq C(\Lambda_1)\|\mathbf{g}^2 + \frac{3}{\pi} \mathbf{G} \nabla_{\mathbf{x}} \cdot \mathscr{W}^{\frac{1}{2}}; L^2(\Lambda_1)\|$$
$$\leq C(\Lambda_1)\|\mathbf{g}^2; L^2(\Lambda_1)\|.$$

With (4.10), we also have

$$\|\nabla \mathbf{v}; H^1(\Lambda_1)\| \leq C\|\mathbf{g}^2; L^2(\Lambda_1)\|.$$

If we apply the relations (4.8) to obtain $v$ with $\nabla \cdot v = g^2$ on $\Lambda_R$, we see

$$\|\nabla_y v_y; L^2(\Lambda_R)\| = R \, \|\nabla_{\mathbf{y}} \mathbf{v}_y; L^2(\Lambda_1)\|$$
$$\|\nabla_z v_y; L^2(\Lambda_R)\| = R^2 \, \|\nabla_z \mathbf{v}_y; L^2(\Lambda_1)\|$$
$$\|\nabla_y v_z; L^2(\Lambda_R)\| = \|\nabla_{\mathbf{y}} \mathbf{v}_z; L^2(\Lambda_1)\| \tag{4.11}$$
$$\|\nabla_z v_z; L^2(\Lambda_R)\| = R \, \|\nabla_z \mathbf{v}_z; L^2(\Lambda_1)\|.$$

Together with (4.9) and Poincaré's inequality this leads to

$$\|v; H^1(\Lambda_R)\| \leq CR\|g^2; L^2(\Lambda_R)\| = CR\|g^2; L^2(\Omega_R)\|. \tag{4.12}$$

Although the support of $g^2$ is contained in $\Omega_R \setminus \Omega_{3R_0}$, the support of $v$ may be larger. Thus we cut $v$ again, using the same function $\chi$ as in formulae (2.2). Put $\chi(x) = \chi(R_0^{-1}(r))$, then clearly the vector field $(1-\chi)v$, extended by zero, belongs to $\mathcal{H}(\Omega_R)$, moreover,

$$\|(1-\chi)v; H^1(\Omega_R)\| \le C(R_0, \chi)\|v; H^1(\Lambda_R)\| \tag{4.13}$$

and

$$\nabla \cdot ((1-\chi)v) = (1-\chi)\, g^2 - (\nabla(1-\chi)) \cdot v = g^2 - (\nabla\chi) \cdot v.$$

The support of $(\nabla\chi) \cdot v$ is contained in the annular domain $\Xi_{2R_0} = \Omega_{2R_0} \setminus \overline{\Omega}_{R_0}$, we calculate the mean-value over $\Xi_{2R_0}$:

$$\int_{\Xi_{R_0}} (\nabla\chi) \cdot v\, dx = \int_{\partial\Xi_{2R_0}} \chi v \cdot n\, do - \int_{\Xi_{2R_0}} \chi(\nabla \cdot v)\, dx. \tag{4.14}$$

The last integral vanishes, since $\nabla \cdot v = 0$ on $\Xi_{2R_0}$. The boundary integral splits into integrals over $\partial\Xi_{R_0} \cap \Sigma_R$, where $v = 0$, and integrals over the lateral surfaces $\Gamma_{R_0} \cup \Gamma_{2R_0}$. On $\Gamma_{2R_0}$ we have $\chi = 0$, while $\chi = 1$ on $\Gamma_{R_0}$. Here we use $\nabla \cdot v = 0$ in $\Lambda_{R_0}$ and $v(y,z) = 0$ for $|z| = \frac{1}{2}$ to see that $\int_{\Gamma_{R_0}} v \cdot n\, do = 0$. Hence with Proposition 4.2 again, we find $\widetilde{v} \in \overset{\circ}{H}{}^1(\Xi_{R_0})^3$ solving $\nabla \cdot \widetilde{v} = (\nabla\chi) \cdot v$ and

$$\|\widetilde{v}; H^1(\Xi_{R_0})\| \le C\|(\nabla\chi) \cdot v; L^2(\Xi_{R_0})\| \le \|v; L^2(\Xi_{R_0})\| \le \|v; H^1(\Lambda_R)\|. \tag{4.15}$$

We extend $\widetilde{v}$ by zero and put $w^2 = (1-\chi)v + \widetilde{v}$, then $w^2 \in \mathcal{H}(\Omega_R)$, $\nabla \cdot w^2 = g^2$, and estimates (4.12), (4.13) and (4.15) lead to

$$\|w^2; H^1(\Omega_R)\| \le C\,R\|g^2; L^2(\Omega_R)\|. \tag{4.16}$$

The final representation of $w$ reads

$$w(x) = w^1(x) + w^2(x) - \frac{3}{\pi}G_{3R_0}\mathscr{W}^{R_0}(x), \tag{4.17}$$

and due to the construction we have $w_{y_i}(x)|_{\Gamma_R} = (G_{3R_0} + \mathbf{G})\bigl(z^2 - \frac{1}{4}\bigr)\frac{1}{R^2}x_i$, $i = 1, 2$, which implies $\|w_y; L^2(\Gamma_R)\| \le C(G_{3R_0} + \mathbf{G})R^{-\frac{1}{2}}$. From here the estimate (4.4) follows with (4.6), (4.7), (4.10) and (4.16) if we observe that $\|\mathscr{W}^{R_0}; H^1(\Omega_R)\| \le CR$ independent of $R \ge R_0$. $\qquad\square$

Lateron we will need the following conclusion from the proof of Lemma 4.3.

**Remark 4.4.** If $\operatorname{supp} g \subset \{(y,z) \in \Omega_R : \frac{1}{2}R \le |y| \le \frac{3}{4}R\}$ and $\int_{\Omega_R} g = 0$, then then the construction in the proof of Lemma leads to a vector field $w \in \overset{\circ}{H}{}^1(\Omega_R)^3$ with $w = 0$ for $|y| > \frac{3}{4}R$.

Indeed, then $G_{3R_0} = 0$ (see (4.6)), $\mathbf{G} = 0$ (see (4.10)), hence the first and the third summand in (4.17) vanish. Further $\operatorname{supp} \mathbf{g}_2 \subset \Lambda_{\frac{3}{4}}$, (see (4.8)),

hence $\mathbf{v}^* \in \overset{\circ}{H}(\Lambda_{\frac{3}{4}})^3$, from here it follows for the second summand in (4.17): $w^2 \in \overset{\circ}{H}(\Omega_{\frac{3}{4}R})^3$.

Further we note that if additionally $g \in \overset{\circ}{H}^1(\Omega_R)$ holds, the application of [33, Lemma 2.3.1] in the proof of Lemma 4.3 actually gives $w \in \overset{\circ}{H}^2(\Omega_R)^3$.

In the next step we derive estimates for the bilinear form $\mathbf{q}_R$. Recall that for $v \in \mathcal{H}(\Omega_R)$, we can define $\overline{v_y}(y)$ and $v'^\#(x)$ by means of (3.9) for all $y \in A_R$ (compare (4.1)) as long as $R > 2R_0$.

**Proposition 4.5.** *The bilinear form $\mathbf{q}_R$ is symmetric and nonnegative, and for $v, w \in \mathcal{H}(\Omega_R)$, the following inequality is valid with a constant $C$ independent of $R \geq R_0$:*

$$|\mathbf{q}_R(v, w)| \leq C \left( \|v_y; H^1(\Xi_R)\| + R^{\frac{1}{2}} \|\overline{v_y}; L^2(\mathbb{S}_R)\| \right)$$
$$\times \left( \|w_y; H^1(\Xi_R)\| + R^{\frac{1}{2}} \|\overline{w_y}; L^2(\mathbb{S}_R)\| \right).$$

*Proof.* The symmetry property $\mathbf{q}_R(v, w) = \mathbf{q}_R(w, v)$ follows immediately from (3.7). Furthermore, only $v_y$, $w_y$ are involved the definition of $\mathbf{q}_R$. Using the notation (4.1) and formula (3.9) again, we obtain $\|\overline{v_y}; H^1(A_R)\| \leq C\|v_y; H^1(\Xi_R)\|$, with a constant independent of $R$. Similar as in (4.8), we put $\mathbf{y} = \frac{y}{R}$, $\overline{\mathbf{v_y}}(\mathbf{y}) = \overline{v_y}(R\mathbf{y})$, $\overline{\mathbf{w_y}}(\mathbf{y}) = \overline{w_y}(R\mathbf{y})$. Formula $(3.8)_1$ leads to

$$|(\Pi_R \overline{v}_r, \overline{w}_r)_{\mathbb{S}_R}| = |(\Pi \overline{\mathbf{v}}_r, \overline{\mathbf{w}}_r)_{\mathbb{S}}| \leq C\|\overline{\mathbf{v}}_r; H^1(A_1)\| \, \|\overline{\mathbf{w}}_r; H^1(A_1)\|$$
$$\leq C \left( R^{-1}\|\overline{v_y}; L^2(A_R)\| + \|\nabla_y \overline{v_y}; L^2(A_R)\| \right)$$
$$\times \left( R^{-1}\|\overline{w}'; L^2(A_R)\| + \|\nabla_y \overline{w}'; L^2(A_R)\| \right).$$

To obtain the last inequality we used similar reasonings as in (4.9) and (4.11). By $(3.8)_2$, we get

$$|(\Pi_R^{-1} (\overline{v}_r)_\bullet, (\overline{w}_r)_\bullet)_{\mathbb{S}_R}| = R^2|(\Pi^{-1}(\overline{\mathbf{v}}_r)_\bullet, (\overline{\mathbf{w}}_r)_\bullet)_{\mathbb{S}}|$$
$$\leq C\,R^2 \, \|\overline{\mathbf{v}}_r; L^2(\mathbb{S})\| \, \|\overline{\mathbf{w}}_r; L^2(\mathbb{S})\|$$
$$= C\,R \, \|\overline{v}_r; L^2(\mathbb{S}_R)\| \, \|\overline{w}_r; L^2(\mathbb{S}_R)\|,$$

while the estimate of the term $R^{-1}|(\overline{v}_\varphi, \overline{w}_\varphi)_{\mathbb{S}_R}|$ is obvious. Collecting all the inequalities gives the estimate. Since

$$(\Pi_R \overline{v}_r, \overline{v}_r)_{\mathbb{S}_R} = \|\overline{\mathbf{v}}_r; H^{\frac{1}{2}}(\mathbb{S})\|^2, \quad (\Pi_R^{-1}(\overline{v}_r)_\bullet, (\overline{v}_r)_\bullet)_{\mathbb{S}_R} = R^2\|(\overline{\mathbf{v}}_r)_\bullet; H^{-\frac{1}{2}}(\mathbb{S})\|^2,$$

we obtain also $\mathbf{q}_R(v, v) \geq 0$ for all $v \in \mathcal{H}(\Omega_R)$. $\qquad\square$

With the previous estimate in mind, we define the following $R$-dependent norms on $\mathcal{H}(\Omega_R)$ and its dual space $\mathcal{H}'(\Omega_R)$:

$$\begin{aligned}
&\|v; \mathcal{H}(\Omega_R)\|^2 = \|v; H^1(\Omega_R)\|^2 + \mathbf{q}_R(v, v) && \text{for } v \in \mathcal{H}(\Omega_R) \\
&\|\Phi; \mathcal{H}'(\Omega_R)\| = \sup\{|\Phi(w)| : \|w; \mathcal{H}(\Omega_R)\| \leq 1\} && \text{for } \Phi \in \mathcal{H}'(\Omega_R).
\end{aligned} \qquad (4.18)$$

Note that for $f \in L^2(\Omega_R)$, and $\Phi(w) = (f, w)_{\Omega_R}$, we obtain

$$\|\Phi; \mathcal{H}'(\Omega_R)\| \leq C\|f; L^2(\Omega_R)\| \tag{4.19}$$

with a constant independent of $R$ and $f$.

**Theorem 4.6.** *For any $\Phi \in \mathcal{H}'(\Omega_R)$, there exists a unique weak solution $U = (V, P) \in \mathcal{H}_\sigma(\Omega_R) \times L^2(\Omega_R)$ to problem (4.2), and the following estimate is valid with a constant $C_S$ independent of $R > R_0$ and $\Phi$:*

$$\|V; \mathcal{H}(\Omega_R)\| + R^{-1}\|P; L^2(\Omega_R)\| \leq C_S \|\Phi; \mathcal{H}'(\Omega_R)\|. \tag{4.20}$$

*Proof.* On $\mathcal{H}_\sigma(\Omega_R)$, we consider the bilinear form $\langle\langle v, w \rangle\rangle = \nu(\nabla v, \nabla w)_{\Omega_R} + \mathbf{q}_R(v, w)$. Since Poincaré's inequality, $\|v; L^2(\Omega_R)\| \leq c\|\nabla v; L^2(\Omega_R)\|$, is valid for $v \in \mathcal{H}(\Omega_R)$ with a constant $c$ independent of $R$ on $\Omega_R$, it is clear that $\langle\langle \cdot \cdot \rangle\rangle$ is coercive and continuous. By means of the Lax Milgram lemma, we find a unique $V \in \mathcal{H}_\sigma(\Omega_R)$ such that

$$\Phi(w) = \nu(\nabla v, \nabla w)_{\Omega_R} + \mathbf{q}_R(v, w) \quad \text{for any } w \in \mathcal{H}_\sigma(\Omega_R). \tag{4.21}$$

The identity (4.21) applied to $w = V$, together with Poincaré's inequality, leads to $\|V; H^1(\Omega_R)\| + \mathbf{q}_R(V, V)^{\frac{1}{2}} \leq C\|\Phi; \mathcal{H}'(\Omega_R)\|$, where $C$ is independent of $R$ and $\Phi$.

The pressure $P$ is obtained by the following well known argument: From Lemma 4.3 we conclude that for any $g \in L^2(\Omega_R)$ there exists a solution $Dg \in \mathcal{H}(\Omega_R)$ to the problem $\nabla \cdot Dg = g$, while inequality (4.4) together with Proposition 4.5 applied to $v = w = Dg$ lead to the estimate

$$\|Dg; \mathcal{H}(\Omega_R)\| \leq C R\|g; L^2(\Omega_R)\|.$$

Thus we obtain a continuous linear functional $F$ on $L^2(\Omega_R)$ by

$$F(g) = \Phi(Dg) - (\nabla V, \nabla Dg)_{\Omega_R} - \mathbf{q}_R(V, Dg), \quad g \in L^2(\Omega_R). \tag{4.22}$$

Moreover, we have

$$|F(g)| \leq \Big( \|\Phi; \mathcal{H}'(\Omega_R)\| \, \|Dg; \mathcal{H}(\Omega_R)\| + \|\nabla V; L^2(\Omega_R)\| \, \|\nabla Dg; L^2(\Omega_R)\|$$
$$+ \mathbf{q}_R(V, V)^{\frac{1}{2}} \mathbf{q}_R(Dg, Dg)^{\frac{1}{2}} \Big)$$
$$\leq C R \,\|\Phi, \mathcal{H}'(\Omega_R)\| \, \|g; L^2(\Omega_R)\|$$

with a constant $C$ independent on $R$. By the Riesz representation theorem there exists a unique $P \in L^2(\Omega_R)$ with $F(g) = (P, g)_{\Omega_R}$ and

$$\|P; L^2(\Omega_R)\| \leq C R \,\|\Phi; \mathcal{H}'(\Omega_R)\|$$

with the same constant $C$ as above. Now, if $w \in \mathcal{H}(\Omega_R)$ is arbitrary, then $w = D(\nabla \cdot w) + w_0$, where $w_0 \in \mathcal{H}_\sigma(\Omega_R)$, and from (4.21) and (4.22) we obtain $F(\nabla \cdot w) = \Phi(w) - (\nabla V, \nabla w)_{\Omega_R} - \mathbf{q}_R(V, w)$, which means that $(V, P)$ is a weak solution to (4.2). $\square$

In order to derive the error estimate for $v^{er} = v^\infty|_{\Omega_R} - v^R$ and $p^{er} = p^\infty|_{\Omega_R} - p^R$, it would be very convenient if we could use $v^\infty - v^R$ as a test function in (4.2). However, this is not possible, since $\widetilde{v}^\infty$ does not fulfill the Dirichlet conditions $(3.13)_{2,3}$ on $\Gamma_R$. We remedy this by cutting off $\widetilde{v}^\infty$ near $\Gamma_R$, but then there appears a nonzero term in the divergence, which we remove with the help of Remark 4.4. We emphasize that Theorem 2.2 implies $v^\infty \in H^2(\Omega_R)$, $p^\infty \in H^1(\Omega_R)$ at least.

For $x = (y, z)$, and $\chi$ as in (2.2) we put $\chi^R(y, z) = \chi(2R^{-1}|y|)$, then

$$\chi^R(y, z) = \begin{cases} 1 & \text{for } |y| < \frac{R}{2} \\ 0 & \text{for } |y| > \frac{3R}{4}, \end{cases} \qquad \begin{aligned} |\nabla_y \chi^R(y, z)| &\leq CR^{-1} \\ |\Delta_y \chi^R(y, z)| &\leq CR^{-2}, \end{aligned} \qquad (4.23)$$

with $C$ independent of $R$. From the representation (2.3) for $v^\infty$, $p^\infty$, it follows that $\widetilde{v}^\infty$ and $\widetilde{p}^\infty$ satisfy the Stokes system (1.2) for $x = (y, z) \in \Omega$ with $|y| > \frac{3}{2}R_0$, in particular $\nabla \cdot \widetilde{v}^\infty = 0$ and $\nabla \cdot (\chi^R \widetilde{v}^\infty) = (\nabla \chi^R) \cdot \widetilde{v}^\infty$ then. Moreover, we have $(\nabla \chi^R) \cdot \widetilde{v}^\infty \in \mathring{H}^1(\Omega_R)$ and $\mathrm{supp}\, (\nabla \chi^R) \widetilde{v}^\infty \subset \{(y, z) \in \Lambda_R : \frac{1}{2}R \leq |y| \leq \frac{3}{4}R\}$. The Gauss' theorem implies for all $T > R > \frac{3}{2}R_0$ ($n$ the external normal vector on the boundary) $\int_{\partial(\Omega_T \setminus \Omega_R)} \widetilde{v}^\infty \cdot n\, do = \int_{\Gamma_T} \widetilde{v}^\infty e_r\, do - \int_{\Gamma_R} \widetilde{v}^\infty e_r\, do = 0$. For fixed $R$ we can pass to the limit $\lim_{T \to \infty} \int_{\Gamma_T} \widetilde{v}^\infty e_r do = 0$ due to the decay properties (2.6), hence $\int_{\Gamma_R} \widetilde{v}^\infty e_r do = 0$ for all $R > \frac{3}{2}R_0$. Thus the same arguments as after (4.14) lead to $\int_{\Omega_R} \nabla \cdot (\chi^R \widetilde{v}^\infty) dx = \int_{\Omega_{\frac{3}{4}R}} \nabla \cdot (\chi^R \widetilde{v}^\infty) dx = 0$. We use Remark 4.4 to get $W^R \in \mathring{H}^2(\Omega_R)^3$ with

$$\nabla \cdot W^R = \nabla \cdot (\chi^R \widetilde{v}^\infty), \qquad W^R(y, z) = 0 \ \text{ for } |y| > \frac{3}{4}R. \qquad (4.24)$$

Now we define the decomposition

$$v^\infty = v^\infty_{ap} + \widetilde{v}^\infty_{cut}, \qquad p^\infty = p^\infty_{ap} + \widetilde{p}^\infty_{cut} \qquad (4.25)$$

with

$$\begin{aligned} v^\infty_{ap} &= V^\infty + \chi^R \widetilde{v}^\infty - W^R, & \widetilde{v}^\infty_{cut} &= (1 - \chi^R)\widetilde{v}^\infty + W^R \\ p^\infty_{ap} &= P^\infty + \chi^R \widetilde{p}^\infty, & \widetilde{p}^\infty_{cut} &= (1 - \chi^R)\widetilde{p}^\infty. \end{aligned} \qquad (4.26)$$

Then $v^\infty_{ap} \in \mathcal{H}_\sigma(\Omega_R)$, while the supports of $\widetilde{v}^\infty_{cut}$, $\widetilde{p}^\infty_{cut}$ are contained in $\Xi_R$. To control finally the error $(v^{er}, p^{er})$ we need various estimates for different terms resulting from (4.25) and (4.26).

**Lemma 4.7.** *Let $N \in \mathbb{N}$ be fixed, $R > 3R_0$, $v^\infty = V^\infty + \widetilde{v}^\infty$ the velocity part of the solution from Theorem 2.2, $\chi^R$, $W^R$ as in (4.23) and (4.24), respectively, and recall (4.18) for $\| \cdot ; \mathcal{H}'(\Omega_R) \|$. For $x = (y, z) \in \Omega_R$, we set*

$$[\Delta, \chi^R]\widetilde{v}^\infty = \Delta(\chi^R \widetilde{v}^\infty) - \chi^R(\Delta \widetilde{v}^\infty).$$

*Then the following estimates are valid with constants $C$ independent of $R$, $\widetilde{v}^\infty$:*

$$\|W^R; H^1(\Omega_R)\| + \|\Delta W^R, \mathcal{H}'(\Omega_R)\| \leq C\|\widetilde{v}_y^\infty; L^2(\Xi_R)\| \qquad (4.27)$$

$$\|(1-\chi^R)\widetilde{v}^\infty; H^1(\Omega_R)\| + \|[\Delta, \chi^R]\widetilde{v}^\infty; L^2(\Omega_R)\| \leq C\|\widetilde{v}^\infty; H^1(\Xi_R)\| \qquad (4.28)$$

$$\|(1-\chi^R)\widetilde{p}^\infty; L^2(\Omega_R)\| \leq C\|\widetilde{p}^\infty; L^2(\Xi_R)\|$$
$$\|(\nabla\chi^R)\widetilde{p}^\infty; L^2(\Omega_R)\| \leq CR^{-1}\|\widetilde{p}^\infty; L^2(\Xi_R)\| . \qquad (4.29)$$

*Proof.* Since $W^R \in \mathring{H}^2(\Omega_R)^3$ vanishes in a neighborhood of $\Gamma_R$, we have $(\Delta W^R, w)_{\Omega_R} = (\nabla W^R, \nabla w)_{\Omega_R}$ for $w \in \mathcal{H}(\Omega_R)$, hence $\|\Delta W^R, \mathcal{H}'(\Omega_R)\| \leq \|\nabla W^R; L^2(\Omega_R)\|$. The definition of $\chi^R$ implies $\partial_z\chi^R = 0$, hence with (4.23) it follows

$$\|\nabla \cdot (\chi^R\widetilde{v}^\infty); L^2(\Omega_R)\| = \|(\nabla_y\chi^R) \cdot \widetilde{v}_y^\infty; L^2(\Xi_R)\| \leq CR^{-1}\|\widetilde{v}_y^\infty; L^2(\Xi_R)\| .$$

The last two inequalities together with (4.4) imply (4.27).

To prove inequality (4.28) we only have to observe that the supports of $1 - \chi^R$, $\nabla\chi^R$ and $\Delta\chi^R$ are contained in $\Xi_R$. Then (4.23) together with the definition of the commutator leads to the assertion, the same arguments work for (4.29). $\qquad\square$

**Theorem 4.8.** *Let $l \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $-1 < \beta < 0$, $N \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $f \in \mathcal{W}_\gamma^{l+2}(\Omega)^3$, $N + l + 4 > \gamma > N + l + 3$ and $(v^\infty, p^\infty)$ the solution to the Stokes problem (1.2) in the layer like domain $\Omega$ as in Theorem 2.2. Further let $(v^R, p^R) \in \mathcal{H}(\Omega_R) \times L_2(\Omega_R)$ be the solution to the weak approximation problem (4.2) with $\Phi(w) = (f, w)_{\Omega_R}$ for $w \in \mathcal{H}(\Omega_R)$ (see Theorem 4.6). Then the following error estimate holds for $R > 3R_0$:*

$$\|v^\infty|_{\Omega_R} - v^R; H^1(\Omega_R)\| + R^{-1}\|p^\infty|_{\Omega_R} - p^R; L_2(\Omega_R)\|$$
$$\leq C R^{3+l-\gamma}\|f; \mathcal{W}_\gamma^{l+2}(\Omega)\|$$
$$\leq C R^{-N}\|f; \mathcal{W}_\gamma^{l+2}(\Omega)\| , \qquad (4.30)$$

*where the constant $C$ is independent of $R$ and $f \in \mathcal{W}_\gamma^{l+2}(\Omega)^3$ (see (2.1) for the definition of the norm).*

*Proof.* We apply the splitting (4.25), (4.26) to $v^\infty$, $p^\infty$. The estimates in Lemma 4.7 lead to

$$\|\widetilde{v}_{cut}^\infty; H^1(\Omega_R)\| + R^{-1}\|\widetilde{p}_{cut}^\infty; L^2(\Omega_R)\|$$
$$\leq C\Big(\|\widetilde{v}^\infty; H^1(\Xi_R)\| + R^{-1}\|\widetilde{p}^\infty; L^2(\Xi_R)\|\Big). \qquad (4.31)$$

Due to the choice of the cut-off functions $\chi^R$ in (4.24), (4.26) and $\chi$ in the asymptotic representation (2.4), we have $\chi^R\chi = \chi$, while $(V^\infty, P^\infty)(y, z)$ satisfies the homogeneous Stokes system for $|y| > \frac{3}{2}R_0$, therefore the pair $(\widetilde{v}_{ap}^\infty, \widetilde{p}_{ap}^\infty)$

fulfills the equations

$$-\nu\Delta\widetilde{v}_{ap}^{\infty} + \nabla\widetilde{p}_{ap}^{\infty} = \chi^R f + [\Delta, \chi^R]\widetilde{v}^{\infty} - \Delta W^R + (\nabla\chi^R)\widetilde{p}^{\infty} \quad \text{in } \Omega_R$$
$$\nabla \cdot \widetilde{v}_{ap}^{\infty} = 0 \quad \text{in } \Omega_R,$$

together with the boundary conditions (recall the notation (1.5)) $\widetilde{v}_{ap}^{\infty} = 0$ on $\Sigma_R$, $M_R(\widetilde{v}_{ap}^{\infty}, \widetilde{p}_{ap}^{\infty}) = 0$ on $\Gamma_R$, here we used also Lemma 3.2 (cf. also Condition I in Section 3). Thus the differences $V := v^R - \widetilde{v}_{ap}^{\infty} \in \mathcal{H}(\Omega_R)$ and $P := p^R - \widetilde{p}_{ap}^{\infty} \in L^2(\Omega_R)$ solve the problem (4.2) with

$$\Phi^{er}(w) = ((1 - \chi^R)f, w)_{\Omega_R} - ([\Delta, \chi^R]\widetilde{v}^{\infty} - \Delta W^R + (\nabla\chi^R)\widetilde{p}^{\infty}, w)_{\Omega_R}.$$

From here it follows with (4.19) and Lemma 4.7

$$\|\Phi^{er}; \mathcal{H}'(\Omega_R)\| \leq C\Big(\|f; L^2(\Xi_R)\| + \|\widetilde{v}^{\infty}; H^1(\Xi_R)\| + R^{-1}\|\widetilde{p}^{\infty}; L^2(\Xi_R)\|\Big).$$

On $\Xi_R$ we have $\frac{R}{2} \leq r \leq R$, thus for any function $\phi \in L^2(\Xi_R)$ and any exponent $\beta \in \mathbb{R}$ we have

$$\|\phi; L^2(\Xi_R)\| \leq CR^{-\beta}\|r^{\beta}\phi; L^2(\Xi_R)\| \leq CR^{K-\beta}\|r^{\beta}\phi; L^2(\Xi_R)\| \quad \text{for } K \in \mathbb{N},$$

where $C$ depends on $\beta$ and $R_0$. From here it follows

$$\|f; L^2(\Xi_R)\| \leq CR^{3+l-\gamma}\|r^{\gamma-l-2}f; L^2(\Xi_R)\|$$
$$\|\widetilde{v}^{\infty}; L^2(\Xi_R)\| \leq CR^{3+l-\gamma}\Big(\|r^{\gamma-l-3}\widetilde{v}_y^{\infty}; L^2(\Xi_R)\| + \|r^{\gamma-l-2}\widetilde{v}_z^{\infty}; L^2(\Xi_R)\|\Big)$$
$$\|\nabla\widetilde{v}^{\infty}; L^2(\Xi_R)\| \leq CR^{3+l-\gamma}\Big(\|r^{\gamma-l-2}\nabla_y\widetilde{v}_y^{\infty}; L^2(\Xi_R)\| + \|r^{\gamma-l-3}\partial_z\widetilde{v}_y^{\infty}; L^2(\Xi_R)\|$$
$$+ \|r^{\gamma-l-1}\nabla_y\widetilde{v}_z^{\infty}; L^2(\Xi_R) + \|r^{\gamma-l-2}\partial\widetilde{v}_z^{\infty}; L^2(\Xi_R)\|\Big)$$
$$R^{-1}\|\widetilde{p}^{\infty}; L^2(\Xi_R)\| \leq C\|r^{-1}\widetilde{p}^{\infty}; L^2(\Xi_R)\|$$
$$\leq CR^{3+l-\gamma}\|r^{\gamma-l-4}\widetilde{p}_y^{\infty}; L^2(\Xi_R)\|, \tag{4.32}$$

which finally leads to estimate (4.30) by means of (2.7) and (4.20).     □

## 5. Strong solutions to the Navier-Stokes problem and their asymptotic properties

The proof for the existence of weak solutions to the Navier-Stokes problem (1.3) is standard using solutions on a sequence of expanding domains (see, e.g., [33, p. 169ff]). In this context we recall that the notion weak solution is related usually to the velocity field alone: A weak solution of (1.3) is $v^{\infty} \in \overset{\circ}{H}^1(\Omega)^3$ with $\nabla \cdot v^{\infty} = 0$ and

$$\nu(\nabla v^{\infty}, \nabla w)_{\Omega} = (v^{\infty} \cdot \nabla v^{\infty}, w)_{\Omega} + (f, w)_{\Omega}$$

for all $w \in C_0^\infty(\Omega)^3$ with $\nabla \cdot w = 0$. Using the results of [19, 20, 27] we derive the existence of uniquely determined strong solutions to (1.3) with asymptotic representation as in Theorem 2.2. However, in contrast to the Stokes problem the number of asymptotic terms as in (2.3), (2.4) is limited for the nonlinear system, even if $f$ has compact support.

**Theorem 5.1.** *Let $l \in \mathbb{N}_0$ be fixed and assume that*

$$f \in \mathcal{W}_\mu^{l+2}(\Omega) \quad \text{for some } \mu \in (l+3, l+4). \tag{5.1}$$

*There exists a number $\varepsilon_0 > 0$ such that for $\|f; L_2(\Omega)\| \le \varepsilon_0$ the Navier-Stokes problem (1.3) admits a unique solution:*

$$v_y^\infty \in H_{loc}^{l+4}(\overline{\Omega}) \cap \mathcal{W}_\mu^{l+3}(\Omega), \qquad v_z^\infty \in H_{loc}^{l+4}(\overline{\Omega}) \cap \mathcal{W}_\mu^{l+2}(\Omega) \tag{5.2}$$

$$p^\infty \in \mathcal{W}_{\mu-1}^{l+3}(\Omega), \qquad\qquad \partial_z p^\infty \in \mathcal{W}_\mu^{l+2}(\Omega). \tag{5.3}$$

*This solution fulfills the estimate*

$$\|v_y^\infty; \mathcal{W}_\mu^{l+3}(\Omega)\| + \|v_z^\infty; \mathcal{W}_\mu^{l+2}(\Omega)\| + \|p^\infty; \mathcal{W}_{\mu-1}^{l+3}(\Omega)\|$$
$$+ \|\partial_z p^\infty; \mathcal{W}_\mu^{l+2}(\Omega)\| \le C \|f; \mathcal{W}_\mu^{l+2}(\Omega)\| \tag{5.4}$$

*with a constant independent of $f$.*

*Proof.* Theorem 4.3 in [20] states the following assertion: (1.3) possesses a unique weak solution $u^\infty$ if $\|f; L^2(\Omega)\| \le \varepsilon_0$ for a suitable $\varepsilon_0$, further there exists a function with $q^\infty \in L_\beta^2(\Omega)$ for any $\beta < -1$ with

$$\nu(\nabla v^\infty, \nabla w)_\Omega - (p^\infty, \nabla \cdot w)_\Omega = (v^\infty \cdot \nabla v^\infty, w)_\Omega + (f, w)_\Omega$$

for all $w \in C_0^\infty(\Omega)^3$. The velocity field $v^\infty$ and the function $q^\infty$ fulfill the estimate $\|\nabla u^\infty; L^2(\Omega)\| + \|q^\infty; L_\beta^2(\Omega)\| \le C(\nu, \beta, \Omega)\|f; L^2(\Omega)\|$. Since the constant functions are contained in $L_\beta^2(\Omega)$ for $\beta < -1$, the pressure $q^\infty$ is determined only up to constant here.

From [27, Theorem 4.2] it follows for $f$ with (5.1) the existence of a solution $(v^\infty, p^\infty)$ to (1.3) with properties (5.2), (5.3). We observe that $\|f; L^2(\Omega)\| \le \|f; \mathcal{W}_\mu^{l+2}(\Omega)\|$ and $v^\infty \in \overset{\circ}{H}{}^1(\Omega)^3$ if (5.1) is valid, hence $u^\infty$ and $v^\infty$ coincide for $\|f; L^2(\Omega)\| \le \varepsilon_0$. Then $p^\infty$ is uniquely determined by the condition $p^\infty \in L_{\mu-4}^2(\Omega)$, since $\mu - 4 > -1$. The estimate (5.4) follows from the a priori estimates used for the proof of [27, Theorem 4.2] and the condition $\|f; L^2(\Omega)\| \le \varepsilon_0$. $\quad\square$

To obtain more information about the asymptotic behavior of the solution $(v^\infty, p^\infty)$ we will use a bootstrap argument similar to the proof of regularity results for solutions to the Navier-Stokes system. With suitable estimates for the nonlinear term at hand one can shift it to the right-hand side of (1.3) and use results for the linear system, thus successively improve the properties of the solutions to the nonlinear problem.

**Theorem 5.2.** *Let $l, N, \gamma$ be fixed with*

$$l \in \mathbb{N}_0, \quad N \in \{1, 2, 3\}, \quad l + 3 + N < \gamma < l + 4 + N. \tag{5.5}$$

*Assume that $\|f; L_2(\Omega)\| \le \varepsilon_0$, $\varepsilon_0$ as in Theorem 5.1, and additionally $f \in \mathcal{W}_\gamma^{l+2}(\Omega)^3$. Then for the solution $(v^\infty, p^\infty)$ to the nonlinear problem asymptotic representations (2.3)–(2.6) of Theorem 2.2 are valid.*

*Proof.* We consider the cases $N = 1, 2, 3$ in (5.5) separately. Since we have the embedding $\mathcal{W}_{\gamma_1}^{l+2}(\Omega) \subset \mathcal{W}_{\gamma_2}^{l+2}(\Omega)$ for all $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 \ge \gamma_2$, for steps 2 and 3 we can use the result of the previous steps to improve the knowledge about the asymptotic behavior.

`Step 1:` $N = 1$. For $v^\infty$ as in (5.2) it follows from [27, Lemma 3.4] that

$$(v^\infty \cdot \nabla)v^\infty \in \mathcal{W}_{2\mu-l-3}^{l+2}(\Omega) \quad \text{for all } \mu \in (l + 3, l + 4). \tag{5.6}$$

Since $2\mu - l - 3$ can be any number in the interval $(l+3, l+5)$, with $2\mu - l - 3 = \gamma$ we then obtain $f - (v^\infty \cdot \nabla)v^\infty \in \mathcal{W}_\gamma^{l+2}(\Omega)$, and the assertion follows from Theorem 2.2.

`Step 2:` $N = 2$. We put $\tilde{\gamma} = 2\mu - l - 3$. Since (5.6) is still true we obtain $f - (v^\infty \cdot \nabla)v^\infty \in \mathcal{W}_{\tilde{\gamma}}^{l+2}(\Omega)$ for all $\tilde{\gamma} \in (l + 3, l + 5)$. We can use Theorem (2.2) again to see that for $r > 2R_0$

$$\begin{aligned}
v^\infty &= v^1 + \tilde{v}^\infty, \quad v_y^1(y, z) = \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)\nabla_y P_1(y), \qquad v_z^1 = 0 \\
P^\infty &= P_1 + \tilde{p}^\infty, \qquad P^1(y) = r^{-1}(a_1\cos(\varphi) + b_1\sin(\varphi)).
\end{aligned} \tag{5.7}$$

The remainders $\tilde{v}^\infty$ and $\tilde{v}^\infty$ fulfill (2.6) with $\gamma$ replaced by $\tilde{\gamma} \in (l + 4, l + 5)$. Clearly we have

$$\begin{aligned}
(v^\infty \cdot \nabla)v^\infty &= (v^1 \cdot \nabla)v^1 + (v^1 \cdot \nabla)\tilde{v}^\infty + (\tilde{v}^\infty \cdot \nabla)v^1 + (\tilde{v}^\infty \cdot \nabla)\tilde{v}^\infty \\
&=: (v^1 \cdot \nabla)v^1 + f_1.
\end{aligned}$$

With $|\partial_z^j \partial_y^\alpha v^1(y, z)| = O(|y|^{-2-|\alpha|})$ elementary but lengthy calculations together with Lemma 3.4 in [27] lead to

$$\begin{aligned}
f_1 &\in \mathcal{W}_{\gamma^*}^{l+2}(\Omega) \qquad \text{for all } \gamma^* < l + 7 \\
(v^1 \cdot \nabla)v^1 &\in \mathcal{W}_{\gamma^*}^{l+2}(\Omega) \qquad \text{for all } \gamma^* < l + 6.
\end{aligned} \tag{5.8}$$

Thus we have again $f - (v^\infty \cdot \nabla)v^\infty \in \mathcal{W}_\gamma^{l+2}(\Omega)$, and we obtain the assertion with $N = 2$ from Theorem 2.2.

`Step 3:` $N = 3$. We assume (5.5) with $N = 3$, recall that we can already use (5.7) and (5.8), from which it is clear that only the term $(v^1 \cdot \nabla)v^1$ prevents

us from applying Theorem 2.2 with $N = 3$. By a proper choice of the angular variable $\varphi$, the main asymptotic term $v_y^1$ can be always reduced to the expression

$$
\begin{aligned}
v_y^1(r, \varphi, z) &= c_1 \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)\nabla_y\left(r^{-1}\sin\varphi\right) \\
&= c_1 \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)r^{-2}\left(-\sin 2\varphi, \cos 2\varphi\right),
\end{aligned}
$$

where $c_1 = (a_1^2 + b_1^2)^{\frac{1}{2}}$. The convective term $(v^1 \cdot \nabla)v^1$ takes the form

$$
\begin{aligned}
\left((v^1 \cdot \nabla)v^1\right)_y &= c_1^2 \frac{1}{2\nu^2}\left(z^2 - \frac{1}{4}\right)^2 r^{-5}\left(-\cos\varphi, -\sin\varphi\right) \\
\left((v^1 \cdot \nabla)v^1\right)_z &= 0.
\end{aligned}
\tag{5.9}
$$

Now we want to use a solution to the Stokes problem in the layer (1.1) with the right-hand side (5.9) and use an ansatz as particular power-law solution

$$
\begin{aligned}
V_y(y, z) &= \mathcal{Z}(z)\, r^{-5}\left(\cos\varphi, \sin\varphi\right) + \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)\nabla_y\left(r^{-4}\mathcal{P}(\varphi)\right) \\
V_z(y, z) &= r^{-6}\, W(\varphi, z) \\
P(y, z) &= r^{-4}\mathcal{P}(\varphi) + r^{-6}Q(\varphi, z)
\end{aligned}
\tag{5.10}
$$

with unknown coefficient functions $\mathcal{Z}$, $\mathcal{P}$, $W$ and $Q$. Inserting (5.10) into the Stokes problem and collecting coefficients at same powers $r^5$, we first arrive to the Dirichlet problem on the interval $\Upsilon = \left(-\frac{1}{2}, \frac{1}{2}\right)$:

$$
-\nu\frac{d^2}{dz^2}\mathcal{Z}(z) = c_1^2\frac{1}{2\nu^2}\left(z^2 - \frac{1}{4}\right)^2, \quad z \in \Upsilon, \quad \mathcal{Z}\left(\pm\frac{1}{2}\right) = 0.
$$

Thus, we obtain the first coefficient in $(5.10)_1$

$$
\mathcal{Z}(z) = -c_1^2\frac{1}{2\nu^3}\left\{\frac{z^6}{30} - \frac{z^4}{24} + \frac{z^2}{32} - 2^{-7}\frac{11}{15}\right\}.
$$

Collecting the coefficients at the powers $r^6$, there appears the one-dimensional Stokes problem with the parameter $\varphi \in [0, 2\pi)$

$$
\begin{aligned}
-\nu\frac{d^2}{dz^2}W(\varphi, z) + \frac{d}{dz}Q(\varphi, z) &= 0, & z \in \Upsilon \\
-\frac{d}{dz}W(\varphi, z) &= r^6\,\nabla_y \cdot V_y(y, z), & z \in \Upsilon \\
W\left(\varphi, \pm\frac{1}{2}\right) &= 0.
\end{aligned}
\tag{5.11}
$$

Equation $(5.11)_2$ and the boundary condition $(5.11)_3$ require the compatibility condition $\int_\Upsilon \nabla_y \cdot V_y(y, z)\, dz = 0$, which with $(5.10)_1$ turns into the Poisson equation for $r \neq 0$:

$$
-\frac{1}{12\nu}\Delta_y\left(r^{-4}\mathcal{P}(\varphi)\right) = -\int_\Upsilon \mathcal{Z}(z)\, dz\,\nabla_y \cdot \left(r^{-5}(\cos\varphi, \sin\varphi)\right) = \frac{1}{140}\frac{c^2}{\nu^3 r^6}.
$$

Here we used also $\int_\Upsilon \mathcal{Z}(z)dz = c_1^2 \frac{1}{\nu^3} \frac{1}{2^4} \frac{1}{35}$ and $(2\nu)^{-1} \int_\Upsilon (z^2 - \frac{1}{4}) = -5(2\nu)^{-1}$. Since we are looking for $P$ decaying at infinity, we obtain $\mathcal{P}(\varphi) = -\frac{3\,c_1^2}{560\,\nu^2}$. From here we get that problem (5.11) admits the solution

$$W(\varphi, z) = -\frac{c_1^2}{6720\,\nu^3} \left( 64\,z^7 - 112\,z^5 + 44\,z^3 - 5\,z \right)$$

$$Q(\varphi, z) = -\frac{c_1^2}{6720\,\nu^2} \left( 448\,z^6 - 560\,z^4 + 132\,z^2 - 5 \right).$$

Put $(u, q) = (v^\infty, p^\infty) - \zeta_0(V, P)$ with $\zeta_0(y, z) = \left( 1 - \chi(R_0^{-1}|y|) \right)$ (see (2.2)), then due to the construction we have

$$\left.\begin{array}{r} -\Delta u(y, z) + \nabla q(y, z) = f(y, z) - f_1(y, z) \\ \nabla \cdot u(y, z) = 0 \end{array}\right\} \quad (y, z) \in \Omega,\ |y| > 2R_0\,.$$

Note that

$$\begin{array}{ll} \zeta_0 V_y \in \mathcal{W}_{\gamma^*-1}^{l+2}(\Omega)^2, & \zeta_0 V_z \in \mathcal{W}_{\gamma^*}^{l+2}(\Omega) \\ \zeta_0 P \in \mathcal{W}_{\gamma^*-1}^{l+3}(\Omega), & \partial_z(\zeta_0 P) \in \mathcal{W}_{\gamma^*}^{l+2}(\Omega) \end{array} \qquad \text{for all } \gamma^* < l + 9.$$

Theorem 2.2 applied to $(u, q)$ leads to the asymptotic representation (2.3), (2.4), where the sum in (2.5) runs to $N = 3$ and the remainders $(\widetilde{u}, \widetilde{q})$ fulfill (2.6). Now the representation of $(u, q)$ gives (2.3), (2.4) for $(v^\infty, p^\infty)$ with $N = 3$, and $(\widetilde{v}^\infty, \widetilde{p}^\infty) = (\widetilde{u}, \widetilde{q}) + \zeta_0(V, P)$. $\qquad\square$

Comparing this result with those of Theorem 2.2 on the Stokes problem we see that in addition to the smallness condition for the data, the decay rate of the remainder $\widetilde{v}^\infty, \widetilde{p}^\infty$ is limited in representation (2.4). Since the next power law term for $p^\infty$ contains $P$ from (5.10) it cannot be harmonic unless $c_1 = 0$, and therefore in the case of the nonlinear problem, the structure of the representation (2.3)–(2.5) is valid only up to $N = 3$ in general.

## 6. Error estimates for the Navier–Stokes problem with ABC

Although Theorem 5.2 does not provide the whole asymptotic series in harmonics for the solution $(v^\infty, p^\infty)$ of the Navier-Stokes problem (1.3), we use the same operator $M_R$ constructed in Section 3 as for the linear problem and formulate the nonlinear problem in the truncated domain $\Omega_R$ as follows:

$$\begin{array}{l} -\nu\Delta v^R + (v^R \cdot \nabla)v^R + \nabla p^R = f \\ \nabla \cdot v^R = 0 \ \text{ in } \Omega_R, \quad v^R = 0 \ \text{ on } \Sigma_R, \quad M_R(v^R, p^R) = 0 \ \text{ on } \Gamma_R\,. \end{array} \tag{6.1}$$

We recall a corollary of the the Banach fixed point principle which is used as a standard argument to solve Navier-Stokes problems with small data.

**Lemma 6.1** (see [25, Lemma 5.1], e.g.)**.** *Let $X, Y$ be Banach spaces, $\mathbf{S} : X \to Y$ a linear invertible operator with $\|\mathbf{S}^{-1}\| \leq C_{\mathbf{S}}$, further let $\mathbf{N} : X \times X \to Y$ be bilinear with $\|\mathbf{N}(\mathbf{u}, \mathbf{v}); Y\| \leq C_{\mathbf{N}} \|\mathbf{u}; X\| \|\mathbf{v}; X\|$. Then for any $\mathbf{f}$ with $\|\mathbf{f}; Y\| \leq (4 C_{\mathbf{S}}^2 C_{\mathbf{N}})^{-1}$ there exists a unique solution $\mathbf{u}$ to*

$$\mathbf{S}\mathbf{u} + \mathbf{N}(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \tag{6.2}$$

*in the ball $\|\mathbf{u}; X\| < (2 C_{\mathbf{S}} C_{\mathbf{N}})^{-1}$, and this solution fulfils $\|\mathbf{u}; X\| \leq 2 C_{\mathbf{S}} \|\mathbf{f}; Y\|$.*

We will apply this lemma to solve problem (6.1) as well as to obtain error estimates, to this end we have to watch carefully the embedding constants of some Sobolev embeddings.

**Lemma 6.2.** *For any $v, V, w \in \mathcal{H}(\Omega_R)$ the following inequalities hold with constants independent of $R \geq R_0$:*

$$\|v; L^4(\Omega_R)\|^2 \leq \|v; L^2(\Omega_R)\| \, \|v; L^6(\Omega_R)\| \leq c \, \|\nabla v; L^2(\Omega_R)\|^2 \tag{6.3}$$

$$\left| \left( (v \cdot \nabla) u, w \right)_{\Omega_R} \right| \leq c \, \|v; L^4(\Omega_R)\| \, \|\nabla u; L^2(\Omega_R)\| \, \|w; L^4(\Omega_R)\|$$
$$\leq C_{\mathbf{N}} \, \|\nabla v; L^2(\Omega_R)\| \, \|\nabla u; L^2(\Omega_R)\| \, \|\nabla w; L^2(\Omega_R)\|. \tag{6.4}$$

*Proof.* The first relation in (6.3) follows from the Hölder inequality while the second one needs the Poincaré's inequality and the inequality

$$\|w; L^6(\Omega_R)\| \leq c \, \|\nabla w; L^2(\Omega_R)\| \quad \text{for all } w \in \mathcal{H}(\Omega_R) \tag{6.5}$$

with a constant independent of $w$ and $R > R_0$. Estimate (6.5) can be verified by extending $v$ by zero on the cylinder $\mathsf{C}_R = \left\{ x = (y, z) : |y| < R, |z| < R \right\}$ and then again by a scaling argument. We change the variables $x \mapsto \mathsf{x} = R^{-1} x$ and define $\mathsf{w}(\mathsf{x}) = w(R\mathsf{x})$. With $\|\mathsf{w}; L^6(\mathsf{C}_1)\| = R^{-\frac{3}{6}} \|w; L^6(\mathsf{C}_R)\|$, $\|\nabla_{\mathsf{x}} \mathsf{w}; L^2(\mathsf{C}_1)\| = R R^{-\frac{3}{2}} \|\nabla w; L^2(\mathsf{C}_R)\|$, and the Sobolev embedding inequality (6.5) on the cylinder $\mathsf{C}_1$ we obtain (6.5) on $\mathsf{C}_R$. Estimate (6.4) then follows from (6.3). $\qquad \square$

**Remark 6.3.** In analogy to Definition 4.1, we call a pair $(v^R, p^R) \in \mathcal{H}_\sigma(\omega_R) \times L^2(\Omega_R)$ a weak solution to (6.1) if, for all $w \in \mathcal{H}(\omega_R)$,

$$(f, w)_{\Omega_R} = \nu(\nabla v^R, \nabla w)_{\Omega_R} + ((v^R \cdot \nabla) v^R, w) - (p^R, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(v^R, w). \tag{6.6}$$

We also recall that the one dimensional Friedrich's inequality implies

$$\|w; L^2(\Omega_R)\| \leq C_F \|\partial_z w; L^2(\Omega_R)\|$$

with a constant $C_F$ independent of $R$. If we put $X = \mathcal{H}_\sigma(\Omega_R)$, $Y = \mathcal{H}'(\Omega_R)$, the operators $\mathbf{S}$ and $\mathbf{N}$ are defined by

$$\begin{aligned} (\mathbf{S}v)(w) &= (\mathbf{S}_0 v)(w) =: \nu(\nabla v, \nabla w)_{\Omega_R} + \mathbf{q}_R(v, w) \\ \mathbf{N}(u, v)(w) &= ((u \cdot \nabla v), w)_{\Omega_R}. \end{aligned} \tag{6.7}$$

Theorem 4.6 implies that $\mathbf{S}^{-1}$ exists and $C_{\mathbf{S}}$ can be chosen independent of $R \geq R_0$, while Lemma 6.2 implies the continuity of the nonlinear operator from $\mathcal{H}_\sigma(\omega_R)$ to $\mathcal{H}'(\Omega_R)$ and that $C_{\mathbf{N}}$ is independent of $R$. Since for $f \in L^2(\Omega)$, we always have $\|f; \mathcal{H}'(\Omega_R)\| \leq C_F \|f; L^2(\Omega_R) \leq C_F \|f; L^2(\Omega)\|$, Lemma 6.1 gives for $C_F \|f; L^2(\Omega)\| < (4C_{\mathcal{S}}^2 C_{\mathbf{N}})^{-1}$ a weak solution $v^R \in \mathcal{H}_\sigma(\Omega_R)$ which is unique in the ball $\|v^R; \mathcal{H}_\sigma\| \leq (2C_{\mathbf{S}} C_{\mathbf{N}})^{-1}$, with (6.6) for all $w \in \mathcal{H}_\sigma(\Omega_R)$. The pressure $p^R$ such that (6.6) is valid for all $w \in \mathcal{H}(\Omega_R)$ can be found in the same way as in Theorem 4.6.

Now we use a similar scheme as in [25], where the problem to find good error estimates is reduced to a nonlinear boundary value for the differences $v^\infty - v^R$, $p^\infty - p^R$.

**Theorem 6.4.** *Let $l \in \mathbb{N}_0$, $N \leq 3$, and $f \in \mathcal{W}_\gamma^{l+2}(\Omega)^3$ with $\gamma \in (l + 3 + N, l + 4 + N)$ as in Theorem 5.2, moreover, let $(v^\infty, p^\infty)$ be the solution of the original problem (1.3). There exist $\varepsilon_1 \in (0, \varepsilon_0]$, $\mathbf{c} > 0$ and $R_1 \geq R_0$ such that, for $\|f; L_2(\Omega)\| \leq \varepsilon_1$ and $R \geq R_1$, problem (6.1) admits a unique solution in the ball:*

$$\|v^R - v^\infty; \mathcal{H}(\Omega_R)\| + \|p^R - p^\infty; L_2(\Omega_R)\| \leq \mathbf{c}.$$

*Moreover, for $R \geq R_1$ the following error estimate holds:*

$$\|v^\infty - v^R; H^1(\Omega_R)\| + R^{-1}\|p^\infty - p^R\| \leq C(f, \Omega) R^{3-l-\gamma} = o(R^{-N}) \text{ as } R \to \infty.$$

*Proof.* `Step 1. Splitting of the solution` $(v^\infty, p^\infty)$. Dealing with an error estimate for the differences $v^{er} = v^R - v^\infty$, $p^{er} = p^R - p^\infty$, like in the proof of Theorem 4.8 we come across the fact that $v^\infty|_{\Omega_R}$ is not contained in $\mathcal{H}_\sigma(\Omega_R)$. Hence we use again the decomposition (4.25), $v^\infty = v_{ap}^\infty + \widetilde{v}_{cut}^\infty$, $p^\infty = p_{ap}^\infty + \widetilde{p}_{cut}^\infty$, where the different summands are defined as in (4.26). If the requirements of Theorem 5.2 are met, then we may use (4.31) and $(4.32)_{3,4}$ and obtain again

$$\|\widetilde{v}_{cut}^\infty; H^1(\Omega_R)\| + R^{-1}\|\widetilde{p}_{cut}^\infty; L^2(\Omega_R)\| \leq C(f, \Omega)R^{3+l-\gamma} \leq C(f, \Omega)R^{-N},$$

as $R$ tends to infinity. Now we have to consider still

$$\mathbf{v}^{er} := v_{ap}^\infty - v^R, \quad \mathbf{p}^{er} := p^\infty - p^R$$

`Step 2. The error system.` Since

$$\nu \Delta v_{ap}^\infty - (v_{ap}^\infty \cdot \nabla)v_{ap}^\infty - \nabla p_{ap}^\infty = f - \nu \Delta \widetilde{v}_{cut}^\infty - \nabla \widetilde{p}_{cut}^\infty - (\widetilde{v}_{cut}^\infty \cdot \nabla)v_{ap}^\infty \\ - (v_{ap}^\infty \cdot \nabla)\widetilde{v}_{cut}^\infty - (\widetilde{v}_{cut}^\infty \cdot \nabla)\widetilde{v}_{cut}^\infty =: f + \mathbf{f}^{er}, \quad (6.8)$$

the remaining error $(\mathbf{v}^{er}, \mathbf{p}^{er})$ has to solve the boundary value problem

$$-\nu \Delta \mathbf{v}^{er} + \nabla \mathbf{p}^{er} + (v_{ap}^\infty \cdot \nabla)\mathbf{v}^{er} + (\mathbf{v}^{er} \cdot \nabla)v_{ap}^\infty = -(\mathbf{v}^{er} \cdot \nabla)\mathbf{v}^{er} + \mathbf{f}^{er}$$

$$\nabla \cdot \mathbf{v}^{er} = 0 \text{ in } \Omega_R$$

$$\mathbf{v}^{er} = 0 \text{ on } \Sigma_R$$

$$M_R(\mathbf{v}^{er}, \mathbf{p}^{er}) = 0 \text{ on } \Gamma_R.$$

A weak solution to this problem is a pair $\mathbf{v}^{er} \in \mathcal{H}_\sigma(\Omega_R)$, $\mathbf{p}^{er} \in L^2(\Omega_R)$ such that for all $w \in \mathcal{H}(\Omega_R)$ it holds

$$
\begin{aligned}
(\mathbf{f}^{er}, w)_{\Omega_R} &= \nu(\nabla \mathbf{v}^{er}, \nabla w)_{\Omega_R} - (\mathbf{p}^{er}, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(\mathbf{v}^{er}, w) \\
&\quad + \big((v_{ap}^\infty \cdot \nabla)\mathbf{v}^{er} + (\mathbf{v}^{er} \cdot \nabla)v_{ap}^\infty, W\big)_{\Omega_R} + \big((\mathbf{v}^{er} \cdot \nabla)\mathbf{v}^{er}, W\big)_{\Omega_R}.
\end{aligned}
\tag{6.9}
$$

Due to the definition of $\chi^R$ (see (4.23)) we obtain $\operatorname{supp} \mathbf{f}^{er} \subset \Xi_R$ for $R > 2R_0$. From inequalities (6.4) and $(4.32)_3$ it follows with (6.8)

$$
\begin{aligned}
(\mathbf{f}^{er}, w)_{\Omega_R} &\leq C_N \|\nabla v_{ap}^\infty; L^2(\Omega_R)\| \, \|\nabla \widetilde{v}_{cut}^\infty; L^2(\Xi_R)\| \, \|\nabla w; L^2(\Omega_R)\| \\
&\leq C_N C(\chi) \|v^\infty; H^1(\Omega)\| \, \|\nabla \widetilde{v}^\infty; L^2(\Xi_R)\| \, \|w; \mathcal{H}(\Omega_R)\| \\
&\leq C_N C(\chi) \, R^{3-l-\gamma} \, \|v^\infty; H^1(\Omega)\| \, \big(\, \|\widetilde{v}_y^\infty; \mathcal{W}_{\gamma-1}^{l+2}(\Xi_R)\| \\
&\quad + \|\widetilde{v}_z^\infty; \mathcal{W}_\gamma^{l+2}(\Omega)\| \,\big) \, \|w; \mathcal{H}(\Omega_R)\|.
\end{aligned}
\tag{6.10}
$$

**Step 3. The linear part of the error system.** We now prove the following assertion: Let $C_\mathbf{N}$ be the constant of (6.4), and

$$
\|\nabla v_{ap}^\infty; L^2(\Omega)\| < \frac{\nu}{2C_\mathbf{N}}.
\tag{6.11}
$$

Then for any $\Phi \in \mathcal{H}'(\Omega_R)$, we obtain a unique $V \in \mathcal{H}_\sigma(\Omega_R)$ with

$$
\nu(\nabla V, \nabla w)_{\Omega_R} + \mathbf{q}_R(V, w) + \big((v_{ap}^\infty \cdot \nabla)V + (V \cdot \nabla)v_{ap}^\infty, w\big)_{\Omega_R} = \Phi(w)
\tag{6.12}
$$

for all $w \in \mathcal{H}(\Omega_R)$. Indeed, the left hand side of (6.12) defines a bilinear form on $\mathcal{H}_\sigma(\Omega_R)$ which is continuous by (6.4) and coercive if (6.11) is fulfilled, since $\big((v_{ap}^\infty \cdot \nabla)V + (V \cdot \nabla)v_{ap}^\infty, V\big)_{\Omega_R} \leq 2C_\mathbf{N}\|\nabla v_{ap}^\infty\|\|\nabla V\|^2$. Thus we obtain a unique $V \in \mathcal{H}_\sigma(\Omega_R)$ such that (6.12) is fulfilled, moreover the inequality above leads to

$$
\Big(\nu - 2C_\mathbf{N}\|\nabla v_{ap}^\infty\|\Big) \, \|\nabla V; L^2(\Omega_R)\| + \mathbf{q}_R(V, V) \leq |\Phi(V)|,
$$

which leads to

$$
\|V; \mathcal{H}(\Omega_R)\| \leq C(\Omega, v^\infty)\|\Phi; \mathcal{H}'(\Omega_R)\|,
\tag{6.13}
$$

here the constant depends neither on $\Phi$ nor on $R$.

**Step 4. Solution of the error system.** First we first treat the problem (6.9) for $\mathbf{v}^{er}$ alone by admitting only test functions $w \in \mathcal{H}_\sigma(\Omega_R)$. To use Lemma 6.1, we set $X = \mathcal{H}_\sigma(\Omega_R)$ and $Y = \mathcal{H}'(\Omega_R)$ again, define $\mathbf{N}$ as in (6.7), but the operator $\mathbf{S} : \mathcal{H}_\sigma(\Omega) \to \mathcal{H}'(\Omega_R)$ by the left hand side of (6.12) now. Hence Problem (6.9) again has the structure (6.2). From Lemma 6.1 it follows: For any $\Phi \in \mathcal{H}'(\Omega_R)$ with

$$
\|\Phi; \mathcal{H}'(\Omega_R)\| \leq (4C(\Omega, v^\infty)^2 C_\mathbf{N})^{-1} \quad \text{(see (6.13) and (6.4)),}
\tag{6.14}
$$

we obtain a unique $\mathbf{v} \in \mathcal{H}_\sigma(\Omega_R)$ with $\|\mathbf{v}; \mathcal{H}(\Omega_R)\| \leq (2C(\Omega, v^\infty)C_\mathbf{N})^{-1}$ such that

$$\Phi(w) = \nu(\nabla\mathbf{v}, \nabla w)_{\Omega_R} + \left((v_{ap}^\infty \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)v_{ap}^\infty, W\right)_{\Omega_R}$$
$$+ ((\mathbf{v} \cdot \nabla)\mathbf{v}, W)_{\Omega_R} + \mathbf{q}_R(\mathbf{v}, w).$$

Moreover, we have with the same constant as in (6.13):

$$\|\mathbf{v}; \mathcal{H}(\Omega_R) \leq 2C(\Omega, v^\infty)\|\Phi; \mathcal{H}'(\Omega_R)\|.$$

Since $\|\nabla v^\infty; L^2(\Omega_R)\| \leq C\|f; L^2(\Omega_R)\|$, there exists $\varepsilon_1$ where $\|f; L^2(\Omega_R)\| \leq \varepsilon_1$ implies the smallness condition (6.11). With $\Phi(w) = (\mathbf{f}^{er}, w)_{\Omega_R}$ it follows from (6.10) that there exists an $R_1$ such that $\|\Phi^{er}; \mathcal{H}'(\Omega_R)\|$ satisfies the smallness condition (6.14) for all $R \geq R_1$, and we obtain $\|\mathbf{v}^{er}; H^1(\Omega_R)\| \leq C_1(\Omega, f) \ R^{3-l-\gamma} = O(R^{-N})$ as $R \to \infty$. Finally, with the same arguments as in the proof of Theorem 4.6 we obtain a unique function $\mathbf{p}^{er} \in L^2(\Omega_R)$ such that (6.9) is fulfilled for all $w \in \mathcal{H}(\Omega_R)$ and $\|\mathbf{p}^{er}; L^2(\Omega_R)\| \leq C_2(\Omega, f) \ R^{3-l-\gamma} = O(R^{-N})$ as $R \to \infty$. $\qquad\square$

**Remark 6.5.** Although at most three terms in the asymptotic representation of the solution $(v^\infty, p^\infty)$ to the Navier-Stokes problem (1.3) are generated by harmonic functions as in (2.5), we used the Steklov-Poincaré operator (3.5) in the ABC. Of course, one can replace the pseudodifferential operators $\Pi_R$ and $\Pi_R^{-1}$ by suitable finite-dimensional approximations. Moreover, one can search for a local operator $B$ in (3.2). If $v^\flat$ is chosen as in (3.12) this means to fix $\overline{B}$ in (3.12) as a differential operator. The simplest ABC of this type are of the form

$$\nu\frac{\partial}{\partial r}\overline{v_r^R} - \overline{p^R} = -2\nu(R^{-1} + 5R)\,\overline{v_r^R}, \quad \nu\frac{\partial}{\partial r}\overline{v_\varphi^R} = -2\nu R^{-1}\,\overline{v_\varphi^R} \quad \text{on } \Gamma_R. \quad (6.15)$$

Then Condition II of Section 3 is fulfilled but Condition I only for $N = 1$. In this case, Theorems 4.8 and 6.4 remain valid with $N = 1$.

We could also try an ansatz which was proposed in [26] to modify the condition (6.15) in the form

$$\nu\frac{\partial}{\partial r}\overline{v_r^R} - \overline{p^R} = -A_1\overline{v_r^R} + B_1\partial_\varphi^2\overline{v_r^R}, \quad \nu\frac{\partial}{\partial r}\overline{v_\varphi^R} = -A_2\overline{v_\varphi^R} + B_2\partial_\varphi^2\overline{v_\varphi^R} \quad \text{on } \Gamma_R.$$

If we choose $A_i$, $B_i$ in such a way that Condition I in Section 3 is fulfilled by $(V^\infty, P^\infty)$ with $N = 2$ in the representation (2.4), we find

$$A_1 = \tfrac{5}{3}\nu(R^{-1} + 7R), \qquad B_1 = \tfrac{\nu}{3}(R^{-1} - 5R),$$
$$A_2 = \tfrac{\nu}{3}R^{-1}, \qquad\qquad B_2 = \tfrac{5\nu}{3}R^{-1}.$$

But now Condition II is violated because $B_1 < 0$ for large $R$. We emphasize that, in principle, it happens only by chance that quadratic forms resulting from the ABC $(3.13)_{2,3}$ and (6.15) are nonnegative: from one side there is no a priori reason to get this property and from the other side there is no free constant to fulfill it artificially!

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# References

[1] Bänsch, E. and Dörfler, W., Adaptive finite elements for exterior domain problems. *Numer. Math.* 80 (1998)(4), 497 – 523.

[2] Bao, W. and Han, H., Error bounds for the finite element approximation of an incompressible material in an unbounded domain. *Numer. Math.* 93 (2003)(3), 415 – 444.

[3] Bayliss, A., Gunzburger, M. and Turkel, E., Boundary conditions for numerical solutions of elliptic equations in exterior domains. *SIAM J. Appl. Math.* 42 (1982), 430 – 451.

[4] Deuring, P. and Kračmar, S., Artificial boundary conditions for the Oseen system in 3D exterior domains. *Analysis* 20 (2000), 65 – 90.

[5] Deuring, P., Calculating Stokes flows around a sphere: Comparison of artificial boundary conditions. *Ann. Univ. Ferrara Sez. VII (N.S.)* 46 (2000), 1 – 9.

[6] Givoli, D. and Patlashenko, I., Optimal local non-reflecting boundary conditions. *Appl. Numer. Math.* 27 (1998)(4), 367 – 384.

[7] Goldstein, C. I., Multigrid methods for elliptic problems in unbounded domains. *SIAM J. Numer. Anal.* 30 (1993), 159 – 183.

[8] Hagstrom, Th., New results on absorbing layers and radiation boundary conditions. In: *Topics in Computational Wave Propagation* (eds.: M. Ainsworth et al.) Lect. Notes Comput. Sci. Eng. 31. Berlin: Springer 2003, pp. 1 – 42.

[9] Han H. and Bao, W., Error estimates for the finite element approximation of problems in unbounded domains. *SIAM J. Numer. Anal.* 37 (2000)(4), 1101 – 1119.

[10] Han, H. and Wen, X., The local artificial boundary conditions for numerical simulations of the flow around a submerged body. *J. Sci. Comput.* 16 (2001)(3), 263 – 286.

[11] Kondratiev, V. A., Boundary value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.* 16 (1967), 227 – 313.

[12] Ladyzenskaja, O. A. and Solonnikov, V. A., Some problems of vector analysis and generalized formulations of boundary value problems for the Navier-Stokes equations. *J. Soviet Math.* 10 (1978), 257 – 286.

[13] Lions, J. L. and Magenes, E., *Non-homogeneous boundary value problems and applications*, Vol. I. Berlin: Springer 1972.

[14] Nazarov, S. A., Asymptotics of the solution to a boundary value problem in a thin cylinder with nonsmooth lateral surface (in Russian). *Izv. Ross. Akad. Nauk., Ser. Mat.* 57 (1993)(1), 202 – 239; English transl.: *Russian Acad. Sci. Izv. Math.* 42 (1994)(1), 183 – 217.

[15] Nazarov, S. A., Asymptotics of the solution of the Neumann problem at a point of tangency of smooth components of the boundary of the domain (in Russian). *Izv. Ross. Akad. Nauk., Ser. Mat.* 58 (1994)(1), 192 – 120; English transl.: *Russian Acad. Sci. Izv. Math.* 44 (1995)(1), 91 – 118.

[16] Nazarov, S. A., On the flow of water under a still stone (in Russian). *Mat. Sb.* 186 (1995)(11), 75 – 110; English transl.: *Sb. Math.* 186 (1995)(11), 1621 – 1658.

[17] Nazarov, S. A., Asymptotic expansions at infinity of solutions of a problem in the theory of elasticity in a layer (in Russian). *Trudy Mosk. Mat. Obsch.* 60 (1998), 3 – 97; English transl.: *Trans. Moscow Math. Soc.* 60 (1999), 1 – 85.

[18] Nazarov, S. A. and Pileckas, K., On the Fredholm property of the Stokes operator in a layer-like domain. *Z. Anal. Anwendungen* 20 (2001), 155 – 182.

[19] Nazarov, S. A. and Pileckas, K. I., The asymptotic properties of the solution to the Stokes problem in domains that are layer-like at infinity. *J. Math. Fluid Mech.* 1(1999)(2), 131 – 167.

[20] Nazarov, S. A. and Pileckas, K. I., On the solvability of the Stokes and Navier–Stokes problem in the domain that are layer-like at infinity. *J. Math. Fluid Mech.* 1 (1999), 78 – 116.

[21] Nazarov, S. A. and Plamenevskiĭ, B. A., *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. Berlin: de Gruyter 1994.

[22] Nazarov, S. A. and Specovius-Neugebauer, M., Approximation of exterior problems. Optimal conditions for the Laplacian. *Analysis* 16 (1996)(4), 305 – 324.

[23] Nazarov, S. A. and Specovius-Neugebauer, M., Approximation of exterior boundary value problems for the Stokes system. *Asymptot. Anal.* 14 (1997)(3), 223 – 255.

[24] Nazarov, S. A. and Specovius-Neugebauer, M., Artificial boundary conditions providing superpower accuracy for the Neumann problem in a layer-like domain. *Zh. Vychisl. Mat. Mat. Fiz.* 43 (2003)10, 1475 – 1486; English transl.: *Comput. Math. Math. Phys.* 43 (2003)(10),1418 – 1429.

[25] Nazarov, S. A. and Specovius-Neugebauer, M., Nonlinear artificial boundary conditions with pointwise error estimates for the exterior three dimensional Navier-Stokes problem. *Math. Nachr.* 252 (2003)(5), 86 – 105.

[26] Nazarov, S. A. and Specovius-Neugebauer, M., Artificial boundary conditions for Petrovsky systems of second order in exterior domains and in other domains of conical type. *Math. Methods Appl. Sci.* 27 (2004)(13), 1507 – 1544.

[27] Pileckas, K., On the asymptotic behavior of solutions of a stationary system of Navier-Stokes equations in a domain of layer type (in Russian). *Mat. Sb.* 193 (2002)(12), 69 – 104; English transl.: *Sb. Math.* 193 (2002)(11–12), 1801 – 1836.

[28] Quarteroni, A. and Valli, A., Theory and application of Steklov-Poincaré operators for boundary-value problems. In: *Applied and Industrial Mathematics* (Venice 1989). Math. Appl. 56. Dordrecht: Kluwer 1991, pp. 179 – 203.

[29] Quarteroni, A. and Valli, A., Theory and application of Steklov-Poincaré operators for boundary-value problems: the heterogeneous operator case. In: *Fourth Intern. Symp. Domain Decomp. Methods Partial Diff. Equations* (Moscow 1990; eds.: R. Glowinski et al.). Philadelphia (PA): SIAM 1991, pp. 58 – 81.

[30] Quarteroni, A. and Valli, A., *Domain decomposition methods for partial differential equations.* Numerical Mathematics and Scientific Computation. Oxford: Clarendon Press 1999.

[31] Ryabenki, V. S. and Tsynkov, S. V., Artificial boundary conditions for the numerical solution of external viscous flow problems. *SIAM J. Numer. Anal.* 32 (1995)(5), 1355 – 1389.

[32] Sanchez-Palencia, E., Approximation of boundary value problems in unbounded domains. In: *Homogenization and applications to material sciences* (Nice 1995; eds.: D. Cioranescu et al.). GAKUTO Int. Ser. Math. Sci. Appl. 9. Tokyo: Gakkotosho 1995, pp. 363 – 381.

[33] Sohr, H., *The Navier-Stokes Equations. An Elementary Functional Analytic Approach.* Basel: Birkhäuser 2001.

[34] Specovius-Neugebauer, M., Approximation of the Stokes Dirichlet problem in domains with cylindrical outlets. *SIAM J. Math. Anal.* 30 (1999), 645 – 677.

[35] Tsynkov, S. V., Numerical solution of problems on unbounded domains. A review. *Appl. Num. Math.* 27 (1998)(4), 465 – 532.