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Unilateral Contact Problems for two Perpendicular Elastic Structures

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Abstract. We consider the problem of unilateral contact between two elastic perpendicular plates. The main focus is on the boundary conditions along the contact zone. We propose a mixed domain formulation. Some limit cases for the considered problem are justified. In particular, a unilateral contact between a plate and a beam is also analyzed.

Keywords. Unilateral contact problem, thin elastic obstacle, elastic plate, Signorini condition, elastic beam, inequality type boundary condition

Mathematics Subject Classification (2000). Primary 35J85, secondary 49J40, 58E35

1. Introduction

Let $\Omega \subset R^2$ be a bounded domain with smooth boundary Γ. Denote by $q = (q_1, q_2)$ a unit normal vector to Γ. The domain Ω is supposed to be representative of the middle surface of an elastic plate. We consider another elastic plate perpendicular to Ω with a middle surface D, $\Omega \cap D = \emptyset$. The boundary of the domain D will be denoted by ∂D. Let $\Omega \cap \partial D = \gamma$ and $\nu = (\nu_1, \nu_2)$ be a unit normal vector to γ located in the plane Ω. Also denote by $n = (n_1, n_2)$ a unit normal internal vector to ∂D located in the plane D (see Fig. 1). We assume that ∂D is a smooth curve, γ is a connected set, $\partial D = \gamma \cup \gamma_0$, $\gamma \cap \gamma_0 = \emptyset$, and γ does not intersect the boundary Γ of the domain Ω.

In this paper we analyze a unilateral contact between the two elastic plates described above. We model the first plate which lies in the horizontal plane in its reference configuration by a Kirchhoff plate while the upright plate is modeled using an elastic solid. The contact may occur along the line γ. The mathematical model will describe a vertical displacement of the first plate and

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a horizontal displacement (in the plane D) of the second plate. In fact, we have a thin elastic obstacle for the first plate and unilateral contact with an elastic structure on the boundary for the second plate. In this case an equilibrium equation for the first plate is considered in the cracked domain $\Omega_\gamma = \Omega \setminus \bar{\gamma}$, and inequality type boundary conditions are imposed on γ. It is well known that crack models with possible contact between crack faces include inequality type boundary conditions (see the book [7]). Meanwhile, it turned out that the inequality type boundary conditions on γ in our case are in fact different.

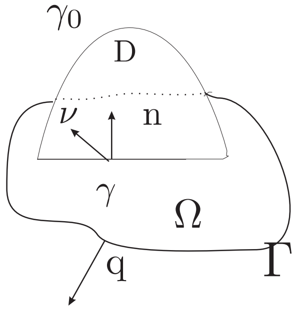

Figure 1: Elastic body in contact with a plate

Note that problems of unilateral contact for elastic plates with rigid obstacles has been analyzed in a number of works ([1–3, 6]). In particular, a thin rigid obstacle is considered in [12]. As for Signorini-type problems for elastic bodies we can find a significant number of publications (see references in [6]). The first question to be dealt with in the contact problem under consideration is concerned with the proper boundary conditions along γ. In this paper we present a complete system of such conditions. A mixed domain formulation will be also proposed. We further analyze the passage to the limits when elasticity moduli of the considered plates converge to infinity, i.e., when the elastic plates converge to the rigid ones. In both cases we arrive at more simple models which have already been analyzed in the literature.

In the last section of this paper we consider a contact problem between an elastic plate and elastic beam, perpendicular to the plate. Both variational and differential formulations of the problem are considered. In particular, a complete system of boundary conditions is given.

Contact problems between structural elements such as plates and beams are important in the understanding of the elastic behavior of multi-link flexible structures. Such structures are crucial in buildings, suspension bridges and space-stations, to mention just a few applications. The problem under consideration in this paper focuses on a exemplary situation, the contact of two such structural elements. Obviously, one may consider more general structural elements, such as Reissner-Mindlin plates and Timoshenko beams together with their asymptotic models when the shear-stiffnes tends to infinity. Moreover, the full problem would also take into account vertical displacements of the upright plate and in-plane deformations of the horizontal plate which might occur upon contact. However, in order to fix ideas and keep the presentation simple, we restrict ourselves to a Kirchhoff plate and a 2-d elastic body, representative of an upright plate undergoing in-plane displacements only. As for mathematical models for multi-link elastic structures we refer the reader to the book [10].

2. Problem formulation

First we give both the differential and variational formulation of the contact problem between the two plates. We search for functions $u(x) = (u_1(x), u_2(x))$, $w(y)$, $x = (x_1, x_2) \in D, y = (y_1, y_2) \in \Omega_\gamma$, such that

$$-\mathrm{div}(A\varepsilon(u)) = g \quad \text{in } D \tag{1}$$
$$\Delta^2 w = f \quad \text{in } \Omega_\gamma \tag{2}$$
$$u = 0 \quad \text{on } \gamma_0 \tag{3}$$
$$w = w_q = 0 \quad \text{on } \Gamma \tag{4}$$
$$un - w \geq 0,\ \sigma_n \leq 0,\ \sigma_\tau = 0,\ \sigma_n(un - w) = 0 \quad \text{on } \gamma \tag{5}$$
$$[w] = [w_\nu] = 0,\ [m(w)] = 0,\ [t^\nu(w)] = \sigma_n \quad \text{on } \gamma. \tag{6}$$

Here $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is a strain tensor, $\sigma = \{\sigma_{ij}\}$ is a stress tensor, $i, j = 1, 2$,

$$\sigma_n = \sigma_{ij} n_j n_i, \qquad \sigma_\tau = \sigma n - \sigma_n \cdot n, \qquad \sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2),$$
$$\sigma n = (\sigma_{1j} n_j, \sigma_{2j} n_j)\ , \qquad \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad i, j = 1, 2,$$

$A = \{a_{ijkl}\}$ is a tensor of elasticity moduli, $i, j, k, l = 1, 2$,

$$a_{ijkl} = a_{jikl} = a_{ijlk}, \qquad a_{ijkl}\xi_{kl}\xi_{ij} \geq c|\xi|^2, \quad c > 0,$$

$[v] = v^+ - v^-$ is a jump of a function v on γ, where $v^\pm$ correspond to the positive and negative (with respect to ν) faces $\gamma^\pm$, respectively. All functions with two lower indices are assumed to be symmetric in those indices, i.e., $\xi_{ij} = \xi_{ji}$ etc. The following notations are also used in (1)–(6):

$$w_\nu = \frac{\partial w}{\partial \nu}, \quad w_q = \frac{\partial w}{\partial q}, \quad m(w) = w_{,ij}\nu_j\nu_i,$$
$$t^\nu(w) = w_{,ijk}s_k s_j \nu_i + w_{,ijj}\nu_i, \quad (s_1, s_2) = (-\nu_2, \nu_1).$$

Summation convention over repeated indices is assumed, $g = (g_1, g_2) \in L^2(D)$, $f \in L^2(\Omega)$ are given functions.

Note that (1)–(2) are equilibrium equations with the Hooke law $\sigma = A\varepsilon(u)$ in (1), $\sigma = \sigma(u)$; $m(w), t^\nu(w)$ are the bending moment and transverse force, respectively, for the first plate. Relations (3)–(4) provide the plates clamping on γ_0 and Γ, respectively. The first inequality in (5) guarantees a mutual nonpenetration between the two elastic plates.

We note the boundary conditions involving moments $m(w)$ and forces $t^\nu(w)$ are somewhat more specialized as those appearing in the literature. In particular, one typically considers formulae

$$m(w) = \mu\Delta w + (1-\mu)\frac{\partial^2 w}{\partial \nu^2}, \quad t^\nu(w) = \frac{\partial}{\partial \nu}\left(\Delta w + (1-\mu)\frac{\partial^2 w}{\partial s^2}\right),$$

where μ corresponds to the Poisson ratio, see e.g. [7,10]. In this case a bilinear form in the Green formula used below would be different. We emphasize that these more complex boundary conditions can be dealt with in the context of the methods developed in this paper, and in fact they can be recovered for the Poisson ratio $\mu = 0$. However, the formulae would be much more involved, and might obscure this first attempt of treating the kind of contact problems under consideration in this paper.

Below we give a variational formulation of the problem (1)–(6) which provides, in particular, the existence of a solution. Next we derive relations (1)–(6) from the variational formulation of the problem.

Let us introduce the following energy spaces

$$H^1_{\gamma_0}(D) := \{v \in H^1(D) \mid v = 0 \text{ on } \gamma_0\}$$
$$H^2_0(\Omega) := \{v \in H^2(\Omega) \mid v = v_q = 0 \text{ on } \Gamma\}.$$

Denote

$$K := \big\{(u,w) \mid u = (u_1,u_2) \in H^1_{\gamma_0}(D),\ w \in H^2_0(\Omega),\ un - w \geq 0 \ \text{ on } \gamma\big\}$$

and consider the energy functional

$$\Pi(u,w) := \frac{1}{2}\int_D \sigma(u)\varepsilon(u) - \int_D gu + \frac{1}{2}\int_\Omega w_{,ij}w_{,ij} - \int_\Omega fw.$$

We can find a solution of the minimization problem

$$\inf_{(u,w)\in K} \Pi(u,w) \tag{7}$$

which is equivalent to the variational inequality problem:

Find $(u,w) \in K$ such that (8)

$$\begin{cases} \displaystyle\int_D \sigma(u)\varepsilon(\bar u - u) - \int_D g(\bar u - u) \\ \displaystyle+ \int_\Omega w_{,ij}(\bar w_{,ij} - w_{,ij}) - \int_\Omega f(\bar w - w) \geq 0 \quad \forall(\bar u, \bar w) \in K. \end{cases} \tag{9}$$

Note that the functional Π is coercive and weakly lower semicontinuous on $[H^1_{\gamma_0}(D)]^2 \times H^2_0(\Omega)$. Moreover, the set K is weakly closed. Hence, the constrained minimization problem (7) has (a unique) solution satisfying the variational inequality (8)–(9).

Theorem 2.1. *Problems* (1)–(6), (7), *and* (8),(9) *are equivalent. Moreover, there exists a unique solution* (u, w) *to* (7), *and hence to* (1)–(6), *and to* (8), (9).

Proof. We proceed to derive relations (1)–(6) from (8)–(9) and clarify in what sense the boundary conditions (5)–(6) hold. First note that equations (1), (2) follow from (9) in the distributional sense. Indeed, it suffices to substitute into (9) test functions $(\bar{u}, \bar{w}) = (u \pm \psi, w \pm \varphi), \psi = (\psi_1, \psi_2) \in C_0^\infty(D), \varphi \in C_0^\infty(\Omega_\gamma)$. This implies (1), (2).

Let us take $(\bar{u}, \bar{w}) = (u + \psi, w)$ as test functions in (9). Here $\psi = (\psi_1, \psi_2) \in H^1_{\gamma_0}(D), \psi_n = \psi n \geq 0$ on γ. This yields

$$\int_D \sigma(u)\varepsilon(\psi) - \int_D g\psi \geq 0. \tag{10}$$

The following Green formula holds [7]:

$$\int_D \sigma(u)\varepsilon(\psi) = -\int_D \mathrm{div}\sigma(u) \cdot \psi - \langle \sigma_n, \psi_n \rangle_{\frac{1}{2},\partial D} - \langle \sigma_\tau, \psi_\tau \rangle_{\frac{1}{2},\partial D}. \tag{11}$$

Here $\langle \cdot, \cdot \rangle_{\frac{1}{2},\partial D}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$, where the space $H^{-\frac{1}{2}}(\partial D)$ is the dual of $H^{\frac{1}{2}}(\partial D)$, $\psi = \psi_n n + \psi_\tau$. We take into account the equilibrium equation

$$-\mathrm{div}\sigma(u) = g \quad \text{in } D.$$

Notice that this equation coincides with (1). Hence (10) implies

$$-\langle \sigma_n, \psi_n \rangle_{\frac{1}{2},\partial D} - \langle \sigma_\tau, \psi_\tau \rangle_{\frac{1}{2},\partial D} \geq 0. \tag{12}$$

We have no restrictions for ψ_τ, consequently, the inequality (12) provides

$$\langle \sigma_\tau, \psi_\tau \rangle_{\frac{1}{2},\partial D} = 0.$$

Since $\psi = 0$ on γ_0, this relation can be written in the form

$$\langle \sigma_\tau, \psi_\tau \rangle^{00}_{\frac{1}{2},\gamma} = 0, \tag{13}$$

where $\langle \cdot, \cdot \rangle^{00}_{\frac{1}{2},\gamma}$ denotes the duality pairing between the space $H^{\frac{1}{2}}_{00}(\gamma)$ and its dual

$H_{00}^{-\frac{1}{2}}(\gamma)$. The norm in the space $H_{00}^{\frac{1}{2}}(\gamma)$ is defined as follows (see e.g. [5]):

$$\|v\|^2_{H_{00}^{\frac{1}{2}}(\gamma)} := \|v\|^2_{H^{\frac{1}{2}}(\gamma)} + \int_\gamma \rho^{-1} v^2,$$

where $\rho(y) = \mathrm{dist}(y, \partial\gamma)$. Consequently, from (13) it follows that

$$\sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2) = 0 \quad \text{in the sense of } H_{00}^{-\frac{1}{2}}(\gamma). \tag{14}$$

To derive (13), we have used the following property of the space $H_{00}^{\frac{1}{2}}(\gamma)$. Let v be a function defined on γ. Denote by $\bar{v}$ an extension of v by zero outside γ, i.e.,

$$\bar{v} = \begin{cases} v & \text{on } \gamma \\ 0 & \text{on } \partial D \setminus \gamma \,. \end{cases}$$

Then $\bar{v} \in H^{\frac{1}{2}}(\partial D)$ if and only if $v \in H_{00}^{\frac{1}{2}}(\gamma)$ (see [7]). The similar property is used in a sequel with respect to the space $H_{00}^{\frac{3}{2}}(\gamma)$.

Inequality (12) also implies

$$\sigma_n \le 0 \quad \text{in the sense of } H_{00}^{-\frac{1}{2}}(\gamma). \tag{15}$$

Consider next an extension of γ into to a closed curve Σ of class $C^{1,1}$ such that $\Sigma \subset \Omega$. In this case the domain Ω is divided into two subdomains Ω_1, Ω_2 with boundaries Σ and $\Sigma \cup \Gamma$, respectively (see Fig. 2).

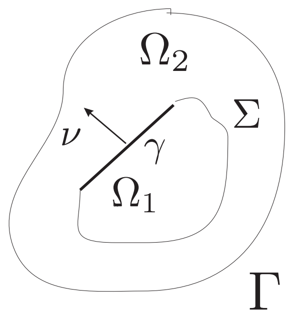

Figure 2: Extension of γ

We choose $(\bar{u}, \bar{w}) = (u, w + \varphi)$ as test functions in (9), $\varphi \le 0$ on γ, $\varphi \in H_0^2(\Omega)$. This provides

$$\int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi \ge 0. \tag{16}$$

Consider the space

$$W := \{v \in H^2(\Omega_1) \mid \Delta^2 v \in L^2(\Omega_1)\}.$$

Then for $v \in W$ we can define $m(v) \in H^{-\frac{1}{2}}(\Sigma)$, $t^\nu(v) \in H^{-\frac{3}{2}}(\Sigma)$, and the following Green formula holds ([7,13]):

$$\int_{\Omega_1} \varphi \Delta^2 v = \int_{\Omega_1} \varphi_{,ij} v_{,ij} + \langle t^\nu(v), \varphi \rangle_{\frac{3}{2},\Sigma} - \langle m(v), \varphi_\nu \rangle_{\frac{1}{2},\Sigma} \quad \forall \varphi \in H^2(\Omega_1). \quad (17)$$

Here $\langle \cdot, \cdot \rangle_{\frac{i}{2},\Sigma}$ means the duality pairing between the space $H^{-\frac{i}{2}}(\Sigma)$ and its dual $H^{\frac{i}{2}}(\Sigma)$, $i = 1, 3$. This Green formula allows us to derive from (16) the inequality

$$-\langle [m(w)], \varphi_\nu \rangle_{\frac{1}{2},\Sigma} + \langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} \geq 0.$$

Since φ_ν is arbitrary on Σ it follows

$$[m(w)] = 0 \quad \text{in the sense of } H^{-\frac{1}{2}}(\Sigma), \quad (18)$$

$$\langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} \geq 0 \quad \forall \varphi \in H_0^2(\Omega), \qquad \varphi \leq 0 \quad \text{on } \gamma. \quad (19)$$

Note that we can substitute $(\bar{u}, \bar{w}) = (u \pm \psi, w \pm \varphi)$ as test functions in (9), where $\psi_n = \varphi$ on $\gamma, \psi = (\psi_1, \psi_2) \in H^1_{\gamma_0}(D), \varphi \in H_0^2(\Omega)$. In this case $\psi_n \in H_{00}^{\frac{1}{2}}(\gamma)$. Assume additionally that $\varphi = 0$ on $\Sigma \setminus \gamma$. In this case $\varphi \in H_{00}^{\frac{3}{2}}(\gamma)$. The space $H_{00}^{\frac{3}{2}}(\gamma)$ is defined as follows [5]:

$$H_{00}^{\frac{3}{2}}(\gamma) := \left\{ v \in H_0^{\frac{3}{2}}(\gamma) \;\middle|\; \int_\gamma \frac{|\nabla v|^2}{\rho} < \infty \right\}, \quad \rho(y) = \operatorname{dist}(y, \partial\gamma).$$

This substitution gives

$$\int_D \sigma(u)\varepsilon(\psi) - \int_D g\psi + \int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi = 0. \quad (20)$$

By (1), (2), (14), (18), the application of the Green formulae (11), (17) in (20) implies $\langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} - \langle \sigma_n, \psi_n \rangle^{00}_{\frac{1}{2},\gamma} = 0$. Since $\varphi = 0$ on $\Sigma \setminus \gamma$ the last relation can be written in the form

$$\langle [t^\nu(w)], \varphi \rangle^{00}_{\frac{3}{2},\gamma} - \langle \sigma_n, \psi_n \rangle^{00}_{\frac{1}{2},\gamma} = 0. \quad (21)$$

Here $\langle \cdot, \cdot \rangle^{00}_{\frac{3}{2},\gamma}$ denotes a duality pairing between $H_{00}^{-\frac{3}{2}}(\gamma)$ and $H_{00}^{\frac{3}{2}}(\gamma)$. Since in our case $\langle \sigma_n, \psi_n \rangle^{00}_{\frac{3}{2},\gamma} = \langle \sigma_n, \psi_n \rangle^{00}_{\frac{1}{2},\gamma}$ from (21) it follows

$$[t^\nu(w)] = \sigma_n \quad \text{in the sense of } H_{00}^{-\frac{1}{2}}(\gamma). \quad (22)$$

We should note the following. Let us take $(\bar{u}, \bar{w}) = (u + \psi, w + \varphi)$ as a test function in (21), where $\psi_n - \varphi \geq 0$ on $\gamma, \psi = (\psi_1, \psi_2) \in H^1_{\gamma_0}(D), \varphi \in H^2_0(\Omega)$. This implies

$$\int_D \sigma(u)\varepsilon(\psi) - \int_D g\psi + \int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi \geq 0.$$

Application of the Green formulae (11), (17) to this inequality, by (1), (2), (14) and (18), gives

$$\langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} - \langle \sigma_n, \psi_n \rangle^{00}_{\frac{1}{2},\gamma} \geq 0 \quad \forall (\psi, \varphi) \in K. \tag{23}$$

In fact, the inequality (23) provides exactly the formulation of the conditions (see (5)–(6)) $\sigma_n \leq 0$, $[t^\nu(w)] = \sigma_n$ on γ. Also note that (15), (19), (22) follow from (23).

By choosing $(\bar{u}, \bar{w}) = (0, 0)$, $(\bar{u}, \bar{w}) = 2(u, w)$ in (9) we can easily derive $\langle [t^\nu(w)], w \rangle_{\frac{3}{2},\Sigma} - \langle \sigma_n, u_n \rangle^{00}_{\frac{1}{2},\gamma} = 0$ which is a precise formulation for the last conditions from (5), (6).

The arguments used show that the first term in (23) does not depend on the choice of Σ. Furthermore, this term is independent of values of φ on $\Sigma \setminus \bar{\gamma}$, ie., if $\varphi^1 = \varphi^2$ on γ, then $\langle [t^\nu(w)], \varphi^1 \rangle_{\frac{3}{2},\Sigma} = \langle [t^\nu(w)], \varphi^2 \rangle_{\frac{3}{2},\Sigma}$.

The system of boundary conditions (3)–(6) is complete, in particular, the variational inequality (8),(9) can be derived from (1)–(6). This concludes the proof of the theorem. $\square$

It is interesting to compare the above division of Ω into the subdomains Ω_1, Ω_2 with more simple approaches used in domain decomposition methods (see [11]).

3. Mixed problem formulation

In this section we provide the mixed formulation for the problem (1)–(6). First rewrite this problem in the following form:

Find functions $u(x) = (u_1(x), u_2(x))$, $\sigma(x) = \{\sigma_{ij}(x)\}$, $i, j = 1, 2$, $w(y)$,

$m(y) = \{m_{ij}(y)\}, i, j = 1, 2,\ x \in D, y \in \Omega_\gamma$, such that

$$-\text{div}\sigma = g \quad \text{in } D \tag{24}$$

$$A^{-1}\sigma = \varepsilon(u) \quad \text{in } D \tag{25}$$

$$\nabla\nabla m = f \quad \text{in } \Omega_\gamma \tag{26}$$

$$m_{ij} = w_{,ij} \quad \text{in } \Omega_\gamma,\ i, j = 1, 2 \tag{27}$$

$$u = 0 \quad \text{on } \gamma_0 \tag{28}$$

$$w = w_q = 0 \quad \text{on } \Gamma \tag{29}$$

$$un - w \geq 0,\ \sigma_n \leq 0,\ \sigma_\tau = 0,\ \sigma_n(un - w) = 0 \quad \text{on } \gamma \tag{30}$$

$$[w] = [w_\nu] = 0,\ [m_\nu] = 0,\ [T^\nu(m)] = \sigma_n \quad \text{on } \gamma, \tag{31}$$

where we use the following notations:

$$\nabla\nabla m = m_{ij,ij}, \quad m_\nu = m_{ij}\nu_j\nu_i,$$
$$T^\nu(m) = m_{ij,k}s_k s_j \nu_i + m_{ij,j}\nu_i, \quad (s_1, s_2) = (-\nu_2, \nu_1).$$

The tensor A^{-1} is obtained by inverting the Hooke law $\sigma = A\varepsilon(u)$.

Introduce the so called admissible set for stresses and moments

$$L := \left\{ (\bar{\sigma}, \bar{m}) \;\middle|\; \begin{array}{l} \bar{\sigma}, \text{div}\bar{\sigma} \in L^2(D),\ \bar{m}, \nabla\nabla\bar{m} \in L^2(\Omega_\gamma), \\ \bar{\sigma}_n \leq 0,\ \bar{\sigma}_\tau = 0,\ \ [\bar{m}_\nu] = 0,\ \ [T^\nu(\bar{m})] = \bar{\sigma}_n \text{ on } \gamma \end{array} \right\}.$$

Here $\bar{\sigma} = \{\bar{\sigma}_{ij}\}, \bar{m} = \{\bar{m}_{ij}\}, i, j = 1, 2$, and boundary conditions for $\bar{\sigma}, \bar{m}$ in the definition of L hold in the following sense:

$$\bar{\sigma}_\tau(\bar{\sigma}_\tau^1, \bar{\sigma}_\tau^2) = 0 \quad \text{in the sense of } H_{00}^{-\frac{1}{2}}(\gamma),$$

$$[\bar{m}_\nu] = 0 \quad \text{in the sense of } H_{00}^{-\frac{1}{2}}(\Sigma).$$

The inequality $\bar{\sigma}_n \leq 0$ and the equality $[T^\nu(\bar{m})] = \bar{\sigma}_n$ hold in the sense

$$\langle [T^\nu(\bar{m})], \bar{w} \rangle_{\frac{3}{2},\Sigma} - \langle \bar{\sigma}_n, \bar{u}_n \rangle_{\frac{1}{2},\gamma}^{00} \geq 0 \ \ \forall (\bar{u}, \bar{w}) \in K.$$

We multiply (25), (27) by $\bar{\sigma} - \sigma$, $\bar{m} - m$, respectively, integrate over D, Ω_γ and sum up. Here $(\bar{\sigma}, \bar{m}) \in L$. This implies the following problem formulation:

We have to find functions $u(x) = (u_1(x), u_2(x)), \sigma(x) = \{\sigma_{ij}(x)\}, i, j = 1, 2,\ w(y), m(y) = \{m_{ij}(y)\}, i, j = 1, 2,\ x \in D, y \in \Omega_\gamma$, such that

$$u \in L^2(D),\ w \in L^2(\Omega_\gamma),\ (\sigma, m) \in L \tag{32}$$

$$-\text{div}\sigma = g \ \text{ in } D \tag{33}$$

$$\nabla\nabla m = f \ \text{ in } \Omega_\gamma \tag{34}$$

$$\left. \begin{array}{l} \displaystyle\int_D A^{-1}\sigma(\bar{\sigma} - \sigma) + \int_D u(\text{div}\bar{\sigma} - \text{div}\sigma) \\ \displaystyle + \int_{\Omega_\gamma} m(\bar{m} - m) - \int_{\Omega_\gamma} w(\nabla\nabla\bar{m} - \nabla\nabla m) \end{array} \right\} \geq 0 \quad \forall\ (\bar{\sigma}, \bar{m}) \in L. \tag{35}$$

The problem (32)–(35) is the mixed formulation of the problem (1)–(6). Note that (1)–(6) is equivalent to (32)–(35). To prove this it suffices to derive (1)–(6) from (32)–(35). In what follows this derivation is given. First note that (35) implies in the distributional sense

$$A^{-1}\sigma = \varepsilon(u) \text{ in } D, \quad m_{ij} = w_{,ij} \text{ in } \Omega_\gamma, \;\; i,j = 1,2. \tag{36}$$

Hence, by (32), the inclusions $u = (u_1, u_2) \in H^1(D)$, $w \in H^2(\Omega_\gamma)$ follow. It is possible to prove the fulfillment of the boundary conditions (4). Let us demonstrate that

$$[w] = [w_\nu] = 0 \text{ on } \gamma. \tag{37}$$

To this end we find a solution $\tilde{w}$ of the problem

$$\Delta^2 \tilde{w} = f \text{ in } \Omega_\gamma \tag{38}$$

$$\tilde{w} = \tilde{w}_q = 0 \text{ on } \Gamma \tag{39}$$

$$m(\tilde{w}) = \varphi, \; t^\nu(\tilde{w}) = \xi \text{ on } \gamma^\pm, \tag{40}$$

where φ, ξ are arbitrary functions from $L^2(\gamma)$. The problem (38)–(40) admits a variational formulation. Indeed, we have to find a function $\tilde{w}$ such that

$$\tilde{w} \in H^2_\Gamma(\Omega_\gamma) \tag{41}$$

$$\int_{\Omega_\gamma} \tilde{w}_{,ij} v_{,ij} - \int_{\Omega_\gamma} fv - \int_\gamma \xi[v] + \int_\gamma \varphi[v_\nu] = 0 \;\; \forall v \in H^2_\Gamma(\Omega_\gamma), \tag{42}$$

where $H^2_\Gamma(\Omega_\gamma) = \{v \in H^2(\Omega_\gamma) \mid v = v_q = 0 \text{ on } \Gamma\}$. We see that the solution $\tilde{w}$ of the problem (41)-(42) satisfies the conditions

$$[m(\tilde{w})] = 0 \text{ in the sense of } H^{-\frac{1}{2}}(\Sigma)$$

$$[t^\nu(\tilde{w})] = 0 \text{ in the sense of } H^{-\frac{3}{2}}(\Sigma).$$

We take in (35) test functions of the form $(\bar{\sigma}, \bar{m}) = (\sigma, m) \pm (0, \tilde{m})$, $\tilde{m} = \{\tilde{m}_{ij}\}, \tilde{m}_{ij} = \tilde{w}_{,ij}, i,j = 1,2$. This provides the relation

$$\int_{\Omega_\gamma} m \cdot \tilde{m} - \int_{\Omega_\gamma} w \cdot \nabla\nabla\tilde{m} = 0$$

which in its own turn, by (36), gives $\langle T^\nu(\tilde{m}), [w]\rangle_{\frac{3}{2},\Sigma} - \langle \tilde{m}_\nu, [w_\nu]\rangle_{\frac{1}{2},\Sigma} = 0$. Meanwhile, from (41), (42) it follows

$$\langle T^\nu(\tilde{m}), [v]\rangle_{\frac{3}{2},\Sigma} - \langle \tilde{m}_\nu, [v_\nu]\rangle_{\frac{1}{2},\Sigma} = \int_\gamma \xi[v] - \int_\gamma \varphi[v_\nu] \quad \forall v \in H^2_\Gamma(\Omega_\gamma).$$

Hence $\int_\gamma \xi[w] - \int_\gamma \varphi[w_\nu] = 0$, and the arbitrariness of φ, ξ proves the fulfillment of the boundary conditions (37) which, in turn, is desired. In particular, we obtain $w \in H_0^2(\Omega)$.

Now we shall prove that the function u from (32)–(35) satisfies the condition

$$u = 0 \quad \text{on} \quad \gamma_0. \tag{43}$$

Recall that $u = (u_1, u_2) \in H^1(D)$ and divide γ_0 into two parts: $\gamma_0 = \gamma_1 \cup \gamma_2$, where γ_i are regular curves, $i = 1, 2$. Denote

$$H^1_{\gamma_1}(D) = \{v \in H^1(D) \mid v = 0 \text{ on } \gamma_1\}.$$

Let $\xi = (\xi_1, \xi_2) \in L^2(\gamma_2)$ be any function. There exists a solution of the problem

$$\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in H^1_{\gamma_1}(D) \tag{44}$$

$$\int_D \sigma(\tilde{u})\varepsilon(v) - \int_D gv + \int_{\gamma_2} \xi v = 0 \quad \forall v = (v_1, v_2) \in H^1_{\gamma_1}(D), \tag{45}$$

where $\sigma(\tilde{u}) = A\varepsilon(\tilde{u})$. It is clear that this solution satisfies the relations

$$\begin{aligned} -\operatorname{div}(A\varepsilon(\tilde{u})) &= g \ \text{ in } D \\ \tilde{u} &= 0 \ \text{ on } \gamma_1 \\ \sigma(\tilde{u})n &= 0 \ \text{ on } \gamma \\ \sigma(\tilde{u})n &= \xi \ \text{ on } \gamma_2. \end{aligned}$$

Denote $\tilde{\sigma} = \sigma(\tilde{u})$ and choose a cut-off function η equal to 1 in a neighborhood of any fixed point of γ_2, $\operatorname{supp} \eta \subset \gamma_2$. In this case $\pm(\eta\tilde{\sigma}, 0) \in L$. We can choose a test function $(\bar{\sigma}, \bar{m}) = (\sigma, m) \pm (\eta\tilde{\sigma}, 0)$ in (35). It gives

$$\int_D A^{-1}\sigma \cdot \eta\tilde{\sigma} + \int_D u \cdot \operatorname{div}(\eta\tilde{\sigma}) = 0$$

and consequently, by (36), $\langle (\eta\tilde{\sigma})n, u\rangle_{\frac{1}{2},\partial D} = 0$. This relation can be written in the form

$$\langle \tilde{\sigma}n, \eta u\rangle_{\frac{1}{2},\partial D} = 0. \tag{46}$$

On the other hand, the identity (45) implies

$$\langle \tilde{\sigma}n, v\rangle_{\frac{1}{2},\partial D} = \int_{\gamma_2} \xi v \quad \forall v = (v_1, v_2) \in H^1_{\gamma_1}(D). \tag{47}$$

Since $\eta u = (\eta u_1, \eta u_2) \in H^1_{\gamma_1}(D)$, from (46), (47) we conclude $\int_{\gamma_2} \xi\eta u = 0$. The arbitrariness of ξ provides $\eta u = 0$ on γ_2 which implies the needed boundary condition (43).

Now we want to prove that the solution of (32)–(35) satisfies the boundary condition

$$un - w \geq 0 \quad \text{on} \;\; \gamma. \tag{48}$$

Consider the solution $\tilde{w}$ of the problem

$$\tilde{w} \in H^2_\Gamma(\Omega_\gamma) \tag{49}$$

$$\int_{\Omega_\gamma} \tilde{w}_{,ij} v_{,ij} - \int_{\Omega_\gamma} fv - \int_{\gamma^+} \varphi v = 0 \quad \forall v \in H^2_\Gamma(\Omega_\gamma), \tag{50}$$

where $\varphi \in L^2(\gamma), \varphi \leq 0$. This solution satisfies the following relations

$$\begin{aligned} \Delta^2 \tilde{w} &= f \;\; \text{in } \Omega_\gamma \\ \tilde{w} = \tilde{w}_q &= 0 \;\; \text{on } \Gamma \\ m(\tilde{w}) &= 0 \;\; \text{on } \gamma^\pm \\ t^\nu(\tilde{w}) &= \varphi \;\; \text{on } \gamma^+ \\ t^\nu(\tilde{w}) &= 0 \;\; \text{on } \gamma^-. \end{aligned}$$

Simultaneously, we find a solution $\tilde{u}$ of the problem

$$\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in H^1_{\gamma_0}(D) \tag{51}$$

$$\int_D \sigma(\tilde{u})\varepsilon(v) - \int_D gv + \int_\gamma \varphi v_n = 0 \quad \forall v = (v_1, v_2) \in H^1_{\gamma_0}(D), \tag{52}$$

where $v_n = vn, \sigma(\tilde{u}) = A\varepsilon(\tilde{u})$. It is clear that $\tilde{u}$ satisfies

$$\begin{aligned} -\mathrm{div}(A\varepsilon(\tilde{u})) &= g \;\; \text{in } D \\ \tilde{u} &= 0 \;\; \text{on } \gamma_0 \\ \sigma_n(\tilde{u}) = \varphi, \;\; \sigma_\tau(\tilde{u}) &= 0 \;\; \text{on } \gamma. \end{aligned}$$

Now we define the tensors $\tilde{\sigma} = \sigma(\tilde{u}), \; \tilde{m} = \{\tilde{m}_{ij}\}, \; \tilde{m}_{ij} = \tilde{w}_{,ij}, i, j = 1, 2$. Then the function $(\bar{\sigma}, \bar{m}) = (\sigma, m) + (\tilde{\sigma}, \tilde{m})$ can be chosen as a test function in (35). Indeed, $\tilde{\sigma}_n \leq 0, \; \tilde{\sigma}_\tau = 0, \;\; [\tilde{m}_\nu] = 0 \;\text{ on }\; \gamma$. Moreover, the identity (50) gives

$$\langle [T^\nu(\tilde{m})], \bar{w} \rangle_{\frac{3}{2}, \Sigma} = \int_\gamma \varphi \bar{w} \;\; \forall \bar{w} \in H^2_0(\Omega),$$

and from (52) it follows

$$-\langle \tilde{\sigma}_n, \bar{u}_n \rangle^{00}_{\frac{1}{2}, \gamma} = -\int_\gamma \varphi \bar{u}_n \quad \forall \bar{u} = (\bar{u}_1, \bar{u}_2) \in H^1_{\gamma_0}(D).$$

Summing the last two relations we arrive at the equality

$$\langle [T^\nu(\tilde{m})], \bar{w}\rangle_{\frac{3}{2},\Sigma} - \langle \tilde{\sigma}_n, \bar{u}_n\rangle^{00}_{\frac{1}{2},\gamma} = \int_\gamma \varphi(\bar{w} - \bar{u}_n)\,. \tag{53}$$

If $\bar{w} - \bar{u}_n \le 0$ on γ, ie., $(\bar{u}, \bar{w}) \in K$, the right-hand side of (53) is nonnegative and thus $(\tilde{\sigma}, \tilde{m}) \in L$, hence, $(\bar{\sigma}, \bar{m}) = (\sigma, m) + (\tilde{\sigma}, \tilde{m}) \in L$. Consequently, a substitution of $(\bar{\sigma}, \bar{m})$ into (35) implies

$$\int_D A^{-1}\sigma\cdot\tilde{\sigma} + \int_D u\cdot \mathrm{div}\tilde{\sigma} + \int_{\Omega_\gamma} m\cdot\tilde{m} - \int_{\Omega_\gamma} w\cdot\nabla\nabla\tilde{m} \ge 0.$$

This inequality yields

$$\langle [T^\nu(\tilde{m})], w\rangle_{\frac{3}{2},\Sigma} - \langle \tilde{\sigma}_n, u_n\rangle^{00}_{\frac{1}{2},\gamma} \ge 0$$

and, by (53), $\int_\gamma \varphi(w - u_n) \ge 0$. Since $\varphi \le 0$ is arbitrary we obtain $w - u_n \le 0$ on γ which completes the proof of (48).

Finally we demonstrate that the solution of (32)–(35) satisfies the boundary condition

$$\sigma_n(un - w) = 0 \ \text{ on } \ \gamma. \tag{54}$$

We can take $(\bar{\sigma}, \bar{m}) = (0, 0)$, $(\bar{\sigma}, \bar{m}) = 2(\sigma, m)$ in (35) as test functions. This provides the relation

$$\int_D A^{-1}\sigma\cdot\sigma + \int_D u\cdot \mathrm{div}\sigma + \int_{\Omega_\gamma} m\cdot m - \int_{\Omega_\gamma} w\cdot\nabla\nabla m = 0$$

and consequently $\langle [T^\nu(m)], w\rangle_{\frac{3}{2},\Sigma} - \langle \sigma_n, u_n\rangle^{00}_{\frac{3}{2},\gamma} = 0$ which means that (54) holds.

Thus the main result of this section can be formulated as follows.

Theorem 3.1. *The mixed formulation* (32)–(35) *of the contact problem between two plates is equivalent to* (1)–(6).

To conclude this section, we note that the smooth and mixed domain formulations in the theory of cracks with possible contact between crack faces can be found in [8, 9].

4. Asymptotic analysis

In practice, the model (1)–(6) includes a number of parameters which may change. This section concerns passages to limits when these parameters converge to limit values. We shall analyze two limit cases:

i) Instead of the law $\sigma = A\varepsilon(u)$ in (1) we consider the family of laws

$$\sigma^\lambda = \frac{A}{\lambda}\,\varepsilon(u), \quad \lambda > 0, \tag{55}$$

and perform the passage to the limit as $\lambda \to 0$.

This problem corresponds to the case where the stiffness of the upright elastic body tends to infinity. Thus, in the limit we expect a rigid thin obstacle for the vertical deformation of the horizontal Kirchhoff plate.

ii) Instead of the equation (2) we consider the family of equations

$$\frac{1}{\lambda}\,\Delta^2 w = f, \ \lambda > 0,$$

and investigate a passage to the limit $\lambda \to 0$.

This asymptotic problem corresponds to an increasing bending stiffness of the horizontal plate. As a result, in the limit one expects a rigid obstacle for the in-plane deformation of the upright elastic body.

First we analyze the case i). For any fixed $\lambda > 0$ one has a unique solution to the following problem:

$$\text{Find } (u^\lambda, w^\lambda) \in K, \text{ such that} \tag{56}$$

$$\left\{ \begin{aligned} &\int_D \sigma^\lambda(u^\lambda)\varepsilon(\bar{u} - u^\lambda) - \int_D g(\bar{u} - u^\lambda) \\ &+ \int_\Omega w^\lambda_{,ij}(\bar{w}_{,ij} - w^\lambda_{,ij}) - \int_\Omega f(\bar{w} - w^\lambda) \geq 0 \quad \forall (\bar{u}, \bar{w}) \in K. \end{aligned} \right. \tag{57}$$

Here $\sigma^\lambda(u^\lambda) = \sigma^\lambda$ are defined from (55). Substituting $(\bar{u}, \bar{w}) = (0, 0)$, $(\bar{u}, \bar{w}) = 2(u, w)$ in (57) as test functions we find

$$\int_D \sigma^\lambda(u^\lambda)\varepsilon(u^\lambda) - \int_D g u^\lambda + \int_\Omega w^\lambda_{,ij} w^\lambda_{,ij} - \int_\Omega f w^\lambda = 0. \tag{58}$$

Relation (58) implies

$$\|w^\lambda\|_{H_0^2(\Omega)} \leq c_1, \ \ \frac{1}{\lambda}\,\|u^\lambda\|^2_{H^1_{\gamma_0}(D)} \leq c_2$$

with constants c_1, c_2 being uniform in λ. It can be assumed that for a subsequence u^λ, w^λ, with the previous notation, as $\lambda \to 0$

$$\begin{aligned} w^\lambda &\to w^0 \ \text{ weakly in } \ H_0^2(\Omega), \\ u^\lambda &\to 0 \quad \text{strongly in } \ H^1_{\gamma_0}(D). \end{aligned}$$

Since $u^\lambda n - w^\lambda \geq 0$ on γ the limit function w^0 satisfies the inequality $w^0 \leq 0$ on γ. Let us take $\bar{w} \in H_0^2(\Omega), \bar{w} \leq 0$ on γ. In this case $(0, \bar{w}) \in K$. Substitute this element $(0, \bar{w})$ into (57) as a test function. It implies

$$\int_\Omega w^\lambda_{,ij}(\bar{w}_{,ij} - w^\lambda_{,ij}) - \int_\Omega f(\bar{w} - w^\lambda) \geq \frac{1}{\lambda}\int_D \sigma(u^\lambda)\varepsilon(u^\lambda) - \int_D g u^\lambda\ .$$

Since $\liminf_{\lambda \to 0} \frac{1}{\lambda}\int_D \sigma(u^\lambda)\varepsilon(u^\lambda) \geq 0$ from the above inequality it follows

$$w^0 \in M \tag{59}$$

$$\int_\Omega w^0_{,ij}(\bar{w}_{,ij} - w^0_{,ij}) - \int_\Omega f(\bar{w} - w^0) \geq 0 \quad \forall\, \bar{w} \in M. \tag{60}$$

Here $M := \{v \in H_0^2(\Omega) \mid v \leq 0 \text{ on } \gamma\}$. It is seen that the problem (59)–(60) describes a contact problem for the plate with a thin rigid obstacle. This obstacle is situated along γ. Like above, we can find a complete system of boundary conditions holding on γ in the problem (59), (60) as follows:

$$[w^0] = [w^0_\nu] = 0, \quad [m(w^0)] = 0 \text{ on } \gamma,$$
$$w^0 \leq 0, \quad [t^\nu(w^0)] \leq 0, \quad [t^\nu(w^0)]w^0 = 0 \text{ on } \gamma.$$

We have thus shown:

Theorem 4.1. *Let $\lambda > 0$ and (u^λ, w^λ) be the unique solution to problem* (56), (57). *If λ tends to zero, the corresponding solution (u^λ, w^λ) tends to $(0, w^0)$ weakly in $H^1_{\gamma_0}(D) \times H^2_0(\Omega)$, where w^0 is the unique solution of the variational inequality* (59), (60). *The latter, in turn, is equivalent to the obstacle problem for the plate.*

Now consider the case ii). For any fixed $\lambda > 0$ there exists a unique solution of the variational inequality:

$$\text{Find } (u^\lambda, w^\lambda) \in K, \text{ such that} \tag{61}$$

$$\begin{cases} \displaystyle\int_D \sigma(u^\lambda)\varepsilon(\bar{u} - u^\lambda) - \int_D g(\bar{u} - u^\lambda) \\ \displaystyle + \frac{1}{\lambda}\int_\Omega w^\lambda_{,ij}(\bar{w}_{,ij} - w^\lambda_{,ij}) - \int_\Omega f(\bar{w} - w^\lambda) \geq 0 \quad \forall (\bar{u}, \bar{w}) \in K. \end{cases} \tag{62}$$

It is seen that (62) implies the relation

$$\int_D \sigma(u^\lambda)\varepsilon(u^\lambda) - \int_D g u^\lambda + \frac{1}{\lambda}\int_\Omega w^\lambda_{,ij} w^\lambda_{,ij} - \int_\Omega f w^\lambda = 0$$

which provides the following estimates being uniform with respect to λ,

$$\frac{1}{\lambda}\|w^\lambda\|^2_{H^2_0(\Omega)} \leq c_3, \quad \|u^\lambda\|_{H^1_{\gamma_0}(D)} \leq c_4.$$

By choosing subsequences we can assume that as $\lambda \to 0$

$$\begin{aligned} w^\lambda &\to 0 \quad \text{strongly in } H_0^2(\Omega) \\ u^\lambda &\to u^0 \quad \text{weakly in } H^1_{\gamma_0}(D). \end{aligned}$$

Obviously, the limit function u^0 satisfies the inequality $u^0 n \geq 0$ on γ. Let us take a test function in (62) in the form $(\bar{u}, 0), \bar{u}n \geq 0$ on γ, $\bar{u} = (\bar{u}_1, \bar{u}_2) \in H^1_{\gamma_0}(D)$. This provides the relation

$$\int_\Omega f w^\lambda + \int_D \sigma(u^\lambda)\varepsilon(\bar{u} - u^\lambda) - \int_D g(\bar{u} - u^\lambda) \geq \frac{1}{\lambda} \int_\Omega w^\lambda_{,ij} w^\lambda_{,ij} \, . \tag{63}$$

By the inequality $\liminf_{\lambda \to 0} \frac{1}{\lambda} \int_\Omega w^\lambda_{,ij} w^\lambda_{,ij} \geq 0$, a passage to the limit in (63) can be performed, hence we arrive at the variational inequality

$$u^0 \in N \tag{64}$$

$$\int_D \sigma(u^0)\varepsilon(\bar{u} - u^0) - \int_D g(\bar{u} - u^0) \geq 0 \quad \forall \bar{u} \in N, \tag{65}$$

where $N := \{v = (v_1, v_2) \in H^1_{\gamma_0}(D) \mid vn \geq 0 \text{ on } \gamma\}$.

One can see that the limit problem (64)–(65) is precisely the Signorini contact problem in the domain D (see [4]). We have shown the following theorem.

Theorem 4.2. *Let $\lambda > 0$ be given, and let (u^λ, w^λ) be the unique solution of the variational inequality* (61),(62). *If λ tends to zero, the corresponding solution (u^λ, w^λ) tends to $(u^0, 0)$ ($u^\lambda \to u^0$ weakly in $H^1_{\gamma_0}(D)$, $w^\lambda \to 0$ strongly in $H_0^2(\Omega)$), where u^0 is the unique solution to the variational inequality* (64), (65), *which, in turn, corresponds to the Signorini problem for the elastic body.*

5. A contact between a plate and a rod

In this section we analyze a contact problem between an elastic plate and an elastic rod (representative of a beam). The middle surface of the plate is denoted by Ω. The properties of Ω are described in Section 1. The beam is situated perpendicular to the plate (see Fig. 3). Let $0 \in \Omega$ be a point of possible contact between the plate and the beam. A middle line of the rod is denoted by α. We assume that α is the interval $(0, 1)$, and the point $x = 0$ is a contact one for the beam. The end point $x = 1$ of the beam is clamped. The boundary Γ of the plate is also clamped. Underline that the beam is assumed to have a nonzero displacement only along the axes x.

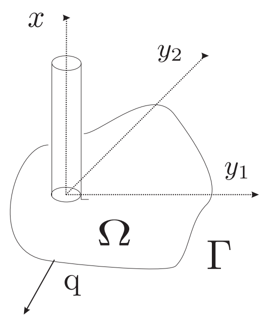

Figure 3: Rod in contact with plate

Consider the Sobolev space $\tilde{H}^1(\alpha) := \{v \in H^1(\alpha) \mid v = 0 \ \text{ at } \ x = 1\}$ and introduce the energy functional on the space $\tilde{H}^1(\alpha) \times H_0^2(\Omega)$

$$G(u,w) := \frac{1}{2}\int_\alpha b u_x^2 - \int_\alpha hu + \frac{1}{2}\int_\Omega w_{,ij}w_{,ij} - \int_\Omega fw,$$

where $f \in L^2(\Omega), h \in L^2(\alpha), b \in L^\infty(\alpha)$ are given functions, $b \geq c_0 > 0, c_0 = const$. Consider the set of admissible displacements

$$S := \{(u,w) \in \tilde{H}^1(\alpha) \times H_0^2(\Omega) \mid u(0) - w(0) \geq 0 \ \}.$$

We use the same notations for zero $x = 0$ and for zero $y = 0$ which should not lead to a confusion.

It is possible to find a solution of the minimization problem

$$\inf_{(u,w)\in S} G(u,w). \tag{66}$$

This solution exists and satisfies the variational inequality:

$$\text{Find } (u,w) \in S, \text{ such that} \tag{67}$$

$$\begin{cases} \displaystyle\int_\alpha b u_x(\bar{u}_x - u_x) - \int_\alpha h(\bar{u} - u) \\ \displaystyle + \int_\Omega w_{,ij}(\bar{w}_{,ij} - w_{,ij}) - \int_\Omega f(\bar{w} - w) \geq 0 \quad \forall(\bar{u},\bar{w}) \in S. \end{cases} \tag{68}$$

For the problem (67)–(68) we can formulate the questions similar to those analyzed in Sections 3, 4 for the problem (8), (9). In particular, the mixed problem formulation can be given. Also it is possible to investigate the passages to limits when elasticity parameters of the plate and the beam are going to infinity. We restrict ourselves to finding a complete system of boundary conditions for the problem (67)–(68). Choose a closed curve Σ of the class $C^{1,1}, \Sigma \subset \Omega$, such that $0 \in \Sigma$. Denote by $\nu = (\nu_1, \nu_2)$ a unite normal vector to the curve Σ. Hence, the domain Ω is divided into two subdomains Ω_1, Ω_2 with boundaries Σ and $\Sigma \cup \Gamma$, respectively. In our considerations ν is oriented towards Ω_2 (see Fig. 2). First of all we write the differential formulation of the problem (67)–(68). Denote $\Omega_0 = \Omega \setminus \{0\}$. We have to find functions $u(x), w(y), x \in \alpha, y = (y_1, y_2) \in \Omega_0$, such that

$$-(bu_x)_x = h \qquad \text{in } \alpha \tag{69}$$
$$\Delta^2 w = f \qquad \text{in } \Omega_0 \tag{70}$$
$$u = 0 \qquad \text{at } x = 1 \tag{71}$$
$$w = w_q = 0 \qquad \text{on } \Gamma \tag{72}$$
$$u(0) - w(0) \geq 0,\ bu_x(0) \leq 0,\ \ bu_x(0)(u(0) - w(0)) = 0 \qquad \text{on } \ \Sigma \tag{73}$$
$$[m(w)] = 0,\ [t^\nu(w)] = bu_x(0)\delta_\Sigma \ \text{ on } \Sigma, \tag{74}$$

where δ_Σ is a distribution on Σ defined by the formula $\delta_\Sigma(\xi) = \xi(0)$. Note that Σ is an arbitrary curve with the above properties.

Theorem 5.1. *The problems* (66), (67)–(68) *and* (69)–(74) *are equivalent. Moreover, there exists a unique solution* (u, w) *to problem* (66), *which, in turn, is the solution to problems* (67)–(68) *and* (69)–(74), *respectively.*

Proof. It can be easily checked that equations (69), (70) follow from (67), (68). This equations hold in the distributional sense. In what follows we derive boundary conditions (73), (74) and clarify in what sense they are fulfilled. We can take in (68) test functions of the form $(\bar{u}, \bar{w}) = (u, w+\varphi)$, $\varphi \in H_0^2(\Omega)$, $\varphi(0) \leq 0$. This gives the inequality

$$\int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi \geq 0.$$

Consequently, applying the Green formula like (17) for the subdomains Ω_1, Ω_2, and (69)–(72) we derive

$$-\langle [m(w)], \varphi_\nu \rangle_{\frac{1}{2},\Sigma} + \langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} \geq 0 \quad \forall \varphi \in H_0^2(\Omega), \ \ \varphi(0) \leq 0$$

which, in its own turn, provides

$$[m(w)] = 0 \ \text{ in the sense of } \ H^{-\frac{1}{2}}(\Sigma), \tag{75}$$
$$\langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} \geq 0 \ \ \forall \varphi \in H_0^2(\Omega), \ \ \varphi(0) \leq 0. \tag{76}$$

In particular, we have $\langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} = 0$ provided that $\varphi(0) = 0$. Hence $\langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma}$ depends only on $\varphi(0)$, and thus $\langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} = k\varphi(0)$, $k =$ const. As we will see below, $k = bu_x(0)$. Note that the values $bu_x(0), bu_x(1)$ make sense since $bu_x, (bu_x)_x \in L^2(\alpha)$.

Substitute next in (68) test functions of the form $(\bar{u}, \bar{w}) = (u + \psi, w)$, where $\psi \in \tilde{H}^1(\alpha)$ is an arbitrary function, $\psi(0) \geq 0$. It provides the relation

$$\int_\alpha bu_x\psi_x - \int_\alpha h\psi \geq 0$$

which, by (69), (71), proves that

$$bu_x(0) \leq 0. \tag{77}$$

Now we substitute in (68) test functions of the form $(\bar{u}, \bar{w}) = (u + \psi, w + \varphi)$, where $(\psi, \varphi) \in S$. In this case the following inequality follows:

$$\int_\alpha bu_x\psi_x - \int_\alpha h\psi + \int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi \geq 0. \tag{78}$$

Consequently, by (69)–(72), (75),

$$\langle [t^\nu(w)], \varphi \rangle_{\frac{3}{2},\Sigma} - bu_x(0)\psi(0) \geq 0 \quad \forall (\psi, \varphi) \in S. \tag{79}$$

This inequality holds for all curves Σ with the properties mentioned above. Relation (79) provides a fulfillment of the following boundary conditions (see (73), (74)) $[t^\nu(w)] = bu_x(0)\delta_\Sigma$ on Σ, $bu_x(0) \leq 0$. Observe that (76), (77) follow from (79).

If $\psi = u$, $w = \varphi$ the left-hand side of (79) is equal to zero. Indeed, let us take $(\bar{u}, \bar{w}) = (0, 0)$, $(\bar{u}, \bar{w}) = 2(u, w)$ in (68) as test functions. This yields

$$\langle [t^\nu(w)], w \rangle_{\frac{3}{2},\Sigma} - bu_x(0)u(0) = 0$$

which proves the fulfillment of the conditions (see (73), (74)) $[t^\nu(w)] = bu_x(0)\delta_\Sigma$ on Σ, $bu_x(0)(u(0) - w(0)) = 0$.

To conclude the section we prove that (71)–(74) is a complete system of boundary conditions. This means that the variational inequality (67)–(68) can be derived from (69)–(74), provided that the solution of (69)–(74) is quite smooth. Indeed, multiply (69), (70) by $\bar{u} - u, \bar{w} - w$, respectively, integrate over α, Ω_0 and sum up, $(\bar{u}, \bar{w}) \in S$. Thus we obtain

$$-\int_\alpha ((bu_x)_x + h)(\bar{u} - u) + \int_{\Omega_0} (\Delta^2 w - f)(\bar{w} - w) = 0. \tag{80}$$

We can divide here the domain Ω_0 into two subdomains Ω_1, Ω_2 by choosing a curve Σ like before. It allows us to use the Green formula (17) for the domains Ω_1, Ω_2. Hence (80), (71), (72), (74) imply the relation

$$\begin{aligned} 0 = &\int_\alpha bu_x(\bar{u}_x - u_x) - \int_\alpha h(\bar{u} - u) + \int_\Omega w_{,ij}(\bar{w}_{,ij} - w_{,ij}) - \int_\Omega f(\bar{w} - w) \\ &+ bu_x(0)(\bar{u}(0) - u(0)) - \langle [t^\nu(w)], \bar{w} - w \rangle_{\frac{3}{2},\Sigma}. \end{aligned} \tag{81}$$

Note that we have changed the integration domain $\Omega_1 \cup \Omega_2$ by Ω in (81). This is possible since $[w] = [w_\nu] = 0$ on Σ. To complete the proof it suffices to notice that, by (73), (74), the sum of two last terms in (81) is nonpositive. Hence, the remaining part is nonnegative which implies (67), (68). This concludes the proof of the theorem. $\square$

Notice that we can rewrite the problem (69)–(74) in the form where the smooth domain Ω is used,

$$-(bu_x)_x = h \qquad \text{in } \alpha \tag{82}$$

$$\Delta^2 w = f + bu_x(o)\delta_0 \qquad \text{in } \Omega \tag{83}$$

$$u = 0 \qquad \text{at } x = 1 \tag{84}$$

$$w = w_q = 0 \qquad \text{on } \Gamma \tag{85}$$

$$u(0) - w(0) \geq 0,\ bu_x(0) \leq 0, \quad bu_x(0)(u(0) - w(0)) = 0\,. \tag{86}$$

Here δ_0 is the Dirac measure, i.e., $\delta_0(\xi) = \xi(0)$, $\xi \in C_0^\infty(\Omega)$. Indeed, if $\varphi(0) = \psi(0)$ in (78) we obtain

$$\int_\alpha bu_x\psi_x - \int_\alpha h\psi + \int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

which implies (83). On the other hand, from (82)–(86) the variational inequality (67)–(68) follows.

6. Extensions and open problems

Remark 6.1. It is plain that the problems considered in this paper are to be seen as exemplary problems.

1. The first extension concerns oblique contact problems, where a plate and an elastic body or a beam which form an arbitrary angle with the plate are in contact. Contact problems of this sort involve vertical as well as in-plane deformations of the elastic object. This problem will be the subject to a forthcoming publication.

2. The problems discussed in this paper can be extended to more complex flexible structures. In particular, one might consider an elastic frame which is in contact with an elastic body or a (system of linked) plate(s).

3. It would be very interesting to extend the analysis of this paper to problem with frictional contact.

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References

[1] Caffarelli, L. A. and Friedman, A., The obstacle problem for the biharmonic operator. *Ann. Scuola Norm. Sup. Pisa*, Ser. IV, 6 (1979), 151 – 184.

[2] Caffarelli, L. A., Friedman, A. and Torelli, A., The two-obstacle problem for the biharmonic operator. *Pacific J. Math.* 103 (1982)(3), 325 – 335.

[3] Dal Maso, G. and Paderni, G., Variational inequalities for the biharmonic operator with varying obstacles. *Ann. Mat. Pura Appl.* 153 (1988), 203 – 227.

[4] Fichera, G., Boundary value problems of elasticity with unilateral constraints. In: *Handbuch der Physik*, Band 6a/2. Berlin: Springer 1972, pp. 391 – 424.

[5] Grisvard, P., *Elliptic Problems in Nonsmooth Domains.* Boston (MA): Pitman 1985.

[6] Khludnev, A. M. and Sokolowski, J., *Modelling and Control in Solid Mechanics.* Basel: Birkhäuser 1997.

[7] Khludnev, A. M. and Kovtunenko, V. A., *Analysis of Cracks in Solids.* Southampton: WIT Press 2000.

[8] Khludnev, A. M., Smooth domain method in equilibrium problem for a plate with a crack. *Siberian Math. J.* 43 (2002)(6), 1388 – 1400.

[9] Khludnev, A. M. and Sokolowski, J., Smooth domain method for crack problems. *Quart. Appl. Math.* 62 (2004)(3), 401 – 422.

[10] Lagnese, J. E., Leugering, G. and Schmidt, E. J. P. G., *Modeling, Analysis and Control of Dynamic Elastic Multi-link Structures.* Boston (MA): Birkhäuser 1994.

[11] Lagnese, J. E. and Leugering, G., *Domain Decomposition Method in Optimal Control of Partial Differential Equations.* Basel: Birkhäuser 2000.

[12] Schild, B., On the coincidence set in biharmonic variational inequalities with thin obstacles. *Ann. Sc. Norm. Super. Pisa Cl. Sci* (4) 13 (1986)(4), 559 – 616.

[13] Temam, R., *Problemes Mathematiques en Plasticite.* Paris: Gauthier-Villars 1983.

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