Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 27 (2008), 179–194

Decay Estimates of Solutions to the Cauchy Problem for a Wave Equation with a Bounded Nonlinear Dissipation

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Abstract. In this paper we prove the global existence and study decay properties of the solutions to the Cauchy problem for a wave equation with a bounded nonlinear dissipative term.

Keywords. Nonlinear wave equation, global existence, decay rate

Mathematics Subject Classification (2000). Primary 35B40, secondary 35L70

1. Introduction

We consider the Cauchy problem for the nonlinear wave equation with a non-linear weak dissipation of the type

$$\begin{cases} u'' - \Delta_x u + \lambda^2(x)u + g(u') = 0 & \text{in } \mathbb{R}^n \times [0, +\infty[\\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{P}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function and λ is a positive function. When we have a bounded domain instead of $\mathbb{R}^n$, and for the case $g(x) = \delta|x|^{m-1}x$ $(m \geq 0)$ (without the term $\lambda^2(x)u$), the decay property of the solutions of (P) was investigated in detail (see [12], and for related works [8, 16]). The previous results all have a serious drawback from the point of view of applications: they never apply if the function g is bounded such that

$$-\infty < \lim_{x\to-\infty} g(x) < \lim_{x\to+\infty} g(x) < +\infty. \tag{Q}$$

If g satisfies at most condition (Q), the dissipation effect by $g(u')$ is weak as $|u'|$ is large and for convenience we call such a term weak dissipation. In [13],

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Nakao considered the particular case $g(x) = \frac{x}{\sqrt{1+x^2}}$ and he studied the decay property of the solutions of the problem.

For the Cauchy problem (P), when $g(x) = \delta|x|^{m-1}x$ $(m \geq 1)$ and $\lambda \equiv 1$, Nakao [11] (see also [15]) proved that the energy decay rate is

$$E(t) \leq (1+t)^{-\frac{2-n(m-1)}{(m-1)}}, \quad t \geq 0.$$

He used a general method introduced in his paper [10] with the condition that the data have compact support. This result was later generalized by the authors [1] to the case of more general function λ.

Our purpose in this paper is to give a precise decay estimate of the energy of solutions to the Cauchy problem (P) with a nonlinear weak dissipation. We use a new method recently introduced by Martinez [9] (see also [1]) to study the decay rate of solutions to the wave equation $u'' - \Delta_x u + g(u') = 0$ in $\Omega \times \mathbb{R}^+$, where Ω is a bounded domain of $\mathbb{R}^n$. This method is based on a new integral inequality that generalizes a result of Haraux [5].

2. Preliminaries

λ and g are functions satisfying the following hypotheses:

λ is a locally bounded measurable function defined on $\mathbb{R}^n$ which satisfies

$$\lambda(x) \geq d(|x|), \tag{1}$$

where d is a decreasing C^1 function such that $\lim_{y\to\infty} d(y) = 0$.

$g : \mathbb{R} \to \mathbb{R}$ is an increasing C^1 function and there exist four constants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_3|v|^m \leq |g(v)| \leq c_4|v|^{\frac{1}{m}} \quad \text{for all } |v| \leq 1 \tag{2}$$

$$c_1 \leq |g(v)| \leq c_2|v|^r \quad \text{for all } |v| \geq 1, \tag{3}$$

where $1 \leq r \leq \frac{n+2}{(n-2)^+}$.

We first state two well known lemmas, and then we state and prove two other lemmas that will be needed later.

Lemma 2.1 ([2]). *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq \frac{2n}{n-2}$ $(n \geq 3)$. Then there is a constant $c_* = c(q)$ such that*

$$\|u\|_q \leq c_*\|u\|_{H^1(\mathbb{R}^n)} \quad \textit{for } u \in H^1(\mathbb{R}^n).$$

Lemma 2.2 (Gagliardo-Nirenberg [2]). *Let* $1 \le r < q \le +\infty$ *and* $p \ge 2$. *Then, the inequality*

$$\|u\|_p \le C\|\nabla_x^m u\|_2^{\theta}\|u\|_r^{1-\theta} \quad \text{for } u \in \mathcal{D}((-\Delta)^{\frac{m}{2}}) \cap L^r$$

holds with some constant $C > 0$ *and* $\theta = (\frac{1}{r} - \frac{1}{p})(\frac{m}{n} + \frac{1}{r} - \frac{1}{2})^{-1}$ *provided that* $0 < \theta \le 1$ *(we assume* $0 < \theta < 1$ *if* $m - \frac{n}{2}$ *is a nonnegative integer).*

Lemma 2.3 ([8]). *Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-increasing function and assume that there are constants* $p \ge 1$ *and* $A > 0$ *such that*

$$\int_S^{+\infty} E^{\frac{p+1}{2}}(t)\,dt \le AE(S), \quad 0 \le S < +\infty,$$

then we have

$$E(t) \le \begin{cases} cE(0)(1+t)^{-\frac{2}{p-1}} & \forall t \ge 0, \quad \text{if } p > 1 \\ cE(0)e^{-\omega t} & \forall t \ge 0, \quad \text{if } p = 1, \end{cases}$$

where c *and* ω *are positive constants independent of the initial energy* $E(0)$.

Lemma 2.4 ([9]). *Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non increasing function and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *an increasing* C^2 *function such that* $\phi(0) = 0$ *and* $\phi(t) \to +\infty$ *as* $t \to +\infty$. *Assume that there exist* $p \ge 1$ *and* $A > 0$ *such that*

$$\int_S^{+\infty} E(t)^{\frac{p+1}{2}}(t)\phi'(t)\,dt \le AE(S), \quad 0 \le S < +\infty,$$

then we have

$$E(t) \le \begin{cases} cE(0)(1+\phi(t))^{-\frac{2}{p-1}} & \forall t \ge 0, \quad \text{if } p > 1 \\ E(t) \le cE(0)e^{-\omega\phi(t)} & \forall t \ge 0, \quad \text{if } p = 1, \end{cases}$$

where c *and* ω *are positive constants independent of the initial energy* $E(0)$.

Proof of Lemma 2.4. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$ (we remark that ϕ^{-1} has a sense by the hypotheses assumed on ϕ). The function f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$, we obtain for $0 \le S < T < +\infty$

$$\int_{\phi(S)}^{\phi(T)} f(x)^{\frac{p+1}{2}}\,dx = \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{\frac{p+1}{2}}\,dx = \int_S^T E(t)^{\frac{p+1}{2}}\phi'(t)\,dt \le Af(\phi(S)).$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that $\int_s^{+\infty} f(x)^{\frac{p+1}{2}}\,dx \le Af(s)$ for $0 \le s < +\infty$. Thanks to Lemma 2.3, we get the desired results. $\square$

Next, let $E(t)$ be the energy associated to the solution of problem (P):

$$E(t) = \|u'\|_2^2 + \|\nabla_x u\|_2^2 + \|\lambda(x)u\|_2^2\,,$$

and we set $\tilde{E}(t) = \|u''(t)\|_2^2 + \|\nabla_x u'(t)\|_2^2 + \|\lambda u'(t)\|_2^2$ and $\tilde{d}(t) = d(L+t)$, where L is a positive constant such that $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \{x \in \mathbb{R}^n, |x| < L\}$.

Now we recall the following local existence theorem

Theorem 2.5 ([14]). *Let $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^n)$. Then under the hypothesis* (1), (2) *and* (3)*, the problem* (P) *admits a unique local solution $u(t)$ on some interval $[0,T[$, $T \equiv T(u_0,u_1) > 0$, in the class $W^{2,\infty}([0,T[;L^2) \cap W^{1,\infty}([0,T[;H^1) \cap L^\infty([0,T[;H^2)$, satisfying the finite propagation property.*

Remark 2.6. The finite propagation property means that if $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \{x \in \mathbb{R}^n, |x| < L\}$ for some $L > 0$, then $\operatorname{supp} u(t) \subset \{x \in \mathbb{R}^n, |x| < L+t\}$. Thus, $d(|x|) \geq d(L+t) \equiv \tilde{d}(t)$ on $\{x \in \mathbb{R}^n, |x| < L+t\}$.

3. Main results

We suppose that

$$\begin{aligned} \int_0^\infty \tilde{d}^2(\tau)\,d\tau &= +\infty\,, && \text{if } m = 1 \\ \int_0^\infty (1+\tau)^{-\frac{n(m-1)}{2}} \tilde{d}^{m+1}(\tau)\,d\tau &= +\infty\,, && \text{if } m > 1\,. \end{aligned} \tag{4}$$

Our main result is the following.

Theorem 3.1. *Assume that $(u_0,u_1) \in H^2 \times H^1(\mathbb{R}^n)$ with compact support, the solution of problem* (P) *admits a global solution and satisfies the following decay estimate:*

$m = 1$*: When $n = 1, 2$, then there exists a positive constant ω such that*

$$E(t) \leq C(E(0))exp\left(1 - \omega \int_0^t \tilde{d}^2(\tau)\,d\tau\right) \quad \forall t > 0. \tag{5}$$

When $n \geq 3$, then there exists a positive constant $C(E(0))$ depending continuously on $E(0)$ such that

$$E(t) \leq \frac{C(E(0))}{\left(\int_0^t \tilde{d}^2(\tau)\,d\tau\right)^{\frac{4}{n-2}}} \quad \forall t > 0. \tag{6}$$

$m > 1$: *When* $m \geq \frac{n}{2}$, *then there exists a positive constant* $C(E(0))$ *depending continuously on* $E(0)$ *such that*

$$E(t) \leq \left(\frac{C(E(0))}{\int_0^t (1+\tau)^{-\frac{n(m-1)}{2}} \tilde{d}^{m+1}(\tau)\, d\tau} \right)^{\frac{2}{(m-1)}} \quad \forall t > 0. \tag{7}$$

When $m < \frac{n}{2}$, *then there exists a positive constant* $C(E(0))$ *depending continuously on* $E(0)$ *such that*

$$E(t) \leq \left(\frac{C(E(0))}{\int_0^t (1+\tau)^{-\frac{n(m-1)}{2}} \tilde{d}^{m+1}(\tau)\, d\tau} \right)^{\frac{4}{(n-2)}} \quad \forall t > 0. \tag{8}$$

Examples.

1) Suppose that $\tilde{d}(t) = \text{const}$. Applying Theorem 3.1 we obtain

$$\begin{aligned}
&E(t) \leq E(0)e^{1-\omega t} && \text{if } m = 1, n = 1, 2\\
&E(t) \leq C(E(0))t^{-\frac{4}{n-2}} && \text{if } m = 1, n \geq 3\\
&E(t) \leq C(E(0))t^{-\frac{2-n(m-1)}{\max(\frac{n}{2},m)-1}} && \text{if } 1 < m < 1 + \tfrac{2}{n}\\
&E(t) \leq C(E(0))(\ln t)^{-\frac{2}{\max(\frac{n}{2},m)-1}} && \text{if } m = 1 + \tfrac{2}{n}.
\end{aligned}$$

2) Suppose that $\tilde{d}(t) = \frac{1}{t^\theta (\ln t)^\beta}$. If we consider $m = 1$ and use a result of J. Dieudonné [3, p. 95] for asymptotic development, we have

$$\begin{aligned}
&E(t) \leq C(E(0))e^{1-\omega t^{1-2\theta}(\ln t)^{-2\beta}} && \text{if } n = 1, 2,\ \theta < \tfrac{1}{2}\\
&E(t) \leq C(E(0))e^{1-\omega(\ln t)^{1-2\beta}} && \text{if } n = 1, 2, \theta = \tfrac{1}{2},\ \beta < \tfrac{1}{2}\\
&E(t) \leq C(E(0))(\ln t)^{-\omega} && \text{if } n = 1, 2,\ \beta = \tfrac{1}{2},\ \theta = \tfrac{1}{2}\\
&E(t) \leq C(E(0))t^{-\frac{4(1-2\theta)}{n-2}}(\ln t)^{\frac{8\beta}{n-2}} && \text{if } n \geq 3,\ \theta < \tfrac{1}{2}\\
&E(t) \leq C(E(0))(\ln t)^{-\frac{4(1-2\beta)}{n-2}} && \text{if } n \geq 3,\ \beta < \tfrac{1}{2},\ \theta = \tfrac{1}{2}\\
&E(t) \leq C(E(0))(\ln \ln t)^{-\frac{4}{n-2}} && \text{if } n \geq 3,\ \beta = \tfrac{1}{2},\ \theta = \tfrac{1}{2}.
\end{aligned}$$

When $m > 1$, then we obtain

$$\begin{aligned}
E(t) &\leq C(E(0)) \left(\int_0^t (1+\tau)^{-\frac{n(m-1)}{2}} \tilde{d}^{m+1}(\tau)\, d\tau \right)^{-\frac{2}{\max\{m,\frac{n}{2}\}-1}}\\
&= C(E(0)) \left(\int_0^t \frac{1}{(1+\tau)^{\frac{n(m-1)}{2}+\theta(m+1)} (\ln \tau)^{\beta(m+1)}}\, d\tau \right)^{-\frac{2}{\max\{m,\frac{n}{2}\}-1}}.
\end{aligned}$$

We have the following cases:
If $n(m-1)+2\theta(m+1)<2$, using again Dieudonné's result, we have

$$\begin{aligned} E(t) &\le C(E(0))\left(t^{1-\frac{n(m-1)}{2}-\theta(m+1)}(\ln t)^{-\beta(m+1)}\right)^{-\frac{2}{\max\{m,\frac{n}{2}\}-1}} \\ &\le C(E(0))\; t^{-\frac{2-n(m-1)-2\theta(m+1)}{\max\{m,\frac{n}{2}\}-1}}(\ln t)^{\frac{2\beta(m+1)}{\max\{m,\frac{n}{2}\}-1}}. \end{aligned}$$

If $n(m-1)+2\theta(m+1)=2$, then

$$\begin{aligned} E(t) &\le C(E(0))\;(\ln t)^{-\frac{2(1-\beta(m+1))}{\max\{m,\frac{n}{2}\}-1}}, \qquad \beta(m+1)<1 \\ E(t) &\le C(E(0))\;(\ln\ln t)^{-\frac{2}{\max\{m,\frac{n}{2}\}-1}}, \quad \beta(m+1)=1. \end{aligned}$$

Lemma 3.2. *Let $u(t)$ be a solution to problem* (P) *on* $[0,T_{\max}[$. *There exists a constant $C(u_0,u_1)$ such that $\|u'\|_{H^1(\mathbb{R}^n)}\le C(u_0,u_1)$.*

Proof. Differentiating the first equation of (P) with respect to t, we get

$$u_{ttt}-\Delta_x u_t+\lambda^2(x)u_t+g'(u_t)u_{tt}=0. \tag{9}$$

Multiplying (9) and (P) by $2u_{tt}$, we get $\frac{d}{dt}\tilde{E}(t)+2\int_{\mathbb{R}^n}g'(u_t(t))|u_{tt}(t)|^2\,dx=0$ and $\frac{d}{dt}E(t)+2\int_{\mathbb{R}^n}g(u_t(t))u_t(t)\,dx=0$. Then, we have

$$\tilde{E}(t)\le\tilde{E}(0) \quad\text{and}\quad E(t)\le E(0). \tag{10}$$

On the other hand, $\|\nabla_x u'\|_{L^2}^2\le\tilde{E}(0)$ and $\|u'\|_{L^2}^2\le E(0)$. Then

$$\forall t\in\mathbb{R}_+:\quad \|u'\|_{H^1(\mathbb{R}^2)}\le\sqrt{E(0)+\tilde{E}(0)}=C(u_0,u_1), \tag{11}$$

which proves the assertion. □

In the proof, we often use the following inequality:

$$\|u(t)\|_2\le\frac{1}{\tilde{d}(t)}\|\lambda(x)u(t)\|_2. \tag{12}$$

Proof of Theorem 3.1.
`A.) Proof of` (5) `and` (6): We consider the case $m=1$, that is,

$$c_1|x|\le|g(x)|\le c_2|x| \qquad \text{for } |x|\le 1 \tag{13}$$
$$c_3\le|g(x)|\le c_4|x|^r \qquad \text{for } |x|\ge 1. \tag{14}$$

`A1.) The case` $n=1$: We denote by c or c_i various positive constants which my be different at different occurences. We multiply the first equation in (P) by

$E^q\phi' u$, where ϕ is a function satisfying all the hypothesis of Lemma 2.4. After integration by parts, we obtain

$$\begin{aligned}
2\int_S^T E^{q+1}\phi'\,dt = &- \left[E^q\phi'\int_{\mathbb{R}^n} uu'\,dx\right]_S^T + \int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_{\mathbb{R}^n} uu'\,dx\,dt \\
&+ 2\int_S^T E^q\phi'\int_{\mathbb{R}^n} u'^2\,dx\,dt - \int_S^T E^q\phi'\int_{\mathbb{R}^n} ug(u')\,dx\,dt \\
\le &- \left[E^q\phi'\int_{\mathbb{R}^n} uu'\,dx\right]_S^T + \int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_{\mathbb{R}^n} uu'\,dx\,dt \\
&+ 2\int_S^T E^q\phi'\int_{\mathbb{R}^n} u'^2\,dx\,dt + c(\varepsilon)\int_S^T E^q\phi'\int_{|u'|\le 1}\frac{1}{\lambda^2(x)}g(u')^2\,dx\,dt \\
&+ \varepsilon\int_S^T E^q\phi'\int_{\mathbb{R}^n}\lambda^2(x)u^2\,dx\,dt + \int_S^T E^q\phi'\int_{|u'|\ge 1} ug(u')\,dx\,dt
\end{aligned}$$

and

$$\begin{aligned}
\int_S^T E^{q+1}\phi'\,dt \le &- \left[E^q\phi'\int_{\mathbb{R}^n} uu'\,dx\right]_S^T + \int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_{\mathbb{R}^n} uu'\,dx\,dt \\
&+ \int_S^T E^q\phi'\int_{|u'|\ge 1} u'^2\,dx\,dt + \int_S^T E^q\phi'\int_{|u'|\ge 1} ug(u')\,dx\,dt \qquad (15) \\
&+ c(\varepsilon)\int_S^T E^q\phi'\int_{|u'|\le 1}\frac{1}{\lambda^2(x)}|u'|^2\,dx\,dt\,.
\end{aligned}$$

Further,

$$\int_S^T E^q\phi'\int_{|u'|\le 1}\frac{1}{\lambda^2(x)}u'^2\,dx\,dt \le C\int_S^T E^q\phi'\int_{\mathbb{R}^n}\left(\frac{1}{\tilde{d}^2(t)}\right)u'g(u')\,dx\,dt$$

and

$$\int_S^T E^q(-E')\,dt = \frac{1}{1+q}\left(E^{1+q}(S) - E^{1+q}(T)\right) \le cE^{1+q}(S)\,.$$

We choose $\phi(t) = \int_0^t \tilde{d}^2(s)\,ds$. It is clear that ϕ is a non decreasing function of class C^2 on $\mathbb{R}_+$. The hypothesis (4) ensures that $\phi(t) \to +\infty$ as $t \to +\infty$. By (12), the definition of E, and the Cauchy-Schwartz inequality we have

$$\begin{aligned}
&-\left[E^q\phi'\int_{\mathbb{R}^n} uu'\,dx\right]_S^T \\
&\quad = E^q(S)\phi'(S)\int_{\mathbb{R}^n} u(S)u'(S)\,dx - E^q(T)\phi'(T)\int_{\mathbb{R}^n} u(T)u'(T)\,dx \\
&\quad \le CE^{q+1}(S)\left[\frac{\phi'(S)}{\tilde{d}(S)} + \frac{\phi'(T)}{\tilde{d}(T)}\right] \\
&\quad \le CE^{q+1}(S)
\end{aligned}$$

and

$$\int_S^T (qE'E^{q-1}\phi' + E^q\phi'') \int_{\mathbb{R}^n} uu'\,dx\,dt \le \int_S^T q|E'|E^q \frac{\phi'(t)}{\tilde{d}(t)}\,dt + \int_S^T E^{q+1}\frac{|\phi''(t)|}{\tilde{d}(t)}\,dt,$$

where $\phi' = \tilde{d}^2(t)$ and $\phi''(t) = 2\tilde{d}(t)\tilde{d}'(t)$. So $\left|\frac{\phi''(t)}{\tilde{d}(t)}\right| \le 2|\tilde{d}'(t)|$. $\tilde{d}(t)$ is bounded, so we obtain

$$\int_S^T E^{q+1}\frac{|\phi''(t)|}{\tilde{d}(t)}\,dt \le -E^{q+1}(S)\left[\tilde{d}(t)\right]_S^T \le E^{q+1}(S)\tilde{d}(S) \le CE^{q+1}(S).$$

Then we deduce from (15) that

$$\begin{aligned}\int_S^T E^{q+1}\phi'\,dt \le CE(S)^{q+1} &+ \int_S^T E^q\phi' \int_{|u'|\ge 1} u'^2\,dx\,dt \\ &+ \int_S^T E^q\phi' \int_{|u'|\ge 1} ug(u')\,dx\,dt\,.\end{aligned} \tag{16}$$

In the last step we use the non-increasing property of E, the Sobolev imbedding $H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and the Cauchy-Schwartz inequality. As $n = 1$, then $H^1(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ and we conclude from the regularity of the solution u that $u' \in L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^n))$. By (13) and (14), we have the two following inequalities:

$$\int_{|u'|>1} (u')^2\,dx \le c\int_{|u'|>1} |u'||u'g(u')|\,dx \le c\|u'\|_{L^\infty(\mathbb{R}^n)}\int_{|u'|>1} |u'g(u')|\,dx \le c(-E')$$

and

$$\int_{|u'|>1} |ug(u')|dx \le c\|u\|_{L^\infty(\mathbb{R}^n)}\int_{|u'|>1} |u'g(u')|dx \le c\|u\|_{H^1(\mathbb{R}^n)}(-E') \le \frac{c}{\tilde{d}(t)}(-E').$$

Choosing $q = 0$ we deduce from (16) that $\int_S^T E(t)\phi'\,dt \le cE(S)$, and we may then complete the proof of this case by applying Lemma 2.4.

`A2.) The case` $n = 2$: We have $|g(y)| \le c_4|y|$ if $|y| \le 1$. Then set $\varepsilon > 0$:

$$\begin{aligned}&\int_S^T E^q\phi'(t) \int_{|u'|\le 1} ug(u')\,dx\,dt \\ &\quad\le \varepsilon \int_S^T E^q\phi'(t) \int_{|u'|\le 1} \lambda^2(x)u^2 + C(\varepsilon)\frac{1}{\lambda^2(x)}g^2(u')\,dx\,dt \\ &\quad\le \varepsilon \int_S^T E^{q+1}\phi'(t)\,dx\,dt + C(\varepsilon)\int_S^T \int_{|u'|\le 1} \frac{1}{\lambda^2(x)}g^2(u')\,dx\,dt \\ &\quad\le \varepsilon \int_S^T E^{1+q}\phi'(t) + C(\varepsilon)\int_S^T E^q\phi'(t)\int_{|u'|\le 1}\frac{1}{\lambda^2(x)}u'^2\,dx\,dt\end{aligned} \tag{17}$$

Next we look at the part $|u'| > 1$: as we work on dimension $n = 2$, we have $\|u\|_{L^{r+1}(\mathbb{R}^n)} \le c\|u\|_{H^1(\mathbb{R}^n)} \le c\frac{\sqrt{E}}{\tilde{d}(t)}$ for all $r \ge 1$. Then set $\epsilon' > 0$:

$$\begin{aligned}
&\int_S^T E^q\phi'(t)\int_{|u'|>1} ug(u')\,dx\,dt \\
&\quad\le \int_S^T E^q\phi'(t)\left(\int_{\mathbb{R}^n}|u|^{r+1}\,dx\right)^{\frac{1}{(r+1)}}\left(\int_{|u'|>1}|g(u')|^{\frac{(r+1)}{r}}\,dx\right)^{\frac{r}{(r+1)}}dt \\
&\quad\le c\int_S^T E^q\phi'(t)\|u\|_{H^1(\mathbb{R}^n)}\left(\int_{|u'|>1}u'g(u')\,dx\right)^{\frac{r}{(r+1)}}dt \\
&\quad\le \int_S^T E^{\frac{2q+1}{2}}\phi'(t)\frac{1}{\tilde{d}(t)}\left(\int_{|u'|>1}u'g(u')\,dx\right)^{\frac{r}{(r+1)}}dt \\
&\quad\le \int_S^T \frac{\phi'(t)}{\tilde{d}(t)}E^{\frac{2q+1}{2}}(-E')^{\frac{r}{(r+1)}}\,dt \\
&\quad\le \int_S^T \frac{\phi'(t)}{\tilde{d}(t)}\left(E^{\frac{2q+1}{2}-\frac{qr}{r+1}}\right)E^{\frac{qr}{(r+1)}}(-E')^{\frac{r}{(r+1)}}\,dt \\
&\quad\le \varepsilon'\int_S^T\phi'(t)E^{(r+1)(\frac{2q+1}{2}-\frac{qr}{r+1})}\,dt + C(\varepsilon')\int_S^T\tilde{d}^{\frac{r-1}{r}}(t)(-E'E^q)\,dt \\
&\quad\le c(\varepsilon')E(S)^{1+q} + \varepsilon' E(0)^{\frac{(r-1)}{2}}\int_S^T E^{q+1}\phi'\,dt.
\end{aligned} \tag{18}$$

Then, choosing ε and ε' small enough, we deduce from (17) and (18) that

$$\begin{aligned}
\int_S^T E^{1+q}\phi'(t)\,dt \le c(\varepsilon')E^{1+q}(S) + c\int_S^T E^q\phi'(t)\int_{|u'|\ge1}u'^2\,dx\,dt \\
+ C(\varepsilon)\int_S^T E^q\phi'(t)\int_{|u'|\le1}\frac{1}{\lambda^2(x)}u'^2(t)\,dt.
\end{aligned}$$

We want to majorize the term $\int_S^T\int_{\mathbb{R}^2}u'^2\,dx\,dt$. Let $R > 0$ and fix $t \ge 0$. Define $S_1^t = \{x \in \mathbb{R}^2 : |u'| \le R\}$ and $S_2^t = \{x \in \mathbb{R}^2 : R < |u'|\}$. By Lemma 2.2, in dimension $n = 2$ we get that there exists a positive constant c depending on Ω such that

$$\forall v \in H^1(\mathbb{R}^2): \quad \|v\|_{L^3(\mathbb{R}^2)} \le \|v\|^{\frac{1}{3}}_{H^1(\mathbb{R}^2)}\|v\|^{\frac{2}{3}}_{L^2(\mathbb{R}^2)}. \tag{19}$$

From the definition of S_2^t, we have $\int_{S_2^t}u'^2\,dx \le \frac{1}{R}\int_{S_2^t}|u'|^3\,dx \le \frac{1}{R}\|u'\|^3_{L^3(\mathbb{R}^2)}$. Then, since u is a strong solution, we can apply (19) to get the estimate $\|u'\|^3_{L^3(\mathbb{R}^2)} \le c\|u'\|_{H^1(\mathbb{R}^2)}\|u'\|^2_{L^2(\mathbb{R}^2)} \le c\|u'\|_{H^1(\mathbb{R}^2)}E(t)$. Consequently,

$$\int_{S_2^t}u'^2\,dx \le \frac{c}{R}\|u'\|_{H^1(\mathbb{R}^2)}E(t). \tag{20}$$

Using (11) and (20), we obtain

$$\begin{aligned}
&\int_S^T E^{1+q}(t)\phi'(t)\,dt\\
&\le cE^{1+q}(S) + c\int_S^T E^q(t)\phi'(t)\int_{\mathbb{R}^2} u'^2\,dx\,dt\\
&\le cE^{1+q}(S) + c\int_S^T E^q(t)\phi'(t)\int_{S_1^t} u'^2\,dx\,dt + \frac{c}{R}\int_S^T E^q(t)\phi'(t)\|u'\|_{H^1(\mathbb{R}^2)}E(t)\,dt\\
&\le cE^{1+q}(S) + c\int_S^T E^q(t)\phi'(t)\int_{S_1^t} u'^2\,dx\,dt + \frac{c}{R}C(u_0,u_1)\int_S^T E^{1+q}(t)\phi'(t)\,dt.
\end{aligned}$$

Now choosing $R>0$ such that $\frac{c}{R}C(u_0,u_1)\le\frac{1}{2}$, we deduce that

$$\frac{1}{2}\int_S^T E^{1+q}(t)\phi'(t)\,dt \le cE(S) + c\int_S^T E^q(t)\phi'(t)\int_{S_1^t} u'^2\,dx\,dt.$$

Next we look at the part S_1^t. We have $|g(y)|\ge c_3|y|$ for all $y\in[-1,1]$. Let

$$\alpha_2 = \inf\left\{\left|\frac{g(y)}{y}\right| : 1\le|y|\le R\right\} > 0.$$

With $\alpha=\min(c_3,\alpha_2)$, we have $|g(y)|\ge\alpha|y|$ if $|y|\le R$. Therefore

$$\begin{aligned}
\int_S^T E^q(t)\phi'(t)\int_{S_1^t} u'^2\,dx\,dt &= \int_S^T E^q(t)\phi'(t)\int_{S_1^t} u'g(u')\frac{u'}{g(u')}\,dx\,dt\\
&\le \frac{1}{\alpha}\int_S^T E^q(t)\phi'(t)\int_{S_1^t} u'g(u')\,dx\,dt\\
&= \frac{c}{\alpha}(E^{1+q}(S)-E^{1+q}(T)).
\end{aligned}$$

Finally, we get

$$\frac{1}{2}\int_S^T E^{1+q}(t)\phi'(t)dt \le cE(S) + \frac{c}{\alpha}(E^{1+q}(S)-E^{1+q}(T)) \le \left(c+\frac{c}{\alpha}\right)E^{1+q}(S). \quad (21)$$

We choose $q=0$. We deduce from (21) that $\int_S^T E(t)\phi'(t)\,dt \le CE(S)$, where C is a positive constant independent of $E(0)$. We may then complete the proof by applying Lemma 2.4.

`A3.) The case` $n\ge 3$: When $n\ge3$, we deduce from (13) and (14) that $c_3|v|^p\le|g(v)|\le c_4|v|^{\frac{1}{p}}$ for all $|v|\le1$, and $c_1\le|g(v)|\le c_2|v|^r$ for all $|v|\ge1$, where p is such that $1 < p$ and $\frac{n}{2} \leq p$, we will later choose the optimal p when $n \geq 3$. By the same techniques we obtain

$$\int_S^T E^{1+q}\phi'(t)\,dt \ \leq cE^{1+q}(S) + C(\epsilon)\int_S^T E^q\phi'(t)\int_{|u'|\geq 1} u'^2\,dx\,dt + \int_S^T E^q\phi'(t)\int_{|u'|\leq 1}\frac{1}{\lambda^2(x)}u'^2(t)\,dt.$$

Putting $\alpha = \frac{2-s}{1-s}$, $s \in (0,1)$, we have

$$\begin{aligned} E^q\phi'\int_{|u'|>1} u'^2\,dx &\leq cE^q\phi'\int_{|u'|>1}|u'|^{\alpha(1-s)}|u'g(u')|^s\,dx \\ &\leq cE^q\phi'\||u'|^{\alpha(1-s)}\|_{\frac{1}{1-s}}\|(u'g(u'))^s\|_{\frac{1}{s}} \\ &= cE^q\phi'\|u'\|_\alpha^{\alpha(1-s)}(-E')^s \\ &\leq \epsilon' E^{\frac{p-1}{2(1-s)}}\phi'\|u'\|_\alpha^\alpha - c(\epsilon')E'. \end{aligned}$$

For $s = \frac{2}{p+1}$ with $q = \frac{p-1}{2}$, we have then

$$E^q\phi'\int_{|u'|>1} u'^2\,dx \leq \epsilon' E^{\frac{p+1}{2}}\phi'\|u'\|_{\frac{2p}{p-1}}^{\frac{2p}{p-1}} - c(\epsilon')E'. \tag{22}$$

As $\frac{n}{2} \leq p$, we have the Sobolev embedding $H^1(\mathbb{R}^n) \subset L^{\frac{2p}{p-1}}(\mathbb{R}^n)$. We deduce from (22) that $E^q\phi'\int_{|u'|>1} u'^2\,dx \leq c\epsilon' E^{\frac{p+1}{2}}\phi' - c(\epsilon')E' \leq c\epsilon' E^{q+1}\phi' - c(\epsilon')E'$. Choosing $\epsilon' = \frac{\epsilon}{c}$, it follows that

$$E^q\phi'\int_{|u'|>1} u'^2\,dx \leq \epsilon E^{q+1}\phi' - c(\epsilon)E'.$$

On the other hand,

$$\begin{aligned} \int_S^T E^q\phi'(t)\int_{|u'|\leq 1}\frac{1}{\lambda^2(x)}u'^2(t)\,dx\,dt &\leq c\int_S^T E^q\phi'(t)\int_{|u'|\leq 1}\frac{1}{\lambda^2(x)}u'g(u')\,dx\,dt \\ &\leq c\int_S^T E^q\frac{\phi'(t)}{\tilde{d}^2(x)}\int_{|u'|\leq 1} u'g(u')\,dx\,dt \\ &\leq c\int_S^T E^q\frac{\phi'(t)}{\tilde{d}^2(x)}(-E')\,dt. \end{aligned}$$

Define $\phi(t) = \int_0^t \tilde{d}^2(s)\,ds$, we have

$$\int_S^T E^q\phi'(t)\int_{|u'|\leq 1}\frac{1}{\lambda^2(x)}u'^2(t)\,dx\,dt \leq \int_S^T E^q(-E')\,dt \leq cE^{q+1}(S).$$

Then $\int_S^T E^{1+q}\phi'(t)\,dt \le cE^{q+1}(S) \le cE(S)$. Using Lemma 2.4, we conclude that

$$E(t) \le \frac{C(E(0))}{\left(\int_0^t \tilde{d}^2(\tau)\,d\tau\right)^{\frac{2}{p-1}}}.$$

When $n \ge 3$, the optimal p such that $\frac{n}{2} \le p$ and $1 < p$ is $p = \frac{n}{2}$. Then we obtain

$$E(t) \le \frac{C(E(0))}{\left(\int_0^t \tilde{d}^2(\tau)\,d\tau\right)^{\frac{4}{n-2}}}.$$

`B:) Proof of` (7) `and` (8)`:` We consider the case $m > 1$, that is,

$$\begin{aligned} c_1|x|^m \le |g(x)| \le c_2|x|^{\frac{1}{m}} \qquad & \text{for } |x| \le 1 \\ c_3 \le |g(x)| \le c_4|x|^r \qquad & \text{for } |x| > 1, \end{aligned} \tag{23}$$

where m is such that $1 < m$ and $\frac{n}{2} \le m$.

If we multiply the first equation of (P) by $E^q\phi' u$, we obtain

$$\begin{aligned} 0 &= \int_S^T E^q\phi' \int_{\mathbb{R}^n} u(u'' - \Delta u + \lambda^2(x)u + g(u'))\,dx\,dt \\ &= \left[E^q\phi' \int_{\mathbb{R}^n} uu'\,dx\right]_S^T - \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_{\mathbb{R}^n} uu'\,dx\,dt \\ &\quad - 2\int_S^T E^q\phi' \int_{\mathbb{R}^n} u'^2\,dx\,dt + \int_S^T E^q\phi' \int_{\mathbb{R}^n} \left(u'^2 + |\lambda u|^2 + |\nabla u|^2\right)\,dx\,dt \\ &\quad + \int_S^T E^q\phi' \int_{\mathbb{R}^n} ug(u')\,dx\,dt \end{aligned}$$

and

$$\begin{aligned} \int_S^T E^{1+q}\phi'(t)\,dt \le cE^{1+q}(S) &+ C(\epsilon)\int_S^T E^q\phi'(t)\int_{|u'|\ge 1} u'^2\,dx\,dt \\ &+ \int_S^T E^q\phi'(t)\int_{|u'|\le 1} u'^2(t)\,dt \\ &+ \int_S^T E^q\phi'(t)\int_{|u'|\le 1} \frac{1}{\lambda^2(x)}g^2(u')\,dx\,dt. \end{aligned}$$

Therefore, by the same estimation as in the case $n = 2$, we obtain

$$\int_S^T E^q\phi' \int_{|u'|>1} |ug(u')|\,dx \le c(\varepsilon')E(S)^{1+q} + \varepsilon' E(0)^{\frac{(r-1)}{2}} \int_S^T \phi' E^{q+1}\,dt.$$

Putting $\alpha = \frac{2-s}{1-s}$, $s \in (0,1)$, we have

$$\begin{aligned} E^q\phi' \int_{|u'|>1} u'^2\,dx &\le cE^q\phi' \int_{|u'|>1} |u'|^{\alpha(1-s)}|u'g(u')|^s\,dx \\ &\le cE^q\phi' \| |u'|^{\alpha(1-s)}\|_{\frac{1}{1-s}} \|(u'g(u'))^s\|_{\frac{1}{s}} \\ &= cE^q\phi' \|u'\|_{\alpha}^{\alpha(1-s)}(-E')^s \\ &\le \epsilon' E^{\frac{m-1}{2(1-s)}}\phi' \|u'\|_{\alpha}^{\alpha} - c(\epsilon')E'. \end{aligned} \tag{24}$$

Choosing $s = \frac{2}{m+1}$ with $q = \frac{m-1}{2}$, we have then

$$E^q\phi' \int_{|u'|>1} u'^2\,dx \le \epsilon' E^{\frac{m+1}{2}}\phi' \|u'\|_{\frac{2m}{m-1}}^{\frac{2m}{m-1}} - c(\epsilon')E'. \tag{25}$$

As $\frac{n}{2} \le m$, we use the Sobolev embedding $H^1(\mathbb{R}^n) \subset L^{\frac{2m}{m-1}}(\mathbb{R}^n)$. We deduce from (25) that $E^q\phi' \int_{|u'|>1} u'^2\,dx \le c\epsilon' E^{\frac{p+1}{2}}\phi' - c(\epsilon')E' \le c\epsilon' E^{q+1}\phi' - c(\epsilon')E'$. Choosing $\epsilon' = \frac{\epsilon}{c}$, it follows that

$$E^q\phi' \int_{|u'|>1} u'^2\,dx \le \epsilon E^{q+1}\phi' - c(\epsilon)E'. \tag{26}$$

By (23), we have the two following inequalities

$$\begin{aligned} \int_S^T E^q\phi' \int_{|u'|\le 1} u'^2\,dx\,dt &\le \int_S^T E^q\phi' \int_{|u'|\le 1} \frac{1}{\lambda^2(x)} u'^2\,dx\,dt \\ &\le 2\int_S^T E^q\phi' \int_{\mathbb{R}^n} \frac{1}{\lambda^2(x)} (u'g(u'))^{\frac{2}{(m+1)}}\,dx\,dt \\ &\le 2\int_S^T E^q\phi' \int_{\{|x|\le L+t\}} \frac{1}{\lambda^2(x)} (u'g(u'))^{\frac{2}{(m+1)}}\,dx\,dt \\ &\le 2\int_S^T E^q\phi' \Big(\frac{1}{\tilde{d}(t)}\Big)^2 (1+t)^{\frac{n(m-1)}{m+1}} \left(\int_{\mathbb{R}^n} u'g(u')\,dx\right)^{\frac{2}{(m+1)}} dt \\ &\le c' \int_S^T E^q\phi' \Big(\frac{1}{\tilde{d}(t)}\Big)^2 (1+t)^{\frac{n(m-1)}{m+1}} (-E')^{\frac{2}{m+1}}\,dt \\ &\le c' \int_S^T E^q\phi' \Big(\frac{1}{\tilde{d}(t)}\Big)^2 (1+t)^{\frac{n(m-1)}{m+1}} (-E')^{\frac{2}{m+1}}\,dt \\ &\le c' \int_S^T E^q\phi' \frac{1}{\tilde{d}^2(t)} (1+t)^{\frac{n(m-1)}{m+1}} (-E')^{\frac{2}{m+1}}\,dt \end{aligned} \tag{27}$$

and

$$\int_S^T E^q\phi' \int_{|u'|\le 1} \frac{1}{\lambda^2(x)} g^2(u')\,dx\,dt \le 2\int_S^T E^q\phi' \int_{\mathbb{R}^n} \frac{1}{\lambda^2(x)} (u'g(u'))^{\frac{2}{(m+1)}}\,dx\,dt.$$

Let $\varepsilon > 0$. Thanks to Young's inequality and to our definitions of m and ϕ, we obtain

$$\begin{aligned}\int_S^T E^q\phi' \int_{|u'|<1} \frac{1}{\lambda^2(x)} u'^2\,dx\,dt \ &\leq 2\,\frac{m-1}{m+1}\,\varepsilon^{\frac{m+1}{m-1}} \int_S^T E^{1+q}(\phi')^{\frac{m+1}{m-1}}(1+t)^n\,dt \\ &\quad + \frac{4}{m+1}\,\frac{1}{\varepsilon^{\frac{(m+1)}{2}}} E(S)\end{aligned}$$

When $m < \frac{n}{2}$, we replace m in (24) by $\frac{n}{2}$, take $q = \frac{\frac{n}{2}-1}{2} = \frac{n-2}{4}$ and obtain from (27)

$$\begin{aligned}&\int_S^T E^q\phi' \int_{|u'|<1} \frac{1}{\lambda^2(x)} u'^2\,dx\,dt \\ &\leq 2\frac{m-1}{m+1}\varepsilon^{\frac{(m+1)}{(m-1)}} \int_S^T E^{1+q} E^{\frac{\frac{n}{2}-m}{m-1}}(\phi')^{\frac{m+1}{m-1}}(1+t)^n\,dt + \frac{4}{m+1}\frac{1}{\varepsilon^{\frac{(m+1)}{2}}}E(S) \\ &\leq 2\frac{m-1}{m+1}\varepsilon^{\frac{(m+1)}{(m-1)}} E^{\frac{\frac{n}{2}-m}{m-1}}(0) \int_S^T E^{1+q}(\phi')^{\frac{m+1}{m-1}}(1+t)^n\,dt + \frac{4}{m+1}\frac{1}{\varepsilon^{\frac{(m+1)}{2}}}E(S). \quad (28)\end{aligned}$$

In all cases we choose ϕ' such that $(\phi')^{\frac{2}{m-1}}\tilde{d}^{-\frac{2(m+1)}{m-1}}(t)(1+t)^n = 1$, thus $\phi(t) = \int_0^t (1+s)^{-\frac{n(m-1)}{2}}\tilde{d}^{m+1}(s)\,ds$. By (12), the definition of E and the Cauchy-Schwartz inequality we have

$$\begin{aligned}-\left[E^q\phi'\int_{\mathbb{R}^n} uu'\,dx\right]_S^T &= E^q(S)\phi'(S)\int_{\mathbb{R}^n} u(S)u'(S)\,dx - E^q(T)\phi'(T)\int_{\mathbb{R}^n} u(T)u'(T)\,dx \\ &\leq CE^{q+1}(S)\left[\frac{\phi'(S)}{\tilde{d}(S)} + \frac{\phi'(T)}{\tilde{d}(T)}\right] \\ &\leq CE^{q+1}(S)\end{aligned}$$

and

$$\int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_{\mathbb{R}^n} uu'\,dx\,dt \leq \int_S^T q|E'|E^q\frac{\phi'(t)}{\tilde{d}(t)}\,dt + \int_S^T E^{q+1}\frac{|\phi''(t)|}{\tilde{d}(t)}\,dt.$$

It holds

$$\phi''(t) = -\frac{n(m-1)}{2}(1+t)^{-\frac{n(m-1)}{2}-1}\tilde{d}^{m+1}(t) + (1+t)^{-\frac{n(m-1)}{2}}(m+1)\tilde{d}^m(t)\tilde{d}'(t).$$

Thus $\left|\frac{\phi''(t)}{\tilde{d}(t)}\right| \leq C\tilde{d}^m(t) - C'\tilde{d}^{m-1}(t)\tilde{d}'(t)$. As $\tilde{d}(t)$ is bounded, we obtain

$$\int_S^T E^{q+1}\frac{|\phi''(t)|}{\tilde{d}(t)}\,dt \leq -CE^{q+1}(S)\left[\tilde{d}^m(t)\right]_S^T \leq CE^{q+1}(S).$$

Then we deduce from (28) that $\int_S^T E^{1+q}\phi'\,dt \le 2CE(S)$, and from Lemma 2.4 $\big($applied with $q = \frac{\max\{m,\frac{n}{2}\}-1}{2}\big)$ we obtain

$$E(t) \le \frac{C}{\phi(t)^{\frac{2}{\max\{m,\frac{n}{2}\}-1}}}\,.$$

To prove global existence in H^2 we use the estimates for second derivatives of $u(t)$ and the energy estimate of $E(t)$. From (10), we have

$$\|u''(t)\| \le \tilde{E}(0)\,. \tag{29}$$

By (2) and (3), we have

$$\begin{aligned}\int_{\mathbb{R}^n} |g(u')|^2\,dx &\le C\int_{|u'|\le 1} |u'|^{\frac{2}{m}}\,dx + C'\int_{|u'|\ge 1} |u'|^{2r}\,dx\\ &\le C(L+t)^{\frac{n(m-1)}{m}} E^{\frac{1}{m}} + C'E^{\frac{2-(n-2)(r-1)}{2}} E_2^{\frac{n(r-1)}{2}}\,.\end{aligned}$$

From (29) and the first equation of the problem (P), we also prove easily that $\|\Delta_x u(t)\|_2 \le C'(t) < \infty$ for all $t \ge 0$. Indeed, we have $\|\Delta_x u(t)\|_2 \le \|u''(t)\|_2 + \|\lambda^2 u(t)\|_2 + \|g(u'(t)\|_2$. □

References

[1] Benaissa, A. and Mokeddem, S., Global existence and energy decay of solutions to the Cauchy problem for a wave equation with a weakly nonlinear dissipation. *Abstr. Appl. Anal.* (2004)(11), 935 – 955.

[2] Brezis, H., *Analyse Fonctionnelle, Théories et Applications.* Collection Mathématique Appliquée pour la Maîtrise. Paris: Masson 1983.

[3] Dieudonné, J., *Calcul Infinitésimal. Collection Methodes.* Paris: Hermann 1968.

[4] Georgiev, V. and Todorova, G., Existence of a solution of the wave equation with nonlinear damping and source terms. *J. Diff. Eqs.* 109 (1994), 295 – 308.

[5] Haraux, A., Two remarks on dissipative hyperbolic problems. *Collége de France Seminar*, Vol. VII (Paris 1983 – 1984)(6). In: Res. Notes Math. 122. Boston (MA): Pitman 1985, pp. 161 – 179.

[6] Ikehata, R., Some remarks on the wave equations with nonlinear damping and source terms. *Nonlinear Anal.* 27 (1996), 1165 – 1175.

[7] Ikehata, R. and Suzuki, T., Stable and unstable sets for evolution equations of parabolic and hyperbolic type. *Hiroshima Math. J.* 26 (1996), 475 – 491.

[8] Komornik, V., *Exact Controllability and Stabilization. The Multiplier Method.* Paris: Masson; Chichester: Wiley 1994.

[9] Martinez, P., A new method to obtain decay rate estimates for dissipative systems. *ESAIM Control Optim. Calc. Var.* 4 (1999), 419 – 444.

[10] Nakao, M., A difference inequality and its applications to nonlinear evolution equations. *J. Math. Soc. Japan* 30 (1978), 747 – 762.

[11] Nakao, M., Energy decay of the wave equation with a nonlinear dissipative term. *Funkcial. Ekvac.* 26 (1983), 237 – 250.

[12] Nakao, M., On the decay of solutions of some nonlinear dissipative wave equations in higher dimensions. *Math. Z.* 193 (1986), 227 – 234.

[13] Nakao, M., Energy decay for the wave equation with a nonlinear weak dissipation. *Diff. Integral Eqs.* 8 (1995), 681 – 688.

[14] Nakao, M. and Ono, K., Global existence to the Cauchy problem of the semilinear wave equation with a nonlinear dissipation. *Funkcial. Ekvac.* 38 (1995), 417 – 431.

[15] Todorova, G., Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms. *J. Math. Anal. Appl.* 239 (1999), 213 – 226.

[16] Yamada, Y., On the decay of solutions for some nonlinear evolution equations of second order. *Nagoya Math. J.* 73 (1979), 69 – 98.

Received December 16, 2005; revised June 21, 2006