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Error Estimates for the Regularization of Optimal Control Problems with Pointwise Control and State Constraints

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Abstract. We discuss a linear-quadratic optimal control problem with pointwise control and state constraints. The state constraints are regularized by a Lavrentiev type regularization. The main results of the paper are estimates for the regularization error and the stability with respect to noisy data.

Keywords. Optimal control, linear-quadratic problems, state constraints, Lavrentiev regularization, perturbations, stability

Mathematics Subject Classification (2000). Primary 49M20, secondary 49K20, 49N10, 90C46

1. Introduction

In this paper we consider the following elliptic control problem with pointwise state constraints and distributed control

$$
(\mathrm{P}) \begin{cases} \min J(y,u) := \dfrac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u\|^2_{L^2(\Omega)} & \\ \text{subject to} \qquad (Ay)(x) = u(x) & \text{in } \Omega \\ \qquad\qquad y(x) = 0 & \text{on } \partial\Omega \\ \qquad\qquad 0 \le u(x) \le b & \text{a.e. in } \Omega \\ \qquad\qquad y_c(x) \le y(x) & \text{a.e. in } \Omega', \end{cases}
$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded domain with $C^{0,1}$ boundary $\partial\Omega$. We assume $\Omega' \subset \Omega$ with $\operatorname{dist}(\Omega', \partial\Omega) > 0$. This is motivated by the investigation of a homogeneous Dirchlet problem. Since the boundary data are fixed a bad choice of y_c would be immediately lead to an empty feasible set. Conversely, a

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function y_c which is bounded below by a positive number in a neighbourhood of the boundary $\partial\Omega$ would be equivalent to the problem under consideration, since the state y is continuous.

The parameters ν and b are fixed positive real numbers and y_d is a fixed function from $L^2(\Omega)$. The state constraint y_c is a function belonging to $L^\infty(\Omega')$ and A is a uniformly elliptic differential operator. More precisely, it has the form

$$Ay(x) = -\sum_{i,j=1}^{n} D_i\big(a_{ij}(x)D_j y(x)\big) + c(x)y(x),$$

where D_i denotes the partial derivative with respect to x_i. Here c is a given function in $L^\infty(\Omega)$ with $c(x) \geq 0$ a.e., and a_{ij} belonging to $C^{0,1}(\bar{\Omega})$, and satisfying the conditions $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$ with the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta\|\xi\|^2 \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n,$$

where θ is some positive constant.

The main difficulty in this problem is the pointwise state constraint $y(x) \geq y_c(x)$. Theory and numerical treatment of such constraints are still a challenge. In this paper we use a Lavrentiev type regularization of the state constraints, introduced in Meyer, Rösch and Tröltzsch [6]. Let us remark that the Lavrentiev regularization is well studied for inverse problems, see Lavrentiev [4], Liu and Nashed [5], Janno [3], Tautenhahn [9], Nair and Tautenhahn [7].

For optimal control problems, the Lavrentiev type regularization is motivated by the following facts. Typically, mixed pointwise control-state constraints have better theoretical properties than state constrained problems. There are several situations known where the existence of measurable and bounded Lagrange multipliers can be shown, see Tröltzsch [10] and Rösch and Tröltzsch [8]. In contrast to this, Lagrange multipliers associated to pointwise state constraints can be expected only in measure spaces. Numerical tests show that the condition numbers of linear systems associated to the regularized problems are essentially smaller than such one for the unregularized problem. The high condition numbers for unregularized problems occur if the state constraint is active on an open subset. On such sets, the optimal control $\bar{u}$ is obtained by $Ay_c = \bar{u}$, i.e., the data have to be differentiated twice. The Lavrentiev regularization of such an operator equation is studied in the theory of inverse problems and corresponding papers were already cited. We will see in this paper that the occurence of the pointwise bounds for the control u stabilizes the problem. Therefore, a control of the regularization parameter with respect to the noice level is not essential. However, first numerical studies show that a reasonable balance between noice level, regularization parameter, and the discretization parameter improves the behavior of the involved numerical methods essentially.

A family of optimal control problems with regularized state constraints is given by

$$
(\mathrm{P}_\varepsilon)\begin{cases}
\min J(y,u) := \dfrac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u\|^2_{L^2(\Omega)} & \\
\text{subject to} \qquad (Ay)(x) = u(x) & \text{in } \Omega \\
\qquad\qquad y(x) = 0 & \text{on } \partial\Omega \\
\qquad\qquad 0 \le u(x) \le b & \text{a.e. in } \Omega \\
\qquad\qquad y_c(x) - \varepsilon u(x) \le y(x) & \text{a.e. in } \Omega'.
\end{cases}
$$

This family is characterized by a regularization parameter $\varepsilon > 0$. Note that Problem (P) is obtained for $\varepsilon = 0$. In this paper we will derive estimates for the regularization error and prove stability properties. The used proving technique was introduced in a paper by Alt [1].

The paper is organized as follows. In Section 2 we state properties of the involved partial differential equations and the optimality conditions. Section 3 contains error estimates for the solutions of the regularized problems (P_ε) with respect to the solution of problem (P). In Section 4, we will study the case of noisy data y_c and y_d. Section 5 is devoted to feasible regularized solutions. The paper ends with numerical tests in Section 6.

2. Optimality conditions

We start with well known results for the weak formulation of the elliptic boundary value problem. A function $y \in V := H^1_0(\Omega)$ is called a weak solution if

$$a(y,v) = (u,v) \quad \forall v \in V \tag{1}$$

is satisfied. Here $a\colon V\times V \to \mathbb{R}$ denotes the bilinear form defined by

$$a(y,u) = \int_\Omega a_{ij}(x)D_iy(x)D_jv(x)\,dx + \int_\Omega c(x)y(x)v(x)\,dx.$$

Lemma 2.1. *Equation* (1) *has a unique solution $y \in V := H^1_0(\Omega)$ for every $u \in L^2(\Omega)$. Moreover, the mapping $u \mapsto y$ is continuous from $L^2(\Omega)$ to V.*

This statement follows immediately from the Lax-Milgram Theorem. It allows us to define a continuous control-to-state mapping $S : L^2(\Omega) \to V$ by $y = Su$ as a weak solution of (1).

Next, we introduce admissible control sets U_{ad} for the Problem (P):

$$U_{ad} := \{u \in L^2(\Omega) \mid 0 \le u(x) \le b \text{ a.e. in } \Omega,\ (Su)(x) \ge y_c(x) \text{ a.e. in } \Omega'\},$$

and the admissible set U_{ad}^{ε} for the modified problem (P_ε):

$$U_{ad}^{\varepsilon} := \{u \in L^2(\Omega) \mid 0 \le u(x) \le b \text{ a.e. in } \Omega,\ \varepsilon u + (Su)(x) \ge y_c(x) \text{ a.e. in } \Omega'\}.$$

In general, these sets may be empty. To avoid this, we have to require the existence of at least one feasible point. However, for our analysis we need a slightly stronger assumption, the existence of a Slater point.

Assumption (A). There exists a control $\hat{u}(x)$ with $0 \le \hat{u}(x) \le b$ and $\hat{y}(x) \ge y_c(x) + \tau$ a.e. in Ω', $\tau > 0$. Here $\hat{y}$ denotes the solution of (1) for the right-hand side $\hat{u}$.

Lemma 2.2. *Assume that* (A) *holds. The optimal control problems* (P) *and* (P_ε) *admit uniquely determined solutions.*

Proof. The objectives of both problems are strictly convex. Moreover, the admissible sets are closed, convex, and bounded in $L^2(\Omega)$. Consequently, the admissible sets are weakly compact in $L^2(\Omega)$. The objective is weakly lower semicontinuous. Therefore, the existence of optimal solutions is guaranteed if the admissible sets are non-empty. This is indeed the case. Because of

$$\hat{y}(x) + \varepsilon\hat{u}(x) \ge \hat{y}(x) \ge y_c(x) + \tau \ge y_c(x),$$

the pair $(\hat{y}, \hat{u})$ is admissible for both problems. The uniqueness of the solution is guaranteed by the strict convexity of the objectives. □

To derive optimality conditions, we introduce an adjoint equation. *All* inequality constraints are contained in the set of admissible controls. Thanks to the non-standard definitions of the admissible sets, the adjoint equation of both problems is the same and contains no Lagrange multipliers. The adjoint equation is given in a weak formulation by

$$a(v, p) = (y - y_d, v) \quad \forall v \in V \tag{2}$$

or in a classical notation

$$\begin{aligned} (A^*p)(x) &= y(x) - y_d(x) && \text{in } \Omega \\ p(x) &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here A^* denotes the adjoint operator; since $a_{ij} = a_{ji}$ we have $A^* = A$.

Lemma 2.3. *The adjoint equation* (2) *admits a unique weak solution* $p \in V$.

In the following we will call the triple $(\bar{y}, \bar{u}, \bar{p})$ *optimal solution of* (P). This means $\bar{u}$ is the optimal control and the corresponding state $\bar{y}$ and the corresponding adjoint state $\bar{p}$ are defined as solutions of (1) and (2). The *optimal solution* $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{p}_\varepsilon)$ *of* (P_ε) is declared in exactly the same way.

Lemma 2.4. *A necessary and sufficient condition for the optimality of* $(\bar{y},\ \bar{u},\ \bar{p})$ *is given by*

$$(\bar{p} + \nu\bar{u},\ u - \bar{u})_{L^2(\Omega)} \geq 0 \quad \textit{for all } u \in U_{ad}. \tag{3}$$

Analogously the optimality condition of $(y_\varepsilon,\ u_\varepsilon,\ p_\varepsilon)$ *is given by*

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon,\ u - \bar{u}_\varepsilon)_{L^2(\Omega)} \geq 0 \quad \textit{for all } u \in U^\varepsilon_{ad}. \tag{4}$$

Since this result is quite standard, we drop the proof.

Notation. If there is no risk of confusion, we denote in the following by $(.\,,.\,) = (.\,,.\,)_{L^2(\Omega)}$ the inner product of $L^2(\Omega)$.

3. Regularization error

In this section we study the error of solutions of the regularized problems (P_ε) with respect to the solution of the original problem (P).

Lemma 3.1. *Assume that* (A) *is satisfied. Then for every* $\varepsilon > 0$*, there exists a constant* $\delta_\varepsilon \in (0, 1)$ *such that* $u_\delta := (1 - \delta)\bar{u}_\varepsilon + \delta\hat{u}$ *is feasible for* (P) *for* $\delta \in [\delta_\varepsilon, 1]$.

Proof. Since $\bar{u}_\varepsilon$ and $\hat{u}$ are feasible for (P_ε), the convex linear combination u_δ fulfills the control constraints $0 \leq u_\delta \leq b$. Consequently, we have only to check the state constraint. Here we know

$$\varepsilon\bar{u}_\varepsilon + \bar{y}_\varepsilon \geq y_c \quad \Rightarrow \quad \bar{y}_\varepsilon \geq y_c - \varepsilon\bar{u}_\varepsilon.$$

According to (A), we have in addition $\hat{y} \geq y_c + \tau$. Due to the linearity of (1), we find $y_\delta = (1 - \delta)\bar{y}_\varepsilon + \delta\hat{y}$, where y_δ denotes the solution of (1) with $u := u_\delta$. This leads to

$$\begin{aligned} y_\delta &= (1 - \delta)\bar{y}_\varepsilon + \delta\hat{y} \\ &\geq (1 - \delta)(y_c - \varepsilon\bar{u}_\varepsilon) + \delta y_c + \delta\tau \\ &\geq y_c - \varepsilon\bar{u}_\varepsilon(1 - \delta) + \delta\tau \\ &\geq y_c - \varepsilon(1 - \delta)b + \delta\tau \end{aligned}$$

using $|u_\varepsilon| \leq b$ in the last line. Consequently, u_δ is feasible for (P) if $\delta\tau - \varepsilon(1-\delta)b$ is positive. This takes place for $\delta \geq \frac{\varepsilon b}{\tau + \varepsilon b}$.Therefore, δ_ε can be defined as $\delta_\varepsilon := \frac{\varepsilon b}{\tau + \varepsilon b}$. □

Lemma 3.2. *For every* $\varepsilon > 0$ *the solution* $\bar{u}$ *is feasible for* (P_ε).

Proof. We have only to check the mixed control-state constraint. Since $\bar{u}$ is feasible for (P),

$$\varepsilon\bar{u} + \bar{y} \geq \bar{y} \geq y_c \;\Rightarrow\; \varepsilon\bar{u} + \bar{y} \geq y_c \quad \text{a.e. in } \Omega$$

is satisfied for every $\varepsilon > 0$. Consequently, $\bar{u}$ is feasible for (P_ε) for arbitrary $\varepsilon > 0$. □

Next, we state the main result of this section.

Theorem 3.3. *Assume that* (A) *is satisfied. Then, there exists a positive constant* C *independent of* ε *with*

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Omega)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} \leq \varepsilon C.$$

Proof. We start with the optimality condition (4). Due to Lemma 3.2, we can test this inequality with $\bar{u}$:

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon, \bar{u} - \bar{u}_\varepsilon) \geq 0. \tag{5}$$

The construction of a test function for (3) is more difficult. Actually, we can not test (3) with $\bar{u}_\varepsilon$ since $\bar{u}_\varepsilon$ may be infeasible for (P). However, from Lemma 3.1 we know the feasibility of u_δ for certain δ. Inserting $u = u_\delta$ in (3), we find

$$(\bar{p} + \nu\bar{u}, u_\delta - \bar{u}) \geq 0 \tag{6}$$

for $\delta \in [\delta_\varepsilon, 1]$. Adding (5) and (6), we find

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon,\ \bar{u} - \bar{u}_\varepsilon) + (\bar{p} + \nu\bar{u},\ u_\delta - \bar{u}) \geq 0.$$

Next, we write this inequality in the form

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon, \bar{u} - \bar{u}_\varepsilon) + (\bar{p} + \nu\bar{u}, u_\delta - \bar{u}_\varepsilon) + (\bar{p} + \nu\bar{u}, \bar{u}_\varepsilon - \bar{u}) \geq 0$$

or

$$(\bar{p} - \bar{p}_\varepsilon,\ \bar{u}_\varepsilon - \bar{u}) + (\bar{p} + \nu\bar{u}, u_\delta - \bar{u}_\varepsilon) + \nu(\bar{u} - \bar{u}_\varepsilon,\ \bar{u}_\varepsilon - \bar{u}) \geq 0. \tag{7}$$

Consider the first term in (7). Since (1) holds true, we obtain $(\bar{p} - \bar{p}_\varepsilon, \bar{u}_\varepsilon - \bar{u}) = a(\bar{p} - \bar{p}_\varepsilon, \bar{y}_\varepsilon - \bar{y})$. On the other hand, (2) implies $a(\bar{p} - \bar{p}_\varepsilon, \bar{y}_\varepsilon - \bar{y}) = (\bar{y} - \bar{y}_\varepsilon, \bar{y}_\varepsilon - \bar{y})$. Hence, we find $(\bar{p} - \bar{p}_\varepsilon, \bar{u}_\varepsilon - \bar{u}) = (\bar{y} - \bar{y}_\varepsilon, \bar{y}_\varepsilon - \bar{y}) = -\|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)}$. Consequently, we can write (7) in the form

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Omega)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} \leq (\nu\bar{u} + \bar{p},\ u_\delta - \bar{u}_\varepsilon).$$

By the Cauchy-Schwarz inequality we arrive at

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Omega)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} \leq \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)}\|u_\delta - \bar{u}_\varepsilon\|_{L^2(\Omega)}. \tag{8}$$

Moreover, we have

$$\|u_\delta - \bar{u}_\varepsilon\|_{L^2(\Omega)} = \|(1-\delta)\bar{u}_\varepsilon + \delta\hat{u} - \bar{u}_\varepsilon\|_{L^2(\Omega)} = \|\delta(\hat{u} - \bar{u}_\varepsilon)\|_{L^2(\Omega)} \le \delta b|\Omega|^{\frac{1}{2}}.$$

For $\delta = \delta_\varepsilon = \frac{\varepsilon b}{\tau+\varepsilon b}$ we obtain

$$\|u_\delta - \bar{u}_\varepsilon\|_{L^2(\Omega)} \le |\Omega|^{\frac{1}{2}} \frac{\varepsilon b^2}{\tau + \varepsilon b} \le |\Omega|^{\frac{1}{2}} \frac{\varepsilon b^2}{\tau}. \tag{9}$$

Combining this inequality with (8), we get

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Omega)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} \le \varepsilon C$$

with $C = \frac{b^2}{\tau}|\Omega|^{\frac{1}{2}} \cdot \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)}$. □

Remark 3.4. It is possible to find an a priori bound $\|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)} \le C'$. We will work out this point in the next section.

4. Stability of the solution with respect to perturbed data

In this section we slightly change our original problem: We investigate perturbed data y_d^σ and y_c^σ instead of the original data y_d and y_c. We are interested in error estimates separating the regularization error and the influence of the noice levels σ_c and σ_d:

$$(\mathrm{P}_\varepsilon^\sigma)\begin{cases} \min J(y,u) := \dfrac{1}{2}\|y - y_d^\sigma\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u\|^2_{L^2(\Omega)} & \\ \text{subject to} \qquad (Ay)(x) = u(x) & \text{in } \Omega \\ \qquad y(x) = 0 & \text{on } \partial\Omega \\ \qquad 0 \le u(x) \le b & \text{a.e. in } \Omega \\ \qquad y_c^\sigma(x) - \varepsilon u(x) \le y(x) & \text{a.e. in } \Omega' \\ \qquad \|y_d - y_d^\sigma\|_{L^2(\Omega)} \le \sigma_d & \\ \qquad \|y_c - y_c^\sigma\|_{L^\infty(\Omega')} \le \sigma_c. & \end{cases}$$

Now, Assumption (A) would be not strong enough to ensure the existence of a feasible control for $(\mathrm{P}_\varepsilon^\sigma)$. This difficulty occurs even for $\varepsilon = 0$ because of perturbations in the data. An additional property of the Slater type point will overcome this problem. The new requirement is that the distance to the constraint $\tilde{\tau}$ is larger than the noice level σ_c in the constraint.

Assumption (B). There exists a control $\tilde{u}(x)$ with $0 \le \tilde{u}(x) \le b$, $\tilde{y}(x) \ge y_c^\sigma(x) + \tilde{\tau}$ a.e. in Ω', and $\tau := \tilde{\tau} - \sigma_c > 0$. Here $\tilde{y}$ denotes the solution of (1) for the right-hand side $\tilde{u}$.

Lemma 4.1. *Let σ_c, σ_d, and ε are fixed nonnegative numbers. Assume that* (B) *is valid. Then the problem* $(\mathrm{P}_\varepsilon^\sigma)$ *admits a uniquely determined solution.*

Proof. As mentioned above, the main difficulty is to show the existence of a feasible point. However, Assumption (B) ensures that $\tilde{u}(x)$ is feasible for $(\mathrm{P}_\varepsilon^\sigma)$:

$$\tilde{y}(x) \geq y_c^\sigma(x) + \tilde{\tau} \geq y_c^\sigma(x) \geq y_c^\sigma(x) - \varepsilon\tilde{u}(x).$$

Moreover, $\tilde{u}(x)$ is also feasible for (P) because of

$$\tilde{y}(x) \geq y_c^\sigma(x) + \tilde{\tau} \geq y_c^\sigma(x) + \sigma_c \geq y_c(x).$$

The remaining part of the proof can be done along the lines of the proof of Lemma 2.2. □

Next, we introduce the set of admissible controls $U_{ad}^{\varepsilon,\sigma}$ via

$$U_{ad}^{\varepsilon,\sigma} := \left\{ u \in L^2(\Omega) \;\middle|\; 0 \leq u(x) \leq b \text{ a.e. in } \Omega,\ \varepsilon u + (Su)(x) \geq y_c(x) \text{ a.e. in } \Omega' \right\}.$$

Similar to Section 3, we will use the notation $(\bar{y}_\varepsilon^\sigma,\ \bar{u}_\varepsilon^\sigma,\ \bar{p}_\varepsilon^\sigma)$ for the *optimal solution of* $(\mathrm{P}_\varepsilon^\sigma)$.

Lemma 4.2. *For the optimality of $(\bar{y}_\varepsilon^\sigma,\ \bar{u}_\varepsilon^\sigma,\ \bar{p}_\varepsilon^\sigma)$, a necessary and sufficient condition is given by*

$$(\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma,\ u - \bar{u}_\varepsilon^\sigma) \geq 0 \quad \textit{for all } u \in U_{ad}^{\varepsilon,\sigma}. \tag{10}$$

This result can be obtained by standard arguments. Therefore, we drop the proof.

Lemma 4.3. *Assume that* (B) *holds. Then for every $\varepsilon > 0$, there exists a constant $\delta^\sigma \in (0,1)$ such that $u_\delta^\sigma := (1-\delta)\bar{u} + \delta\tilde{u}$ is feasible for* $(\mathrm{P}_\varepsilon^\sigma)$ *for* $\delta \in [\delta^\sigma, 1]$.

Proof. We check only the mixed control-state constraint. Let y_δ^σ be the solution of (1) for right-hand side u_δ^σ. We start with

$$\begin{aligned} y_\delta^\sigma &= (1-\delta)\bar{y} + \delta\tilde{y} \\ &\geq (1-\delta)y_c + \delta(y_c^\sigma + \tilde{\tau}) \\ &\geq (1-\delta)(y_c^\sigma - \sigma_c) + \delta y_c^\sigma + \delta\tilde{\tau} \\ &\geq y_c^\sigma + \delta\tilde{\tau} - (1-\delta)\sigma_c\,. \end{aligned}$$

Consequently, u_δ^σ is feasible for $(\mathrm{P}_\varepsilon^\sigma)$ for $\delta \in [\delta^\sigma, 1]$ with $\delta^\sigma := \frac{\sigma_c}{\sigma_c + \tilde{\tau}}$. □

Lemma 4.4. *Assume that* (B) *holds. Then for every $\varepsilon > 0$, there exists a positive constant ρ_ε^σ such that $u_\rho^\sigma := (1-\rho)\bar{u}_\varepsilon^\sigma + \rho\tilde{u}$ is feasible for (P) for every ρ in $[\rho_\varepsilon^\sigma, 1]$.*

Proof. Again, we check only the mixed control-state constraint. Let y^σ_ρ be the solution of (1) for right-hand side u^σ_ρ. Using (B), we find

$$\begin{aligned} y^\sigma_\rho &= (1-\rho)\bar y^\sigma_\varepsilon + \rho\tilde y \\ &\geq (1-\rho)(y^\sigma_c - \varepsilon\bar u^\sigma_\varepsilon) + \rho(y^\sigma_c + \tilde\tau) \\ &\geq y^\sigma_c - (1-\rho)\varepsilon\bar u^\sigma_\varepsilon + \rho\tilde\tau \\ &\geq y_c - \sigma_c - (1-\rho)\varepsilon b + \rho\tilde\tau \\ &\geq y_c - (1-\rho)(\varepsilon b + \sigma_c) + \rho\tau. \end{aligned}$$

Consequently, u^σ_ρ is feasible for (P) for $\rho \in [\rho^\sigma_\varepsilon, 1]$ with $\rho^\sigma_\varepsilon := \frac{\varepsilon b + \sigma_c}{\varepsilon b + \sigma_c + \tau}$. □

Theorem 4.5. *Assume that Assumption* (B) *is satisfied. Then, there exist positive constants* C_1 *and* C_2 *independent of* σ_d *and* ε *with*

$$\nu\|\bar u - \bar u^\sigma_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|\bar y - \bar y^\sigma_\varepsilon\|^2_{L^2(\Omega)} \leq C_1\varepsilon + C_2\sigma_c + \frac{1}{2}\sigma_d^2.$$

Proof. We start with the optimality condition (10). Due to Lemma 4.3, we can test this inequality with u^σ_δ for $\delta \in [\delta^\sigma, 1]$,

$$(\bar p^\sigma_\varepsilon + \nu\bar u^\sigma_\varepsilon, u^\sigma_\delta - \bar u^\sigma_\varepsilon) \geq 0. \tag{11}$$

According to Lemma 4.4, we can test the optimality condition (3) with u^σ_ρ for $\rho \in [\rho^\sigma_\varepsilon, 1]$,

$$(\bar p + \nu\bar u, u^\sigma_\rho - \bar u) \geq 0. \tag{12}$$

Adding (11) and (12), we find

$$(\bar p^\sigma_\varepsilon + \nu\bar u^\sigma_\varepsilon, u^\sigma_\delta - \bar u^\sigma_\varepsilon) + (\bar p + \nu\bar u, u^\sigma_\rho - \bar u) \geq 0.$$

Next we rewrite this inequality as

$$(\bar p^\sigma_\varepsilon + \nu\bar u^\sigma_\varepsilon, u^\sigma_\delta - \bar u) + (\bar p^\sigma_\varepsilon + \nu\bar u^\sigma_\varepsilon, \bar u - \bar u^\sigma_\varepsilon) + (\bar p + \nu\bar u, u^\sigma_\rho - \bar u^\sigma_\varepsilon) + (\bar p + \nu\bar u, \bar u^\sigma_\varepsilon - \bar u) \geq 0.$$

Combining the second and the last term, the inequality can be written in the form

$$(\bar p^\sigma_\varepsilon + \nu\bar u^\sigma_\varepsilon, u^\sigma_\delta - \bar u) + (\bar p + \nu\bar u, u^\sigma_\rho - \bar u^\sigma_\varepsilon) + \nu(\bar u - \bar u^\sigma_\varepsilon, \bar u^\sigma_\varepsilon - \bar u) + (\bar p - \bar p^\sigma_\varepsilon, \bar u^\sigma_\varepsilon - \bar u) \geq 0. \tag{13}$$

We start with the estimation of last term in (13). Since (1) holds true, we have

$$(\bar p - \bar p^\sigma_\varepsilon, \bar u^\sigma_\varepsilon - \bar u) = a(\bar y^\sigma_\varepsilon - \bar y, \bar p - \bar p^\sigma_\varepsilon)\,. \tag{14}$$

On the other hand, the adjoint equation (2) implies

$$a(\bar y^\sigma_\varepsilon - \bar y, \bar p - \bar p^\sigma_\varepsilon) = (\bar y - \bar y^\sigma_\varepsilon, \bar y^\sigma_\varepsilon - \bar y) + (y^\sigma_d - y_d, \bar y^\sigma_\varepsilon - \bar y). \tag{15}$$

From (14) and (15), we conclude

$$(\bar{p} - \bar{p}_\varepsilon^\sigma, u_\varepsilon^\sigma - \bar{u}) = -\|\bar{y}_\varepsilon^\sigma - \bar{y}\|_{L^2(\Omega)}^2 + (y_d - y_d^\sigma, \bar{y} - \bar{y}_\varepsilon^\sigma). \tag{16}$$

Consequently, we can write (13) in the form

$$\begin{aligned} &\nu\|\bar{u} - \bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)}^2 + \|\bar{y} - \bar{y}_\varepsilon^\sigma\|_{L^2(\Omega)}^2 \\ &\quad \le (\nu\bar{u} + \bar{p}, u_\rho^\sigma - \bar{u}_\varepsilon^\sigma) + (\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma, u_\delta^\sigma - \bar{u}) + (y_d - y_d^\sigma, \bar{y} - \bar{y}_\varepsilon^\sigma). \end{aligned} \tag{17}$$

Next, we estimate the three inner products using the special choice $\rho = \rho_\varepsilon^\sigma$. We start with

$$\begin{aligned} (\nu\bar{u} + \bar{p}, u_\rho^\sigma - \bar{u}_\varepsilon^\sigma) &\le \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)} \|u_\rho^\sigma - \bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)} \\ &= \rho\|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)} \|\tilde{u} - \bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)} \\ &\le \frac{\varepsilon b + \sigma_c}{\varepsilon b + \sigma_c + \tau} b|\Omega|^{\frac{1}{2}} \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)} \\ &\le \frac{\varepsilon b + \sigma_c}{\tau} b|\Omega|^{\frac{1}{2}} \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)}. \end{aligned} \tag{18}$$

Estimating the second term, we find

$$\begin{aligned} (\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma, u_\delta^\sigma - \bar{u}) &\le \|\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)} \|u_\delta^\sigma - \bar{u}\|_{L^2(\Omega)} \\ &\le \delta\|\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)} \|\tilde{u} - \bar{u}\|_{L^2(\Omega)} \\ &\le \frac{\sigma_c}{\sigma_c + \tilde{\tau}} b|\Omega|^{\frac{1}{2}} \|\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)} \\ &\le \frac{\sigma_c}{\tilde{\tau}} b|\Omega|^{\frac{1}{2}} \|\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)} \end{aligned} \tag{19}$$

for the setting $\delta = \delta^\sigma$. The remaining third term will be estimated by Young's inequality

$$(y_d - y_d^\sigma, \bar{y} - \bar{y}_\varepsilon^\sigma) \le \frac{1}{2}\sigma_d^2 + \frac{1}{2}\|\bar{y} - \bar{y}_\varepsilon^\sigma\|_{L^2(\Omega)}^2 . \tag{20}$$

Inserting (18),(19), and (20) in (17), we end up with

$$\nu\|\bar{u} - \bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\bar{y}_\varepsilon^\sigma - \bar{y}\|_{L^2(\Omega)}^2 \le C_1\varepsilon + C_2\sigma_c + \frac{1}{2}\sigma_d^2$$

with $C_1 = \frac{b^2}{\tau}|\Omega|^{\frac{1}{2}} \cdot \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)}$, $C_2 = \frac{b}{\tau}|\Omega|^{\frac{1}{2}} \cdot \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)} + \frac{b}{\tilde{\tau}}|\Omega|^{\frac{1}{2}} \cdot \|\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)}$. This completes the proof. □

Remark 4.6. Note that the constants C_1 and C_2 depend on σ_c via τ. However, if we fix a certain $\tilde{\sigma}_c := \sigma_c$ and a corresponding τ, then the statement of Theorem 4.5 becomes independent of σ_c for $\sigma_c \in [0, \tilde{\sigma}_c]$.

Corollary 4.7. *Assume that* (B) *is satisfied. Then there exists a uniform bound* K *such that*

$$\|\nu\bar{u}+\bar{p}\| \le K$$
$$\|\nu u_\varepsilon^\sigma + p_\varepsilon^\sigma\| \le K.$$

Proof. Since $\tilde{u}$ is feasible for (P) and ($\mathrm{P}_\varepsilon^\sigma$), we know

$$J(\bar{y},\bar{u}) \le J(\tilde{y},\tilde{u}) =: J$$
$$J(y_\varepsilon^\sigma, u_\varepsilon^\sigma) \le J(\tilde{y},\tilde{u}) = J.$$

The first inequality means $\frac{1}{2}\|\bar{y}-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|\bar{u}\|^2_{L^2(\Omega)} \le J$ and implies

$$\|\bar{y}-y_d\|_{L^2(\Omega)} \le \sqrt{2J} \tag{21}$$
$$\|\bar{u}\|_{L^2(\Omega)} \le \sqrt{\frac{2J}{\nu}}. \tag{22}$$

Using (21), we obtain $\|\bar{p}\|_{L^2(\Omega)} \le \bar{C}\|\bar{y}-y_d\|_{L^2(\Omega)} \le \bar{C}\sqrt{2J}$. Finally we arrive at

$$\|\nu\bar{u}+\bar{p}\| \le \nu\|\bar{u}\| + \|\bar{p}\| \le \sqrt{2\nu J} + \bar{C}\sqrt{2J}.$$

Proceeding as above for $\|\nu u_\varepsilon^\sigma + p_\varepsilon^\sigma\|_{L^2(\Omega)}$ we receive the inequality

$$\|\nu u_\varepsilon^\sigma + p_\varepsilon^\sigma\|_{L^2(\Omega)} \le \sqrt{2\nu J} + \bar{C}\sqrt{2J}.$$

Hense, we can choose $K := \sqrt{2\nu J} + \bar{C}\sqrt{2J}$. □

5. Feasible regularized solutions

Until now, the solutions of the introduced regularized problems are in general not feasible for (P). In this section, we will discuss two ways to construct feasible solutions. The first way is motivated by the proving technique of the last sections. Here, we constructed feasible solutions for (P) based on the solutions of the regularized problems. We start with the solution of the problem (P_ε).

Lemma 5.1. *The following estimate for the regularization is valid:*

$$\nu\|\bar{u}-u_\delta\|^2_{L^2(\Omega)} + \|\bar{y}-y_\delta\|^2_{L^2(\Omega)} \le c\varepsilon$$

provided that the parameter δ *is chosen as* $\delta = \delta_\varepsilon = \frac{\varepsilon b^2}{\tau+\varepsilon b}$. *Here,* u_δ *is the function introduced in Lemma* 3.1.

Proof. The desired property is obtained by Theorem 3.3 and the triangle inequality. Moreover, we conclude from (9)

$$\|u_\delta - \bar{u}_\varepsilon\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}} \frac{\varepsilon b^2}{\tau + \varepsilon b} \leq c\varepsilon.$$

A similar estimate is valid for the state. □

Next, we present a similar result for ($\mathrm{P}_\varepsilon^\sigma$)

Lemma 5.2. *The following estimate for the regularization is valid:*

$$\nu\|\bar{u} - u_\rho^\sigma\|^2_{L^2(\Omega)} + \|\bar{y} - y_\rho^\sigma\|^2_{L^2(\Omega)} \leq c_1\varepsilon + c_2\sigma_c + \frac{1}{2}\sigma_d^2$$

provided that the parameters ρ *is chosen as* $\rho = \rho_\varepsilon^\sigma = \frac{\varepsilon b + \sigma_c}{\varepsilon b + \sigma_c + \tau}$. *Here,* u_δ *is the function introduced in Lemma* 3.1.

This lemma can be shown with similar arguments like for Lemma 5.1. Let us remark that an alternative error estimation technique is available for the regularized Problem (P_ε):

Corollary 5.3. *Since* (y_δ, u_δ) *is feasible for* (P) *and* $(\bar{y}, \bar{u})$ *is feasible for* (P_ε) *for every* $\varepsilon > 0$ *the following inequality is satisfied:*

$$J(\bar{y}_\varepsilon, \bar{u}_\varepsilon) \leq J(\bar{y}, \bar{u}) \leq J(y_\delta, u_\delta).$$

This relation can also be used to obtain the results of Theorem 3.3 *and Lemma* 5.1.

Let us discuss a second approach to construct feasible regularized solutions for (P). We replace (P_ε) by (P'_ε):

$$(\mathrm{P}'_\varepsilon)\begin{cases} \min J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} & \\ \text{subject to} \qquad (Ay)(x) = u(x) & \text{in } \Omega \\ \qquad y(x) = 0 & \text{on } \partial\Omega \\ \qquad 0 \leq u(x) \leq b & \text{a.e. in } \Omega \\ \qquad y_c(x) + \varepsilon u(x) \leq y(x) & \text{a.e. in } \Omega', \end{cases}$$

where only the sign in the mixed constrained has changed.

Now the situation with respect to the feasibility changes: Feasible controls for (P'_ε) are automatically feasible for (P). A drawback of this approach is that the existence of a feasible point for (P'_ε) can only be guaranteed for sufficiently small ε. More precisely, if Assumption (A) is satisfied, then the existence of a feasible control can be shown for $\varepsilon \leq \frac{\tau}{\|\hat{u}\|_{L^\infty(\Omega)}}$.

Theorem 5.4. *Assume that* (A) *is satisfied. Then the estimate*

$$\nu\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Omega)}+\|\bar{y}-\bar{y}_\varepsilon\|^2_{L^2(\Omega)}\le \varepsilon C$$

is valid with a positive constant C independent of ε for $\varepsilon \le \frac{\tau}{\|\hat{u}\|_{L^\infty(\Omega)}}$.

The result can be obtained along the lines of Theorem 3.3. We have to change the problem once more if we want to deal with noisy data:

$$(\mathrm{P}_\varepsilon^{\sigma'})\begin{cases}\min J(y,u):=\dfrac{1}{2}\|y-y_d^\sigma\|^2_{L^2(\Omega)}+\dfrac{\nu}{2}\|u\|^2_{L^2(\Omega)} & \\ \text{subject to} \qquad (Ay)(x)=u(x) & \text{in } \Omega\\ y(x)=0 & \text{on } \partial\Omega\\ 0\le u(x)\le b & \text{a.e. in } \Omega\\ y_c^\sigma(x)+\varepsilon u(x)+\sigma_c\le y(x) & \text{a.e. in } \Omega'\\ \|y_d-y_d^\sigma\|_{L^2(\Omega)}\le\sigma_d & \\ \|y_c-y_c^\sigma\|_{L^\infty(\Omega')}\le\sigma_c. & \end{cases}$$

The additional addend σ_c ensures the feasibility of the solution of $(\mathrm{P}_\varepsilon^{\sigma'})$ for (P). The existence of feasible solution for $(\mathrm{P}_\varepsilon^{\sigma'})$ can be ensured if Assumption (B) is satisfied and if ε is sufficiently small, i.e., $\varepsilon \le \frac{\tau}{\|\tilde{u}\|_{L^\infty(\Omega)}}$ with the quantity τ defined in Assumption (B).

Theorem 5.5. *Assume that* (B) *is satisfied. Then the estimate*

$$\nu\|\bar{u}-\bar{u}_\varepsilon^{\sigma'}\|^2_{L^2(\Omega)}+\|\bar{y}-\bar{y}_\varepsilon^{\sigma'}\|^2_{L^2(\Omega)}\le c_1\varepsilon+c_2\sigma_c+\frac{1}{2}\sigma_d^2$$

is valid for $\varepsilon \le \frac{\tau}{\|\tilde{u}\|_{L^\infty(\Omega)}}$.

This result can be obtained along the lines of the proof of Theorem 4.5.

6. Numerical tests

We slightly modify the problem for the numerical tests:

$$J(y,u)=\frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)}+\frac{\nu}{2}\|u-u_d\|^2_{L^2(\Omega)}$$

and

$$\begin{aligned}(-\Delta y)(x)&=u(x)+f(x) && \text{in } \Omega,\\ y(x)&=0 && \text{on } \partial\Omega,\end{aligned}$$

that means we introduce functions f and u_d. This allows us to construct an example where the exact solution is known for the unregularized problem.

For this purpose we need a different formulation of the optimality system. This formulation was introduced by Casas [2] and contains a Borel measure μ as additional source term of an adjoint partial differential equation:

$$\begin{aligned} -\Delta\hat{p} &= \bar{y} - \bar{y}_d - \mu \quad \text{in } \Omega \\ \hat{p} &= 0 \qquad\qquad\quad\;\; \text{on } \partial\Omega\,. \end{aligned}$$

The optimality condition reads now

$$\bar{u}(x) = \Pi_{[a,b]}\left(-\frac{1}{\nu}\hat{p}(x) + u_d\right),$$

where the pointwise projection $\Pi_{[a,b]}$ is defined by

$$\Pi_{[a,b]}f(x) := \max\big(a, \min(b, f(x))\big).$$

Moreover the following condition for the multiplier have to be fullfilled

$$\int_{\Omega'} (y - y_c)\, d\mu = 0, \quad \mu \geq 0.$$

The following example is constructed in such a way that these necessary and sufficient optimality conditions are satisfied. Next, we present a complete list of functions appearing in our example. It is easy to show that these data satisfy the optimality system.

The optimal state is given by

$$\bar{y}(x) = \sin \pi x_1 \sin \pi x_2.$$

The example depends on a parameter $c \in (0,1)$. In our tests we always set $c = 0.6$. The function y_c is given by

$$y_c(x) = \begin{cases} \bar{y}(x) & \text{if} \quad \bar{y}(x) \geq c \\ 2\bar{y}(x) - c & \text{if} \quad \bar{y}(x) < c. \end{cases}$$

The desired state y_d is a discontinuous function

$$y_d(x) = \begin{cases} \bar{y}(x) - 1 & \text{if} \quad \bar{y}(x) \geq c \\ (4\nu\pi^4 + 1)\bar{y}(x) & \text{if} \quad \bar{y}(x) < c. \end{cases}$$

The adjoint state $\hat{p}$ is a function with a kink:

$$\hat{p}(x) = \begin{cases} -2\pi^2\nu c & \text{if} \quad \bar{y}(x) \geq c \\ -2\pi^2\nu\bar{y}(x) & \text{if} \quad \bar{y}(x) < c. \end{cases}$$

Consequently, the Lagrange multiplier contains a line measure concentrated on the curve $\bar{y}(x) = c$. The function part of the Lagrange multiplier is given by

$$\mu(x) = \begin{cases} 1 & \text{if} \quad \bar{y}(x) \geq c \\ 0 & \text{if} \quad \bar{y}(x) < c. \end{cases}$$

For convenience, we define a function v by

$$v(x) = 2\pi^2 \sin \pi x_1 \sin \pi x_2 - \kappa$$

with a positive κ (in the computations we used $\kappa = 5$). Next we set

$$f(x) = \begin{cases} v + \kappa & \text{if} \quad v(x) \leq 0 \\ \kappa & \text{if} \quad v(x) \in (0, b) \\ v + \kappa - b & \text{if} \quad v(x) \geq b \end{cases}$$

and

$$\bar{u}(x) = \begin{cases} 0 & \text{if} \quad v(x) \leq 0 \\ v & \text{if} \quad v(x) \in (0, b) \\ b & \text{if} \quad v(x) \geq b. \end{cases}$$

Moreover, we have the function

$$u_d(x) = \begin{cases} -2\pi^2 c + v(x) & \text{if} \quad \bar{y}(x) \geq c \\ -\kappa & \text{if} \quad \bar{y}(x) < c. \end{cases}$$

The constant $b = 100$ is chosen in a way that the upper bound is not active.

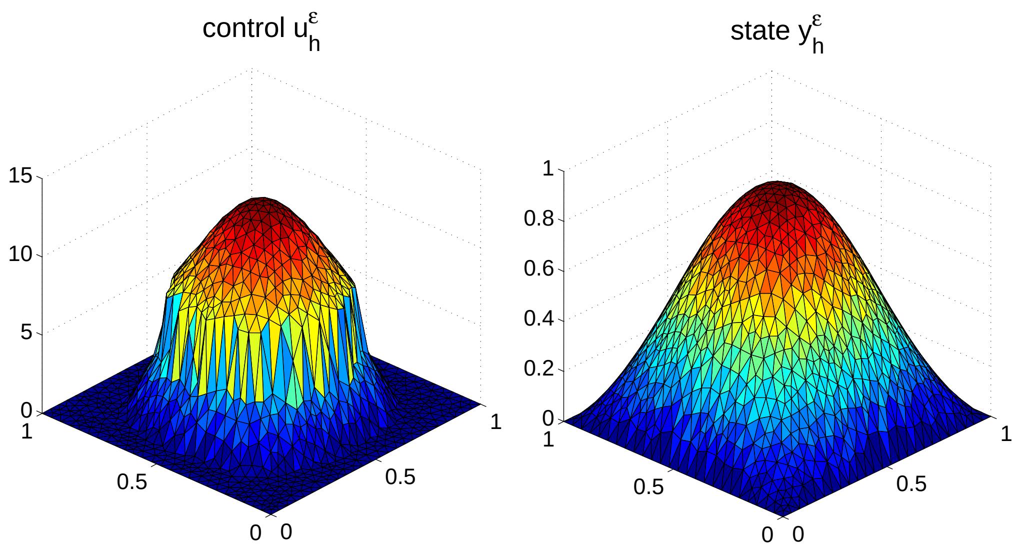

Figure 1: Optimal numerical solution for $\varepsilon = 0.005$

We used a uniform finite element mesh with triangles. Moreover, we discretized the control, state and adjoint state by piecewise linear functions. The mesh size was $h = 0.04$. Furthermore, we used $\nu = 0.1$. The regularization parameter ε varies in an interval where the regularization error is larger than the discretization error. Figure 1 shows the computed optimal control and the optimal state for $\varepsilon = 0.005$. The computed error behavior is presented in Figure 2 and Table 1.

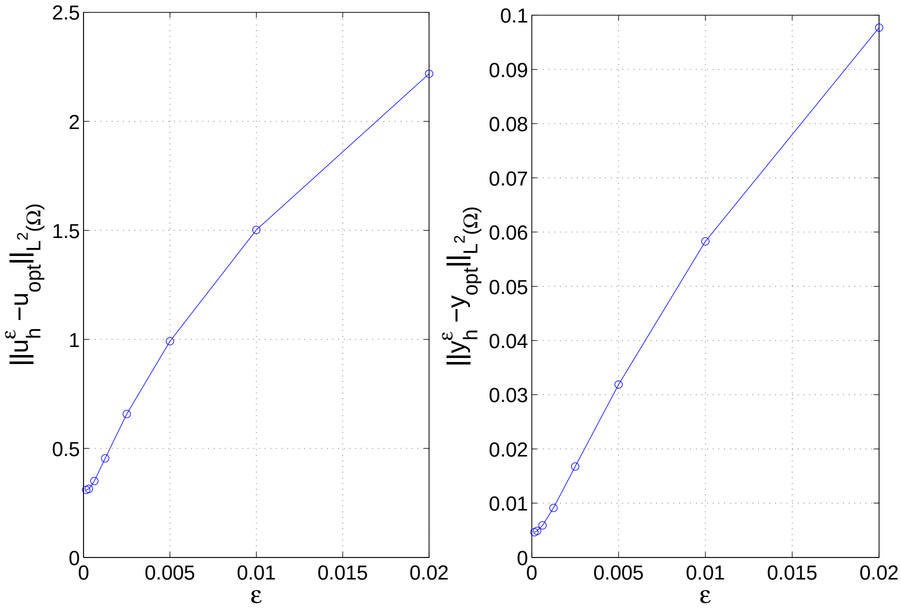

Figure 2: Error behaviour with respect to ε

ε	$\|\bar{u} - u_h^\varepsilon\|$	$\frac{\|\bar{u} - u_h^\varepsilon\|}{\sqrt{\varepsilon}}$	$\|\bar{y} - y_h^\varepsilon\|$	$\frac{\|\bar{y} - y_h^\varepsilon\|}{\sqrt{\varepsilon}}$
$2^1 \cdot 10^{-2}$	$2.2180e + 0$	15.684	$9.7706e - 2$	0.69089
$2^0 \cdot 10^{-2}$	$1.5018e + 0$	15.018	$5.8289e - 2$	0.58289
$2^{-1} \cdot 10^{-2}$	$9.9099e - 1$	14.015	$3.1875e - 2$	0.45079
$2^{-2} \cdot 10^{-2}$	$6.5758e - 1$	13.152	$1.6745e - 2$	0.33490
$2^{-3} \cdot 10^{-2}$	$4.5411e - 1$	12.844	$9.1220e - 3$	0.25801
$2^{-4} \cdot 10^{-2}$	$3.5025e - 1$	14.010	$5.9091e - 3$	0.23636
$2^{-5} \cdot 10^{-2}$	$3.1421e - 1$	17.774	$4.8770e - 3$	0.27588

Table 1: ε-dependency ($\|\cdot\|$ means $\|\cdot\|_{L^2(\Omega)}$)

Next, we perturb the function y_c. We add some random pertubations to y_c such that $\|y_c^\sigma - y_c\|_{L^\infty(\Omega)} \le \sigma_c$ is valid. We set the regularization parameter $\varepsilon = 2^{-8} \cdot 10^{-2}$ in order to decrease the regularization error generated by ε. Figure 3 and Table 2 illustrate the expected behavior of the numerical solutions with respect to σ_c.

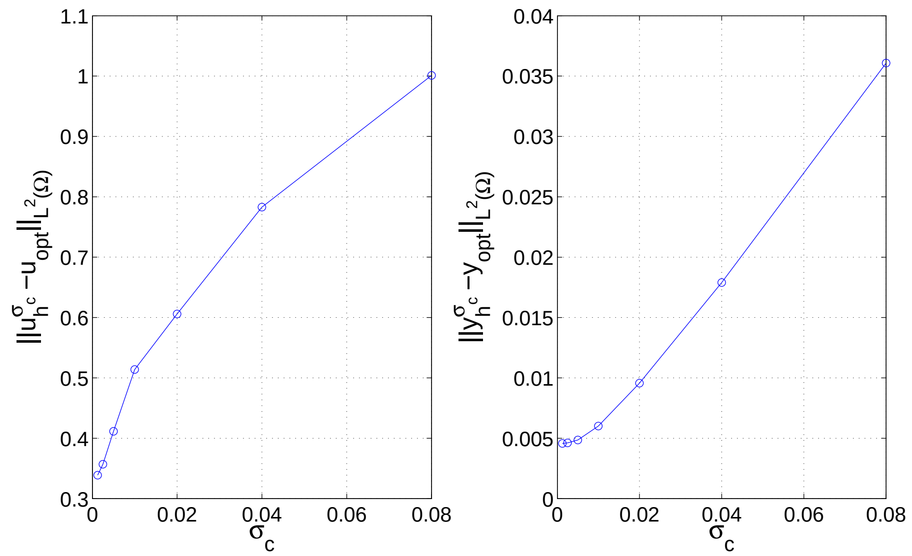

Figure 3: Error behavior with respect to σ_c

σ_c	$\|\bar{u} - u_h^{\varepsilon,\sigma_c}\|$	$\dfrac{\|\bar{u} - u_h^{\varepsilon,\sigma_c}\|}{\sqrt{\sigma_c}}$	$\|\bar{y} - y_h^{\varepsilon,\sigma_c}\|$	$\dfrac{\|\bar{y} - y_h^{\varepsilon,\sigma_c}\|}{\sqrt{\sigma_c}}$
$2^3 \cdot 10^{-2}$	$1.0011e+0$	3.5394	$3.6082e-2$	0.1276
$2^2 \cdot 10^{-2}$	$7.8281e-1$	3.9140	$1.7899e-2$	0.0895
$2^1 \cdot 10^{-2}$	$6.0590e-1$	4.2844	$9.5663e-3$	0.0676
$2^0 \cdot 10^{-2}$	$5.1377e-1$	5.1377	$6.0098e-3$	0.0601
$2^{-1} \cdot 10^{-2}$	$4.1156e-1$	5.8204	$4.8575e-3$	0.0687
$2^{-2} \cdot 10^{-2}$	$3.5701e-1$	7.1401	$4.6160e-3$	0.0923

Table 2: σ_c-dependency ($\|\cdot\|$ means $\|\cdot\|_{L^2(\Omega)}$)

Let us summarize our numerical experiences. The computational results show the theoretical expected behavior. This concerns the dependence on ε as well as the dependence on σ_c. The tests deliver numerical results in a range where the discretization error is small with respect to the investigated influences.

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