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Implicit Difference Methods for Quasilinear Differential Functional Equations on the Haar Pyramid

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Abstract. We present a new class of numerical methods for quasilinear first order partial functional differential equations. The numerical methods are difference schemes which are implicit with respect to the time variable. The existence of approximate solutions is proved by using a theorem on difference inequalities. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type for given operators. Numerical experiments are presented. Results obtained in this paper can be applied to differential integral problems and to equations with deviated variables.

Keywords. Initial problems on the Haar pyramid, implicit difference methods, stability and convergence, interpolating operators

Mathematics Subject Classification (2000). 65M10, 65M15, 35R10

1. Introduction

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X to Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let E be the Haar pyramid

$$E = \{(t,x) = (t,x_1,\dots,x_n) \in \mathbb{R}^{1+n} : t \in [0,a], x \in [-b+Mt, b-Mt]\},$$

where $a > 0$, $b = (b_1,\dots,b_n)$, $M = (M_1,\dots,M_n) \in \mathbb{R}^n_+$, $\mathbb{R}_+ = [0,+\infty)$, and $b > Ma$. Write

$$E_0 = [-d_0, 0] \times [-d, d], \quad E_t = (E_0 \cup E) \cap ([-d_0, t] \times \mathbb{R}^n), \quad 0 < t \leqslant a,$$

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where $d_0 \in \mathbb{R}_+$. Put $\Omega = E \times C(E_0 \cup E, \mathbb{R})$ and suppose that $f : \Omega \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$, $g : \Omega \to \mathbb{R}$, $\varphi : E_0 \to \mathbb{R}$ are given functions. We consider the differential functional equation

$$\partial_t z(t,x) = \sum_{i=1}^{n} f_i(t,x,z)\partial_{x_i} z(t,x) + g(t,x,z), \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{for } (t,x) \in E_0. \tag{2}$$

A function $v : E_0 \cup E \to \mathbb{R}$ is a *classical solution* of above problem if

(i) $v \in C(E_0 \cup E, \mathbb{R})$ and v is of class C^1 on E;

(ii) v satisfies (1) on E and initial condition (2) holds.

The function $f : \Omega \to \mathbb{R}^n$ is said to satisfy the Volterra condition if for each $(t,x) \in E$ and for $z, \bar{z} \in C(E_0 \cup E, \mathbb{R})$ such that $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E_t$ we have $f(t,x,z) = f(t,x,\bar{z})$. Note that the Volterra condition for f means that the value of f at the point (t,x,z) of the space Ω depends on (t,x) and on the restriction of z to the set E_t. In the same way we define the Volterra condition for the function g. In the present paper we assume that f and g satisfy the Volterra condition and we consider classical solutions of (1), (2).

The Haar pyramid is a natural set for the existence and uniqueness of initial problems for differential and functional differential equations (see [8,12,15,16]). We are interested in the construction of a method for the approximation of solutions to problem (1), (2) with solutions of difference functional equations and in an estimation of the difference between these solutions.

In recent years, a number of papers concerning difference methods for functional differential equations have been published. It is easy to construct an explicit difference method for a nonlinear differential equation which satisfies the consistency conditions on all sufficiently regular solutions of a considered problem. The main task in these investigations is to find a finite difference equation which is stable. The method of difference inequalities or simple theorems on recurrent inequalities are used in the investigation of the stability of difference functional problems.

Convergence results are also based on a general theorem on an error estimate of approximate solutions to functional difference equations of the Volterra type with initial or initial boundary conditions and with an unknown function of several variables.

The problems mentioned above have an extensive bibliography. It is not our aim to show a full review of papers concerning numerical methods for partial functional differential equations. We shall mention only those which contain such review; they are [2,3,6,7,13,14] and the monograph [8]. The papers [9–11] initiated the investigations of implicit difference methods for first order partial differential equations.

The numerical method of lines for nonlinear functional differential equations was considered in [1]. By using a discretization in spatial variables, the nonlinear equations are replaced by sequences of initial problems for ordinary functional differential equations. The question of under what conditions the solutions of ordinary equations tend to solutions of original problems is investigated. The proofs of the convergence of the numerical methods of lines are based on differential inequalities technique.

In the paper we present another approach to the difference methods for problem (1), (2). We prove that there is a class of implicit difference methods for (1), (2) which are convergent. The stability of difference schemes is investigated by using a comparison method with nonlinear estimates of the Perron type for given functions.

The paper is organized as follows. In Section 2 we propose an implicit difference method of the Euler type for problem (1), (2). This leads to an implicit difference functional problem. The existence and uniqueness of solutions of such problems are considered in Section 3. The method of difference inequalities is used. The next section deals with a theorem on the error estimate for approximate solutions of implicit difference functional equations. A convergence result and an error estimate are presented in Section 4. In the next section we discuss the problem of the existence of solutions of implicit difference schemes on rectangular domain. More precisely, we show that there is a class of initial problems (1), (2) for which approximate solutions can be calculated easily. Numerical examples are given in the last part of the paper.

Note that differential equations with deviated variables and differential integral equations can be obtained from (1) by specializing the operators f and g. Existence result for problem (1), (2) are given in [8, Chapter 2].

2. Discretization of quasilinear equations

We will denote by $F(X,Y)$ the class of all functions defined on X and taking values in Y where X and Y are arbitrary sets. For $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ we write $\|x\| = \sum_{i=1}^{n} |x_i|$ and $x \diamond y = (x_1 y_1, \ldots, x_n y_n)$. For a function $z \in C(E_0 \cup E, \mathbb{R})$ and for a point $t \in [0,a]$ we put

$$\|z\|_t = \max\{|z(\tau,x)| : (\tau,x) \in E_t\}.$$

We define a mesh on the set $E_0 \cup E$ in the following way. Let (h_0, h'), $h' = (h_1, \ldots, h_n)$, stand for steps of the mesh. Let us denote by H the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbb{N}$ and $K = (K_1, \ldots, K_n) \in \mathbb{N}^n$ with the properties $K_0 h_0 = d_0$ and $K \diamond h' = b$. For $h \in H$ and for $(r, m) \in \mathbb{Z}^{1+n}$, where $m = (m_1, \ldots, m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \quad x^{(m)} = m \diamond h', \quad x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}).$$

Let $N_0 \in \mathbb{N}$ be defined by the relations $N_0 h_0 \leqslant a < (N_0 + 1)h_0$. Write

$$R_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbb{Z}^{1+n}\}$$

and

$$E_h = E \cap R_h^{1+n}, \quad E_{h.0} = E_0 \cap R_h^{1+n},$$
$$E_{h.r} = (E_{h.0} \cup E_h) \cap ([-d_0, t^{(r)}] \times \mathbb{R}^n), \quad 0 \leqslant r \leqslant N_0.$$

Moreover we put

$$E'_h = \{(t^{(r)}, x^{(m)}) \in E_h : (t^{(r+1)}, x^{(m)}) \in E_h\},$$
$$I_h = \{t^{(r)} : 0 \leqslant r \leqslant N_0\}, \quad I'_h = I_h \setminus \{t^{(N_0)}\}.$$

For a function $z : E_{h.0} \cup E_h \to \mathbb{R}$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ and

$$\|z\|_{h.i} = \max\{|z^{(r,m)}| : (t^{(r)}, x^{(m)}) \in E_{h.i}\},$$

where $0 \leqslant i \leqslant N_0$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$, $1 \leqslant i \leqslant n$, where 1 is the i-th coordinate.

Classical difference methods for (1), (2) consist in replacing partial derivatives ∂_t and $(\partial_{x_1}, \ldots, \partial_{x_n}) = \partial_x$ with difference operators δ_0 and $(\delta_1, \ldots, \delta_n) = \delta$, respectively. Moreover, because equation (1) contains the functional variable we need an interpolating operator $T_h : F(E_{h.0} \cup E_h, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$. This leads to the difference equation

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_i(t^{(r)}, x^{(m)}, T_h[z]) \delta_i z^{(r,m)} + g(t^{(r)}, x^{(m)}, T_h[z]) \tag{3}$$

with the initial condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{h.0}\,, \tag{4}$$

where $\varphi_h : E_{h.0} \to \mathbb{R}$ is a given function. The following examples of equations (3) are considered in the literature. Write

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\Big[z^{(r+1,m)} - z^{(r,m)}\Big] \tag{5}$$

and

$$\delta_j z^{(r,m)} = \frac{1}{h_j}\Big[z^{(r,m+e_j)} - z^{(r,m)}\Big] \quad \text{for } 1 \leqslant j \leqslant \kappa$$
$$\delta_j z^{(r,m)} = \frac{1}{h_j}\Big[z^{(r,m)} - z^{(r,m-e_j)}\Big] \quad \text{for } \kappa + 1 \leqslant j \leqslant n$$

where $0 \leqslant \kappa \leqslant n$ is fixed. The numerical method (3), (4) with the above given δ_0 and δ is known as the Euler method. The Lax difference scheme is the second important example. It is obtained by putting

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\Big[z^{(r+1,m)} - \frac{1}{2n}\sum_{i=1}^{n}\Big(z^{(r,m+e_i)} + z^{(r,m-e_i)}\Big)\Big]$$
$$\delta_j z^{(r,m)} = \frac{1}{2h_j}\Big[z^{(r,m+e_j)} - z^{(r,m-e_j)}\Big], \quad 1 \leqslant j \leqslant n.$$

The stability of difference equations generated by first order partial functional differential equations is strictly connected with the so-called Courant-Friedrichs-Levy (CFL) conditions, see [4]. The (CFL) condition for equation (1) and for the Euler difference method has the form

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j}|f_j(t,x,z)| \geqslant 0 \quad \text{on } \Omega_h. \tag{6}$$

The (CFL) condition for the Lax scheme has the form

$$1 - n\,\frac{h_0}{h_j}\,|f_j(t,x,z)| \geqslant 0, \quad 1 \leqslant j \leqslant n, \quad \text{on } \Omega_h. \tag{7}$$

Regularity assumptions in stability theorems for f and g with respect to the functional variable are the same for the both above difference methods. It is assumed that f and g satisfy the Lipschitz condition with respect to the functional variable. Nonlinear estimates of the Perron type for f and g with respect to the functional variable can be also adopted.

Suppose that the functions f and g are bounded on Ω. The (CFL) conditions (6) and (7) state that there are requirements for the steps h_0 and $h' = (h_1, \ldots, h_n)$ in convergence theorems for (3), (4).

In the paper we present a new approach to the numerical solving of (1), (2). We prove that under natural assumptions on given functions and on the mesh there is a class of implicit difference schemes for (1), (2) which is convergent. The aim of the paper is to show that there are difference methods for (1), (2) for which the (CFL) conditions can be omitted.

We formulate the implicit difference method of the Euler type for (1), (2). Put $\Omega_h = E'_h \times F(E_{h.0} \cup E_h, \mathbb{R})$. Suppose that functions $f_h : \Omega_h \to \mathbb{R}^n$, $f_h = (f_{h.1}, \ldots, f_{h.n})$, $g_h : \Omega_h \to \mathbb{R}$, $\varphi_h : E_{h.0} \to \mathbb{R}$ are given. We will approximate classical solutions of problem (1), (2) by means of solutions of the implicit difference problem

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_{h.i}(t^{(r)}, x^{(m)}, z)\delta_i z^{(r+1,m)} + g_h(t^{(r)}, x^{(m)}, z) \tag{8}$$
$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{h.0}. \tag{9}$$

It is important in our considerations that the operator $\delta z = (\delta_1, \ldots, \delta_n)$ appears in (8) at the point $(t^{(r+1)}, x^{(m)})$. The function f_h is said to satisfy the Volterra condition if for each $(t^{(r)}, x^{(m)}) \in E'_h$ and for $z, \bar{z} \in F(E_{h.0} \cup E_h, \mathbb{R})$ such that $z|_{E_{h.r}} = \bar{z}|_{E_{h.r}}$ we have $f(t^{(r)}, x^{(m)}, z) = f(t^{(r)}, x^{(m)}, \bar{z})$. In the same way we formulate the Volterra condition for g_h.

The difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ are defined in the following way:

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\Big(z^{(r+1,m)} - z^{(r,m)}\Big) \tag{10}$$

$$\delta_i z^{(r+1,m)} = \frac{1}{h_i}\Big(z^{(r+1,m+e_i)} - z^{(r+1,m)}\Big) \quad \text{if } f_{h.i}(t^{(r)}, x^{(m)}, z^{(r,m)}) \geqslant 0 \tag{11}$$

$$\delta_i z^{(r+1,m)} = \frac{1}{h_i}\Big(z^{(r+1,m)} - z^{(r+1,m-e_i)}\Big) \quad \text{if } f_{h.i}(t^{(r)}, x^{(m)}, z^{(r,m)}) < 0. \tag{12}$$

The corresponding explicit difference scheme has the form

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_{h.i}(t^{(r)}, x^{(m)}, z)\delta_i z^{(r,m)} + g_h(t^{(r)}, x^{(m)}, z). \tag{13}$$

If f_h and g_h satisfy the Volterra condition, then it is clear that there exists exactly one solution of equation (13) with initial condition (9).

We prove that under natural assumptions on the functions f_h and g_h there exists exactly one solution $u_h : E_{h.0} \cup E_h \to \mathbb{R}$ of problem (8), (9) with difference operators defined by (10)–(12). Let

$$\Delta^{(r)} = \{x^{(m)} : x^{(m)} \in [-b + Mt^{(r)}, b - Mt^{(r)}]\}, \quad 0 \leqslant r \leqslant N_0.$$

Write

$$E^{+}_{h.i}[\varepsilon] = \{(t^{(r)}, x^{(m)}) \in E'_h : b_i - M_i t^{(r)} - \varepsilon \leqslant x_i^{(m_i)} \leqslant b_i - M_i t^{(r)}\}$$
$$E^{-}_{h.i}[\varepsilon] = \{(t^{(r)}, x^{(m)}) \in E'_h : -b_i + M_i t^{(r)} \leqslant x_i^{(m_i)} \leqslant -b_i + M_i t^{(r)} + \varepsilon\},$$

where $1 \leqslant i \leqslant n$ and $\varepsilon > 0$ is such that $\varepsilon < b_i - M_i a$ for $1 \leqslant i \leqslant n$.

Assumption $H[f_h]$**.** Suppose that the function $f_h : \Omega_h \to \mathbb{R}^n$ satisfies the Volterra condition and

1) $h' \leqslant M h_0$,
2) there exists $\varepsilon > 0$ such that

$$f_{h.i}(t^{(r)}, x^{(m)}, z) \leqslant 0 \quad \text{for } (t^{(r)}, x^{(m)}, z) \in E^{+}_{h.i}[\varepsilon] \times F(E_{h.0} \cup E_h, \mathbb{R})$$
$$f_{h.i}(t^{(r)}, x^{(m)}, z) \geqslant 0 \quad \text{for } (t^{(r)}, x^{(m)}, z) \in E^{-}_{h.i}[\varepsilon] \times F(E_{h.0} \cup E_h, \mathbb{R}).$$

Remark 2.1. Comments on the condition 2) of Assumption $H[f_h]$ are given in Section 4 for f_h defined by

$$f_h(t,x,z) = f(t,x,T_h[z]), \quad (t,x,z) \in \Omega_h\,,$$

where $T_h : F(E_{h.0} \cup E_h, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$ is an interpolating operator.

We will need the following lemma on difference inequalities generated by problem (8), (9).

Lemma 2.2. *Suppose that Assumption $H[f_h]$ is satisfied and $h \in H$, $f_h : \Omega_h \to \mathbb{R}^n$, $z_h : E_{h.0} \cup E_h \to \mathbb{R}$.*

(I) *If z_h satisfies the implicit difference inequality*

$$z_h^{(r+1,m)} \leqslant h_0 \sum_{i=1}^{n} f_{h.i}(t^{(r)}, x^{(m)}, z_h)\delta_i z_h^{(r+1,m)} \quad \text{on } E'_h \tag{14}$$

and $z_h^{(r,m)} \leqslant 0$ on $E_{h.0}$, then $z^{(r+1,m)} \leqslant 0$ on E_h.

(II) *If z_h satisfies the implicit difference inequality*

$$z_h^{(r+1,m)} \geqslant h_0 \sum_{i=1}^{n} f_{h.i}(t^{(r)}, x^{(m)}, z_h)\delta_i z_h^{(r+1,m)} \quad \text{on } E'_h \tag{15}$$

and $z_h^{(r,m)} \geqslant 0$ on $E_{h.0}$, then $z^{(r+1,m)} \geqslant 0$ on E_h.

Proof. Consider the case (I). Suppose that $0 \leqslant r \leqslant K-1$ is fixed and $z_h^{(j,m)} \leqslant 0$ on $E_{h.r}$ and there exists $x^{(\widetilde{m})} \in \Delta^{(r+1)}$ such that

$$z_h^{(r+1,\widetilde{m})} = \max\{z_h^{(r+1,m)} : -b + Mt^{(r+1)} \leqslant x^{(m)} \leqslant b - Mt^{(r+1)}\}$$

and

$$z_h^{(r+1,\widetilde{m})} > 0. \tag{16}$$

Write

$$J_+^{(r,m)}[z_h] = \{i : 1 \leqslant i \leqslant n \text{ and } f_{h.i}(t^{(r)}, x^{(m)}, z_h) \geqslant 0\} \tag{17}$$

$$J_-^{(r,m)}[z_h] = \{1, \ldots, n\} \setminus J_+^{(r,m)}[z_h]. \tag{18}$$

It follows from (14) that

$$\begin{aligned} z_h^{(r+1,\widetilde{m})} \leqslant h_0 &\sum_{i \in J_+^{(r,\widetilde{m})}[z_h]} \frac{1}{h_i} f_{h.i}(t^{(r)}, x^{(\widetilde{m})}, z_h)\Big[z_h^{(r+1,\widetilde{m}+e_i)} - z_h^{(r+1,\widetilde{m})}\Big] \\ &+ h_0 \sum_{i \in J_-^{(r,\widetilde{m})}[z_h]} \frac{1}{h_i} f_{h.i}(t^{(r)}, x^{(\widetilde{m})}, z_h)\Big[z_h^{(r+1,\widetilde{m})} - z_h^{(r+1,\widetilde{m}-e_i)}\Big] \leqslant 0. \end{aligned}$$

We thus get $z_h^{(r+1,\widetilde{m})} \leqslant 0$ which contradicts (16). Then assertion (14) follows by induction with respect to r. In a similar way we prove that (15) holds in the case (II). This completes the proof. □

Lemma 2.3. *Suppose that Assumption $H[f_h]$ is satisfied and $h \in H$, $f_h : \Omega_h \to \mathbb{R}^n$, $g_h : \Omega_h \to \mathbb{R}$. Then difference functional problem* (8), (9) *with δ_0 and δ defined by* (10)–(12) *has exactly one solution $z_h : E_{h.0} \cup E_h \to \mathbb{R}$.*

Proof. Suppose that $0 \leqslant r \leqslant N_0 - 1$ is fixed and z_h is known on the set $E_{h.r}$. Let $(t^{(r)}, x^{(m)}) \in E'_h$ and $f_{h.i}(t^{(r)}, x^{(m)}, z^{(r,m)}) \geqslant 0$. Then

$$\delta_i z^{(r+1,m)} = \frac{1}{h_i}\Big[z^{(r+1,m+e_i)} - z^{(r+1,m)}\Big].$$

It follows from Assumption $H[f_h]$ that $x^{(m)} \in \Delta^{(r+1)}$ and the difference expression $\delta_i z^{(r+1,m)}$ is well defined. The same conclusion can be drawn in case when $f_{h.i}(t^{(r)}, x^{(m)}, z) < 0$.

The homogeneous problem corresponding to (8), (9) has the form

$$z^{(r+1,m)} = h_0 \sum_{i=1}^{n} f_{h.i}(t^{(r)}, x^{(m)}, z^{(r,m)})\delta_i z^{(r+1,m)} \tag{19}$$

$$z^{(r+1,m)} = 0 \quad \text{on } E_{h.0}. \tag{20}$$

It follows from Lemma 2.2 that the initial problem (19), (20) has exactly one zero solution. Therefore the problem (8), (9) has exactly one solution $z_h^{(r+1,m)}$, $x^{(m)} \in \Delta^{(r+1)}$ with any choice of the function $g_h : \Omega_h \to \mathbb{R}$. Consequently the function z_h is defined and it is unique on $E_{h.r+1}$. Since z_h is given on $E_{h.0}$ the proof is completed by induction. □

3. Approximate solutions of functional difference equations

We start with a theorem on the error estimate of approximate solutions for implicit difference functional equations. Let us denote by F_h the Niemycki operator corresponding to (8), i.e.,

$$F_h[z]^{(r,m)} = \sum_{i=1}^{n} f_{h.i}(t^{(r)}, x^{(m)}, z)\delta_i z^{(r+1,m)} + g_h(t^{(r)}, x^{(m)}, z).$$

Then we consider the difference functional equation

$$\delta_0 z^{(r,m)} = F_h[z]^{(r,m)} \tag{21}$$

with initial boundary condition (9). Suppose that $v_h : E_{h.0} \cup E_h \to \mathbb{R}$ and $\gamma, \alpha_0 : H \to \mathbb{R}_+$ are such functions that

$$|\delta_0 v_h^{(r,m)} - F_h[v_h]^{(r,m)}| \leqslant \gamma(h) \quad \text{on } E'_h \tag{22}$$

$$|\varphi_h^{(r,m)} - v_h^{(r,m)}| \leqslant \alpha_0(h) \quad \text{on } E_{h.0} \tag{23}$$

and

$$\lim_{h\to 0} \gamma(h) = 0, \quad \lim_{h\to 0} \alpha_0(h) = 0. \tag{24}$$

The function v_h satisfying the above relations can be considered as an approximate solution of problem (9), (21).

Assumption $H[\sigma_h]$**.** The function $\sigma_h : I'_h \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

1) $\sigma_h(t, \cdot)$ is continuous and nondecreasing for each $t \in I'_h$,
2) $\sigma_h(t, 0) = 0$ for $t \in I'_h$, and for each $\bar{c} \geqslant 1$ the difference problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \bar{c} \sigma_h(t^{(r)}, \eta^{(r)}), \quad 0 < r \leqslant N_0 - 1 \tag{25}$$

$$\eta^{(0)} = 0 \tag{26}$$

is stable in the following sense: if $\gamma, \alpha_0 : H \to \mathbb{R}_+$ are such functions that $\lim_{h\to 0} \gamma(h) = 0$, $\lim_{h\to 0} \alpha_0(h) = 0$ and $\eta_h : I_h \to \mathbb{R}_+$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \bar{c} \sigma_h(t^{(r)}, \eta^{(r)}) + h_0 \gamma(h), \quad 0 < r \leqslant N_0 - 1 \tag{27}$$

$$\eta^{(0)} = \alpha_0(h), \tag{28}$$

then there exists a function $\beta : H \to \mathbb{R}_+$ such that $\eta_h^{(r)} \leqslant \beta(h)$ for $0 \leqslant r \leqslant N_0$ and $\lim_{h\to 0} \beta(h) = 0$.

Assumption $H[f_h, g_h]$**.** Assumption $H[f_h]$ holds true and there is a function $\sigma_h : I'_h \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying Assumption $H[\sigma_h]$ and such that

$$\|f_h(t^{(r)}, x^{(m)}, z) - f_h(t^{(r)}, x^{(m)}, \bar{z})\| \leqslant \sigma_h(t, \|z - \bar{z}\|_{h.r}) \tag{29}$$

$$|g_h(t^{(r)}, x^{(m)}, z) - g_h(t^{(r)}, x^{(m)}, \bar{z})| \leqslant \sigma_h(t, \|z - \bar{z}\|_{h.r}) \tag{30}$$

on $E_{h.0} \cup E_h$.

Remark 3.1. It follows from (29), (30) that f_h and g_h satisfy the Volterra condition.

Theorem 3.2. *Suppose that Assumption* $H[f_h, g_h]$ *and*

1) $h \in H$, $\varphi_h : E_{h.0} \to \mathbb{R}$ *is a given function and* $u_h : E_{h.0} \cup E_h \to \mathbb{R}$ *is the solution of the problem* (8), (9),
2) *the functions* $v_h : E_{h.0} \cup E_h \to \mathbb{R}$, $\gamma, \alpha_0 : H \to \mathbb{R}_+$ *are such that the conditions* (22)–(24) *are satisfied,*
3) *there is* $c_0 \in \mathbb{R}_+$ *such that the estimates*

$$|\delta_i v_h^{(r,m)}| \leqslant c_0 \quad \text{on } E_h, \quad 1 \leqslant i \leqslant n, \tag{31}$$

are satisfied. Then there exists a function $\alpha : H \to \mathbb{R}_+$ *such that*

$$|u_h^{(r,m)} - v_h^{(r,m)}| \leqslant \alpha(h) \quad \text{on } E_h \tag{32}$$

and $\lim_{h\to 0} \alpha(h) = 0$.

Proof. Let the function $\Gamma_h : E'_h \to \mathbb{R}$ be defined by the relation

$$\delta_0 v_h^{(r,m)} = F_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)}. \tag{33}$$

It follows from (22) that $|\Gamma_h^{(r,m)}| \leqslant \gamma(h)$ on E'_h. Let $J_+^{(r,m)}$, $J_-^{(r,m)}$ be the sets defined by (17), (18) and

$$\begin{aligned}\Lambda_h^{(r,m)} = h_0 &\sum_{i \in J_+^{(r,m)}[u_h]} \frac{1}{h_i} f_{h.i}(t^{(r)}, x^{(m)}, u_h)(u_h - v_h)^{(r+1,m+e_i)} \\ &- h_0 \sum_{i \in J_-^{(r,m)}[u_h]} \frac{1}{h_i} f_{h.i}(t^{(r)}, x^{(m)}, u_h)(u_h - v_h)^{(r+1,m-e_i)}.\end{aligned}$$

Since u_h satisfies the difference problem (8), (9) then we have

$$\begin{aligned}(u_h - v_h)^{(r+1,m)}&\Big[1 + h_0 \sum_{i=1}^{n} \frac{1}{h_i} |f_{h.i}(t^{(r)}, x^{(m)}, u_h)|\Big] \\ &= (u_h - v_h)^{(r,m)} + \Lambda_h^{(r,m)} \\ &\quad + h_0 \sum_{i=1}^{n} \frac{1}{h_i} \Big[f_{h.i}(t^{(r)}, x^{(m)}, u_h) - f_{h.i}(t^{(r)}, x^{(m)}, v_h)\Big] \delta_i v_h^{(r+1,m)} \\ &\quad + h_0 \Big[g_h(t^{(r)}, x^{(m)}, u_h) - g_h(t^{(r)}, x^{(m)}, v_h)\Big] - h_0 \Gamma_h^{(r,m)}.\end{aligned} \tag{34}$$

Write $\varepsilon_h^{(r)} = \max\{|(u_h - v_h)^{(i,m)}| : (t^{(i)}, x^{(m)}) \in E_{h.r}\}$ for $0 \leqslant r \leqslant N_0$. It follows that

$$|\Lambda_h^{(r,m)}| \leqslant h_0 \varepsilon_h^{(r+1)} \sum_{i=1}^{n} \frac{1}{h_i} |f_{h.i}(t^{(r)}, x^{(m)}, u_h)|, \quad (t^{(r)}, x^{(m)}) \in E'_h. \tag{35}$$

We conclude from Assumption $H[f_h, g_h]$ and from (31), (34), (35) that the function ε_h satisfies the recurrent inequality

$$\varepsilon_h^{(r+1)} \leqslant \varepsilon_h^{(r)} + h_0(1 + c_0)\sigma_h(t^{(r)}, \varepsilon_h^{(r)}) + h_0 \gamma(h), \tag{36}$$

where $0 \leqslant r \leqslant N_0 - 1$ and $\varepsilon_h^{(0)} \leqslant \alpha_0(h)$. Consider the solution $\eta_h : I_h \to \mathbb{R}_+$ of the difference problem

$$\begin{aligned}\eta^{(r+1)} &= \eta^{(r)} + h_0(1 + c_0)\sigma_h(t^{(r)}, \eta^{(r)}) + h_0 \gamma(h), \quad 0 \leqslant r \leqslant N_0 - 1 \\ \eta^{(0)} &= \alpha_0(h).\end{aligned}$$

It follows from the monotonicity of σ_h and (36) that $\varepsilon_h^{(r)} \leqslant \eta_h^{(r)}$ for $0 \leqslant r \leqslant N_0$. Then we obtain the assertion of Theorem 3.2 from the stability of the problem (25), (26). □

4. Difference schemes for quasilinear equations

We give examples of functions f_h and g_h corresponding to f and g and we prove that the implicit difference methods are convergent. We adopt additional assumptions for the mesh $E_{h.0} \cup E_h$. We assume that the steps of the mesh satisfy the condition: $h' = Mh_0$. Then we can write the definitions of the sets $E_{h.0}$ and E_h in the following way:

$$E_{h.0} = \{(t^{(r)}, x^{(m)}) : \; -K_0 \leqslant r \leqslant 0, \; -K \leqslant m \leqslant K\}$$
$$E_h = \{(t^{(r)}, x^{(m)}) : \; 0 \leqslant r \leqslant N_0, \; |m_r| \leqslant K_r - r \text{ for } r = 1, \ldots, n\}.$$

Assumption $H[T_h]$**.** There is an operator $T_h : F(E_{h.0} \cup E_h, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$ such that

1) for w, $\tilde{w} \in F(E_{h.0} \cup E_h, \mathbb{R})$ we have

$$\|T_h[w] - T_h[\tilde{w}]\|_{t^{(r)}} \leqslant \|w - \tilde{w}\|_{h.r}, \quad 0 \leqslant r \leqslant N_0, \tag{37}$$

2) there is $\mu > 0$ such that for each function $v : E_{h.0} \cup E_h \to \mathbb{R}$, which is of class C^2, there is $c_0 \in \mathbb{R}_+$ with the property

$$\|v - T_h[v_h]\|_{t^{(r)}} \leqslant c_0 h_0^{\mu}, \quad 0 \leqslant r \leqslant N_0, \tag{38}$$

where v_h is the restriction of v to the set $E_{h.0} \cup E_h$.

Remark 4.1. Condition (37) states that T_h satisfies the Lipschitz condition with the constant $L = 1$. It is clear that relation (37) implies the Volterra condition for T_h.

Assumption (38) means that the function v is approximated by $T_h[v_h]$ and the error of this approximation is estimated by $c_0 h_0^{\mu}$.

Remark 4.2. An example of the operator T_h satisfying Assumption $H[T_h]$ is given in [8], see also [6]. Let $\tilde{C}$ be such a constant that for a function $v : E_0 \cup E \to \mathbb{R}$ we have

$$|\partial_{tt} v(t,x)| \leqslant \tilde{C}, \quad |\partial_{tx_j} v(t,x)| \leqslant \tilde{C}, \quad |\partial_{x_i x_j} v(t,x)| \leqslant \tilde{C}, \quad i,j = 1 \ldots, n,$$

on $E_0 \cup E$ and $C_0 = \frac{\tilde{C}}{2}(1 + \|M\|)^2$. The operator T_h considered in [8] satisfies condition (38) for $\mu = 2$ and for the above C_0.

Now we approximate solutions of the problem (1), (2) with solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_i(t^{(r)}, x^{(m)}, T_h[z]) \delta_i z^{(r+1,m)} + g(t^{(r)}, x^{(m)}, T_h[z]) \tag{39}$$

with initial condition (9), where δ_0 is defined by (5) and

$$\delta_j z^{(r,m)} = \frac{1}{h_j}\Big[z^{(r,m+e_j)} - z^{(r,m)}\Big] \quad \text{if } f(t^{(r)}, x^{(m)}, T_h[z]) \geqslant 0 \tag{40}$$

$$\delta_j z^{(r,m)} = \frac{1}{h_j}\Big[z^{(r,m)} - z^{(r,m-e_j)}\Big] \quad \text{if } f(t^{(r)}, x^{(m)}, T_h[z]) < 0\,, \tag{41}$$

where $1 \leqslant j \leqslant n$. Write

$$E_i^+[\varepsilon] = \{(t,x) \in E : b_i - M_i t - \varepsilon < x_i \leqslant b_i - M_i t\}$$
$$E_i^-[\varepsilon] = \{(t,x) \in E : -b_i + M_i t \leqslant x_i < -b_i + M_i t + \varepsilon\},$$

where $1 \leqslant i \leqslant n$ and $\varepsilon > 0$ satisfies the condition $\varepsilon < b_i - M_i a$ for $1 \leqslant i \leqslant n$.

Assumption $H[f,g]$**.** Suppose that

1) the functions $f : \Omega \to \mathbb{R}^n$ and $g : \Omega \to \mathbb{R}$ are continuous and they satisfy the Volterra condition,
2) there is $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that
   (i) σ is continuous and it is nondecreasing with respect to both variables,
   (ii) $\sigma(t,0) = 0$ for $t \in [0,a]$ and for each $c \geqslant 1$ the maximal solution of the Cauchy problem
   $$\omega'(t) = c\sigma(t, \omega(t)), \quad \omega(0) = 0,$$
   is $\tilde{\omega}(t) = 0$ for $t \in [0,a]$,
3) the estimates
   $$\|f(t,x,z) - f(t,x,\bar{z})\| \leqslant \sigma(t, \|z - \bar{z}\|_t)$$
   $$|g(t,x,z) - g(t,x,\bar{z})| \leqslant \sigma(t, \|z - \bar{z}\|_t)$$
   are satisfied for (t,x,z), $(t,x,\bar{z}) \in \Omega$,
4) there is $\varepsilon > 0$ such that
   $$f_i(t,x,z) \leqslant 0 \quad \text{for } (t,x,z) \in E_i^+[\varepsilon] \times C(E_0 \cup E, \mathbb{R})$$
   $$f_i(t,x,z) \geqslant 0 \quad \text{for } (t,x,z) \in E_i^-[\varepsilon] \times C(E_0 \cup E, \mathbb{R}),$$
   where $1 \leqslant i \leqslant n$.

We give comments on the condition 4) of Assumption $H[f,g]$.

Remark 4.3. Suppose that Assumption $H[f,g]$ is satisfied and $z \in C(E_0 \cup E, \mathbb{R})$. Let us consider the Cauchy problem

$$\eta'(\tau) = -f(\tau, \eta(\tau), z), \quad \eta(t) = x\,, \tag{42}$$

where $(t,x) \in E$. The solution $g[z](\cdot,t,x) = (g_1[z](\cdot,t,x), \ldots, g_n[z](\cdot,t,x))$ of (42) is the bicharacteristic of equation (1) corresponding to z. Condition 4) of Assumption $H[f,g]$ states that

(i) for each $(t,x) \in E_i^+[\varepsilon]$, $1 \leqslant i \leqslant n$, there is $\varepsilon_0 > 0$ such that the function $g_i[z](\cdot,t,x) : [t-\varepsilon_0, t] \to \mathbb{R}$ is increasing,

(ii) for each $(t,x) \in E_i^-[\varepsilon]$, $1 \leqslant i \leqslant n$, there is $\varepsilon_0 > 0$ such that the function $g_i[z](\cdot,t,x) : [t-\varepsilon_0, t] \to \mathbb{R}$ is decreasing.

This property of bicharacteristics is important in the construction of implicit difference schemes for (1), (2). The difference operator $(\delta_1, \ldots, \delta_n)$ used in the paper satisfies the conditions:

(i) if the function $g_i[z](\cdot,t,x)$ is increasing on $[t-\varepsilon_0, t]$, then we put

$$\delta_i z(t,x) = \frac{1}{\tau}\Big[z(t,x) - z(t, x-\tau e_i)\Big],$$

where $1 \leqslant i \leqslant n$ and $\tau > 0$,

(ii) if $g_i[z](\cdot,t,x)$ is decreasing on $[t-\varepsilon_0, t]$, then we put

$$\delta_i z(t,x) = \frac{1}{\tau}\Big[z(t, x+\tau e_i) - z(t,x)\Big],$$

where $1 \leqslant i \leqslant n$ and $\tau > 0$.

Remark 4.4. Write

$$\begin{aligned} \partial_0 E_i^+ &= \{(t,x) \in E : x_i = b_i - M_i t\} \\ \partial_0 E_i^- &= \{(t,x) \in E : x_i = -b_i + M_i t\}, \end{aligned}$$

where $1 \leqslant i \leqslant n$. Suppose that there is $\widetilde{\varepsilon} > 0$ such that

$$\begin{aligned} &f_i(t,x,w) < -\widetilde{\varepsilon} \quad \text{for } (t,x,w) \in \partial_0 E_i^+ \times C(E_0 \cup E, \mathbb{R}) \\ &f_i(t,x,w) > \widetilde{\varepsilon} \qquad \text{for } (t,x,w) \in \partial_0 E_i^- \times C(E_0 \cup E, \mathbb{R}), \end{aligned}$$

where $1 \leqslant i \leqslant n$. Then condition 4) of Assumption $H[f,g]$ is satisfied.

Remark 4.5. Suppose that the function $f : \Omega \to \mathbb{R}$ satisfies the condition

$$x \diamond f(t,x,w) \leqslant \theta_{(n)} \quad \text{for } (t,x,w) \in \Omega, \tag{43}$$

where $\theta_{(n)} = (0, \ldots, 0) \in \mathbb{R}^n$. Then condition 4) of Assumption $H[f,g]$ is satisfied.

Now we formulate the main result of the paper.

Theorem 4.6. *Suppose that Assumptions $H[T_h]$ and $H[f,g]$ are satisfied and*

1) *$h \in H$, and there is a function $\alpha_0 : H \to \mathbb{R}_+$ such that*

$$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| \leqslant \alpha_0(h) \;\; \textit{on } E_{h.0} \quad \textit{and} \quad \lim_{h\to 0} \alpha_0(h) = 0, \tag{44}$$

2) $h' = Mh_0$ *and the function* $u_h : E_{h.0} \cup E_h \to \mathbb{R}$ *is a solution of the problem* (9), (39) *with the operator* δ *defined by* (40), (41),
3) *the function* $v : E_0 \cup E \to \mathbb{R}$ *is a solution of* (1), (2) *and* v *is of class* C^2, *and* v_h *is the restriction of function* v *to the set* $E_{h.0} \cup E_h$.

Then there is $\varepsilon > 0$ *and a function* $\alpha : H \to \mathbb{R}_+$ *such that for* $\|h\| < \varepsilon$ *we have*

$$|u_h^{(r,m)} - v_h^{(r,m)}| \leqslant \alpha(h) \ \text{ on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{45}$$

Proof. We prove that the functions $f_h(t,x,z) = f(t,x,T_h[z])$ and $g_h(t,x,z) = g(t,x,T_h[z])$ satisfy all assumptions of the Theorem 3.1. First we show that the problem (25), (26) is stable in the sense of Assumption $H[\sigma_h]$. Let the functions α_0, $\gamma : H \to \mathbb{R}_+$ are such that $\lim_{h\to 0} \alpha_0(h) = 0$, $\lim_{h\to 0} \gamma(h) = 0$. Consider the solution $\eta_h : I_{h.0} \cup I_h \to \mathbb{R}_+$ of the following difference problem:

$$\eta^{(r+1)} = \eta^{(r)} + h_0 c\sigma(t^{(r)}, \eta^{(r)}) + h_0\gamma(h), \quad 0 \leqslant r \leqslant N_0 - 1 \tag{46}$$

$$\eta^{(0)} = \alpha_0(h). \tag{47}$$

Denote by $\omega_h : [0,a] \to \mathbb{R}_+$ the maximal solution of the problem

$$\omega'(t) = c\sigma(t, \omega(t)) + \gamma(h), \quad \omega(0) = \alpha_0(h).$$

There exists $\varepsilon > 0$ such that the solution ω_h is defined on $[0,a]$ for $\|h\| < \varepsilon$ and $\lim_{h\to 0} \omega_h(t) = 0$ uniformly on $[0,a]$. The function ω_h is convex on $[0,a]$, therefore it satisfies the difference inequality

$$\omega_h^{(r+1,m)} \geqslant \omega_h^{(r)} + h_0 c\sigma(t^{(r)}, \omega_h^{(r)}) + h_0\gamma(h), \quad 0 \leqslant r \leqslant N_0 - 1.$$

Since η_h satisfies (46), (47), then we have $\eta_h^{(r)} \leqslant \omega_h^{(r)}$ for $0 \leqslant r \leqslant N_0$. This proves the stability of (25), (26). Moreover we have

$$\begin{aligned}\|f_h(t,x,z) - f_h(t,x,\bar z)\| &= \|f(t,x,T_h[z]) - f(t,x,T_h[\bar z])\| \\ &\leqslant \sigma(t, \|T_h[z-\bar z]\|_t) \\ &= \sigma(t, \|z - \bar z\|_{h.r})\end{aligned}$$

and

$$|g_h(t,x,z) - g_h(t,x,\bar z)| \leqslant \sigma(t, \|z - \bar z\|_{h.r}).$$

Let us denote by $\widetilde{F}_h$ the Niemycki operator corresponding to (39), i.e.,

$$\widetilde{F}_h[z]^{(r,m)} = \sum_{i=1}^{n} f_i(t^{(r)}, x^{(m)}, T_h[z])\delta_i z^{(r+1,m)} + g(t^{(r)}, x^{(m)}, T_h[z]).$$

Consider the function $\widetilde{\Gamma}_h : E'_h \to \mathbb{R}$ defined by the relation $\delta_0 v_h^{(r,m)} = \widetilde{F}_h[v_h]^{(r,m)} + \widetilde{\Gamma}_h^{(r,m)}$. We prove that there is $\widetilde{\gamma} : H \to \mathbb{R}_+$ such that

$$|\widetilde{\Gamma}_h^{(r,m)}| \leqslant \widetilde{\gamma}(h) \ \text{ on } E'_h \quad \text{and} \quad \lim_{h\to 0} \widetilde{\gamma}(h) = 0. \tag{48}$$

Write

$$\Gamma_{h.0}^{(r,m)} = \delta_0 v_h^{(r,m)} - \partial_t v^{(r,m)} + \sum_{i=1}^{n} f_i(t^{(r)}, x^{(m)}, v)\Big[\partial_{x_i} v^{(r,m)} - \delta_i v_h^{(r,m)}\Big]$$

$$\Gamma_{h.1}^{(r,m)} = \sum_{i=1}^{n} \Big[f_i(t^{(r)}, x^{(m)}, v) - f_i(t^{(r)}, x^{(m)}, T_h[v_h])\Big]\delta_i v_h^{(r+1,m)}$$
$$+ g(t^{(r)}, x^{(m)}, v) - g(t^{(r)}, x^{(m)}, T_h[v_h]).$$

Then $\widetilde{\Gamma}_h^{(r,m)} = \Gamma_{h.0}^{(r,m)} + \Gamma_{h.1}^{(r,m)}$ on E_h'. It follows easily that there is $\gamma_0 : H \to \mathbb{R}_+$ such that $|\Gamma_{h.0}^{(r,m)}| \leqslant \gamma_0(h)$ on E_h' and $\lim_{h\to 0} \gamma_0(h) = 0$. Suppose that $c_0 \in \mathbb{R}_+$ is defined by the relations $|\partial_{x_i} v(t,x)| \leqslant c_0$, $(t,x) \in E$, $1 \leqslant i \leqslant n$. It follows from Assumptions $H[f,g]$ and $H[T_h]$ that there is $\bar{c} \in \mathbb{R}_+$ such that

$$|\Gamma_{h.1}^{(r,m)}| \leqslant (1+c_0)\sigma(t^{(r)}, \quad \|v - T_h[v_h]\|_{t^{(r)}}) \leqslant (1+c_0)\sigma(a, \bar{c}h_0^{\mu}).$$

Then condition (48) is satisfied with $\tilde{\gamma}(h) = \gamma_0(h) + (1+c_0)\sigma(a, \bar{c}h_0^{\mu})$. The assertion of Theorem 4.6 then follows from Theorem 3.2. □

Remark 4.7. Suppose that Assumption $H[f,g]$ is satisfied with $\sigma(t,\xi) = L\xi$, $(t,\xi) \in [0,a] \times \mathbb{R}_+$, where $L \in \mathbb{R}_+$. Then we have assumed that f and g satisfy the Lipschitz condition with respect to the functional variable. In this case we have the estimates

$$|u_h^{(r,m)} - v_h^{(r,m)}| \leqslant \begin{cases} \alpha_0(h)e^{cLa} + \gamma(h)\frac{e^{cLa}-1}{cL} & \text{if } L > 0 \\ \alpha_0(h) + \gamma(h)a & \text{if } L = 0. \end{cases}$$

The above estimates are obtained by solving problem (46), (47).

5. Solutions of implicit difference equations on rectangular domains

Suppose that Assumption $H[f,g]$ is satisfied with $\Omega = E \times C(E_0 \cup E, \mathbb{R})$ where $E = [0,a] \times [-b,b]$. Assume also that condition (43) is satisfied. Then the natural domain of the existence of solutions to problem (1), (2) is the set $E_0 \cup ([0,a] \times [-b,b])$. We give a simple method for solving of the implicit difference problem (9), (39) in this case.

Suppose that $0 \leqslant r \leqslant N_0$ is fixed and that solution u_h of (9), (39) is defined on the set $E_{h.r}$. We first compute $z_h^{(r+1,m)}$ for $\theta_{(n)} \leqslant m \leqslant K$. According to assumption (43) we have

$$f_i(t^{(r)}, x^{(m)}, T_h[z_h]) \leqslant 0, \quad 1 \leqslant i \leqslant n,$$

where $0 \leqslant r < N_0$ and $\theta_{(n)} \leqslant m \leqslant K$, and

$$f_i(t^{(r)}, x^{(m)}, T_h[z_h]) = 0, \quad 1 \leqslant i \leqslant n,$$

for such $m = (m_1, \ldots, m_n)$ that $m_i = 0$. We conclude from (40), (41) that equation (39) for $\theta_{(n)} \leqslant m \leqslant K$ has the form

$$\begin{aligned} &z_h^{(r+1,m)}\Big[1 - h_0 \sum_{i=1}^{n} \frac{1}{h_i} f_i(t^{(r)}, x^{(m)}, T_h[z_h])\Big] \\ &= -h_0 \sum_{i=1}^{n} \frac{1}{h_i} f_i(t^{(r)}, x^{(m)}, T_h[z_h]) z_h^{(r+1,m-e_i)} + z_h^{(r,m)} + h_0 g(t^{(r)}, x^{(m)}, T_h[z_h]). \end{aligned} \tag{49}$$

We deduce from (49) that $z_h^{(r+1,m)}$ may be computed for $m = \theta_{(n)}$ and for $m = (m_1, 0, \ldots, 0) \in Z^n$ where $1 \leqslant m_1 \leqslant K_1$. Our next goal is to determine $z_h^{(r+1,m)}$ for $m = (m_1, m_2, 0, \ldots, 0) \in Z^n$ where $0 \leqslant m_1 \leqslant K_1$, $0 \leqslant m_2 \leqslant K_2$. We conclude from (49) that the numbers $z_h^{(r+1,m)}$ for the above m exist and that they are unique.

Suppose that the solution $z_h^{(r+1,m)}$ is computed for $m = (m_1, \ldots, m_\kappa, 0, \ldots, 0) \in Z^n$ where $0 \leqslant m_i \leqslant K_i$ and $1 \leqslant \kappa \leqslant n-1$ is fixed. Repeated application of (49) enables us to calculate $z_h^{(r+1,m)}$ for $m = (m_1, \ldots, m_{\kappa+1}, 0, \ldots, 0) \in Z^n$ where $0 \leqslant m_i \leqslant K_i$ and $1 \leqslant i \leqslant \kappa+1$. It follows from the above considerations that the solution $z_h^{(r+1,m)}$ of (9), (39) may be calculated for $\theta_{(n)} \leqslant m \leqslant K$.

In a similar way the solution $z_h^{(r+1,m)}$ may be computed on the set $E_{h.r+1} \cap (X_1 \times X_2 \times \ldots \times X_n)$ where $X_i = \mathbb{R}_+$ or $X_i = R_-$, $R_- = (-\infty, 0]$. Then our problem is solved by induction with respect to r, $0 \leqslant r \leqslant N_0$.

Remark 5.1. It is easy to see that condition (43) may be replaced by the following assumption: there exists $\widetilde{x}$, $-b < \widetilde{x} < b$ such that $(x - \widetilde{x}) \diamond f(t, x, w) \leqslant \theta_{(n)}$ for $(t, x, w) \in \Omega$.

6. Numerical examples

We give now numerical examples concerning our theory for quasilinear differential equations. To find an approximate solutions we use implicit difference method of the Euler type and explicit difference method with difference operators given by the Lax scheme.

Example 6.1. For $n = 1$ we put

$$E = \{(t, x) : t \in [0, 0.5], \quad x \in [-1, 1]\}. \tag{50}$$

Consider the differential equation

$$\begin{aligned}\partial_t z(t,x) =& -x\Big[1+\cos\big(z(t,0.5x)-e^{tx}\big)\Big]\partial_x z(t,x)\\ &+\sin\big(z(t,-0.5x)-e^{-tx}\big)+2x(1+2t)z(t,x)\end{aligned} \tag{51}$$

with the initial condition

$$z(0,x)=1, \quad x\in[-1,1]. \tag{52}$$

Note that condition (43) is satisfied for equation (51). The exact solution of this problem is known. It is $v(t,x)=e^{2tx}$. Let $h=(h_0,h_1)$, stand for steps of the mesh on E.

Let us denote by $u_h : E_h \to \mathbb{R}$ the solution of the implicit difference problem corresponding to (51), (52). Write

$$\eta_h^{(r)} = \frac{1}{2K+1}\sum_{m=-K}^{K}\Big|u_h^{(r,m)}-v^{(r,m)}\Big|, \tag{53}$$

where $K\in\mathbb{N}$ is defined by the condition $Kh_1=1$. The numbers $\eta_h^{(r)}$ are the arithmetical mean of the errors with fixed $t^{(r)}$. The values of the function η_h are listed in Table 1.

	$h=(10^{-2},10^{-3})$	$h=(2\cdot10^{-3},2\cdot10^{-4})$
$t=0.1$	0.001842	0.000371
$t=0.2$	0.003532	0.000714
$t=0.3$	0.005214	0.001055
$t=0.4$	0.006989	0.001416
$t=0.5$	0.008937	0.001813

Table 1: The error η_h.

The results shown in Table 1 are consistent with our mathematical analysis. We consider also an approximate solution z_h of (51), (52) which is obtained by using the classical Lax scheme. The domain of z_h is the set

$$\widetilde{E}_h = \Big\{(t^{(r)},x^{(m)}) : 0\leqslant r\leqslant N_0,\ x^{(m)}\in[-1+t^{(r)},1-t^{(r)}]\Big\}, \tag{54}$$

and $N_0h_0 \leqslant 0.5 < (N_0+1)h_0$. Write

$$\widetilde{\eta}_h = \max\Big\{|z_h^{(r,m)}-v^{(r,m)}| : (t^{(r)},x^{(m)})\in\widetilde{E}_h\Big\}. \tag{55}$$

In the considered cases the values of $\widetilde{\eta}_h$ are bigger than 10^2.

Example 6.2. Let $E \subset R^2$ be defined by (50). Consider the differential integral equation

$$\begin{aligned}\partial_t z(t,x) = &-2\sin x\left[1+\sin\left(3\int_{-x}^{x} z(t,s)ds - 2xz(t,x)\right)\right]\partial_x z(t,x)\\ &+\int_0^t z(\tau,x)\,d\tau + x^2(1+t) + 4x(e^t-1)\sin x\end{aligned} \tag{56}$$

with the initial condition

$$z(0,x) = 0, \quad x \in [-1,1]. \tag{57}$$

Note that equation (56) satisfies (43). The exact solution of this problem is known. It is $v(t,x) = (e^t-1)x^2$. Put $h = (h_0, h_1)$ stand for the steps of the mesh on E.

Let us denote by $u_h : E_h \to \mathbb{R}$ the solution of the implicit difference problem corresponding to (56), (57). Let η_h be defined by (53). The numbers $\eta_h^{(r)}$ are the arithmetical mean of the errors with fixed $t^{(r)}$. The values of the function η_h are listed in Table 2.

	$h = (10^{-2}, 10^{-3})$	$h = (2\cdot 10^{-3}, 2\cdot 10^{-4})$
$t = 0.1$	0.001466	0.002330
$t = 0.2$	0.003891	0.002330
$t = 0.3$	0.007605	0.002330
$t = 0.4$	0.012503	0.010204
$t = 0.5$	0.018385	0.016133

Table 2: The error η_h.

The results given in Table 2 are consistent with our theoretical results. We consider also an approximate solution z_h of (56),(57) which is obtained by using the classical Lax scheme. The domain of z_h is the set $\widetilde{E}_h$ defined by (54). Let $\widetilde{\eta}_h$ be defined by (55). In the considered cases the values of $\widetilde{\eta}_h$ are bigger than 10^2.

The above examples show that there are implicit difference schemes for problem (1), (2) which are convergent and the corresponding classical difference methods are not convergent. This is due to the fact that we need the (CFL) conditions in the classical case and we do not need special assumptions on the steps of the mesh in the case of implicit difference schemes.

Note that there is a natural class of differential equations (1) for which the implicit difference problems can be easily solved. This class of equations is described in Section 5.

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