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About Solutions of Poisson's Equation with Transition Condition in Non-Smooth Domains

Anja Schlömerkemper

Abstract. Starting from integral representations of solutions of Poisson's equation with transition condition, we study the first and second derivatives of these solutions for all dimensions $d \geq 2$. This involves derivatives of single layer potentials and Newton potentials, which we regularize smoothly. On smooth parts of the boundary of the non-smooth domains under consideration, the convergence of the first derivative of the solution is uniform; this is well known in the literature for regularizations using a sharp cut-off by balls. Close to corners etc. we prove convergence in L^1 with respect to the surface measure. Furthermore we show that the second derivative of the solution is in L^1 on the bulk.
The interface problem studied in this article is obtained from the stationary Maxwell equations in magnetostatics and was initiated by work on magnetic forces.

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1. Introduction

We derive regularity results for solutions of Poisson's equation with transition condition on bounded non-smooth domains in $d \geq 2$ dimensions. In particular, we study integral representations of the solutions in terms of Newton potentials and single layer potentials. Our interest is twofold. Firstly, we consider smoothly regularized versions of the gradients of the potentials and calculate the limits in the bulk and on surfaces as well as on interfaces (cf. Theorem 4.2). This part is closely related to several results known in the literature, where sharp cut-offs are chosen in order to regularize, see below for details. Moreover, we study the second gradient of solutions of Poisson's equation. We prove that the second gradient is an L^1 function on the bulk (Theorem 5.3). In order to show this we study the second gradient of the single layer potential on the bulk (Theorem 5.2).

A. Schlömerkemper: Max-Planck-Institute for Mathematics in the Sciences, Inselstraße 22–26, D-04103 Leipzig, Germany; schloem@mis.mpg.de

The main methods of the proofs go back to corresponding ones in [25] for $d = 3$ dimensions and a simpler geometric setting as well as stronger assumptions on the regularity of the domains. In [25] it is assumed among other things that the 3-dimensional domain Ω is a union of two domains which are nested such that the boundary of the inner domain does not intersect the boundary of Ω. Moreover, in [25], all domains are required to be Lipschitz continuous and piecewise C^2.

In this article, the bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is assumed to be the union of two disjoint bounded domains A and B in $\mathbb{R}^d$ which have some part of their boundaries in common, but do not have to be nested. The domains are primarily assumed to be Lipschitz continuous and piecewise $C^{1,\alpha}$, $0 < \alpha \leq 1$, see Assumption $\mathcal{A}_1$ and Definition 2.1 for details.

The mathematical difficulties in the case $d = 2$ arise since the fundamental solution of Laplace's equation is basically different to the three-dimensional one, cf. (6). In the case $d > 3$ the fundamental solution has the same structure as the three-dimensional one, which allows for a straightforward generalization of the previous estimates. The difficulty in this case lies in finding appropriate assumptions on the domains, see in particular Assumption $\mathcal{A}_1$(iv) and Definition 2.1(iii). The case of having non-nested sets is also taken care off in Assumption $\mathcal{A}_1$; see also the proof of Theorem 5.2 on the second gradient of the single layer potential, where we assume $\alpha = 1$ to obtain the desired bounds. Otherwise the generalization from piecewise C^2 to piecewise $C^{1,\alpha}$ is rather straightforward, cf. the proofs of Propositions 3.2 and 3.3.

In this article we consider two different smooth regularizations: one is analogous to the one considered in [25], cf. the beginning of Section 3; the other regularization is defined in (16). The latter regularization is motivated by earlier work in $d = 2$ dimensions, cf. [26] and see also [22]. It has the advantage that the smooth cut-off function η is multiplied by the gradient of the fundamental solution of Laplace's equation, ∇N, which is homogeneous of degree minus one if $d = 2$. In the former regularization, η is multiplied by N, which is only homogeneous of degree $2 - d$ in the case $d \geq 3$.

With respect to applications in the sciences, the work in this article is motivated by studies of magnetic forces in continuous media in [22, 23], where the regularity results of this paper are applied for $d = 2, 3$. Typical terms in magnetic force formulae are of the form $\int_A (M(x) \cdot \nabla) H(x)\, dx$ and $\int_{\partial A} (M \cdot n)(x) H^-(x)\, ds_x$, respectively, where $M : \mathbb{R}^d \to \mathbb{R}^d$ denotes the magnetization, which is a given datum, and $H : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic field, a solution of the stationary Maxwell equations, which has inner trace H^-. The outer normal to ∂A is denoted by n. In order to show for instance the existence of such integrals, some regularity results on H and the gradient of H (Theorem 5.3) are needed. Furthermore, the convergence results for the regularized potentials

(Theorem 4.2) are applied in the derivation of the force formulae, cf. [22, 25] as well as (37).

The stationary Maxwell equations in magnetostatics lead to the Poisson equation. Maxwell's equations read $\operatorname{curl} H = 0$ and $\operatorname{div} H = -\operatorname{div} M$, where M and H are again a given magnetization and the magnetic field, respectively. The first equation allows to write H as the gradient of some potential $u : \mathbb{R}^d \to \mathbb{R}$. We set $H = -\nabla u$. Then Maxwell's equations become Poisson's equation for u, i.e., $-\Delta u = -\operatorname{div} M$, where $\Delta = \sum_{j=1}^{d} \partial_j^2$.

In physical applications, M is supported on the closure of a bounded domain $\Omega \subset \mathbb{R}^d$. The normal component of the magnetization might jump at $\partial\Omega$ and at interfaces, while the magnetization is smooth otherwise. Let Γ be such an interface. Then we have the following transition condition in addition to Poisson's equation: $[H \cdot n_\Gamma] = -[M \cdot n_\Gamma]$, where $[a] := a^+ - a^-$, $a \in \mathbb{R}$, denotes the difference between outer and inner traces. Moreover, n_Γ denotes a normal to Γ. We refer to Section 2 for details about Γ and M and the precise assumptions on them which we require in this article. In terms of u, the transition condition reads $[\nabla u \cdot n_\Gamma] = [M \cdot n_\Gamma]$. A similar transition condition holds at the boundary of Ω, cf. (2).

Integral representations of solutions u of Poisson's equation with transition condition are given by a sum of a Newton potential and a single layer potential, cf. (11). We are interested in regularity results for $H = -\nabla u$ and its derivatives. Therefore it is natural to study the first and second derivatives of the potentials in the bulk and on surfaces. Gradients of the potentials have been studied extensively in the past. For the nowadays classical results in domains with smooth boundaries we refer to the monographs by Kellogg [11] and Mikhlin [18]. For more general results see, e.g., [8, 19] and the references below.

Single layer potentials are studied also in a different context. These potentials and the so-called double layer potentials occur when one solves boundary value problems of Laplace's equation (and generalizations) by boundary integral equations. To solve those integral equations, certain properties of the layer potentials are proved in order to ensure invertibility. See for instance [12, Chapter 2, Section 2] for further references and an overview of this potential technique in smooth domains as well as in C^1 and in Lipschitz domains. The results in Lipschitz domains trace back to the work by Verchota [28], which we partly apply after equation (11) and in the proofs of Section 3.

For recent results in the potential theory on Lipschitz domains we refer to [16] for higher regularity results in fractional Sobolev spaces, where the dependence of the regularity on the regularity of the domain is considered. Moreover, there has been recent work on potential theory on Lipschitz domains in Riemannian manifolds, see, e.g., [20] for Sobolev-Besov space results and references therein. The last-mentioned works strive for maximal regularity results. Here, however, our goal is different and the analysis can be based on the work by Verchota [28]: We prove regularity of the solution of Poisson's equation for a large class of domains being of interest in applications. Whether the regularity assumptions on the domains are optimal remains an open problem.

Li and Vogelius [14] and Li and Nirenberg [13] derived estimates for the gradient of solutions to divergence form elliptic equations and systems, respectively, with discontinuous coefficients. Those works were stimulated by questions arising in the context of composite materials. The interfaces were supposed to be $C^{1,\alpha}$ and to satisfy further assumptions. Here, we require only weaker assumptions on interfaces and boundaries. In particular, the domains might have corners and edges.

It is well known that the solutions of Laplace's equation (and of more general elliptic equations) have singularities near corner points and edges, see for instance [9] and references therein. To investigate such singularities of solutions of, e.g., transmission problems for the Laplace equation across a Lipschitz interface, single and double layer potentials are studied, see, e.g., [5] and [17, 21]. The latter authors work in Hilbert space settings in two and three dimensions and give some numerical examples. For further studies of layer potentials and their first gradients in the context of numerical simulations (boundary element methods) and thus for Hilbert space settings we refer to [24, 27].

Costabel and Dauge [2] analyzed singularities of solutions of the time-dependent Maxwell equation on polyhedral domains in a Hilbert space setting. However, here we are interested in solutions of the stationary Maxwell equations, which are in some L^1 and $W^{1,1}$ spaces, respectively. Moreover, we do not want to restrict the boundary data to be continuous since applications as for instance in micromagnetism (see, e.g., [3, 10]) have boundary data $M \cdot n$ that are only piecewise continuous, which is in particular due to the non-smoothness of the domain. In this article we therefore do not want to restrict $M \cdot n$ to be in the Hilbert space $H^{-\frac{1}{2}}$. We assume that the boundary data are in L^∞ on the boundary.

The outline of the paper is as follows. Section 2 gives the precise assumptions on the domains. Moreover, Poisson's equation with the transition conditions as well as the solution of this in terms of Newton and single layer potentials are summarized. Then, regularized potentials are introduced and some properties are asserted. In Section 3 we consider the gradient of the single layer potential for different smooth regularizations and relate this to the sharp cut-off by balls often used in the literature. The topic of Section 4 is the convergence of the regularized gradient of the solution of Poisson's equation and its integral representation formulae (Theorem 4.2). In Section 5 we prove that the second gradient of the single layer potential is an L^1 function on the bulk (Theorem 5.2). Finally, we conclude that the second gradient of the solution of Poisson's equation is an L^1 function on the bulk (Theorem 5.3).

2. Notation and preliminaries

Throughout this article we suppose that the following assumptions on the domains hold.

Assumption $\mathcal{A}_1$

(i) A and B are bounded domains in $\mathbb{R}^d$, $d \geq 2$, such that $A \cap B = \emptyset$. Moreover A and B have some boundary in common, i.e., its $d-1$ dimensional Hausdorff measure $\mathcal{H}^{d-1}(\partial A \cap \partial B)$ is strictly greater than zero.

(ii) A and B are Lipschitz domains, i.e., locally, the boundary of A (and B, respectively) is the graph of a Lipschitz continuous function, and A (and B, respectively) is on one side of the boundary only.

(iii) ∂A and ∂B are piecewise $C^{1,\alpha}$ for $0 < \alpha \leq 1$, cf. Definition 2.1 below.

(iv) ∂A and ∂B satisfy the neighbourhood estimate, cf. Definition 2.2 below.

We phrase the definition of a piecewise $C^{1,\alpha}$ boundary for ∂A only; the definition for ∂B runs analogously.

Definition 2.1. The boundary ∂A is said to be *piecewise* $C^{1,\alpha}$, $0 < \alpha \leq 1$, if there exist finitely many pairwise disjoint sets $U_i \subset \partial A$ which are relatively open in ∂A and have the following properties:

(i) U_i is a connected, orientable $C^{1,\alpha}$ submanifold of $\mathbb{R}^d$ and the outer normal n to ∂A restricted to U_i is $C^{0,\alpha}$ up to the boundary,

(ii) $\partial A \subset \bigcup_i \bar{U}_i$, and

(iii) the relative boundary ∂U_i is a finite union of connected $C^{1,\alpha}$ submanifolds of $\mathbb{R}^d$. If $d = 2$, ∂U_i is required to be a union of finitely many points in $\mathbb{R}^d$.

Next we give the definition of the neighbourhood estimate for ∂A. It holds analogously for ∂B.

Definition 2.2. Let ∂U_i be as in Definition 2.1. We say that ∂A satisfies the *neighbourhood estimate* if there exists a constant $c > 0$ such that for any ∂U_i, the d dimensional volume of $\{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial U_i) \leq r\}$ is bounded by cr^2 for small enough r.

This definition is obvious in $d = 2$ since ∂U_i is a union of finitely many points; it is natural for higher dimensions, see [26, Remark 5] for a discussion of this. As an aside, this definition is equivalent to saying that ∂U_i has finite $(d-2)$ dimensional upper Minkowski content, see, e.g., [15, p. 79]. We apply Assumption $\mathcal{A}_1$(iv) and Definition 2.1(iii) in the proofs of Lemma 3.8 and Theorem 5.2 below.

Assumption $\mathcal{A}_1$ includes for instance two- and three-dimensional polygonal domains. The domains A and B might be nested, but they do not have to be nested, cf. Figure 1. We set $\Gamma := \partial A \cap \partial B$ and $\Omega := \operatorname{int}(\overline{A} \cup \overline{B})$.

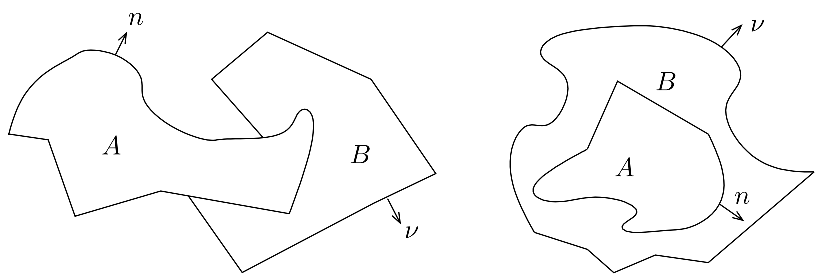

Figure 1: Sketches of possible configurations.

The common boundary Γ can be considered as an interface within Ω. Since ∂A and ∂B are piecewise $C^{1,\alpha}$ by assumption, also Γ is piecewise $C^{1,\alpha}$. Thus Γ can be written as a finite union of $C^{1,\alpha}$ submanifolds of $\mathbb{R}^d$ as in Definition 2.1.

Next we summarize our assumptions on the data, i.e., on the right hand side of the Poisson equation (1) and of the transition condition (2) below.

Assumption $\mathcal{A}_2$ The support of $M : \mathbb{R}^d \to \mathbb{R}^d$ is in $\overline{\Omega}$. If M is restricted to A, $M_{|A} \in W^{1,\infty}(A; \mathbb{R}^d)$. Similarly, $M_{|B} \in W^{1,\infty}(B; \mathbb{R}^d)$.

For brevity we set $w := -\operatorname{div} M$ on $\mathbb{R}^3 \setminus (\Gamma \cup \partial\Omega)$. By Assumption $\mathcal{A}_2$, we have $w \in L^\infty(\mathbb{R}^d \setminus \Gamma \cup \partial\Omega)$. Furthermore, let n denote the outer normal to ∂A; this is defined almost everywhere due to Assumption $\mathcal{A}_1$. The outer and inner traces

$$g^\pm := (M \cdot n)^\pm = M^\pm \cdot n$$

on ∂A are in $L^\infty(\partial A)$ (see, e.g., [1, A 6.6]). Similarly, the traces

$$g^{\nu^\pm} := (M \cdot \nu)^{\nu^\pm} = M^{\nu^\pm} \cdot \nu$$

on ∂B are in $L^\infty(\partial B)$. Here, ν alludes to the outer normal ν to ∂B. For definiteness we define the jump of the traces at the interface Γ throughout the paper as follows: $[a] := a^+ - a^- = -(a^{\nu^+} - a^{\nu^-})$, $a \in \mathbb{R}$. By Assumption $\mathcal{A}_2$, $[g] \in L^\infty(\Gamma)$. Note that we sometimes also write g instead of g^- for brevity, if this meaning is obvious from the context.

Remark 2.3. By Assumption $\mathcal{A}_2$ we have $M_{|A} \in W^{1,\infty}(A, \mathbb{R}^d)$. There exists an $\widetilde{M} \in C^{0,1}(\overline{A}, \mathbb{R}^d)$ which equals M almost everywhere by an embedding theorem (see, e.g., [1, Satz 8.5]). Thus we have $g^\pm = \widetilde{M}^\pm \cdot n$ almost everywhere. Since n is piecewise $C^{0,\alpha}$ by Assumption $\mathcal{A}_1$, also $g^\pm$ is piecewise $C^{0,\alpha}$ almost everywhere.

We consider Poisson's equation

$$-\Delta u = w \quad \text{in } \mathbb{R}^3 \setminus (\Gamma \cup \partial\Omega) \tag{1}$$

with transition conditions

$$[\nabla u \cdot n] = [g] \quad \text{on } \Gamma$$
$$[\nabla u \cdot \nu] = \begin{cases} -g^- & \text{on } \partial A \setminus \Gamma \\ -g^{\nu^-} & \text{on } \partial B \setminus \Gamma\,. \end{cases} \tag{2}$$

Note that we sometimes tacitly assume that u decreases sufficiently fast at infinity so that it is a unique solution.

The aim is to study integral representations of a solution u_Ω of (1) and (2) and of its gradients. In order to do so, we firstly consider Poisson's equation on the domain A:

$$-\Delta u = w_{|A} \quad \text{in } \mathbb{R}^d \setminus \partial A \tag{3}$$

with transition condition

$$[\nabla u \cdot n] = -g^- \quad \text{on } \partial A. \tag{4}$$

A solution of these equations is denoted by u_A. Similarly, u_B is defined. By linearity of Poisson's equation we then have

$$u_\Omega = u_A + u_B \quad \text{and thus} \quad \nabla u_\Omega = \nabla u_A + \nabla u_B. \tag{5}$$

Next we study u_A and its first and second gradient. Solutions of Poisson's equation can be represented in terms of the fundamental solution of Laplace's equation. For this fix a point $y \in \mathbb{R}^d$. Then the normalized fundamental solution of Laplace's equation $-\Delta u = 0$ is given by (see, e.g., [8, p. 17] and note the different sign conventions)

$$N(x-y) := \begin{cases} -\frac{1}{2\pi} \ln |x-y| & \text{if } d = 2 \\ \frac{1}{d(d-2)\omega_d} |x-y|^{2-d} & \text{if } d \geq 3 \end{cases} \qquad \text{for all } x \in \mathbb{R}^d, x \neq y. \tag{6}$$

Here ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. That is, $\omega_d = \frac{2\pi^{\frac{d}{2}}}{d\,\Gamma(\frac{d}{2})}$, where $\Gamma(\cdot)$ is the Gamma-function.

We can write (3) and (4) equivalently in the form

$$-\Delta u = w\mathcal{L}^d_{|A} + g^- \mathcal{H}^{d-1}_{|\partial A} \quad \text{on } \mathbb{R}^d, \tag{7}$$

where $\mathcal{L}^d$ denotes the d dimensional Lebesgue measure and $\mathcal{H}^{d-1}$ is as before the $d-1$ dimensional Hausdorff measure. The equation is understood in the sense of distributions. Since the right hand side of (7) is a distribution with compact support on $\mathbb{R}^d$, there exists a solution u_A of (7) with the integral representation

$$u_A(x) = \int_A w(y) N(x-y)\,dy + \int_{\partial A} g^-(y) N(x-y)\,ds_y \tag{8}$$

for all $x \in \mathbb{R}^d$ with $s_y = \mathcal{H}^{d-1}$ denoting the surface measure with respect to y, see, e.g., [4, p. 73].

The following abbreviations will be used throughout this article:

$$\mathscr{V}_A(w)(x) := \int_A w(y)N(x-y)\,dy \tag{9}$$

$$\mathscr{S}_{\partial A}(g)(x) := \int_{\partial A} g(y)N(x-y)\,ds_y, \tag{10}$$

where g is understood here as the inner trace of g with respect to ∂A. Then (8) becomes

$$u_A(x) = \mathscr{V}_A(w)(x) + \mathscr{S}_{\partial A}(g)(x), \quad x \in \mathbb{R}^d. \tag{11}$$

As usual in the literature, we call $\mathscr{V}_A$ Newton potential and $\mathscr{S}_{\partial A}$ single layer potential. It is well known that the volume potential exists under Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ (see, e.g., [18, Chapter 11, §6]).

As already mentioned in the introduction, there are several results about single layer potentials known in the literature. (Be aware of different sign conventions.) Here we follow mainly Verchota [28]. Verchota proved invertibility of layer potentials for Laplace's equation in certain spaces for bounded Lipschitz domains in $\mathbb{R}^d$, $d \geq 2$. For this he summarized results about the existence of layer potentials in his Section 1. We apply some of these results on the existence of single layer potentials and their normal and tangential derivatives, see below for details. Note that the results in [28] are phrased under the assumption that $g \in L^p(\partial A)$ for $1 < p < \infty$. This holds in our setting since $g \in L^\infty(\partial A)$ and $\mathcal{H}^{d-1}(\partial A) < \infty$ by Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$.

By [28, Lemma 1.8], the single layer potential (10) exists on $\mathbb{R}^d$ and in $L^p(\partial A)$, $1 < p < \infty$. Moreover, the jump of the traces at ∂A is zero. Since we have $\mathcal{H}^{d-1}(\partial A) < \infty$ by Assumption $\mathcal{A}_1$, the single layer potential also exists in $L^1(\partial A)$. Next we study an approximation of the single layer potential, which prepares for later approximations of the derivatives of the single layer potential studied in Section 3.

In the works cited above, the singularity is truncated sharply, i.e., by a sharp cut-off. Here we also consider a smooth cut-off, which is used in applications, cf. [22, 25], and is interesting on its own. Let $\eta : [0,\infty] \to \mathbb{R}$ be a smooth function such that $\eta(r) = 0$ if $0 \leq r \leq \frac{1}{2}$ and $\eta(r) = 1$ if $r \geq 1$. We set

$$u_A^{(\delta)}(x) := \mathscr{V}_A^{(\delta)}(w)(x) + \mathscr{S}_{\partial A}^{(\delta)}(g)(x), \quad x \in \mathbb{R}^d, \tag{12}$$

where

$$\mathscr{V}_A^{(\delta)}(w)(x) := \int_A w(y)\eta\left(\frac{|x-y|}{\delta}\right)N(x-y)\,dy \tag{13}$$

$$\mathscr{S}_{\partial A}^{(\delta)}(g)(x) := \int_{\partial A} g(y)\eta\left(\frac{|x-y|}{\delta}\right)N(x-y)\,ds_y\,. \tag{14}$$

The single layer potential is approximated uniformly by $\mathscr{S}_{\partial A}^{(\delta)}(g)$ on compact subsets of the $C^{1,\alpha}$ submanifolds of the boundary and pointwise else. A proof of this can, e.g., easily be adapted from the proof of an analogous statement for $d = 3$ in [25, p. 265]. Furthermore, we note that $\mathscr{V}_A^{(\delta)}(w)$ converges uniformly to the Newton potential $\mathscr{V}_A(w)$ on $\mathbb{R}^d$, which follows from a straightforward calculation which yields $|\mathscr{V}_A(w)(x) - \mathscr{V}_A^{(\delta)}(w)(x)| \le c\delta^2|\ln\delta|$ if $d = 2$, and $|\mathscr{V}_A(w)(x) - \mathscr{V}_A^{(\delta)}(w)(x)| \le c\delta^2$ if $d \ge 3$. In the following sections we use these regularizations and others to study the gradient of the solutions of the above Poisson equation.

3. About the gradient of the single layer potential

In this section we study approximations of the gradient of the single layer potential on Ω and Γ, respectively. One approximation is based on the definition of $\mathscr{S}_{\partial A}^{(\delta)}(g)$ in (14): We take the gradient of this and calculate the limit as $\delta \to 0$. Secondly we study a regularization, where ∇N is multiplied by a smooth function η defined as above, i.e., we consider

$$(\nabla\mathscr{S}_{\partial A})^{(\delta)}(g)(x) = \int_{\partial A} g(y)R^{(\delta)}(x-y)\,ds_y \tag{15}$$

with

$$R^{(\delta)}(x-y) := \eta\left(\frac{|x-y|}{\delta}\right)\nabla N(x-y). \tag{16}$$

Note that $(\nabla\mathscr{S}_{\partial A})^{(\delta)}(g) \neq (\nabla\mathscr{S}_{\partial A}^{(\delta)})(g)$ in general. The advantage of this regularization is that η is multiplied with ∇N, which is homogeneous of degree minus one if $d = 2$, cf. a corresponding remark in the introduction. Similarly we set

$$(\nabla_{\tan}\mathscr{S}_{\partial A})^{(\delta)}(g)(x) = \int_{\partial A} g(y)R_{\tan}^{(\delta)}(x-y)\,ds_y$$

$$(n\cdot\nabla\mathscr{S}_{\partial A})^{(\delta)}(g)(x) = \int_{\partial A} g(y)n(x)\cdot R^{(\delta)}(x-y)\,ds_y,$$

where $R_{\tan}^{(\delta)}(x-y)$ and $n(x)\cdot R^{(\delta)}(x-y)$ are defined accordingly to (16). Note that these functions are in general different from $(\nabla_{\tan}\mathscr{S}_{\partial A}^{(\delta)})(g)$ and $(n\cdot\nabla\mathscr{S}_{\partial A}^{(\delta)})(g)$, respectively.

The following proposition is a straightforward extension of an analogous statement [25, p. 253] from three dimensions to $d \geq 2$, to a more general geometric setting, and to both kind of regularizations introduced above.

Proposition 3.1. *Let Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold. Then*

$$(\nabla \mathscr{S}_{\partial A}^{(\delta)})(g) \longrightarrow \nabla \mathscr{S}_{\partial A}(g) \quad \text{in } L^1(\Omega) \text{ as } \delta \to 0$$
$$(\nabla \mathscr{S}_{\partial A})^{(\delta)}(g) \longrightarrow \nabla \mathscr{S}_{\partial A}(g) \quad \text{in } L^1(\Omega) \text{ as } \delta \to 0.$$

Proof. Since $|\nabla \eta(\frac{|x-y|}{\delta})| \leq \frac{c}{\delta} \chi_{[\frac{\delta}{2},\delta]}(|x-y|)$, we have

$$\left|\nabla\left(\eta\left(\frac{|x-y|}{\delta}\right) N(x-y)\right)\right| \\ \leq \frac{c}{\delta} |N(x-y)| \chi_{[\frac{\delta}{2},\delta]}(|x-y|) + c|\nabla N(x-y)| \chi_{[\frac{\delta}{2},\infty)}(|x-y|).$$

If $d \geq 3$, we can bound the first term by $c|\nabla N(x-y)|\chi_{[\frac{\delta}{2},\delta]}(|x-y|)$. Thus, if $d \geq 3$, both regularizations lead to the bound $c|\nabla N(x-y)|\chi_{[\frac{\delta}{2},\infty)}(|x-y|)$.

We split Ω in a set close to ∂A and the complement of this. To do so, we set $T_t := \{x \in \Omega : \operatorname{dist}(x, \partial A) < t\}$ for some fixed $t > 0$. The volume of T_t can be estimated with the help of the coarea formula (see, e.g., [6, Section 3.4]). This yields that the volume of T_t is bounded by a constant times t times $\mathcal{H}^{d-1}(\partial A)$, which is finite by assumption.

If $x \in \Omega \backslash T_t$, it holds $|\nabla N(x-y)| \leq c t^{1-d}$ for all $y \in \partial A$. Moreover, $\frac{c}{\delta}|N(x-y)|\chi_{[\frac{\delta}{2},\infty)}(|x-y|) \leq \frac{c}{\delta}|\ln t|\chi_{[\frac{\delta}{2},\infty)}(|x-y|)$ if $d = 2$. Hence $(\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x)$ as well as $(\nabla \mathscr{S}_{\partial A}^{(\delta)})(g)(x)$ converge uniformly to $\nabla \mathscr{S}_{\partial A}(g)(x) = \int_{\partial A} g(y) \nabla N(x-y)\, ds_y$ for all $x \in \Omega \setminus T_t$ as $\delta \to 0$. Therefore it is natural to consider

$$\int_\Omega |(\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x) - \nabla \mathscr{S}_{\partial A}(g)(x)|\, dx \\ \leq \int_{\Omega \backslash T_t} |(\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x) - \nabla \mathscr{S}_{\partial A}(g)(x)|\, dx + \int_{T_t} |(\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x)|\, dx \\ + \int_{T_t} |\nabla \mathscr{S}_{\partial A}(g)(x)|\, dx\,,$$

and correspondingly for $(\nabla \mathscr{S}_{\partial A}^{(\delta)})(g)$. It remains to show that for all $\varepsilon > 0$ there exists a $t > 0$ such that the latter two integrals are bounded by $\frac{\varepsilon}{2}$. Since $|(\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x)| \leq c \int_{\partial A} |\nabla N(x-y)|\, ds_y$, an integration over T_t yields

$$\int_{T_t} (\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x)\, dx \leq c \int_{T_t} \int_{\partial A} \frac{1}{|x-y|^{d-1}}\, ds_y\, dx = c \int_{\partial A} \int_{T_t} \frac{1}{|x-y|^{d-1}}\, dx\, ds_y.$$

By Hölder's inequality, we obtain for $y \in \partial A$

$$\int_{T_t} \frac{1}{|x-y|^{d-1}}\,dx \le \left(\int_{T_t} 1\,dx\right)^{\frac{1}{2d-1}} \left(\int_{T_t} \frac{1}{|x-y|^{\frac{2d-1}{2}}}\,dx\right)^{\frac{2d-2}{2d-1}}. \tag{17}$$

The latter integral can be estimated by $c\int_0^{\mathrm{diam}(\Omega)} r^{-\frac{1}{2}}\,dr$ and thus is bounded independently of t. Moreover we know that the volume of T_t is bounded by $ct\mathcal{H}(\partial A)$. Hence $\int_{T_t} |(\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x)|\,dx \le ct^{\frac{1}{2d-1}}$, which can be chosen to be smaller than $\frac{\varepsilon}{2}$. Since $|\nabla \mathscr{S}_{\partial A}(g)(x)|$ is bounded by a constant times $\int_{\partial A} \frac{1}{|x-y|^{d-1}}\,ds_y$, we obtain analogously $\int_{T_t} |\nabla \mathscr{S}_{\partial A}(g)(x)|\,dx \le \frac{\varepsilon}{2}$.

To finish the proof, we need to show similar estimates for $(\nabla \mathscr{S}_{\partial A}^{(\delta)})(g)$. If $d \ge 3$, we know that $|(\nabla \mathscr{S}_{\partial A}^{(\delta)})(g)(x)| \le c\int_{\partial A} |\nabla N(x-y)|\,ds_y$ and thus we can proceed as before. It remains to consider the case $d = 2$ and to estimate the term

$$\frac{c}{\delta}\int_{T_t}\int_{\partial A} |\ln|x-y||\chi_{[\frac{\delta}{2},\delta)}(|x-y|)ds_y dx \le \frac{c}{\delta}\int_{\partial A}\int_{T_t} |\ln|x-y||\chi_{[\frac{\delta}{2},\delta)}(|x-y|)dx ds_y.$$

Hölder's inequality yields

$$\begin{aligned} &\frac{1}{\delta}\int_{T_t} |\ln|x-y||\chi_{[\frac{\delta}{2},\infty)}(|x-y|)\,dx \\ &\le \frac{1}{\delta}\left(\int_{T_t} |x-y||\ln|x-y||^2\chi_{[\frac{\delta}{2},\delta)}(|x-y|)\,dx\right)^{\frac{1}{2}}\left(\int_{T_t} |x-y|^{-1}\,dx\right)^{\frac{1}{2}} \\ &\le ct^{\frac{1}{6}}. \end{aligned} \tag{18}$$

Indeed, the second integral on the right hand side is bounded by $ct^{\frac{1}{3}}$, cf. (17). The first integral is bounded by $c\int_{\frac{\delta}{2}}^{\delta} r^2 \ln^2 r\,dr = c\big[\frac{r^3\ln^2 r}{3} - \frac{2}{3}r^3\big(\frac{\ln r}{3} - \frac{1}{9}\big)\big]_{\frac{\delta}{2}}^{\delta} \le c\delta^3\ln^2\delta$. Now, with $\frac{1}{\delta}(\delta^3\ln^2\delta)^{\frac{1}{2}} < 1$ for small δ, we obtain the asserted bound in (18) and the asserted convergence in $L^1(\Omega)$ as $\delta \to 0$. □

Next we consider normal and tangential derivatives of the single layer potential on the boundaries and the interface Γ. As mentioned earlier, we follow [28] mainly. Verchota assumes that A is a bounded Lipschitz domain (with a fixed regular family of cones) and that $g \in L^p(\partial A)$, $1 < p < \infty$; both requirements are satisfied here by Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$. According to, e.g., Verchota [28, Theorem 1.11], the normal component of the gradient of $\mathscr{S}_{\partial A}(g)$ at ∂A is

$$\big(n\cdot\nabla \mathscr{S}_{\partial A}(g)\big)^{\pm}(x) = \lim_{\beta\to 0} n(x)\cdot(\nabla \mathscr{S}_{\partial A})(g)(x \pm \beta n(x)) \tag{19}$$

$$= \mp\frac{1}{2}g(x) - p.v.\int_{\partial A} g(y)n(x)\cdot\frac{(x-y)}{|x-y|^d}\,ds_y \tag{20}$$

$$:= \mp\frac{1}{2}g(x) - \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} g(y)n(x)\cdot\frac{(x-y)}{|x-y|^d}\,ds_y \tag{21}$$

for almost every $x \in \partial A$. The principal value on the right hand side of (20) exists in $L^p(\partial A)$, $1 < p < \infty$, and pointwise almost everywhere. Throughout the paper, Cauchy principal values (*p.v.*) of integrals are defined analogously as in (21).

If A is a smooth domain (for instance $C^{1,\alpha}$, $0 < \alpha \leq 1$) and g is a continuous function, the convergence of the traces in (19) is uniform in $x \in \partial A$ (see, e.g., [7, Chapter 3.F], [18, Section 18.7]). It is well known that this breaks down if the domain has edges or corners (see, e.g., [11, p. 157]). In this article we consider Lipschitz domains which are merely piecewise $C^{1,\alpha}$, cf. Definition 2.1. Thus we do not have uniform convergence in general. However, we still have uniform convergence on compactly embedded subsets of the smooth parts of the boundary, cf. Lemma 3.5 below.

While the normal component of the gradient of the single layer potential jumps, its tangential component is continuous across ∂A. It follows from [28, Theorem 1.6] that

$$\big(\nabla_{\tan}\mathscr{S}_{\partial A}(g)\big)^{+}(x) = \big(\nabla_{\tan}\mathscr{S}_{\partial A}(g)\big)^{-}(x) = p.v.\int_{\partial A} g(y)\nabla_{\tan}N(x-y)\,ds_y \quad (22)$$

exists in $L^p(\partial A)$, $1 < p < \infty$, and for almost every $x \in \partial A$.

Kellog [11, p. 162] proves uniform convergence of the tangential derivative in three-dimensional C^2 domains under the assumption that the surface density g is uniformly Hölder continuous. He also mentions that this result applies to compactly embedded subsets of C^2 submanifolds [11, p. 160].

In the following proposition we consider, under Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$, the convergence of $(\nabla_{\tan}\mathscr{S}^{(\delta)}_{\partial A})(g)$ to the tangential derivative of the single layer potential as $\delta \to 0$. In Proposition 3.3 below we then prove an analogous statement for the convergence of $(\nabla_{\tan}\mathscr{S}_{\partial A})^{(\delta)}(g)$. The proofs are generalizations of the proof of an analogous statement in [25, Section 6.3] under the assumption that A is a C^2 domain in $\mathbb{R}^3$.

Proposition 3.2. *Let Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold and let $U_i \subset \partial A$ be one of the $C^{1,\alpha}$ submanifolds in Definition* 2.1*. Furthermore, let U be a compactly embedded subset of U_i. Then*

$$\nabla_{\tan}\mathscr{S}_{\partial A}(g)(x) = \lim_{\delta\to 0}\big(\nabla_{\tan}\mathscr{S}^{(\delta)}_{\partial A}\big)(g)(x)$$

uniformly for all $x \in U$.

Proof. Let $\mathcal{U}$ be a neighbourhood of $x \in U$ and let $\psi : \mathcal{U}' \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$ be a parameterization of $\partial A \cap \mathcal{U}$ such that $\psi(x') = x$ for fixed $x' \in \mathcal{U}'$, cf. Figure 2.

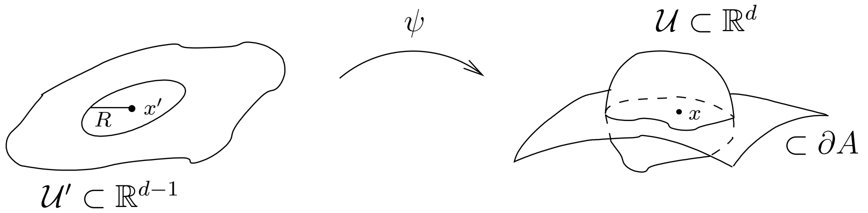

Figure 2: Parameterization of $\partial A \cap \mathcal{U}$ in the proof of Proposition 3.2.

We choose $\mathcal{U}$ and ψ such that $\psi^{-1}(\partial A \cap \mathcal{U}) = B_R(x')$ with some constant $R > 0$, where $B_R(x')$ denotes the $d-1$ dimensional ball of radius R about x', which reduces to the interval $[x'-R, x'+R]$ if $d = 2$. Notice that $D\psi(x')$ maps $\mathbb{R}^{d-1}$ on the tangent space at $x = \psi(x')$; if $d = 2$, $D\psi(x')$ simply is $\frac{d}{dx'}\psi(x')$. Therefore, the uniform convergence of $(\nabla_{\tan}\mathscr{S}_{\partial A}^{(\delta)})(g)(x)$ as $\delta \to 0$ is equivalent to uniform convergence of

$$\nabla_x \mathscr{S}_{\partial A}(g)(x) \cdot D\psi(x')v = \lim_{\delta \to 0} \int_{\partial A} g(y) \nabla\left(\eta\left(\frac{|x-y|}{\delta}\right) N(x-y)\right) \cdot D\psi(x')v \, ds_y$$

for all $v \in \mathbb{R}^{d-1}$. Note that $\nabla_{x'}(\mathscr{S}_{\partial A}(g) \circ \psi)(x')v = \nabla_x \mathscr{S}_{\partial A}(g)(x) \cdot D\psi(x')v$, where $\nabla_{x'} = \frac{d}{dx'}$ if $d = 2$. Hence we obtain the proposition if we show that

$$\nabla_{x'}\big(\mathscr{S}_{\partial A}^{(\delta)}(g) \circ \psi\big) \longrightarrow \nabla_{x'}\big(\mathscr{S}_{\partial A}(g) \circ \psi\big) \tag{23}$$

uniformly as $\delta \to 0$. We rewrite $\mathscr{S}_{\partial A}^{(\delta)}(g)$ as

$$\begin{aligned}
&\mathscr{S}_{\partial A}^{(\delta)}(g)(x) \\
&= \int_{\partial A \cap \mathcal{U}} g(y)\eta\left(\frac{|x-y|}{\delta}\right) N(x-y)\, ds_y + \int_{\partial A \setminus \mathcal{U}} g(y)\eta\left(\frac{|x-y|}{\delta}\right) N(x-y)\, ds_y \\
&= \int_{\partial A \cap \mathcal{U}} g(y)\eta\left(\frac{|x-y|}{\delta}\right) N(x-y)\, ds_y + \int_{\partial A \setminus \mathcal{U}} g(y) N(x-y)\, ds_y,
\end{aligned}$$

where the second equality holds for all δ smaller than the minimal distance between x and $\partial\mathcal{U}$ by the definition of η. Hence we only need to consider the first term, which we denote by $\mathscr{S}_{\partial A}^{(1,\delta)}(g)$, to prove (23). By a change of variables we obtain

$$\begin{aligned}
\mathscr{S}_{\partial A}^{(1,\delta)}(g)(x) &= \int_{B_R(x')} \eta\left(\frac{|\psi(x') - \psi(y')|}{\delta}\right) g(\psi(y')) N(\psi(x') - \psi(y')) J_\psi(y')\, d^{d-1}y' \\
&=: Q_\delta(x').
\end{aligned}$$

Correspondingly we define $Q(x')$. Since $g \circ \psi$ and J_ψ are bounded on $B_R(x')$,

$$\big|Q(x') - Q_\delta(x')\big| \le \begin{cases} c\,\big|\int_0^{C\delta} \ln r\, dr\big| & \text{if } d = 2 \\ c\int_{B_{C\delta}(x')} \frac{1}{|x'-y'|^{d-2}}\, d^{d-1}y' & \text{if } d \ge 3 \end{cases}$$
$$\le \begin{cases} c\delta|\ln(C\delta)|, & \text{if } d = 2 \\ c\delta, & \text{if } d \ge 3\,. \end{cases}$$

Thus $Q_\delta(x')$ converges uniformly to $Q(x')$ as $\delta \to 0$. Next we show that $\nabla_{x'} Q_\delta(x')$ converges uniformly to $\nabla_{x'} Q(x')$ as $\delta \to 0$ by proving that $\nabla_{x'} Q_\delta(x')$ is a Cauchy sequence in C^0 as $\delta \to 0$. Hence Q is C^1 and $\nabla_{x'} Q_\delta$ converges uniformly to $\nabla_{x'} Q$.

To prove that $\nabla_{x'} Q_\delta(x')$ is a Cauchy sequence in C^0 as $\delta \to 0$, let $\varepsilon \in [\frac{\delta}{2}, \delta)$ initially. We have

$$Q_\varepsilon(x') - Q_\delta(x') = \int_{B_R(x')} f(|\psi(x') - \psi(y')|) g(\psi(y')) J_\psi(y')\, d^{d-1}y', \tag{24}$$

where $f(t) := \big(\eta(\frac{t}{\varepsilon}) - \eta(\frac{t}{\delta})\big)\widetilde{N}(t)$ with $\widetilde{N}(|x-y|) := N(x-y)$. The support of f is contained in $[\delta/4, \delta]$, and, for small enough $\delta > 0$,

$$\begin{aligned} |f'(t)| &\le \left|\frac{d}{dt}\left(\eta\Big(\frac{t}{\varepsilon}\Big) - \eta\Big(\frac{t}{\delta}\Big)\right)\right| |\widetilde{N}(t)| + \left|\eta\Big(\frac{t}{\varepsilon}\Big) - \eta\Big(\frac{t}{\delta}\Big)\right| |\widetilde{N}'(t)| \\ &\le \frac{c}{t}|\widetilde{N}(t)| + ct^{1-d} \\ &\le c \begin{cases} \frac{|\ln t|}{t} & \text{if } d = 2 \\ t^{1-d} & \text{if } d \ge 3 \end{cases}. \end{aligned} \tag{25}$$

Moreover, $f \in C^\infty$, and $f(|\psi(x') - \psi(y')|)$ vanishes for all y' in a neighborhood of $\partial B_R(x')$. When we differentiate (24), we can therefore commute differentiation and integration. Hence

$$\begin{aligned} &\big|\nabla_{x'} Q_\varepsilon(x') - \nabla_{x'} Q_\delta(x')\big| \\ &\quad = \left| \int_{B_R(x')} \nabla_{x'} f(|\psi(x') - \psi(y')|) g(\psi(y')) J_\psi(y')\, d^{d-1}y' \right| \qquad (26) \\ &\quad \le \int_{B_R(x')} \big|\nabla_{x'} f(|\psi(x') - \psi(y')|) + \nabla_{y'} f(|\psi(x') - \psi(y')|)\big| \\ &\qquad\qquad \times |g(\psi(y')) J_\psi(y')|\, d^{d-1}y' \qquad (27) \\ &\qquad + \left| \int_{B_R(x')} \nabla_{y'} f(|\psi(x') - \psi(y')|) g(\psi(y')) J_\psi(y')\, d^{d-1}y' \right|. \qquad (28) \end{aligned}$$

Since ψ is $C^{1,\alpha}$, $|D\psi(x') - D\psi(y')| \le c|x'-y'|^\alpha$. This in turn is bounded by $c|\psi(x')-\psi(y')|^\alpha$. By (25) we then obtain for small enough δ that the integral in (27) is bounded by

$$
\begin{aligned}
c \int_{B_R(x')} & \left| f'(|\psi(x')-\psi(y')|) \frac{\psi(x')-\psi(y')}{|\psi(x')-\psi(y')|} (D\psi(x') - D\psi(y')) \right| \, d^{d-1}y' \\
& \le c \begin{cases} \frac{|\ln|\delta/4||}{\delta} \delta^\alpha \int_{c\delta/4}^{c\delta} dr & \text{if } d=2 \\ \delta^{1-d+\alpha} \int_{c\delta/4}^{c\delta} r^{d-2}\, dr & \text{if } d \ge 3 \end{cases} \\
& \le c \begin{cases} \delta^\alpha |\ln \delta| & \text{if } d=2 \\ \delta^\alpha & \text{if } d \ge 3\,. \end{cases}
\end{aligned} \tag{29}
$$

Hence it remains to estimate (28). By adding and subtracting $J_\psi(x')$ we can make again use of ψ being $C^{1,\alpha}$. We have

$$
\begin{aligned}
\Bigg| \int_{B_R(x')} & \nabla_{y'} f(|\psi(x')-\psi(y')|) g(\psi(y')) \big(J_\psi(y') - J_\psi(x')\big) \, d^{d-1}y' \Bigg| \\
& \le \Bigg| \int_{B_R(x')} \big(\nabla_{y'} f(|\psi(x')-\psi(y')|) + \nabla_{x'} f(|\psi(x')-\psi(y')|) \big) \\
& \qquad \times \; g(\psi(y')) \big(J_\psi(y') - J_\psi(x')\big) \, d^{d-1}y' \Bigg| \\
& \quad + \Bigg| \int_{B_R(x')} \nabla_{x'} f(|\psi(x')-\psi(y')|) g(\psi(y')) \big(J_\psi(y') - J_\psi(x')\big) \, d^{d-1}y' \Bigg| .
\end{aligned}
$$

The first term on the right hand side can be estimated as (27), but simpler. The second term can be bounded by

$$
c|D\psi(x')| \int_{B_R(x')} |f'(|\psi(x')-\psi(y')|)| \, |y'-x'|^\alpha \, d^{d-1}y' \le c \begin{cases} \delta^\alpha |\ln \delta| & \text{if } d=2 \\ \delta^\alpha & \text{if } d \ge 3\,, \end{cases}
$$

analoglously to (29). The bound on $D\psi(x')$ is uniform since $x = \psi(x') \in U$, which is compactly embedded in U_i by assumption. To estimate (28), we consider $J_\psi(x') \int_{B_R(x')} \nabla_{y'} f(|\psi(x')-\psi(y')|) g(\psi(y')) d^{d-1}y'$. We add and subtract $g(\psi(x'))$. By Remark 2.3, $g \in C^{0,\alpha}$ almost everywhere. Hence

$$
|g(\psi(y')) - g(\psi(x'))| \le c|\psi(y') - \psi(x')|^\alpha \quad \text{for a.e. } \psi(x'), \psi(y') \in \mathcal{U}.
$$

Since $D\psi(y')$ is uniformly bounded on the ball $B_R(x')$, we can estimate the term $\int_{B_R(x')} \nabla_{y'} f(|\psi(x')-\psi(y')|) \big(g(\psi(y')) - g(\psi(x'))\big) \, d^{d-1}y'$ as in (29). Finally, since f is zero on the boundary, an integration by parts of the remaining term, $J_\psi(x') g(\psi(x')) \int_{B_R(x')} \nabla_{y'} f(|\psi(x')-\psi(y')|) \, d^{d-1}y'$, yields that it vanishes. Hence (26) is bounded by $c\delta^\alpha |\ln \delta|$ and $c\delta^\alpha$, respectively.

Recall that we have assumed $\varepsilon \in [\frac{\delta}{2}, \delta)$. We obtain the same estimates for arbitrary $0 < \varepsilon < \delta$ by summing a geometric series. Hence $\nabla_{x'} Q_\delta(x')$ is a Cauchy sequence as $\delta \to 0$, and $\nabla_{x'} Q_\delta$ and therefore $\nabla_{x'}\big(\mathscr{S}_{\partial A}^{(1,\delta)}(g) \circ \psi\big)$ converge uniformly as $\delta \to 0$. Together with the uniform convergence $\mathscr{S}_{\partial A}^{(\delta)}(g) \to \mathscr{S}_{\partial A}(g)$ this proves (23) and hence the proposition. □

Proposition 3.3. *Let Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold and let $U_i \subset \partial A$ be one of the $C^{1,\alpha}$ submanifolds in Definition* 2.1. *Furthermore, let U be a compactly embedded subset of U_i. Then*

$$\nabla_{\tan}\mathscr{S}_{\partial A}(g)(x) = \lim_{\delta\to 0}(\nabla_{\tan}\mathscr{S}_{\partial A})^{(\delta)}(g)(x)$$

uniformly for all $x \in U$.

Proof. We use the same parametrization ψ of the boundary as in the proof of Proposition 3.2. Since (23) holds, we only need to prove that

$$\left|\nabla_{x'}\big(\mathscr{S}_{\partial A}^{(\delta)}(g) \circ \psi\big)(x') \cdot v - \int_{\partial A} g(y)\eta\Big(\frac{|x-y|}{\delta}\Big)\nabla N(x-y)\cdot D\psi(x')v\,ds_y\right| \leq \begin{cases} c\delta^\alpha |\ln\delta| & \text{if } d=2 \\ c\delta^\alpha & \text{if } d \geq 3\,, \end{cases} \tag{30}$$

where

$$\nabla_{x'}(\mathscr{S}_{\partial A}^{(\delta)}(g)\circ\psi)(x')\cdot v = \int_{\partial A}\nabla_{x'}\Big(\eta\Big(\frac{|\psi(x')-y|}{\delta}\Big)\Big)\cdot v\, g(y)N(\psi(x')-y)ds_y + \int_{\partial A}\eta\Big(\frac{|\psi(x')-y|}{\delta}\Big)g(y)\nabla N(x-y)\cdot D\psi(x')v ds_y.$$

Recall that $x = \psi(x')$. Thus (30) follows if the first integral is bounded by $c\delta^\alpha|\ln\delta|$ and $c\delta^\alpha$, respectively. Note that the first integral vanishes trivally if $|y-x| > \delta$. Similarly as in the proof of Proposition 3.2 we rewrite this integral using the change of variables $y = \psi(y')$. We obtain that it is bounded by $c|I|$ with

$$I := \int_{B_{c\delta}(x')} h(\psi(x') - \psi(y'))\cdot D\psi(x')v\, j(y')\, d^{d-1}y',$$

where $h(w) = \frac{1}{\delta}\eta'\big(\frac{|w|}{\delta}\big)\frac{w}{|w|}N(w)$ and $j(y') = g(\psi(y'))J_\psi(y')$. Since $g \in C^{0,\alpha}$ almost everywhere and $\psi \in C^{1,\alpha}$, we have $|j(y') - j(x')| \leq c|x'-y'|^\alpha$ almost everywhere for some constant $c > 0$. Since $\eta'(\frac{\cdot}{\delta})$ is supported on $[\frac{\delta}{2}, \delta]$, we have $|h(w)| \leq c\frac{|\ln\delta|}{\delta}$ if $d = 2$ and $|h(w)| \leq c\delta^{1-d}$ else. Thus

$$\left|\int_{B_{c\delta}(x')} h(\psi(x')-\psi(y'))\cdot D\psi(x')v\,\big(j(y')-j(x')\big)\,d^{d-1}y'\right| \leq \begin{cases} c\delta^\alpha|\ln\delta| & \text{if } d=2 \\ c\delta^\alpha & \text{if } d\geq 3. \end{cases}$$

In order to show that I is bounded by $c\delta^\alpha|\ln\delta|$ and $c\delta^\alpha$, respectively, it remains to show that

$$\int_{B_{c\delta}(x')} h(\psi(y')-\psi(x'))\cdot D\psi(x')v\, j(x')\, d^{d-1}y' \tag{31}$$

is at most bounded by $c\delta^\alpha|\ln\delta|$ and $c\delta^\alpha$, respectively. Set $z'=x'-y'$. Since ψ is $C^{1,\alpha}$, $\psi(x')-\psi(y')=D\psi(x')z'+\mathcal{O}(|z'|^{1+\alpha})$. Moreover we know that h is smooth and $|Dh(w)|\le c\delta^{-2}|\ln\delta|$ if $d=2$ and $|Dh(w)|\le c\delta^{-d}$ else. Thus

$$h(\psi(x')-\psi(y'))=h(D\psi(x')z')+\begin{cases}\mathcal{O}(\delta^{\alpha-1}|\ln\delta|) & \text{if } d=2\\ \mathcal{O}(\delta^{1+\alpha-d}) & \text{if } d\ge 3\end{cases}$$

and therefore we have

$$I=\int_{B_{c\delta}(0)} h(D\psi(x')z')\cdot D\psi(x')v\, j(x')\, d^{d-1}z'+\begin{cases}\mathcal{O}(\delta^{\alpha}|\ln\delta|) & \text{if } d=2\\ \mathcal{O}(\delta^{\alpha}) & \text{if } d\ge 3\,.\end{cases}$$

The integral on the right hand side vanishes since h is antisymmetric and the domain of integration is invariant under $z'\mapsto -z'$. Hence (30) is proved, which finishes the proof of Proposition 3.3. □

Remark 3.4. The above proof can also be adapted to show convergence of the corresponding Cauchy principal integrals as defined in (22).

Next we come back to the normal derivative of the single layer potential. Mikhlin [18, Satz 18.7.1] proved uniform convergence of the normal derivative of the single layer potential on closed Ljapunov-surfaces and for continuous g. He used Cauchy principal integrals in his proof. The proof can be adapted in a straightforward way such that we also have uniform convergence on compactly embedded subsets of $C^{1,\alpha}$ submanifolds. Here we show that the same convergence result also holds for the smooth regularizations defined above.

Lemma 3.5. *Let Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold and let U_i be one of the $C^{1,\alpha}$ submanifolds in Definition* 2.1. *Furthermore, let U be a compactly embedded subset of U_i. Then*

$$\begin{aligned}\big(n\cdot\nabla\mathscr{S}_{\partial A}(g)\big)^{\pm}(x) &= \mp\frac{1}{2}g(x)-p.v.\int_{\partial A} g(y)\frac{n(x)\cdot(x-y)}{|x-y|^d}\,ds_y\\ &= \mp\frac{1}{2}g(x)+\lim_{\delta\to 0}\big(n\cdot\nabla\mathscr{S}^{(\delta)}_{\partial A}\big)(g)(x) \qquad (32)\\ &= \mp\frac{1}{2}g(x)+\lim_{\delta\to 0}\big(n\cdot\nabla\mathscr{S}_{\partial A}\big)^{(\delta)}(g)(x) \qquad (33)\end{aligned}$$

uniformly for all $x\in U$.

Proof. As commented on above, the first equation follows from [18, Satz 18.7.1] and Remark 2.3. To prove (32) and (33), we apply a central estimate in the proof of the convergence of the normal derivative of the single layer potential (see, e.g., [18, Section 18.1]): Since $U \subset \partial A$ is $C^{1,\alpha}$, $|n(x)\cdot(x-y)| \leq c|x-y|^{1+\alpha}$ for all $x, y \in U$. Hence

$$\begin{aligned}
\Big| \int_{\partial A} &\chi_{[\varepsilon,\infty)}(|x-y|)g(y)n(x)\cdot\nabla N(x-y)\,ds_y - (n\cdot\nabla\mathscr{S}_{\partial A})^{(\delta)}(g)(x)\Big| \\
&\leq c\int_{\partial A}\big|\chi_{[\varepsilon,\infty)}(|x-y|)n(x)\cdot\nabla N(x-y) - n(x)\cdot R^{(\delta)}(x-y)\big|\,ds_y \\
&\leq c\int_{\partial A}\chi_{[\min\{\varepsilon,\frac{\delta}{2}\},\max\{\varepsilon,\delta\}]}(|x-y|)\big|n(x)\cdot\nabla N(x-y)\big|\,ds_y, \\
&\leq c\int_{\min\{\varepsilon,\frac{\delta}{2}\}}^{\max\{\varepsilon,\delta\}}\frac{r^\alpha}{r^{d-1}}r^{d-2}\,dr \\
&\leq c\max\{\varepsilon,\delta\}^\alpha,
\end{aligned}$$

which tends to zero as $\varepsilon, \delta \to 0$ and proves (33). To prove (32), it remains to estimate

$$\begin{aligned}
\int_{\partial A}&\left|n(x)\cdot\nabla\eta\left(\frac{|x-y|}{\delta}\right)\right||N(x-y)|\,ds_y \\
&\leq \frac{c}{\delta}\int_{\partial A}|x-y|^\alpha\chi_{[\frac{\delta}{2},\delta]}(|x-y|)|N(x-y)|\,ds_y, \qquad (34)
\end{aligned}$$

where we used that $\big|n(x)\cdot\nabla\eta(\frac{|x-y|}{\delta})\big| \leq \frac{c}{\delta}|n(x)\cdot\frac{x-y}{|x-y|}|\chi_{[\frac{\delta}{2},\delta]}(|x-y|)$. If $d=2$, (34) is bounded by $c\delta^\alpha|\ln\delta|$; if $d\geq 3$, (34) is bounded by $c\delta^\alpha$. Both bounds tend to zero as $\delta \to 0$. □

By writing the gradient as a linear combination of normal and tangential derivatives, we obtain

$$\big(\nabla\mathscr{S}_{\partial A}(g)\big)^{\pm}(x) = \mp\frac{1}{2}g(x)n(x) + \mathscr{B}_{\partial A}(g)(x) \qquad (35)$$

in $L^p(\partial A)$, $1 < p < \infty$, and pointwise for almost every $x \in \partial A$, where

$$\mathscr{B}_{\partial A}(g)(x) := p.v.\int_{\partial A} g(y)\nabla N(x-y)\,ds_y \qquad (36)$$

is defined for any function $g \in L^p(\partial A)$, $1 < p < \infty$ and for any $x \in \mathbb{R}^d$. Again, since $\mathcal{H}^{d-1}(\partial A)$ is bounded, this statement also holds if g is assumed to be in $L^\infty(\partial A)$.

Remark 3.6.

(i) If $x \in \mathbb{R}^d \setminus \partial A$, the principal value in (36) becomes trivial, i.e., $\mathscr{B}_{\partial A}(g) = \nabla \mathscr{S}_{\partial A}(g)$.

(ii) On ∂A, $\mathscr{B}_{\partial A}(g)$ equals the average of the inner and outer traces of $\nabla \mathscr{S}_{\partial A}$, i.e., $\mathscr{B}_{\partial A}(g) = \frac{1}{2}\big((\nabla \mathscr{S}_{\partial A}(g))^+ + (\nabla \mathscr{S}_{\partial A}(g))^-\big)$.

(iii) From what we showed above, it follows that we could equivalently define $\mathscr{B}_{\partial A}(g)$ by using the smooth regularizations, i.e., by replacing the right hand side in (36) with $\lim_{\delta\to 0} \big(\nabla \mathscr{S}^{(\delta)}_{\partial A}\big)(g)(x)$ or $\lim_{\delta \to 0}(\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x)$.

Set $(\nabla \mathscr{S}_{\partial A}(g))(x) = (\nabla_{\tan} \mathscr{S}_{\partial A}(g))(x) + (n \cdot \nabla \mathscr{S}_{\partial A}(g))(x) n(x)$. Then Proposition 3.2 and Lemma 3.5 yield the following corollary. Recall that pointwise convergence holds for all $x \in \partial A$ according to [28].

Corollary 3.7. *Let Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold and let $U_i \subset \partial A$ be one of the $C^{1,\alpha}$ submanifolds in Definition* 2.1. *Furthermore, let U be a compactly embedded subset of U_i. Then*

$$(\nabla \mathscr{S}_{\partial A}(g))^{\pm}(x) = \mp \frac{1}{2} g(x) n(x) + \mathscr{B}_{\partial A}(g)(x)$$

for all $x \in U$.

Next we consider the parts of ∂A which are close to edges and corners. From [28] we also know that $\mathscr{B}_{\partial A}(g)$ exists in $L^p(\partial A)$, $1 < p < \infty$, and hence in $L^1(\partial A)$ by Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$. A simple proof of this in the context of our geometrical setting and for the regularization as in (15) close to edges and corners follows easily from [25, p. 257]; it makes in particular use of Assumption $\mathcal{A}_1$(iv) and yields the following lemma.

Lemma 3.8. *Let $r_0 > 0$ and let $U_i \subset \partial A$ be as in Definition* 2.1. *Set $\Lambda = \bigcup_i \partial U_i$ and $\mathcal{V}_{r_0} = \{x \in \partial A : \operatorname{dist}(x, \Lambda) < r_0\}$. Then*

$$\begin{aligned} \big(\nabla \mathscr{S}^{(\delta)}_{\partial A}\big)(g) &\longrightarrow \mathscr{B}_{\partial A}(g) \quad \text{in } L^1(\mathcal{V}_{r_0}) \text{ as } \delta \to 0 \\ (\nabla \mathscr{S}_{\partial A})^{(\delta)}(g) &\longrightarrow \mathscr{B}_{\partial A}(g) \quad \text{in } L^1(\mathcal{V}_{r_0}) \text{ as } \delta \to 0 \,. \end{aligned}$$

Note that $g \in L^\infty(\partial A)$ by Assumption $\mathcal{A}_2$. Hence the convergence of the gradient of the single layer potential in $L^1(\partial A)$ implies

$$\begin{aligned} \lim_{\delta\to 0} \int_{\partial A} g(y) \big(\nabla \mathscr{S}^{(\delta)}_{\partial A}\big)(g)(x-y)\, ds_y &= \lim_{\delta \to 0} \int_{\partial A} g(y) (\nabla \mathscr{S}_{\partial A})^{(\delta)}(g)(x-y)\, ds_y \\ &= \int_{\partial A} g(y) \mathscr{B}_{\partial A}(g)(x-y)\, ds_y, \end{aligned} \tag{37}$$

which is a useful statement in, e.g., the context of magnetic forces [22, 25], cf. also the remarks in the introduction.

4. Approximation of the gradient of solutions of the Poisson equation

First we consider the approximation of the Newton potential $\mathscr{V}_A(w)$ defined in (9). Recall (13) for the definition of $\mathscr{V}_A^{(\delta)}(w)$. The Newton potential $\mathscr{V}_{\Omega\setminus\Gamma}(w)$ and its regularization $\mathscr{V}_{\Omega\setminus\Gamma}^{(\delta)}(w)$ are defined accordingly. Since Γ is a set of d-dimensional Lebesgue measure zero, one might want to write $\mathscr{V}_{\Omega\setminus\Gamma}(w) = \mathscr{V}_\Omega(w)$. But we stick to the more complicated notation as this reminds us of the definition of $w = \operatorname{div} M$ on $\mathbb{R}^d \setminus (\Gamma \cup \partial\Omega)$ and the regularity assumptions on M, cf. Assumption $\mathcal{A}_2$. Furthermore, we set

$$\begin{aligned}(\nabla\mathscr{V}_{\Omega\setminus\Gamma})^{(\delta)}(w)(x) &= \int_{\Omega\setminus\Gamma} w(y)\eta\Big(\frac{|x-y|}{\delta}\Big)\nabla N(x-y)\,dy\\ &= \int_{\Omega\setminus\Gamma} w(y)R^{(\delta)}(x-y)\,dy\,.\end{aligned}$$

Proposition 4.1. *Let Assumptions* $\mathcal{A}_1$(i)–(ii) *and* $\mathcal{A}_2$ *hold. Then*

$$\begin{aligned}(\nabla\mathscr{V}_{\Omega\setminus\Gamma}^{(\delta)})(w)(x) &\longrightarrow \nabla\mathscr{V}_{\Omega\setminus\Gamma}(w)(x) \quad \textit{uniformly in } x\in\mathbb{R}^d \textit{ as } \delta\to 0\\ (\nabla\mathscr{V}_{\Omega\setminus\Gamma})^{(\delta)}(w)(x) &\longrightarrow \nabla\mathscr{V}_{\Omega\setminus\Gamma}(w)(x) \quad \textit{uniformly in } x\in\mathbb{R}^d \textit{ as } \delta\to 0.\end{aligned}$$

Proof. Let $x \in \mathbb{R}^d$. By construction, $1-\eta(\frac{|x-y|}{\delta})$ is supported in the ball $B_\delta(x)$ and its derivative has support in $B_\delta(x)\setminus B_{\frac{\delta}{2}}(x)$. Moreover, $|\nabla(1-\eta(\frac{|x-y|}{\delta}))| \le \frac{c}{\delta}\chi_{B_\delta(x)\setminus B_{\frac{\delta}{2}}(x)}(y)$. Hence

$$\begin{aligned}&\left|\nabla\Big(\Big(1-\eta\Big(\frac{|x-y|}{\delta}\Big)\Big)N(x-y)\Big)\right|\\ &\qquad\le \frac{c}{\delta}|N(x-y)|\chi_{B_\delta(x)\setminus B_{\frac{\delta}{2}}(x)}(y) + |\nabla N(x-y)|\chi_{B_\delta(x)}(y).\end{aligned}$$

Since w is essentially bounded on $\Omega\setminus\Gamma$ by assumption, we obtain

$$\begin{aligned}&\big|(\nabla\mathscr{V}_{\Omega\setminus\Gamma})(w)(x) - (\nabla\mathscr{V}_{\Omega\setminus\Gamma}^{(\delta)})(w)(x)\big|\\ &\qquad\le c\int_{\Omega\setminus\Gamma}\left|\nabla\Big(\Big(1-\eta\Big(\frac{|x-y|}{\delta}\Big)\Big)N(x-y)\Big)\right|\,dy\\ &\qquad\le \frac{c}{\delta}\int_{B_\delta(x)\setminus B_{\frac{\delta}{2}}(x)}|N(x-y)|\,dy + c\int_{B_\delta(x)}|\nabla N(x-y)|\,dy\\ &\qquad\le \begin{cases} c\delta|\ln\delta| & \text{if } d=2\\ c\delta & \text{if } d\ge 3\end{cases}\end{aligned}$$

uniformly for all $x\in\mathbb{R}^d$ for small $\delta>0$. This also yields that $|\nabla\mathscr{V}_{\Omega\setminus\Gamma}(w)(x) - (\nabla\mathscr{V}_{\Omega\setminus\Gamma})^{(\delta)}(w)(x)| \le c\int_{B_\delta(x)}|\nabla N(x-y)|\,dy$ converges uniformly. □

Next we make use of the splitting of u_Ω in $u_A + u_B$, cf. (5), and put together the above results for the single layer potentials. Here we consider the integral representation of the solution u_Ω of Poisson's equation (1)–(2).

Theorem 4.2. *Let Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold. Let u_Ω be a solution of Poisson's equation* (1) *with transition condition* (2) *and let $u_\Omega^{(\delta)}$ be its regularization analogously to* (12). *Furthermore define $(\nabla u_\Omega)^{(\delta)}$ correspondingly to* (15), *i.e.,*

$$(\nabla u_\Omega)^{(\delta)}(x) = \int_{\Omega\setminus\Gamma} w(y) R^{(\delta)}(x-y)\, dy + \int_\Gamma g(y) R^{(\delta)}(x-y)\, ds_y + \int_{\partial A\setminus\Gamma} g(y) R^{(\delta)}(x-y)\, ds_y + \int_{\partial B\setminus\Gamma} g(y) R^{(\delta)}(x-y)\, ds_y.$$

Then $(\nabla u_\Omega^{(\delta)})$ as well as $(\nabla u_\Omega)^{(\delta)}$ converge to

$$\nabla u_\Omega = \nabla \mathscr{V}_{\Omega\setminus\Gamma}(w) + \nabla \mathscr{S}_\Gamma(-[g]) + \nabla \mathscr{S}_{\partial A\setminus\Gamma}(g^-) + \nabla \mathscr{S}_{\partial B\setminus\Gamma}(g^{\nu^-}) \quad \text{in } L^1(\Omega) \tag{38}$$

and to

$$\overline{\nabla u_\Omega} = \nabla \mathscr{V}_{\Omega\setminus\Gamma}(w) + \mathscr{B}_\Gamma(-[g]) + \mathscr{B}_{\partial A\setminus\Gamma}(g^-) + \mathscr{B}_{\partial B\setminus\Gamma}(g^{\nu^-}) \quad \text{in } L^1(\Gamma \cup \partial\Omega)$$

as $\delta \to 0$. The convergence is uniform on compactly embedded subsets of the $C^{1,\alpha}$ submanifolds of Γ and $\partial\Omega$. Moreover, the gradient in the tangential direction at $x \in \partial\Omega \cup \Gamma$ is continuous across the interface and is given by

$$(\nabla_{\tan} u_\Omega)^\pm(x) = \nabla_{\tan} \mathscr{V}_{\Omega\setminus\Gamma}(w)(x) + \nabla_{\tan} \mathscr{S}_\Gamma(-[g])(x) + \nabla_{\tan} \mathscr{S}_{\partial A\setminus\Gamma}(g^-)(x) + \nabla_{\tan} \mathscr{S}_{\partial B\setminus\Gamma}(g^{\nu^-})(x)$$

for almost every $x \in \partial\Omega \cup \Gamma$. The gradient in the normal direction jumps at the interface and reads

$$(n \cdot \nabla u_\Omega)^\pm(x) = (n \cdot \nabla \mathscr{V}_{\Omega\setminus\Gamma})(w)(x) + (n \cdot \nabla \mathscr{S}_\Gamma)(-[g])(x) + (n \cdot \nabla \mathscr{S}_{\partial A\setminus\Gamma})(g^-)(x) + (n \cdot \nabla \mathscr{S}_{\partial B\setminus\Gamma})(g^{\nu^-})(x)$$

for almost every $x \in \partial\Omega \cup \Gamma$. The inner and outer traces of the gradient of u_Ω are given by

$$(\nabla u_\Omega)^\pm(x) = \nabla \mathscr{V}_{\Omega\setminus\Gamma}(w)(x) + \mathscr{B}_\Gamma(-[g])(x) + \mathscr{B}_{\partial A\setminus\Gamma}(g^-)(x) + \mathscr{B}_{\partial B\setminus\Gamma}(g^{\nu^-})(x) \mp \frac{1}{2} \begin{cases} -[g](x) n(x) & \text{a.e. if } x \in \Gamma \\ g^-(x) n(x) & \text{a.e. if } x \in \partial A \setminus \Gamma \\ g^{\nu^-}(x) \nu(x) & \text{a.e. if } x \in \partial B \setminus \Gamma\,. \end{cases}$$

Thus there holds

$$\overline{\nabla u_\Omega}(x) = \frac{1}{2}\big((\nabla u_\Omega)^+(x) + (\nabla u_\Omega)^-(x)\big)$$
$$= \begin{cases} [g](x)n(x) + (\nabla u_\Omega)^-(x) & a.e.\ if\ x \in \Gamma \\ -g^-(x)n(x) + (\nabla u_\Omega)^-(x) & a.e.\ if\ x \in \partial A \setminus \Gamma \\ -g^{\nu^-}(x)\nu(x) + (\nabla u_\Omega)^-(x) & a.e.\ if\ x \in \partial B \setminus \Gamma\,. \end{cases}$$

All the traces mentioned as well as the gradients of the potentials can be approximated by the use of either smooth regularization.

Proof. By (5) and (8), the integral representation of u_Ω reads

$$u_\Omega(x) = (u_A + u_B)(x)$$
$$= \mathscr{V}_{\Omega\setminus\Gamma}(w)(x) + \mathscr{S}_\Gamma(-[g])(x) + \mathscr{S}_{\partial A\setminus\Gamma}(g^-)(x) + \mathscr{S}_{\partial B\setminus\Gamma}(g^{\nu^-})(x),$$

where we assume as before that the standard normal to Γ is n. Similarly we have formulae for $u_\Omega^{(\delta)}$, $(\nabla u_\Omega^{(\delta)})$ and $(\nabla u_\Omega)^{(\delta)}$. The statements in the theorem then follow from Propositions 3.1 and 4.1 for the convergence in $L^1(\Omega)$, and from Propositions 3.2 and 3.3 as well as Lemma 3.5, Lemma 3.8 and Corollary 3.7 for the convergence on the boundaries. Note that these assertions hold analogously for the single layer potentials $\mathscr{S}_\Gamma(-[g])$ and $\mathscr{S}_{\partial B\setminus\Gamma}(g^{\nu^-})$ as well as for their gradients. □

5. About the second gradient of a solution of Poisson's equation

As already mentioned in the introduction, the second gradient of solutions of Poisson's equation is of interest for instance in the context of magnetic forces. There, a typical expression is an integral over a bounded domain A of a vector-valued $W^{1,\infty}$ function M times the second gradient of a solution of Poisson's equation, $\nabla H = -\nabla(\nabla u)$, i.e., $\int_A (M(x)\cdot\nabla)H(x)\,dx$. We prove existence of such expressions by showing that the second gradient of u is an L^1 function on the bulk, i.e., on A in the above example. This then also allows for instance to integrate by parts.

Recall that the gradient of u jumps at interfaces and surfaces. Hence the distributional derivative of the gradient of u is not integrable on these interfaces and surfaces, respectively. Here we show integrability of the second gradient of u on the bulk up to interfaces and surfaces (and not only local integrability away from interfaces and surfaces).

For a study of second derivatives of the Newton potential we use a well-known L^p estimate, see, e.g., [8, Section 9.4] or [19, Chapter XI §11], to obtain the following lemma. Though Γ is a set of d dimensional Lebesgue measure zero, we write $L^1(\Omega \setminus \Gamma)$ to remind us of the regularity assumption on M and thus on $w = -\operatorname{div} M$ at the interface Γ, cf. Assumption $\mathcal{A}_2$.

Lemma 5.1. *Let Assumptions* $\mathcal{A}_1$(i)–(ii) *and* $\mathcal{A}_2$ *hold. Then*

$$\nabla(\nabla \mathscr{V}_{\Omega\setminus\Gamma})(w) \equiv \nabla^2 \mathscr{V}_{\Omega\setminus\Gamma}(w) \equiv \big(\partial_i \partial_j \mathscr{V}_{\Omega\setminus\Gamma}(w)\big)_{i,j=1,\dots,d} \in L^1(\Omega \setminus \Gamma).$$

Proof. Since Ω is a bounded domain, we have $w \in L^p(\Omega \setminus \Gamma)$, $1 < p < \infty$, by Assumption $\mathcal{A}_2$. Hence, a special case of the Calderon-Zygmund inequality, see, e.g., [8, Theorem 9.9], yields $\nabla^2 \mathscr{V}_{\Omega\setminus\Gamma}(w) \in L^p(\Omega\setminus\Gamma)$, which implies $\nabla^2 \mathscr{V}_{\Omega\setminus\Gamma}(w) \in L^1(\Omega \setminus \Gamma)$ by the boundedness of Ω. □

The proof of the following theorem is in the line of corresponding assertions in [25, Section 4] for $d = 3$ and a simpler geometric setting. Here we make use of Assumption $\mathcal{A}_1$(iv) and Definition 2.1(iii) and have to bear in mind that the sets A and B do not have to be nested, cf. Figure 3. In Remark 3.6(i) we noted that $\mathscr{B}_{\partial A}(g^-)(x) = \nabla \mathscr{S}_{\partial A}(g^-)(x)$ if $x \in \mathbb{R}^d \setminus \partial A$, which holds similarly for $\mathscr{B}_\Gamma(-[g])$ and $\mathscr{B}_{\partial B}(g^{\nu^-})$. However, we stick to the $\mathscr{B}$s for brevity in the following theorem.

Theorem 5.2. *Let Assumptions* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *hold and assume that* $\alpha = 1$. *Then*

$$\nabla \mathscr{B}_\Gamma(-[g]),\ \nabla \mathscr{B}_{\partial A}(g^-),\ \nabla \mathscr{B}_{\partial B}(g^{\nu^-}) \in L^1(\Omega \setminus \Gamma).$$

Proof. First we consider $\nabla \mathscr{B}_\Gamma(-[g])$. Let $x \in \Omega\setminus\Gamma$. Since $\mathcal{H}^{d-1}(\partial A \cap B(x,\rho)) \le c\rho^{d-1}$, we have

$$\begin{aligned}
\big|(\nabla \mathscr{B}_\Gamma(-[g]))(x)\big| &\le c \int_{\partial A} \frac{1}{|x-y|^d}\, ds_y \\
&= \int_{\partial A} \left(\int_0^\infty \chi_{|x-y|<\rho} \frac{d}{\rho^{d+1}}\, d\rho \right) ds_y \qquad (39)\\
&= c \int_{\operatorname{dist}(x,\partial A)}^\infty \frac{d}{\rho^{d+1}} \mathcal{H}^{d-1}(\partial A \cap B(x,\rho))\, d\rho \\
&\le c \int_{\operatorname{dist}(x,\partial A)}^\infty \frac{1}{\rho^2}\, d\rho \\
&\le \frac{c}{\operatorname{dist}(x,\partial A)}. \qquad (40)
\end{aligned}$$

We split $\Omega\setminus\Gamma$ into the following three sets, cf. Figure 3. Fix some $0 < r_0 < 1$ and $\varepsilon > 0$ and denote by Λ the union of all the boundaries of the $C^{1,1}$ submanifolds V_i of $\partial(\Omega \setminus \Gamma)$, cf. Definition 2.1. Then we set

$$(\Omega \setminus \Gamma)^{(1)} := \big\{x \in \Omega \setminus \Gamma \,:\, \operatorname{dist}(x,\partial A) \ge r_0^{1+\varepsilon}\big\}$$
$$(\Omega \setminus \Gamma)^{(2)} := \big\{x \in \Omega \setminus \Gamma \,:\, r_0^{1+\varepsilon} \ge \operatorname{dist}(x,\partial A) \ge \operatorname{dist}(x,\Lambda)^{1+\varepsilon}\big\}$$
$$(\Omega \setminus \Gamma)^{(3)} := \big\{x \in \Omega \setminus \Gamma \,:\, \operatorname{dist}(x,\partial A) \le \operatorname{dist}(x,\Lambda)^{1+\varepsilon},\ \operatorname{dist}(x,\partial A) \le r_0^{1+\varepsilon}\big\}.$$

By (40) and the definition of $(\Omega \setminus \Gamma)^{(1)}$ we have $\int_{(\Omega\setminus\Gamma)^{(1)}} |(\nabla \mathscr{B}_\Gamma(-[g]))(x)|\, dx \le \frac{c}{r_0^{1+\varepsilon}} \le c$.

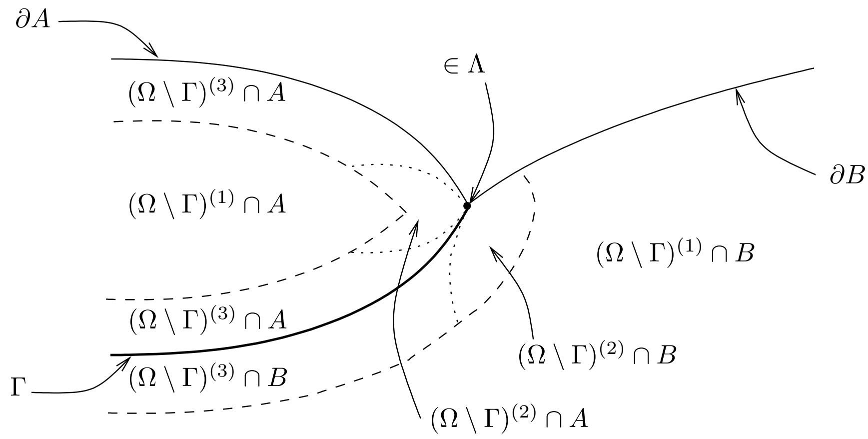

Figure 3: Sketch of the sets $(\Omega \setminus \Gamma)^{(1)}$, $(\Omega \setminus \Gamma)^{(2)}$ and $(\Omega \setminus \Gamma)^{(3)}$ as defined in the proof of Theorem 5.2. In this sketch we consider a geometric setting where (a portion of) Λ happens to be in the relative boundary of Γ.

Next we estimate $\nabla \mathscr{B}_\Gamma(-[g])$ integrated over $(\Omega \setminus \Gamma)^{(2)}$. The definition of $(\Omega \setminus \Gamma)^{(2)}$ and (40) yield $|(\nabla \mathscr{B}_\Gamma(-[g]))(x)| \le \frac{c}{\operatorname{dist}(x,\Lambda)^{1+\varepsilon}}$ for all $x \in (\Omega \setminus \Gamma)^{(2)}$. The volume of $(\Omega \setminus \Gamma)^{(2)}$ is bounded by the volume of $\{x \in \Omega \setminus \Gamma : \operatorname{dist}(x, \Lambda) \le r_0\}$. The latter volume can be estimated by cr_0^2. Indeed, let V_i denote the $C^{1,1}$ submanifolds as in Definition 2.1. Then the volume of $\{x \in \Omega \setminus \Gamma : \operatorname{dist}(x, \Lambda) \le r_0\}$ is bounded by the finite sum of the volumes of $\{x \in \Omega \setminus \Gamma : \operatorname{dist}(x, \partial V_i) \le r_0\}$. These are in turn bounded by cr_0^2 by the neighbourhood estimate, cf. Definition 2.2. Thus, by similar estimates as in (39)–(40),

$$\begin{aligned}\int_{(\Omega\setminus\Gamma)^{(2)}} \frac{1}{\operatorname{dist}(x,\Lambda)^{1+\varepsilon}}\, dx &= \int_0^\infty \frac{1+\varepsilon}{\rho^{2+\varepsilon}} \int_{(\Omega\setminus\Gamma)^{(2)}} \chi_{\operatorname{dist}(x,\Lambda)<\rho}\, dx\, d\rho \\ &\le c\int_0^{r_0} \frac{1+\varepsilon}{\rho^{2+\varepsilon}} \rho^2\, d\rho + c\int_{r_0}^\infty \frac{1+\varepsilon}{\rho^{2+\varepsilon}}\, d\rho \\ &= c\frac{1+\varepsilon}{1-\varepsilon} r_0^{1-\varepsilon} + c\frac{1}{r_0^{1+\varepsilon}} \le c,\end{aligned}$$

which proves the integrability of $\nabla \mathscr{B}_\Gamma(-[g])$ on $(\Omega \setminus \Gamma)^{(2)}$.

It remains to show the integrability on $(\Omega \setminus \Gamma)^{(3)}$. Let $x \in (\Omega \setminus \Gamma)^{(3)}$ and set $r = \min\{\operatorname{dist}(x, \Lambda), r_0\}$. Then we obtain, similarly to the derivation of (40),

$$\left| \int_{\Gamma\setminus B(x,r)} [g](y) \nabla^2 N(x-y)\, ds_y \right| \le c \int_{\Gamma\setminus B(x,r)} \frac{1}{|x-y|^d}\, ds_y \le \frac{c}{r} \le \frac{c}{\operatorname{dist}(x,\partial A)^{\frac{1}{1+\varepsilon}}},$$

which is integrable on $(\Omega \setminus \Gamma)^{(3)}$. Indeed, the coarea formula yields that the volume of $\{x \in (\Omega \setminus \Gamma)^{(3)} : \operatorname{dist}(x, \partial A) < \rho\}$ is bounded by $c\rho$ for $\rho < r_0^{1+\varepsilon}$. Thus we have

$$\begin{aligned}
\int_{(\Omega\setminus\Gamma)^{(3)}} \frac{1}{\operatorname{dist}(x,\partial A)^{\frac{1}{1+\varepsilon}}}\,dx &= \int_{(\Omega\setminus\Gamma)^{(3)}} \int_0^\infty \chi_{\operatorname{dist}(x,\partial A)<\rho}\,\frac{\frac{1}{1+\varepsilon}}{\rho^{\frac{1}{1+\varepsilon}+1}}\,d\rho\,dx \\
&\le c\int_0^{r_0^{1+\varepsilon}} \frac{1}{\rho^{\frac{1}{1+\varepsilon}}}\,d\rho + c\int_{r_0^{1+\varepsilon}}^\infty \frac{1}{\rho^{\frac{1}{1+\varepsilon}+1}}\,d\rho \\
&= \frac{c}{1-\frac{1}{1+\varepsilon}} r_0^{(1+\varepsilon)(1-\frac{1}{1+\varepsilon})} + \frac{c}{r_0^{(1+\varepsilon)\frac{1}{1+\varepsilon}}} \le c.
\end{aligned} \tag{41}$$

To finish the proof of the integrability on $(\Omega \setminus \Gamma)^{(3)}$, we need to estimate the integral $\int_{\Gamma\cap B(x,r)}[g](y)\nabla^2 N(x-y)\,ds_y$, which we write as

$$[g](x)\int_{\Gamma\cap B(x,r)} \nabla^2 N(x-y)\,ds_y + \int_{\Gamma\cap B(x,r)} \big([g](y)-[g](x)\big)\nabla^2 N(x-y)\,ds_y. \tag{42}$$

This allows us to integrate the first term by parts. Before we do so, we consider the second integral in (42). Recall that we assume $\alpha = 1$ and that $[g]$ is in $C^{0,1}$ almost everywhere, cf. Remark 2.3. Hence a similar estimate as in (40) yields that the second integral in (42) is bounded by a constant times

$$\begin{aligned}
\int_{\Gamma\cap B(x,r)} \frac{1}{|x-y|^{d-1}}\,ds_y &\le c\int_{\operatorname{dist}(x,\Gamma A)}^r \frac{1}{\rho^d}\rho^{d-1}\,d\rho \\
&\le c\ln\frac{1}{\operatorname{dist}(x,\Gamma)} \\
&\le c\ln\frac{1}{\operatorname{dist}(x,\partial A)}.
\end{aligned} \tag{43}$$

As this is bounded by $\operatorname{dist}(x,\partial A)^{-\frac{1}{1+\varepsilon}}$, we obtain the integrability on $(\Omega \setminus \Gamma)^{(3)}$ of the second term in (42) by (41).

We next estimate the first term in (42), which takes some time. Without loss of generality we assume that $\Gamma \cap B(x,r)$ is connected and contained in one of the $C^{1,1}$ submanifolds V_i; otherwise apply the following arguments to each of the connected components restricted to one of the $C^{1,1}$ submanifolds.

We parameterize the boundary $\Gamma \cap B(x,r)$ by a function $\psi : \mathbb{R}^{d-1} \to \mathbb{R}^d$, $y' \mapsto \psi(y') = (y', \psi_d(y')) = y$ such that either (i) there is a $\tilde{c}$ with $0 < \tilde{c} < 1$ and $B(x', \tilde{c}r) \subset \psi^{-1}(\Gamma \cap B(x,r))$, where x' is the orthogonal projection of x on $\mathbb{R}^{d-1}$, or (ii) there is a $\tilde{c}$ with $0 < \tilde{c} < 1$ and $B(x', \tilde{c}r) \cap \psi^{-1}(\Gamma \cap B(x,r)) = \emptyset$. (That is, we exclude the case $x' \in \partial(\psi^{-1}(\Gamma \cap B(x,r))$ by choosing an appropriate parameterization.) Case (ii) is of interest when $x \in (\Omega \setminus \Gamma)^{(3)}$ is close to the relative boundary $\partial\Gamma$; it is not needed in the proofs of the statements for $\nabla\mathscr{B}_{\partial A}(g^-)$ and $\nabla\mathscr{B}_{\partial B}(g^{\nu^-})$ below. If (ii) holds, we always have $|y'-x'|>\tilde{c}r$. Since $|x-\psi(y')|^d \geq |x'-y'|^d$, then $\int_{\Gamma\cap B(x,r)} \nabla^2 N(x-y)\,ds_y$ is bounded by

$$\left|\int_{\psi^{-1}(\Gamma\cap B(x,r))} \nabla^2 N(x-\psi(y'))J_\psi(y')\,d^{d-1}y'\right| \leq c\int_{\tilde{c}r}^{r} \rho^{-d}\rho^{d-2}\,d\rho \\ \leq \frac{c}{r} \leq \frac{c}{\operatorname{dist}(x,\Gamma)^{\frac{1}{1+\varepsilon}}}\,, \tag{44}$$

which is integrable on $(\Omega\setminus\Gamma)^{(3)}$ by (41). If (i) holds, we split the integral $\int_{\psi^{-1}(\Gamma\cap B(x,r))} \nabla^2 N(x-\psi(y'))J_\psi(y')\,d^{d-1}y'$ as follows:

$$\int_{\psi^{-1}(\Gamma\cap B(x,r))\setminus B(x',\tilde{c}r)} \nabla^2 N(x-\psi(y'))J_\psi(y')\,d^{d-1}y' \\ + \int_{B(x',\tilde{c}r)} \nabla^2 N(x-\psi(y'))J_\psi(y')\,d^{d-1}y'. \tag{45}$$

The first integral in (45) can be estimated similarly as in (44) and thus is also bounded by $c\operatorname{dist}(x,\Gamma)^{-\frac{1}{1+\varepsilon}}$. Hence it is integrable on $(\Omega\setminus\Gamma)^{(3)}$ by (41).

Next we estimate the second integral in (45). To do so, we write $\nabla^2 N(x-\psi(y'))$ in terms of tangential and normal derivatives. Let $(t_1,\dots,t_{d-1},n)$ be an orthonormal basis at $\psi(y')$, where n is the normal at $\psi(y')$ to ∂A. The normal is $C^{0,1}$ on $\partial A\cap B(x,r)$ and hence on $\Gamma\cap B(x,r)$. We denote the tangential and normal derivatives of N by $\nabla_{t_i}N = t_i\cdot\nabla N$, $i=1,\dots,d-1$, and $\nabla_n N = n\cdot\nabla N$, respectively.

Recall that N is a fundamental solution of Laplace's equation, i.e., $\Delta N=0$. Hence $\nabla_n^2 N = -\sum_{i=1}^{d-1}\nabla_{t_i}^2 N$. We therefore only need to consider second derivatives of the form $(\nabla_{t_i}\nabla_n)N$ and $(\nabla_{t_i}\nabla_{t_j})N$ with $i,j=1,\dots,d-1$. By the product rule we have $(\nabla_{t_i}\nabla_n)N = \nabla_{t_i}(\nabla_n N) - (\nabla N)\nabla_{t_i}n$ and

$$(\nabla_{t_i}\nabla_{t_j})N = \nabla_{t_i}(\nabla_{t_j}N) - (\nabla N)\nabla_{t_i}t_j, \tag{46}$$

respectively. We write the tangential derivative in terms of y' so that we can make use of Assumption $\mathcal{A}_1$. There is an invertible matrix $(a_{ij})_{i,j=1,\dots,d-1}$ such that $t_i(\psi(y')) = a_{ik}(y')\partial_k\psi(y')$. Since we assume here that $\Gamma\cap B(x,r)$ is piecewise $C^{1,1}$, $a_{ik}(\cdot)$ is $C^{0,1}$, as is the tangent vector t_i. Hence $|\nabla_{t_i}t_j|\leq c$ almost everywhere. Thus the second term on the right hand side of (46) is bounded by $c|\nabla N|$. Similarly we can bound $(\nabla N)\nabla_{t_i}n$ by $c|\nabla N|$.

Next we write the first term on the right hand side of (46) in terms of y':

$$\big(\nabla_{t_i}(\nabla_{t_j}N)\big)(x-\psi(y')) = \big(a_{ik}(y')(\partial_k\psi)(y')\cdot\nabla(\nabla_{t_j}N)\big)(x-\psi(y')) \\ = -\big(a_{ik}(y')\partial_k(\nabla_{t_j}N)\big)(x-\psi(y')).$$

Similarly we obtain $\nabla_{t_i}(\nabla_n N)(x-\psi(y')) = -\big(a_{ik}(y')\partial_k(\nabla_n N)\big)(x-\psi(y'))$. This allows us to integrate by parts:

$$\int_{B(x',\tilde{c}r)} (\nabla_{t_i}(\nabla_n N))(x-\psi(y'))J_\psi(y')\,d^{d-1}y'$$
$$= \int_{B(x',\tilde{c}r)} \partial_k\big(a_{ik}(y')J_\psi(y')\big)(\nabla_n N)(x-\psi(y'))\,d^{d-1}y'$$
$$- \int_{\partial B(x',\tilde{c}r)} a_{ik}(y')(\nabla_n N)(x-\psi(y'))J_\psi(y')\,ds_y^{d-2}$$

(with $\partial B(x',\tilde{c}r) = \{-|\tilde{c}r - x'|, |\tilde{c}r - x'|\}$ if $d=2$). Therefore we obtain for all $d \geq 2$ by using the bounds $|\partial_k a_{ik}(y')| \leq c$ and $|\partial_k J_\psi(y')| \leq c$

$$\int_{B(x',\tilde{c}r)} \nabla^2 N(x-\psi(y'))J_\psi(y')\,d^{d-1}y'$$
$$\leq c\int_{B(x',\tilde{c}r)} |\nabla N(x-\psi(y'))|\,d^{d-1}y' + c\int_{\partial B(x',\tilde{c}r)} |\nabla N(x-\psi(y'))|\,d^{d-2}s_{y'}$$
$$\leq \int_{B(x',\tilde{c}r)} \frac{c}{|x-\psi(y')|^{d-1}}\,d^{d-1}y' + \int_{\partial B(x',\tilde{c}r)} \frac{c}{|x-\psi(y')|^{d-1}}\,d^{d-2}s_{y'}.$$

Now we use again that $x \in \Omega\setminus\Gamma$ and thus $|x-\psi(y')| \geq \operatorname{dist}(x,\Gamma) \geq \operatorname{dist}(x,\partial A) > 0$. Hence the first integral is bounded by $c\ln\frac{1}{\operatorname{dist}(x,\partial A)}$, which is integrable on $(\Omega\setminus\Gamma)^{(3)}$, see (43). Since $|x-\psi(y')| \geq |x'-y'|$, the boundary integral can be estimated by $\int_{\partial B(x',\tilde{c}r)} \frac{c}{|x'-y'|^{d-1}}\,d^{d-2}y'$, which in turn is bounded by $c\frac{r^{d-2}}{r^{d-1}} = \frac{c}{r}$. Recall that $r = \min\{\operatorname{dist}(x,\Lambda), r_0\}$ and $x \in (\Omega\setminus\Gamma)^{(3)}$. Hence $r^{-1} \leq \operatorname{dist}(x,\partial A)^{-\frac{1}{1+\varepsilon}}$, which is integrable on $(\Omega\setminus\Gamma)^{(3)}$, see (41). Hence $\nabla\mathscr{B}_\Gamma(-[g]) \in L^1(\Omega\setminus\Gamma)$.

Finally, notice that $|\nabla\mathscr{B}_{\partial A}(g^-)(x)| \leq \int_{\partial A}\frac{1}{|x-y|^d}\,ds_y$. To show that the gradient of $\mathscr{B}_{\partial A}(g^-)$ is in $L^1(\Omega\setminus\Gamma)$, we thus can proceed analogously as from (39) onwards. It remains to prove that $\nabla\mathscr{B}_{\partial B}(g^{\nu^-}) \in L^1(\Omega\setminus\Gamma)$. This also can be shown analogously to the above proof, but now we change the definitions of the sets $(\Omega\setminus\Gamma)^{(i)}$, $i=1,2,3$: We replace $\operatorname{dist}(x,\partial A)$ with $\operatorname{dist}(x,\partial B)$ and then proceed as before. □

Finally, we conclude that the second gradient of the solution of Poisson's equation with transition condition is in $L^1(\Omega\setminus\Gamma)$.

Theorem 5.3. *Let Assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold and assume that $\alpha = 1$. Then*

$$\nabla(\nabla u_\Omega) \in L^1(\Omega\setminus\Gamma).$$

Proof. This follows with Lemma 5.1 and Theorem 5.2 together with (38) and Remark 3.6(i). □

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