

Discontinuous Irregular Oblique Derivative Problems for Nonlinear Elliptic Equations of Second Order

Guochun Wen and Zuoliang Xu

Abstract. In this paper, the discontinuous irregular oblique derivative problems (or discontinuous Poincaré boundary value problems) for nonlinear elliptic equations of second order in multiply connected domains are discussed by using a complex analytic method. Firstly the uniqueness of solutions for such boundary value problems is proved and a priori estimates of their solutions are given, and then by the Leray-Schauder theorem, the existence of solutions of the above problems is verified. As a special case the result about the continuous irregular oblique derivative problem for the nonlinear equations is derived.

Keywords. Nonlinear elliptic equations, discontinuous irregular oblique derivative problems, complex analytic method

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1. Formulation of the problem

It is known that there are many problems in mechanics and physics, whose boundary conditions are discontinuous (see [1,3]). In [1,2], H. Begehr and G. C. Wen studied discontinuous Riemann-Hilbert problem and oblique derivative problem for nonlinear elliptic complex equations, and proved the existence and uniqueness of these boundary value problems. In [3], V. N. Monakhov considered the Poincaré problem with discontinuous coefficients in the boundary condition for a quasilinear second order elliptic equation by using function theoretic methods. This paper deals with the discontinuous irregular oblique

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derivative problems for nonlinear elliptic equations of second order. The results in this paper are further developments of the theory of nonlinear elliptic complex equations as stated in [1–7].

Let D be an $(N + 1)$ -connected bounded domain in the complex plane \mathbb{C} with the boundary $\Gamma = \bigcup_{j=0}^N \Gamma_j \in C^2_\mu$ ($0 < \mu < 1$), where Γ_j ($j = 1, \dots, N$) are located in the domain $D_0 = \{|z| < 1\}$ bounded by $\Gamma_0 = \Gamma_{N+1}$, and $0 \in D$. Without loss of generality, we can assume that D is a circular domain in $|z| < 1$ with the boundary $\Gamma_{N+1} = \{|z| = 1\}$ and $\Gamma_j = \{|z - a_j| = r_j\}$, $j = 1, \dots, N$. Introduce the second order nonlinear elliptic equation in the complex form (see [6])

$$\begin{cases} u_{z\bar{z}} = F(z, u, u_z, u_{zz}) \\ F = \operatorname{Re}[Qu_{zz} + A_1u_z] + \hat{A}_2u + A_3 \\ Q = Q(z, u, u_z, u_{zz}) \\ A_j = A_j(z, u, u_z), \quad j = 1, 2, 3 \\ \hat{A}_2 = A_2 + |u|^\sigma, \end{cases} \tag{1}$$

where σ is a positive number. Moreover we consider the equation of second order

$$u_{xx} + b(u)u_{xy} + u_{yy} - |u|^\sigma u = 0 \text{ in } D$$

with the coefficient $b(u) = u^2$ if $|u| \leq 1$ and $b(u) = u^{-2}$ if $|u| > 1$, whose complex form is

$$u_{z\bar{z}} = \operatorname{Re}[Qu_{zz}] + \hat{A}_2u \text{ in } D,$$

where $Q = Q(u) = -\frac{1}{2}ib(u)$, $\hat{A}_2 = \frac{1}{4}|u|^\sigma$ with the condition $|Q| \leq \frac{1}{2} < 1$. It is clear that the above complex equation is a special case of (1).

Suppose that (1) satisfies the following

Conditions C^* :

- 1) $Q(z, u, u_z, U)$, $A_j(z, u, u_z)$, $j = 1, 2, 3$, are continuous in $u \in \mathbb{R}$, $u_z \in \mathbb{C}$ for almost every $z \in D$, $U \in \mathbb{C}$, and $Q = 0$, $A_j = 0$, $j = 1, 2, 3$, for $z \notin D$.
- 2) The above functions are measurable in $z \in D$ for any continuously differentiable function $u(z)$ in D^* and any measurable function $U(z) \in L_{p_0}(\bar{D})$, and satisfy

$$L_p[A_1(z, u, u_z), \bar{D}] \leq k_0, L_p[A_2(z, u, u_z), \bar{D}] \leq k_0, L_p[A_3(z, u, u_z), \bar{D}] \leq k_1.$$

Here p_0, p ($2 < p_0 \leq p$), k_0, k_1 are non-negative constants, and $D^* = \bar{D} \setminus T$, $T = \{z_1, \dots, z_{m+1}\}$, z_1, \dots, z_{m+1} (or $z_{K'}$, K' is an integer as stated in (2) below) are distinct points arranged according to the positive direction on Γ , \bar{D} is any closed subdomain in D^* .

3) The equation (1) satisfies the uniform ellipticity condition

$$|F(z, u, u_z, U_1) - F(z, u, u_z, U_2)| \leq q_0|U_1 - U_2|, \quad \hat{A}_2 \geq 0 \text{ in } D,$$

for almost every point $z \in D$, any continuously differentiable function $u(z)$ in D^* and $U_1, U_2 \in \mathbb{C}$, where $q_0 < 1$ is a non-negative constant. We can assume that there exists at least one point of discontinuity on Γ_j ($1 \leq j \leq N + 1$). Because for any component Γ_j ($1 \leq j \leq N + 1$) if $\lambda(z)$ on such component is continuous, we can choose a point on this component as a point of discontinuity. It is obvious that in this case the partial index K_j of $\lambda(z)$ on Γ_j ($1 \leq j \leq N + 1$) is unchanged. Denote by Γ'_k the curve on Γ from z_{k-1} to z_k (if there is only one point of discontinuity on Γ_j ($1 \leq j \leq N + 1$), then $z_k = z_{k-1}$ ($1 \leq k \leq m + 1$)), and in general Γ'_k includes the initial point z_{k-1} and does not include the terminal point z_k ($k = 2, \dots, m + 2$).

Problem P. Find a continuous solution $u(z)$ of (1) in \overline{D} , whose partial derivatives u_x, u_y in $D^* = \overline{D} \setminus T$ are continuous, and satisfies the boundary conditions

$$\frac{\partial u}{\partial \nu} + 2a_1(z)u(z) = 2a_2(z), \quad z \in \Gamma^* = \Gamma \setminus T, \quad u(z_k) = b_k, \quad k = 0, 1, \dots, K', \quad (2)$$

where $\nu (= \nu_1 + i\nu_2 \neq 0)$ can be arbitrary at every point on Γ^* , $K' (\geq J')$ is a non-negative integer, $z_0 \in \Gamma_{J+1}$, the numbers $J, J' (\geq 0)$ will be stated below. It is easy to see that the boundary condition (2) can be rewritten in the complex form

$$\operatorname{Re}[\overline{\lambda(z)}u_z] + a_1(z)u = a_2(z), \quad z \in \Gamma^*, \quad u(z_k) = b_k, \quad k = 0, 1, \dots, K', \quad (3)$$

in which $\lambda(z) = \cos(\nu, x) + i \sin(\nu, x) = e^{i(\nu, x)} \neq 0$, (ν, x) is the angle between ν and the x -axis. It is easy to see that $u(z_k) = b_k$ ($k = 0, 1, \dots, K'$) are definite, because $u(z)$ is continuous in \overline{D} and u_x, u_y or u_z at z_j ($j = 0, 1, \dots, K'$) maybe the points of discontinuity. Suppose that $\lambda(z), a_1(z), a_2(z)$ satisfy the conditions

$$\begin{aligned} C_\alpha[\lambda(z), \Gamma'_k] &\leq k_0, & |b_k| &\leq k_2, \quad k = 0, 1, \dots, K', \\ C_\alpha[a_1(z), \Gamma'_k] &\leq k_0, & 0 &\leq a_1(z) \cos(\nu, n) \text{ on } \Gamma^* = \Gamma \setminus T, \\ C_\alpha[R_k(z)a_2(z), \Gamma'_k] &\leq k_2, & k &= 1, \dots, m + 1, \end{aligned} \quad (4)$$

in which n is the outward normal vector on Γ^* and

$$R_k(z) = |z - z_{k-1}|^{\beta_{k-1}}|z - z_k|^{\beta_k},$$

$\beta_k < 1$ ($k = 1, \dots, m + 1$), and α ($\frac{1}{2} < \alpha < 1$), k_2 are non-negative constants.

For convenience, we assume that $\cos(\nu, n) \equiv 0, a_1(z) \equiv 0$ on Γ'_k ($k = 1, \dots, J$), and the above identical formulas on Γ'_k ($J + 1 \leq k \leq m + 1$) do not hold. But if L is the longest curve on Γ'_k ($J + 1 \leq k \leq m + 1$) such that $\cos(\nu, n) = 0, a_1(z) = 0$, then we require that it appears one of the following four cases:

1. One end point of L belongs to $\tilde{T} = \{z_0, z_1, \dots, z_{K'}\}$ and $\int_{\Gamma_j} a_2(z)ds = 0$, if $\cos(\nu, n) = 0$, $a_1(z) = 0$ on Γ_j , $1 \leq j \leq J' (\leq N + 1)$.
2. The inequality $\cos(\nu, n) \cos(\nu, s) > 0$ in the neighborhood of the initial point of L on $\Gamma'_k \setminus L$ holds.
3. The inequality $\cos(\nu, n) \cos(\nu, s) < 0$ in the neighborhood of the terminal point of L on $\Gamma'_k \setminus L$ holds, where s is the tangent vector of Γ'_k identical to the positive direction of Γ , and $\Gamma' = \bigcup_{k=1}^J \Gamma'_k$, $\Gamma'' = \bigcup_{k=J+1}^{m+1} \Gamma'_k$.
4. If the initial and terminal points of L do not conform to the conditions in cases 2 and 3, then there is one point condition which belongs to \tilde{T} . Problem P with homogeneous boundary condition of (2) or (3) is called Problem P_0 .

The number $K = \frac{1}{2}(K_1 + \dots + K_{m+1})$ is called the index of Problem P , where

$$\begin{aligned}
 K_k &= \left[\frac{\phi_k}{\pi} \right] + J_k, & J_k &= 0 \text{ or } 1, \\
 e^{i\phi_k} &= \frac{\lambda(z_k - 0)}{\lambda(z_k + 0)}, & \gamma_k &= \frac{\phi_k}{\pi} - K_k, & k &= 1, \dots, m + 1.
 \end{aligned}
 \tag{5}$$

Here we choose the index $K \geq N - \frac{1}{2}$ and denote $K' = 2K - 2N + J' + 1$.

Problem Q. In addition we consider the discontinuous boundary value problem: Equation (1) and boundary condition (2) are replaced by

$$\begin{cases}
 u_{z\bar{z}} = G(z, u, u_z, u_{zz}) \\
 G = \text{Re}[Qu_{zz} + A_1u_z] + A_3 \\
 Q = Q(z, u, u_z, u_{zz}) \\
 A_j = A_j(z, u, u_z), \quad j = 1, 3.
 \end{cases}
 \tag{6}$$

and

$$\begin{aligned}
 \text{Re}[\overline{\lambda(z)}u_z] &= a_2(z), \quad z \in \Gamma^* \\
 u(z_0) &= b_0 \\
 \text{Im}[\overline{\lambda(z)}u_z]_{z=z'_k} &= b'_k, \quad k = 1, \dots, K',
 \end{aligned}$$

respectively, in which $z'_k \notin T$ ($k = 1, \dots, K'$) are distinct points on Γ^* as stated in [6], b'_k ($k = 1, \dots, K'$) are real constants satisfying the conditions $|b'_k| \leq k_2$, $k = 1, \dots, K'$, such that the above boundary value problem is well-posed. The above boundary value problem with homogeneous boundary condition is called Problem Q_0 .

It is clear that Problem P is a discontinuous irregular oblique derivative problem. If $\cos(\nu, n) > 0$ on Γ , then Problem P is the discontinuous third boundary value problem (Problem III or O). If $\cos(\nu, n) = 1$, $a_1(z) = 0$ on Γ , then Problem P is the discontinuous Neumann boundary problem (Problem II). If $\cos(\nu, n) = 0$, $a_2(z) \equiv 0$ on Γ , Problem P is equivalent to the discontinuous first boundary value problem (Problem I).

2. Uniqueness and estimates of solutions

First of all we prove the following result.

Theorem 2.1. *Suppose that equation (1) satisfies Conditions C^* . Then Problem P_0 for equation (1) with the condition $A_3 = 0$ in D has only the trivial solution.*

Proof. Let $u(z)$ be any solution of Problem P_0 for equation (1) with $A_3 = 0$. From Conditions C^* , it is easily seen that $u(z)$ is a solution of the uniformly elliptic equation:

$$u_{z\bar{z}} = \operatorname{Re}[Qu_{zz} + A_1u_z] + \hat{A}_2u, \quad |Q| \leq q_0 < 1, \quad \hat{A}_2 = A_2 + |u|^\sigma \geq 0 \text{ in } D, \quad (7)$$

and satisfies the boundary conditions

$$\frac{\partial u}{\partial \nu} + 2a_1(z)u(z) = 0 \text{ on } \Gamma^*, \quad u(z_j) = 0, \quad j = 0, 1, \dots, K'.$$

Substitute the solution $u(z)$ into the coefficients of equation (7), we can find a solution $\Psi(z)$ of (7) satisfying the condition $\Psi(z) = 1$ on Γ . Thus the function $U(z) = \frac{u(z)}{\Psi(z)}$ is a solution of the equation

$$U_{z\bar{z}} = \operatorname{Re}[QU_{zz} + A_0U_z], \quad A_0 = -2(\log \Psi)_{\bar{z}} + 2Q(\log \Psi)_z + A_1,$$

satisfying the boundary conditions

$$\frac{\partial U}{\partial \nu} + a_1^*(z)U(z) = 0 \text{ on } \Gamma^*, \quad U(z_j) = 0, \quad j = 0, 1, \dots, K', \quad (8)$$

where $a_1^*(z) = a_1(z) + \frac{\partial \Psi / \partial \nu}{\Psi(z)}$, $a_1^*(z) \cos(\nu, n) \geq 0$ on Γ^* . If $M = \max_{\bar{D}} U(z) > 0$ in D , then there exists a point $z^* \in \Gamma$ such that $M = U(z^*) = \max_{\bar{D}} U(z) > 0$. For the case 1, when $z^* \in \Gamma'$, noting that $\cos(\nu, n) \equiv 0$, $a_1(z) \equiv 0$, $\frac{\partial \Psi(z)}{\partial \nu} \equiv 0$ on Γ' , we have $\frac{\partial U}{\partial \nu} \equiv 0$, $U(z) \equiv M$ on Γ_j ($1 \leq j \leq J'$). This contradicts the point conditions in (8). For the cases 2 and 3, when $z^* \in \Gamma''$, if $\cos(\nu, n) > 0$ at z^* , from [6, Chap. III, Cor. 2.10], we have $\frac{\partial U}{\partial \nu} > 0$ at z^* . This contradicts the condition (8) on Γ'' . If $\cos(\nu, n) = 0$ and $a_1^*(z^*) \neq 0$ at z^* , then $\frac{\partial U}{\partial \nu} + a_1^*(z)U \neq 0$ at z^* . This is also impossible. Denote by L the longest curve of Γ including the point z^* , such that $\cos(\nu, n) = 0$ and $a^*(z) = 0$, thus $u(z) = M$ on L . From the point conditions in (8), any point of $\tilde{T} = \{z_0, z_1, \dots, z_{K'}\}$ cannot be an end point of L , then there exists a point $z' \in \Gamma'' \setminus \{L \cup \tilde{T}\}$, such that at z' , $\cos(\nu, n) > 0$ (< 0), $\frac{\partial U}{\partial n} > 0$, $\cos(\nu, s) > 0$ (< 0), $\frac{\partial U}{\partial s} \geq 0$, or $\cos(\nu, n) < 0$ (> 0), $\frac{\partial U}{\partial n} > 0$, $\cos(\nu, s) > 0$ (< 0), $\frac{\partial U}{\partial s} \leq 0$. Hence

$$\frac{\partial U}{\partial \nu} = \cos(\nu, n) \frac{\partial U}{\partial n} + \cos(\nu, s) \frac{\partial U}{\partial s} > 0, \quad \text{or } < 0 \text{ at } z'$$

holds, where s is the tangent vector at $z' \in \Gamma''$ identical to the positive direction of Γ , and then

$$\frac{\partial U}{\partial \nu} + a_1^* U > 0, \quad \text{or} \quad \frac{\partial U}{\partial \nu} + a_1^* U < 0 \quad \text{at} \quad z',$$

it is also impossible. For the case 4, when $z^* \in \Gamma''$, it will be occurred the similar case as in the case 1. This shows that $u(z)$ cannot attain its maximum M at a point $z^* \in \Gamma$. Similarly we can prove that $u(z)$ cannot attain its minimum at a point $z_* \in \Gamma$, hence $u(z) = 0$ on Γ , thus $u(z) = 0$ in \bar{D} . \square

By a similar way as stated before, we can prove the uniqueness theorem of solutions of Problem P for equation (1) with $\sigma = 0$ as follows.

Corollary 2.2. *Suppose that equation (1) with $\sigma = 0$ satisfies Conditions C^* and the following condition: for any real functions $u_j(z) \in C^1(D^*), V_j(z) \in L_{p_0}(\tilde{D})(j = 1, 2)$, the equality*

$$\begin{aligned} &F(z, u_1, u_{1z}, V_1) - F(z, u_2, u_{2z}, V_2) = \\ &\operatorname{Re}[\tilde{Q}(V_1 - V_2) + \tilde{A}_1(u_1 - u_2)_z] + \tilde{A}_2(u_1 - u_2) \quad \text{in } D \end{aligned} \tag{9}$$

holds, where $|\tilde{Q}| \leq q_0$ in D , $\tilde{A}_1, \tilde{A}_2 \in L_{p_0}(\bar{D})$. Then Problem P_0 for equation (1) has at most one solution.

Theorem 2.3. *Let equation (1) satisfy Conditions C^* . Then any solution of Problem P for (1) satisfies the estimates*

$$\begin{aligned} \hat{C}_\delta^1[u, \bar{D}] &= C_\delta[|u|^{\sigma+1}, \bar{D}] + C_\delta[X(z)u_z, \bar{D}] \leq M_1 \\ \|u\|_{\tilde{W}_{p_0}^2(D)} &= \|X(z)u_z\|_{W_{p_0}^1(D)} \leq M_1, \hat{C}_\delta^1[u, \bar{D}] \leq M_2(k_1 + k_2), \end{aligned} \tag{10}$$

where $X(z)$ is given as

$$X(z) = \prod_{j=1}^{m+1} |z - z_j|^{\eta_j}, \quad \eta_j = \begin{cases} |\gamma_j| + \tau, & \text{if } \gamma_j < 0, \beta_j \geq |\gamma_j| \\ |\beta_j| + \tau, & \text{if } \gamma_j \geq 0, \text{ or } \gamma_j < 0, \beta_j < |\gamma_j|. \end{cases} \tag{11}$$

Here γ_j ($j = 1, \dots, m + 1$) are real constants as stated in (5), τ, δ (with $\delta \leq \min[\beta, \tau], 0 < \beta \leq \alpha$) are sufficiently small positive constants, $k = (k_0, k_1, k_2)$, $M_1 = M_1(q_0, p_0, \alpha, k, p, D)$, $M_2 = M_2(q_0, p_0, \alpha, k_0, p, D)$ are non-negative constants.

Proof. By using the reduction to absurdity, we shall prove that any solution $u(z)$ of Problem P satisfies the estimate of boundedness

$$\hat{C}^1[u, \bar{D}] = C[|u|^{\sigma+1}, \bar{D}] + C[X(z)u_z, \bar{D}] \leq M_3, \tag{12}$$

where $M_3 = M_3(q_0, p_0, \alpha, k, p, D)$ is a non-negative constant. Suppose that (12) is not true, then there exist sequences of coefficients $\{A_j^{(m)}\}$ ($j = 1, 2, 3$), $\{Q^{(m)}\}$, $\{\lambda^{(m)}(z)\}$, $\{a_j^{(m)}\}$ ($j = 1, 2$), $b_j^{(m)}$ ($j = 0, 1, \dots, K'$), which satisfy Conditions C^* and (4), such that $\{A_j^{(m)}\}$ ($j = 1, 2, 3$), $\{Q^{(m)}\}$, $\{\lambda^{(m)}(z)\}$, $\{a_j^{(m)}\}$ ($j = 1, 2$) and $\{b_j^{(m)}\}$ ($j = 0, 1, \dots, K'$) in \overline{D} , Γ weakly converge or uniformly converge to $A_j^{(0)}$ ($j = 1, 2, 3$), $Q^{(0)}$, $\lambda^{(0)}(z)$, $a_j^{(0)}$ ($j = 1, 2$), $b_j^{(0)}$ ($j = 0, 1, \dots, K'$), respectively. Moreover the corresponding boundary value problems

$$u_{z\bar{z}} - \operatorname{Re}[Q^{(m)}u_{zz} + A_1^{(m)}u_z] - \hat{A}_2^{(m)}u = A_3^{(m)}, \quad \hat{A}_2^{(m)} = A_2^{(m)} + |u|^\sigma, \quad (13)$$

and

$$\frac{\partial u}{\partial \nu} + 2a_1^{(m)}(z)u = 2a_2^{(m)}(z) \quad \text{on } \Gamma^*, \quad u(z_j) = b_j^{(m)}, \quad j = 0, 1, \dots, K', \quad (14)$$

have the solutions $\{u^{(m)}(z)\}$ with unbounded $\hat{C}[u^{(m)}(z), \overline{D}]$ ($m = 1, 2, \dots$). We can choose a subsequence of $\{u^{(m)}(z)\}$ denoted by $\{u^{(m)}(z)\}$ again, such that $h_m = \hat{C}[u^{(m)}(z), \overline{D}] \rightarrow \infty$ as $m \rightarrow \infty$, and assume that $h_m \geq \max[k_1, k_2, 1]$. It is obvious that $\tilde{u}^{(m)}(z) = \frac{u^{(m)}(z)}{h_m}$ ($m = 1, 2, \dots$) are solutions of the boundary value problems

$$\tilde{u}_{z\bar{z}} - \operatorname{Re}[Q^{(m)}\tilde{u}_{zz} + A_1^{(m)}\tilde{u}_z] - \hat{A}_2^{(m)}\tilde{u} = \frac{A_3^{(m)}}{h_m}, \quad (15)$$

and

$$\frac{\partial \tilde{u}}{\partial \nu} + 2a_1^{(m)}(z)\tilde{u} = \frac{2a_2^{(m)}(z)}{h_m} \quad \text{on } \Gamma^*, \quad \tilde{u}(z_j) = \frac{b_j^{(m)}}{h_m}, \quad j = 0, 1, \dots, K'. \quad (16)$$

We can see that the functions in the above equation and the boundary conditions satisfy the Conditions C^* , (4), and

$$\begin{aligned} \frac{|\tilde{u}|^{\sigma+1}}{h_m} &\leq 1, & L_\infty \left[\frac{A_3^{(m)}}{h_m}, \overline{D} \right] &\leq 1, \\ \left| \frac{a_2^{(m)}}{h_m} \right| &\leq 1, & \left| \frac{b_j^{(m)}}{h_m} \right| &\leq 1, \quad j = 0, 1, \dots, K'. \end{aligned} \quad (17)$$

Hence from [6, Chap. III, Thm. 10], we can obtain the estimate

$$\hat{C}_\delta^1[\tilde{u}^{(m)}(z), \overline{D}] \leq M_4, \quad \|X(z)\tilde{u}_z^{(m)}\|_{\tilde{W}_{p_0}^1(D)} \leq M_4, \quad (18)$$

in which $M_4 = M_4(q_0, p_0, \beta, k, D)$ is a non-negative constant. Thus from the sequence of functions $\{\tilde{u}^{(m)}(z)\}$, we can choose a subsequence denoted by $\{\tilde{u}^{(m)}(z)\}$ again, which in \overline{D} uniformly converges to $\tilde{u}_0(z)$, and $X(z)\tilde{u}_x^{(m)}$, $X(z)\tilde{u}_y^{(m)}$ in \overline{D}

are uniformly convergent and $\tilde{u}_{xx}^{(m)}, \tilde{u}_{yy}^{(m)}, \tilde{u}_{xy}^{(m)}$ in \tilde{D} are weakly convergent. This shows that $\tilde{u}_0(z)$ is a solution of the boundary value problem

$$\tilde{u}_{0z\bar{z}} - \operatorname{Re}[Q^{(0)}\tilde{u}_{0zz} + A_1^{(0)}\tilde{u}_{0z}] - \hat{A}_2^{(0)}\tilde{u}_0 = 0, \tag{19}$$

and

$$\frac{\partial \tilde{u}_0}{\partial \nu} + 2a_1^{(0)}(z)\tilde{u}_0 = 0 \text{ on } \Gamma^*, \quad u_0(z_j) = 0, \quad j = 0, 1, \dots, K'. \tag{20}$$

We see that the coefficient $A_3^{(0)}$ in equation (19) is equal to 0 and boundary condition (20) is homogeneous. On the basis of Theorem 2.1, the solution $\tilde{u}_0(z) = 0$, however, from $\hat{C}^1[\tilde{u}^{(m)}(z), \bar{D}] = 1$, we can derive that there exists a point $z^* \in \bar{D}$, such that $[|\tilde{u}_0(z)|^{\sigma+1} + |X(z)u_z]|_{z=z^*} \neq 0$. This is impossible. This shows that the first two estimates in (10) are true. Moreover it is not difficult to verify the third estimate in (10). \square

3. Solvability

In this section, we mainly prove the solvability of Problem P for equation (24) below. In the following we first discuss the solution of equation (6).

Theorem 3.1. *Let equation (1) in D satisfy Conditions C^* . Then Problems P and Q for equation (6) are solvable, and the solution $u(z)$ satisfies*

$$\hat{C}_\delta^1[u, \bar{D}] = C_\delta[|u|^{\sigma+1}, \bar{D}] + C_\delta[X(z)w(z), \bar{D}] \leq M_5, \tag{21}$$

in which $M_5 = M_5(p_0, \delta, k, D)$ is a non-negative constant, and $X(z)$ is as stated in (11). Besides the solution of Problems P and Q for (6) are unique, if the condition (9) holds.

Proof. It is clear that Problem Q for (6) is equivalent to Problem A for the first order complex equation

$$w_{\bar{z}} = \operatorname{Re}[Qw_z + A_1w] + A_3 \tag{22}$$

with the boundary conditions

$$\begin{aligned} \operatorname{Re}[\overline{\lambda(z)}w(z)] &= a_2(z), \quad z \in \Gamma^* \\ u(z_0) &= b_0 \\ \operatorname{Im}[\overline{\lambda(z'_j)}w(z'_j)] &= b'_j, \quad j = 1, \dots, K', \end{aligned}$$

and the relation

$$u(z) = 2\operatorname{Re} \int_{z_0}^z w(z)dz + b_0 \text{ in } D \tag{23}$$

(see [6, Chap. 6]). From Theorems 2.3 and 2.4 [6, Chap. IV], we see that Problem A for (22) has a solution $w(z)$ satisfying the estimate (21), and $u(z)$ in (23) is the solution of Problem Q for (22). Next let $u_0(z)$ be a solution of Problem Q for the linear case of equation (22). If $u_0(z)$ satisfies the point conditions $u(z_j) = b_j, j = 0, 1, \dots, K'$ in (3), then the solution is also a solution of Problem P for the equation (22). Otherwise we can find $K' = 2K - 2N + J' + 1$ linearly independent solutions $u_k(z)$ ($k = 1, \dots, K'$) of Problem Q for the homogeneous equation of (22) satisfying the boundary conditions

$$\begin{aligned} \operatorname{Re}[\overline{\lambda(z)}u_{kz}] &= 0, \quad z \in \Gamma^* \\ u_k(z_0) &= 0 \\ \operatorname{Im}[\overline{\lambda(z)}u_{kz}]|_{z=z'_j} &= \delta_{jk}, \quad j = 1, \dots, K'. \end{aligned}$$

It is easy to see that

$$J = \begin{vmatrix} u_1(z_1) & \dots & u_{K'}(z_1) \\ \vdots & \ddots & \vdots \\ u_1(z_{K'}) & \dots & u_{K'}(z_{K'}) \end{vmatrix} \neq 0.$$

Hence we can find K' real constants d_k ($k = 1, \dots, K'$), which are not all equal to 0, such that $u(z) = u_0(z) + \sum_{k=1}^{K'} d_k u_k(z)$ is a solution of Problem P for (22). Moreover, by using the method of parameter extension and the Schauder fixed-point theorem, we can find a solution of Problem P for the general equation (6) (see [6, Chap. III, Sec. 4]). Besides we can also prove the other part of the theorem. \square

In order to prove the existence of solutions of Problem P for equation (1), we first introduce the nonlinear elliptic equation of second order

$$u_{z\bar{z}} = F_m(z, u, u_z, u_{zz}), \quad F_m = \operatorname{Re}[Q_m u_{zz} + A_{1m} u_z] + \hat{A}_{2m} u + A_3 \quad (24)$$

in D , with the coefficients $\hat{A}_2 = A_2 + |u|^\sigma$ and

$$Q_m = \begin{cases} Q, \\ 0, \end{cases} \quad A_{jm} = \begin{cases} A_j, \\ 0, \end{cases} \quad j = 1, 3, \quad \hat{A}_{2m} = \begin{cases} \hat{A}_2 & \text{in } D_m, \\ 0 & \text{in } \mathbb{C} \setminus D_m, \end{cases}$$

where $D_m = \{z \in D | \operatorname{dist}(z, \Gamma) \geq \frac{1}{m}\}$, m is a positive integer.

Theorem 3.2. *Let the equation (1) satisfy Conditions C^* , and $u(z)$ be any solution of Problem P for equation (24). Then $u(z)$ can be expressed in the form*

$$u(z) = U(z) + \tilde{v}(z) = U(z) + \hat{v}(z) + v(z), \quad (25)$$

in which $\tilde{v}(z) = \hat{v}(z) + v(z)$ is a solution of (24) with the homogeneous first boundary condition

$$\tilde{v}(z) = 0 \quad \text{on } \partial D_0 = \{|z| = 1\},$$

where

$$v(z) = HF_m = \frac{2}{\pi} \iint_{D_0} F_m(\zeta, u, u_\zeta, u_{\zeta\zeta}) \ln |z - \zeta| d\sigma_\zeta,$$

which is called Problem I. $U(z)$ is a solution of Problem I for $U_{z\bar{z}} = 0$ in D satisfying the boundary condition (29) below, and $U(z), \tilde{v}(z)$ satisfy the estimates

$$\hat{C}_\delta^1[U, \bar{D}] = C_\delta[|U|^{\sigma+1}, \bar{D}] + C_\delta[X(z)U_z, \bar{D}] \leq M_6, \quad \hat{C}_\delta^1[\tilde{v}, \bar{D}_0] \leq M_7, \quad (26)$$

where $\delta, M_j = M_j(\delta, k_1, p, D_m)$ ($j = 6, 7$) are non-negative constants.

Proof. It is clear that the solution $u(z)$ can be expressed as (25). On the basis of Theorem 2.3, it is easy to see that $U(z)$ satisfies the first estimate in (26), and then we know that $\tilde{v}(z)$ satisfies the second estimate of (26). \square

Theorem 3.3. *If the equation (1) satisfies Conditions C^* , then Problem P for the equation (24) has a solution.*

Proof. We introduce the equation and the boundary condition with the parameter $t \in [0, 1]$:

$$u_{z\bar{z}} = tF_m(z, u, u_z, u_{z\bar{z}}) \quad \text{in } D_0, \quad (27)$$

$$\begin{aligned} \operatorname{Re}[\overline{\lambda(z)}u_z] - a_2(z) &= -ta_1(z)u(z), \quad z \in \Gamma^* \\ u(z_0) &= b_0 \\ u(z_j) &= b_j, \quad j = 1, \dots, K'. \end{aligned} \quad (28)$$

Denote by B_M a bounded open set in the Banach space $B = \hat{W}_{p_0}^2(D_0) = \hat{C}_\delta^1(\bar{D}_0) \cap \tilde{W}_{p_0}^2(D_0)$ ($0 < \delta \leq \alpha$), where $\hat{C}_\delta^1(\bar{D}_0)$ and $\tilde{W}_{p_0}^2(D_0)$ are as stated before, the elements of which are real functions $V(z) (\in B)$ satisfying the inequalities $\hat{C}_\delta^1[V, \bar{D}_0] < M_8 = M_7 + 1$, in which M_7 is a non-negative constant as stated in (26). We choose any function $V(z) \in \overline{B_M}$ and substitute it into the appropriate position in the right hand side of (24). Afterwards we make an integral $v(z) = H\rho$ as follows:

$$v(z) = H\rho = \frac{2}{\pi} \iint_{D_0} \rho(\zeta) \log |z - \zeta| d\sigma_\zeta,$$

where $D_0 = \{|z| < 1\}$, $\rho(z) = V_{z\bar{z}}$. Next we find a solution $\hat{v}(z)$ of the boundary value problem in D_0 :

$$\begin{aligned} \hat{v}_{z\bar{z}} &= 0 && \text{in } D_0 \\ \hat{v}(z) &= -v(z) && \text{on } \partial D_0. \end{aligned}$$

Denote $\tilde{v}(z) = v(z) + \hat{v}(z)$. Moreover by using a similar way as in [5] and [6], we can find a solution $U(z)$ of the boundary value problem in D_0 :

$$U_{z\bar{z}} = 0 \quad \text{in } D,$$

and

$$\begin{cases} \operatorname{Re}[\overline{\lambda(z)}(U + \tilde{v})_z] + ta_1(z)(U + \tilde{v}) = a_2(z) & \text{on } \Gamma^* \\ U(z_0) + \tilde{v}(z_0) = b_0 \\ U(z_j) + \tilde{v}(z_j) = b_j, \quad j = 1, \dots, K'. \end{cases} \quad (29)$$

Now we discuss the equation

$$\tilde{V}_{z\bar{z}} = tF_m(z, u, u_z, U_{zz} + \tilde{v}_{zz}), \quad 0 \leq t \leq 1, \quad (30)$$

and boundary condition

$$\tilde{V}(z) = 0 \quad \text{on } \partial D_0, \quad 0 \leq t \leq 1,$$

where $u(z) = U(z) + \tilde{v}(z)$. By Conditions C^* and the principle of contracting mappings, the above Problem I for the equation (30) in D_0 has a unique solution $\tilde{V}(z)$. Denote by $\tilde{V} = S(V, t)$ ($0 \leq t \leq 1$) the mapping from V onto \tilde{V} . Furthermore, if $u(z)$ is a solution of Problem P in D_0 for the equation

$$u_{z\bar{z}} = tF_m(z, u, u_z, u_{zz}), \quad 0 \leq t \leq 1, \quad (31)$$

and boundary conditions

$$\begin{cases} \operatorname{Re}[\overline{\lambda(z)}u_z] - a_2(z) = -ta_1(z)u(z) & \text{on } \Gamma^*, \\ u(z_0) = b_0, \\ u(z_j) = b_j, \quad j = 1, \dots, K', \quad 0 \leq t \leq 1, \end{cases}$$

then it follows from Theorem 2.3, that the solution $u(z)$ of Problem P for (31) satisfies (10). Consequently $\tilde{V}(z) = u(z) - U(z) \in B_M$. Set $B_0 = B_M \times [0, 1]$. In the following, we shall verify that the mapping $\tilde{V} = S(V, t)$ satisfies the three conditions of Leray-Schauder theorem:

1. For every $t \in [0, 1]$, $\tilde{V} = S(V, t)$ continuously maps the Banach space B into itself, and it is completely continuous in B_M . Besides, for every function $V(z) \in \overline{B_M}$, $S(V, t)$ is uniformly continuous with respect to $t \in [0, 1]$.

In fact, we arbitrarily choose $V_n(z) \in \overline{B_M}$, $n = 1, 2, \dots$, it is clear that from $\{V_n(z)\}$ there exists a subsequence $\{V_{n_k}(z)\}$, such that $\{V_{n_k}(z)\}$, $\{V_{n_k z}(z)\}$ and corresponding functions $\{U_{n_k}(z)\}$, $\{U_{n_k z}(z)\}$ uniformly converge to $V_0(z)$, $V_{0z}(z)$, $U_0(z)$, $U_{0z}(z)$ in $\overline{D_0}$, $\overline{D_1}$, respectively. We can find a solution $\tilde{V}_0(z)$ of Problem D for the equation $\tilde{V}_{0z\bar{z}} = tF_m(z, u_0, u_{0z}, U_{0zz} + \tilde{v}_{0zz})$, $0 \leq t \leq 1$,

where $u_0(z) = U_0(z) + v_0(z) + \hat{v}_0(z) = S_1(V_0, t)$. From $\tilde{V}_{n_k} = S(V_{n_k}, t)$ and $\tilde{V}_0 = S(V_0, t)$, we have

$$(\tilde{V}_{n_k} - \tilde{V}_0)_{z\bar{z}} = t[F_m(z, u_{n_k}, u_{n_k z}, U_{n_k z z} + \tilde{v}_{n_k z z}) - F_m(z, u_0, u_0, U_{0 z z} + \tilde{v}_{0 z z}) + C_{n_k}(z)], \quad 0 \leq t \leq 1,$$

where $C_{n_k} = F_m(z, u_{n_k}, u_{n_k z}, U_{n_k z z} + \tilde{v}_{0 z z}) - F_m(z, u_0, u_0, U_{0 z z} + \tilde{v}_{0 z z})$ in D_0 . By Theorem 2.3 and the conditions of equation (24), we can verify that

$$L_2[C_{n_k}, \overline{D_0}] \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{32}$$

In fact, by Conditions C^* and the conditions of equation (24), it is not difficult to see that C_{n_k} converges to 0 for almost every point $z \in D_0$. Thus for two arbitrary sufficiently small positive constants ε_1 and ε_2 , there exists a subset D_* in D_0 and a positive integer N , so that $\text{meas } D_* < \varepsilon_1$ and $|C_{n_k}| < \varepsilon_2$ in $\overline{D_0} \setminus D_*$ for $n > N$. By the Hölder inequality and the Minkowski inequality, we have

$$\begin{aligned} L_{p_0}[C_{n_k}, \overline{D_0}] &\leq L_{p_0}[C_{n_k}, D_*] + L_{p_0}[C_{n_k}, \overline{D_0} \setminus D_*] \\ &\leq L_{p_1}[C_{n_k}, D_*] L_{p_2}[1, D_*] + \varepsilon_2 (\text{meas } D)^{\frac{1}{p_0}} \\ &\leq L_{p_1}[C_{n_k}, D_*] \varepsilon_1^{\frac{1}{p_2}} + \varepsilon_2 \pi^{\frac{1}{p_0}} \\ &\leq M_0 \varepsilon_1^{\frac{1}{p_2}} + \varepsilon_2 \pi^{\frac{1}{p_0}} = \varepsilon, \end{aligned}$$

in which $M_0 = \sup_{n_k} L_{p_1}[C_{n_k}, D_*]$, p_1 is a constant such that $p_0 < p_1 < \min(p, \frac{1}{1-\alpha})$ and $p_2 = \frac{p_0 p_1}{p_1 - p_0}$. Therefore the formula (32) is true. Similarly to (10)–(20), we can derive that

$$\|\tilde{V}_{n_k} - \tilde{V}_0\|_{\tilde{W}_{p_0}^2(D_0)} \leq \frac{L_2[C_{n_k}, \overline{D_0}]}{[1 - q_0]},$$

where $q_0 < 1$. It is not difficult to show that $\|\tilde{V}_{n_k} - \tilde{V}_0\|_{\tilde{W}_{p_0}^2(D_0)} \rightarrow 0$ as $k \rightarrow \infty$. Moreover, from Theorem 2.3, we can verify that there exists a subsequence of $\{\tilde{V}_{n_k}(z) - \tilde{V}_0(z)\}$, denoted by $\{\tilde{V}_{n_k}(z) - \tilde{V}_0(z)\}$ again, such that $\hat{C}_\delta^1[\tilde{V}_{n_k} - \tilde{V}_0, \overline{D_0}] \rightarrow 0$ as $k \rightarrow \infty$. This shows the complete continuity of $\tilde{V} = S(V, t)$ ($0 \leq t \leq 1$) in $\overline{B_M}$. By using a similar method, we can prove that $\tilde{V} = S(V, t)$ ($0 \leq t \leq 1$) continuously maps $\overline{B_M}$ into B , and $\tilde{V} = S(V, t)$ is uniformly continuous with respect to $t \in [0, 1]$ for $V \in \overline{B_M}$.

2. For $t = 0$, from Theorem 2.3 and the estimates in (26), it is clear that $\tilde{V} = S(V, 0) \in B_M$.

3. From Theorem 2.3 and (26), we see that $\tilde{V} = S(V, t)$ ($0 \leq t \leq 1$) has no solution $V(z)$ on the boundary $\partial B_M = \overline{B_M} \setminus B_M$.

Hence by the Leray-Schauder theorem, we know that Problem P for the equation (27) and boundary condition (28) with $t = 1$, namely (24) has a solution $u(z) = U(z) + \tilde{V}(z) = U(z) + v(z) + \hat{v}(z) = S_1(V, 1) \in B_M$. \square

Theorem 3.4. *Under the same conditions as in Theorem 2.3, Problem P for the equation (1) has a solution.*

Proof. By Theorem 2.3 and Theorem 3.2, Problem P for the equation (24) possesses a solution $u_m(z)$, and the solution $u_m(z)$ of Problem P for (24) satisfies the estimate (10), where $m = 1, 2, \dots$. Thus, we can choose a subsequence $\{u_{m_k}(z)\}$, such that $\{u_{m_k}(z)\}, \{u_{m_k z}(z)\}$ in \bar{D} uniformly converge to $u_0(z), u_{0z}(z)$, respectively. Obviously, $u_0(z)$ satisfies the boundary condition (2) of Problem P. Finally we can verify that $u_0(z)$ is a solution of equation (1) (see [6, Chap. III, Sec. 3]). \square

Remark 3.5. In the above discussion, it is assumed that $K \geq N - \frac{1}{2}$. Actually the index K can be equal to $N - 1$. In this case the boundary condition (2) should be replaced by

$$\frac{\partial u}{\partial \nu} + 2a_1(z)u(z) = 2a_2(z) + h(z), \quad z \in \Gamma^*, \quad u(z_0) = b_0,$$

where $h(z) = 0$ on $\Gamma'_k, k = 1, \dots, m$, and $h(z) = h_0$ on Γ'_{m+1} , h_0 is an undetermined real constant. As stated in the proof of Theorem 2.1, if the positive maximum M of $U(z)$ is taken at a point $z^* \in \Gamma'_{m+1}$, then $\frac{\partial U}{\partial \nu} + 2a_1^*(z)U(z) = h_0 \geq 0$; and if the negative minimum m of $U(z)$ is taken at a point $z_* \in \Gamma'_{m+1}$, then $\frac{\partial U}{\partial \nu} + 2a_1^*(z)U(z) = h_0 \leq 0$, thus $h_0 = 0$. Hence $U(z)$ cannot attain its positive maximum and negative minimum on Γ'_{m+1} . This shows that the uniqueness theorem for $K = N - 1$ is also valid. Therefore the above theorems for $K = N - 1$ are true. The case $K < N - 1$ remains to be further discussed.

Remark 3.6. Finally we explain that the continuous irregular oblique derivative problem is a special case of the above Problem P, namely we find the continuously differentiable solution $u(z)$ of equation (1) in \bar{D} satisfying the boundary conditions

$$\frac{\partial u}{\partial \nu} + 2a_1(z)u(z) = 2a_2(z), \quad z \in \Gamma, \quad u(z_j) = b_j, \quad j = 0, 1, \dots, K', \quad (33)$$

where $\nu (= \nu_1 + i\nu_2 \neq 0)$ can be arbitrary at every point on Γ , $K' (= 2K - 2N + J + 1 \geq 0), J$ are non-negative integers as stated below. It is clear that the boundary condition (33) can be rewritten in the complex form

$$\operatorname{Re}[\overline{\lambda(z)}u_z] + a_1(z)u = a_2(z), \quad z \in \Gamma, \quad u(z_j) = b_j, \quad j = 0, 1, \dots, K', \quad (34)$$

in which $\lambda(z) = \cos(\nu, x) + i \sin(\nu, x) = e^{i(\nu, x)} \neq 0$, (ν, x) is the angle between ν and the x -axis, $z_j (\in \Gamma_j, j = 0, 1, \dots, K')$ are distinct points on Γ . Suppose that $\lambda(z), a_1(z), a_2(z)$ satisfy the conditions

$$\begin{aligned} C_\alpha[\lambda(z), \Gamma] &\leq k_0, & |b_j| &\leq k_2, \quad j = 0, 1, \dots, K', \\ C_\alpha[a_1(z), \Gamma] &\leq k_0, & 0 &\leq a_1(z) \cos(\nu, n) \quad \text{on } \Gamma, \\ C_\alpha[a_2(z), \Gamma] &\leq k_2, \end{aligned}$$

in which α ($\frac{1}{2} < \alpha < 1$), k_2 are non-negative constants. The boundary $\partial D = \Gamma$ can be divided into two parts, namely $E^+ \subset \{z \in \partial D \mid \cos(\nu, n) \geq 0, a_1 \geq 0\}$ and $E^- \subset \{z \in \partial D \mid \cos(\nu, n) \leq 0, a_1 \leq 0\}$, and $E^+ \cap E^- = \emptyset$, $E^+ \cup E^- = \Gamma$, $\overline{E^+} \cap \overline{E^-} = E^0$. For every component $L = \Gamma_j$ ($0 \leq j \leq N$) of Γ , there are three cases: 1. $L \subset E^+$. 2. $L \subset E^-$. In these cases, if $\cos(\nu, n) \equiv 0, a_1(z) \equiv 0$ on Γ_j ($1 \leq j \leq J, J \leq N+1$), and the above identical formulas on Γ_j ($J < j \leq N+1$) do not hold, then we need the conditions $\int_{\Gamma_j} a_2(z) ds = 0$ ($1 \leq j \leq J$), and $u(z_j) = b_j, j = 0, 1, \dots, K' (\geq J)$, in which z_j, b_j ($j = 0, 1, \dots, K'$) are as stated before, and denote $\Gamma' = \bigcup_{j=1}^J \Gamma_j, \Gamma'' = \bigcup_{j=J+1}^{N+1} \Gamma_j$. 3. There exists at least a point on each component of $L^+ = E^+ \cap L$ and $L^- = E^- \cap L$, such that $\cos(\nu, n) \neq 0$ at the point, and $E^0 \cap L \in \{z_0, z_1, \dots, z_{K'}\}$ such that every component of L^+, L^- includes its initial point and does not include its terminal point; and $z_j \in \overline{L^+} \cap L^-$ ($0 \leq j \leq K'$), when the direction of ν at z_j is equal to the direction of L ; and $z_j \in L^+ \cap \overline{L^-}$ ($0 \leq j \leq K'$), when the direction of ν at z_j is opposite to the direction of L ; and $\cos(\nu, n)$ changes the sign once on the two components with the end point z_j ($0 \leq j \leq K'$); we may assume that $u(z_j) = b_j, j = 0, 1, \dots, K'$. The above boundary value problem for (1) is called Problem P' , and Problem P' with homogeneous boundary condition of (33) or (34) is called Problem P'_0 . The number $K = \frac{1}{2}(K_1 + \dots + K_{N+1})$, where $K_j = \Delta_{\Gamma_j} \arg \lambda(z), j = 1, \dots, N+1$, is called the index of Problem P' . We can choose $K' = 2K - 2N + J + 1$. As a special case of Problem P in Sections 1-3, we see that Problem P' for the equation (1) has a solution.

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