Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 27 (2008), 339–352
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Kernel Dimension of Singular Integral Operators with Piecewise Continuous Coefficients on the Unit Circle

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Abstract. In this paper we propose a method to compute the kernel dimension of a Fredholm singular integral operator $aP + bQ$ with piecewise continuous coefficients a, b. We show that the kernel dimension can be extracted from the singular value behavior of a polynomial collocation method. The results are illustrated by numerical experiments.

Keywords. Cauchy singular integral operators, kernel dimension, singular values, splitting property

Mathematics Subject Classification (2000). 45E10, 15A18, 65R20, 47B35

1. Introduction

Let $\mathbb{T}$ denote the unit circle $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$, let $L^2 := L^2(\mathbb{T})$ be the Hilbert space of all square integrable functions on the unit circle $\mathbb{T}$, and let L^2_N refer to the Hilbert space of all column-vectors of length N with components from L^2. A bounded function $a : \mathbb{T} \to \mathbb{C}$ is called piecewise continuous if it has one-sided limits $a(t \pm 0)$ for all $t \in \mathbb{T}$. We denote by PC the C^*-algebra of all piecewise continuous functions and by $PC_{N\times N}$ the C^*-algebra of all $N \times N$–matrices with entries from PC.

In this paper we deal with the singular integral operators

$$(Au)(t) := (cu)(t) + \frac{d(t)}{\pi i} \int_{\mathbb{T}} \frac{u(\tau)}{\tau - t} d\tau,$$

where the coefficients c, d are piecewise continuous matrix functions. It is well known (see e.g. [7]) that for all functions $c, d \in PC_{N\times N}$ the operator A belongs

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This work was supported by the Deutsche Forschungsgemeinschaft, DFG project SI 474110-1.

to $\mathcal{L}(L^2_N)$ where $\mathcal{L}(L^2_N)$ stand for the C^*-algebra of all bounded and linear operators acting L^2_N. Moreover, since $S^* = S$ and S^2 equals the identity operator $I \in \mathcal{L}(L^2_N)$, where

$$(Su)(t) := \frac{1}{\pi i} \int_{\mathbb{T}} \frac{u(\tau)}{\tau - t} d\tau,$$

the operators $P := \frac{1}{2}(I + S)$ and $Q := \frac{1}{2}(I - S)$ are orthogonal projectors, and the operator A can be rewritten as $A = aP + bQ$, where $a = c + d, b = c - d$.

For the special singular integral operator $aP + Q$ one has the identity

$$aP + Q = (PaP + Q)(I + QaP),$$

where $I + QaP$ is an invertible operator whose inverse is $I - QaP$. Thus the Fredholm properties of $aP + Q$ and of $PaP/_{\mathrm{im}}P$ coincide. The last operator is the Toeplitz operator T (see Section 2). In particular we have $\dim \ker T(a) = \dim \ker(aP + Q)$. This equality is used in Section 5. Moreover, one gets the following matrix representation of $PaP + Q$ with respect to the standard basis $\{t^k\}_{k \in \mathbb{Z}}$ in L^2_N

$$PaP + Q = \begin{pmatrix} \ddots & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & a_0 & a_{-1} & a_{-2} & \\ & & & a_1 & a_0 & a_{-1} & \ddots \\ & & & a_2 & a_1 & a_0 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\{a_k\}_{k \in \mathbb{Z}}$ is the sequence of the Fourier coefficients of a function $a \in PC_{N \times N}$.

Thus, the singular integral operator $aP+Q$ is closely related to the Toeplitz operator $T(a)$ defined in Section 2. Moreover, taking into account that the Fredholmness of the operator $aP+bQ$ implies the invertibility of the coefficients a and b, and writing $aP + bQ$ as $b(b^{-1}aP + Q)$, we conclude that the Fredholm properties of the singular integral operator $aP + bQ$ are completely the same as Fredholm properties of the Toeplitz operator $T(b^{-1}a)$. In particular, the kernel dimensions of the operators $aP + bQ$ and $T(b^{-1}a)$ coincide.

A method to compute the kernel dimension of a Fredholm Toeplitz operator $T(a)$ was proposed in [9, 10]. This method is based on the observation that the behaviour of the singular values of modified finite sections of the Toeplitz operator is subject to the so-called k-splitting property (see Theorem 2.1). Moreover, it was shown the k-splitting property can be clearly observed for smooth symbols a (see Theorem 2.2). However, the main difficulty in working with Toeplitz operators is the computation of the Fourier coefficients.

In this paper we adapt this algorithm to singular integral operators with piecewise continuous coefficients. We prove that the singular values of the modified collocation method have the k-splitting property (see Theorem 3.5). Moreover, we estimate the convergence speed of $s_k(A_{n,r})$ to zero by the order of the smoothness of the coefficients (see Corollary 4.3). Finally, in Section 5 we present several numerical examples.

It turns out that the estimates for the convergence speed of the k-th singular values to zero for the collocation method are the same as for the finite section method (compare estimates (4) and (5) with (2) and (3)). The numerical experiments confirm these theoretical results (see Section 5). This observation leads to the conclusion that to compute the kernel dimension of a Fredholm Toeplitz operator and/or of a Fredholm singular integral operator one can use the collocation method instead of the finite section method.

2. Toeplitz operators and their finite sections

In this section we briefly present results for Toeplitz operators which are taken from [9] and [10]. Let ℓ^2_N be the Hilbert space of all sequences $x : \mathbb{Z}_+ \to \mathbb{C}^N, \mathbb{Z}_+ = \{i \in \mathbb{Z} : i \geq 0\}$, such that

$$\|x\|_{\ell^2_N} := \left(\sum_{i=0}^{\infty} \|x_i\|^2_{\mathbb{C}^N} \right)^{\frac{1}{2}} < \infty ,$$

where $\|x_i\|^2_{\mathbb{C}^N} = \|(x_i^1, x_i^2, \ldots, x_i^N)\|^2_{\mathbb{C}^N} = |x_i^1|^2 + |x_i^2|^2 + \ldots + |x_i^N|^2$.

The block Toeplitz operator $T(a) : \ell^2_N \to \ell^2_N$ is defined by the matrix representation

$$T(a) = \left(a_{i-j}\right)_{i,j=0}^{\infty} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $\{a_k\}_{k\in\mathbb{Z}}$ is the sequence of the Fourier coefficients of a function $a \in PC_{N\times N}$. It is well known (see e.g. [2]) that for all piecewise continuous functions a the Toeplitz operator $T(a)$ is a linear bounded operator on ℓ^2_N, i.e. $T(a) \in \mathcal{L}(\ell^2_N)$, and $\|T(a)\|_{\mathcal{L}(\ell^2_N)} = \|a\|_\infty$.

Further, given a function $a \in L^\infty_{N\times N}$ and a number $r \in \{0, 1, \ldots\}$ let us denote by $T_{n,r}(a), n \geq r$, the following rectangular truncated Toeplitz matrices

$$T_{n,r}(a) = \begin{pmatrix} a_0 & a_{-1} & \cdots\cdots & a_{-n+r} \\ a_1 & a_0 & \cdots\cdots & a_{-n+r+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n-1} & \cdots\cdots & a_r \end{pmatrix}.$$

We recall that the singular values of a square matrix $A_n \in \mathbb{C}^{nN\times nN}$ are defined as square roots of the spectrum points of $A_n^*A_n$. We denote the (ordered) singular values of A_n by $s_1(A_n), s_2(A_n), \ldots s_{nN}(A_n)$, i.e., we have $0 \le s_1(A_n) \le s_2(A_n) \le \ldots \le s_{nN}(A_n) = \|A_n\|$. Note also that the singular values can be also defined as approximation numbers, that is

$$s_k(A_n) = \operatorname{dist}(A_n, \mathcal{F}_{n,k}) := \inf\{\|A_n - F\| \ : \ F \in \mathcal{F}_{n,k}\}, \tag{1}$$

where $\mathcal{F}_{n,k}$ denotes the collection of all matrices from $\mathbb{C}^{nN\times nN}$ having the image of the dimension at most $(nN-k)$.

To evaluate the singular values of $T_{n,r}(a)$ we extend these matrices to square $nN \times nN$ matrices by filling in zeros in the remaining places.

Theorem 2.1 (see [10, Theorem 3.1])**.** *Let $a \in PC_{N\times N}$. If the Toeplitz operator $T(a)$ is Fredholm, then the singular values of $T_{n,r}(a)$ have the k-splitting property, that is*

$$\lim_{n\to\infty} s_k\left(T_{n,r}(a)\right) = 0 \qquad \text{and} \qquad \liminf_{n\to\infty} s_{k+1}\left(T_{n,r}(a)\right) > 0$$

with $k = \dim\ker T(a) + \dim\ker T(t^r\tilde{a})T(t^{-r}E)$, where $E = \operatorname{diag}(1,1,\ldots,1)$ is the identity $N\times N$ matrix, and the function $\tilde{a}$ is defined by $\tilde{a}(t) = a(\frac{1}{t})$.

Note also that if r is large enough then the kernel dimension of the operator $T(t^r\tilde{a})T(t^{-r}E)$ can be controlled, more precisely it is equal to Nr. Moreover one can estimate the convergence speed of $s_k(T_{n,r}(a))$ to zero.

Theorem 2.2 (see [10, Theorem 4.1 and Corollary 4.2])**.** *Let $a \in PC_{N\times N}$. If the Toeplitz operator $T(a)$ is Fredholm, then*

$$s_k\left(T_{n,r}(a)\right) \le C \max\left(\|Q_{n-r}\varphi_1\|, \ldots, \|Q_{n-r}\varphi_l\|, \|Q_n\psi_1\|, \ldots, \|Q_n\psi_m\|\right)$$

with $k = \dim\ker T(a) + \dim\ker T(t^r\tilde{a})T(t^{-r}E)$, where the constant C does not depend on n, where $\{\varphi_i\}_{i=1}^l$ and $\{\psi_i\}_{i=1}^m$ are some orthonormal bases of $\ker T(a)$ and $\ker T(t^r\tilde{a})T(t^{-r}E)$, respectively, and where $Q_n \in \mathcal{L}(\ell_N^2)$ are orthogonal projectors defined by

$$(x_0, x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots) \mapsto (0, 0, \ldots, 0, x_n, x_{n+1}, \ldots).$$

In particular, if the Fourier coefficients a_k of the function a fulfil $\sum_{k\in\mathbb{Z}} |k|^\alpha \|a_k\| < \infty$ for some $\alpha > 0$, then

$$s_k\left(T_{n,r}(a)\right) = O\left((n-r)^{-\alpha}\right) \quad \text{as } n \to \infty. \tag{2}$$

If, in addition, the function a is rational, then there is a $\rho > 0$ such that

$$s_k\left(T_{n,r}(a)\right) = O\left(e^{-\rho(n-r)}\right) \qquad \text{as } n \to \infty. \tag{3}$$

Remark. The estimates (2), (3) were obtained for the first time by A. Böttcher in [1] for the particular case $N = 1, r = 0$. It seems that Böttchers proof is restricted to $N = 1$.

3. A polynomial collocation method for singular integral operators

First, for any $n \in \mathbb{Z}_+$ we introduce the Fourier projection $P_n \in \mathcal{L}(L^2_N)$ by the rule

$$a = \sum_{k\in\mathbb{Z}} a_k t^k \mapsto \sum_{k=-n}^{n} a_k t^k,$$

and the Lagrange interpolation operator L_n associated to the points $t_j = \exp\frac{2\pi ij}{2n+1}$, $j = 0, 1, \ldots, 2n$ (that is L_n assigns to a function a its Lagrange interpolation polynomial $L_n a \in \operatorname{im} P_n$, uniquely determined by the conditions $(L_n a)(t_j) = a(t_j), j = 0, \ldots, 2n$).

Given $r \in \mathbb{Z}_+$, we define the operators

$$A_{n,r} := L_n(aP + bQ)P_n(P_n - W_n P_{r-1} W_n), \qquad n \in \mathbb{Z}_+,$$

where $P_{-1} := 0$, and $W_n \in \mathcal{L}(L^2_N)$ is the flip operator acting by the rule $W_n a = \sum_{k=0}^{n} a_{n-k} t^k + \sum_{k=-n}^{-1} a_{-n-k-1} t^k$. Note that if $r = 0$ then we get a polynomial collocation method $A_{n,0}$ for the solution of singular integral operators with piecewise continuous coefficients.

To analyze the behavior of the singular values of the operators $A_{n,r}$ we will use the results of our paper [8], where some abstract theory is presented. To this end we denote by $\mathcal{F}$ the set of all sequences $\{A_n\}$ of linear operators $A_n \in \mathcal{L}(\operatorname{im} P_n)$ for which there exist operators $W_1\{A_n\}, W_2\{A_n\} \in \mathcal{L}(L^2_N)$ such that

$$A_n P_n \to W_1\{A_n\} \qquad \text{and} \qquad W_n A_n W_n \to W_2\{A_n\}$$
$$A_n^* P_n \to W_1\{A_n\}^* \qquad \text{and} \qquad W_n A_n W_n \to W_2\{A_n\}^*$$

hold in the sense of strong convergence for $n \to \infty$. If we define

$$\lambda_1\{A_n\} + \lambda_2\{B_n\} := \{\lambda_1 A_n + \lambda_2 B_n\}, \qquad \{A_n\}\{B_n\} := \{A_n B_n\},$$

and

$$\|\{A_n\}\|_{\mathcal{F}} := \sup\Big\{\|A_n P_n\|_{\mathcal{L}(L^2_N)} \;:\; n \in \mathbb{Z}_+\Big\},$$

then it is not hard to see that $\mathcal{F}$ becomes a C^*- algebra with the unit element $\{P_n\}$.

Obviously the set $\mathcal{G}$ of all sequences $\{G_n\}$ with $\|G_n P_n\| \to 0$ as $n \to \infty$ forms a closed two-sided ideal of $\mathcal{F}$. Moreover, one can prove that the sequences $\{P_n K P_n\}$ and $\{W_n K W_n\}$ belong to the algebra $\mathcal{F}$ for any compact operator $K \in \mathcal{L}(L^2_N)$, i.e. for any $K \in \mathcal{K}(L^2_N)$. We denote by $\mathcal{J}$ the set of all sequences $\{J_n\} \in \mathcal{F}$ of the form $J_n = P_n K P_n + W_n L W_n + G_n$, where $K, L \in \mathcal{K}(L^2_N)$ and $\{G_n\} \in \mathcal{G}$.

The algebra $\mathcal{F}$ plays an important role in the theory of polynomial approximation methods for singular integral operators in the space L^2_N. A few historical remarks can be found in the Notes and Comments to Chapter 7 in [7]. In particular the collocation method was analyzed by Junghanns and Silbermann in [6] using Banach algebra techniques. The related results are also reflected in Chapter 7 of [7]. This source will be used for citations.

Lemma 3.1 (see [7, Proposition 7.6]). *$\mathcal{J}$ is a closed two-sided ideal of $\mathcal{F}$.*

In our specific setting a result of [8] reads as follows.

Theorem 3.2 (see [8, Theorem 5.2] or [5, Chapter 6]). *Let $\{A_n\} \in \mathcal{F}$. If the coset $\{A_n\} + \mathcal{J}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{J}$, then the operators $W_1\{A_n\}, W_2\{A_n\}$ are Fredholm, and the singular values of A_n have the k-splitting property, that is*

$$\lim_{n\to\infty} s_k(A_n) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} s_{k+1}(A_n) > 0$$

with $k = \dim\ker W_1\{A_n\} + \dim\ker W_2\{A_n\}$.

In [7] it was shown that the sequence $\{A_{n,0}\} = \{L_n(aP + bQ)P_n\}$ of the collocation method belongs to the C^*-algebra $\mathcal{F}$ for any $a, b \in PC_{N\times N}$. Moreover, there one got a criteria for the invertibility of the coset $\{A_{n,0}\} + \mathcal{J}$ in the algebra $\mathcal{F}/\mathcal{J}$.

Theorem 3.3 (see [7, Theorem 7.22]). *Let $a, b \in PC_{N\times N}$.*

(a) *The sequence $\{A_{n,0}\}$ belongs to the C^*-algebra $\mathcal{F}$. In particular,*

$$W_1\{A_{n,0}\} = aP + bQ \quad \textit{and} \quad W_2\{A_{n,0}\} = \tilde{a}P + \tilde{b}Q,$$

where the functions $\tilde{a}, \tilde{b}$ are defined by $\tilde{a}(t) = a(\frac{1}{t}), \tilde{b}(t) = b(\frac{1}{t})$.

(b) *The coset $\{A_{n,0}\} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$ if and only if the operator $W_1\{A_{n,0}\} = aP + bQ$ is Fredholm.*

Since for each $r \in \mathbb{Z}_+$ the sequence $\{W_n P_{r-1} W_n\}$ belongs to the ideal $\mathcal{J}$ we can extend this results to the operators $A_{n,r}$.

Corollary 3.4. *Let $a, b \in PC_{N\times N}$ and let $r \in \mathbb{Z}_+$.*

(a) *The sequence $\{A_{n,r}\}$ belongs to the C^*-algebra $\mathcal{F}$. In particular,*

$$W_1\{A_{n,r}\} = aP + bQ \quad \textit{and} \quad W_2\{A_{n,r}\} = (\tilde{a}P + \tilde{b}Q)Q_{r-1},$$

where $Q_{r-1} := I - P_{r-1}$.

(b) *The coset $\{A_{n,r}\} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$ if and only if the operator $W_1\{A_{n,r}\} = aP + bQ$ is Fredholm.*

Now combining Theorem 3.2 and Corollary 3.4 we arrive at the following result.

Theorem 3.5. *Let $a, b \in PC_{N\times N}$. If the singular integral operator $aP + bQ$ is Fredholm, then the singular values of $A_{n,r}$ have the k-splitting property with $k = k(A_{n,r}) := \dim\ker(aP + bQ) + \dim\ker(\tilde{a}P + \tilde{b}Q)Q_{r-1}$.*

Moreover, Lemma 3.2 from [8] implies that if r is large enough then the kernel dimension of the operator $(\tilde{a}P + \tilde{b}Q)Q_{r-1}$ is equal to the rank of the projector P_{r-1}, that is to $N(2r-1)$. Thus Theorem 3.5 offers a way to compute the kernel dimension of a Fredholm singular integral operator $(aP + bQ)$ with piecewise continuous coefficients a, b. Notice that r is large enough if $K(A_{n,r+1}) = k(A_{n,r}) = 2N$.

4. Convergence speed of $s_k(A_{n,r})$ to zero

In this section we will exclusively deal with smooth coefficients a, b. Therefore we denote by $C_{N\times N} \subset PC_{N\times N}$ the algebra of all continuous matrix functions on $\mathbb{T}$, by $\mathcal{H}^s_{N\times N} \subset C_{N\times N}$ $(s > 0)$ the Hölder-Zygmund spaces (for the definition see, e.g., [7, Definition 2.34]), and by $\mathcal{R}_{N\times N} \subset C_{N\times N}$ the algebra of all rational matrix functions on $\mathbb{T}$.

By the results of the previous section we have to determine the number of the singular values of $A_{n,r}$ tending to zero. This suggests us to investigate the convergence speed of $s_k(A_{n,r})$ to zero. To this end we use again general results of our paper [8].

Lemma 4.1 (see [8, Section 6]). *Let $a, b \in PC_{N\times N}$. If the singular integral operator $aP + bQ$ is Fredholm, then*

$$s_k(A_{n,r}) \le C \max\big(\|A_{n,r}\varphi_1\|, \ldots, \|A_{n,r}\varphi_l\|, \|W_n A_{n,r} W_n \psi_1\|, \ldots, \|W_n A_{n,r} W_n \psi_m\|\big)$$

with $k = \dim\ker(aP + bQ) + \dim\ker(\tilde{a}P + \tilde{b}Q)Q_{r-1}$, where the constant C does not depend on n, and $\{\varphi_i\}_{i=1}^l$ and $\{\psi_i\}_{i=1}^m$ are some orthonormal bases of $\ker(aP + bQ)$ *and* $\ker(\tilde{a}P + \tilde{b}Q)Q_{r-1}$, *respectively.*

Thus, we have to estimate the norms $\|A_{n,r}\varphi\|$ and $\|W_n A_{n,r} W_n \psi\|$, where $\varphi \in \ker(aP + bQ), \psi \in \ker(\tilde{a}P + \tilde{b}Q)Q_{r-1}$, and $\|\varphi\| = \|\psi\| = 1$.

First for each continuous function $f \in C_{N\times N}$ we put

$$E_n(f) := \inf_{p \in \mathcal{R}^n_{N\times N}} \|f - p\|_\infty, \quad n \in \mathbb{Z}_+,$$

where $\mathcal{R}^n_{N\times N}$ is the set of all trigonometric polynomials p on $\mathbb{T}$ of the form $p(t) = \sum_{k=-n}^n p_k t^k$, $p_k \in \mathbb{C}^{N\times N}$. It is well-known (see e.g. [11, Chapter 3.13]) that, for any $f \in C_{N\times N}$ and $n \in \mathbb{Z}_+$, there is a polynomial $p_n(f) \in \mathcal{R}^n_{N\times N}$ such that $E_n(f) = \|f - p_n(f)\|_\infty$.

Further we fix an $\alpha \in (0,1)$, and denote by $[\alpha n]$ the integer part of $\alpha n, n \in \mathbb{Z}_+$. Taking into account that $(P_n - W_n P_{r-1} W_n)P_{n-[\alpha n)} = P_{n-[\alpha n]}$ whenever

$[\alpha n] > r$, we get

$$\begin{aligned}
\|A_{n,r}\varphi\| &= \|L_n(aP+bQ)P_n(P_n - W_nP_{r-1}W_n)\varphi\| \\
&\le \|L_n(aP+bQ)P_nP_{n-[\alpha n]}\varphi\| \\
&\quad + \|L_n(aP+bQ)P_n(P_n - W_nP_{r-1}W_n)Q_{n-[\alpha n]}\varphi\| \\
&\le \big(\|a\|_\infty + \|b\|_\infty\big)\|Q_{n-[\alpha n]}\varphi\| \\
&\quad + \|L_n(p_{[\alpha n]}(a)P + p_{[\alpha n]}(b)Q)P_nP_{n-[\alpha n]}\varphi\| \\
&\quad + \|L_n\left(\big[a - p_{[\alpha n]}(a)\big]P + \big[b - p_{[\alpha n]}(b)\big]Q\right)P_nP_{n-[\alpha n]}\varphi\| \\
&\le \|L_n(p_{[\alpha n]}(a)P + p_{[\alpha n]}(b)Q)P_{n-[\alpha n]}\varphi\| \\
&\quad + \big(\|a\|_\infty + \|b\|_\infty\big)\|Q_{n-[\alpha n]}\varphi\| + E_{[\alpha n]}(a) + E_{[\alpha n]}(b)
\end{aligned}$$

for any $a, b \in C_{N\times N}$ and all n large enough. Here we used that (see [7, 7.3(c)]) $\|L_naPP_n\|, \|L_naQP_n\| \le \|a\|_\infty$, for all $a \in C_{N\times N}$, and $\|P_n - W_nP_{r-1}W_n\| \le 1$, for all $n \in \mathbb{Z}_+$. Now we note that the Lagrange interpolation operator L_n is exact for any trigonometric polynomial of degree n. Hence for all sufficiently large n

$$\begin{aligned}
\|A_{n,r}\varphi\| &\le \|(p_{[\alpha n]}(a)P + p_{[\alpha n]}(b)Q)P_{n-[\alpha n]}\varphi\| \\
&\quad + \big(\|a\|_\infty + \|b\|_\infty\big)\|Q_{n-[\alpha n]}\varphi\| + E_{[\alpha n]}(a) + E_{[\alpha n]}(b) \\
&\le \|(aP+bQ)P_{(n-[\alpha n])}\varphi\| \\
&\quad + \|([a - p_{[\alpha n]}(a)]P + [b - p_{[\alpha n]}(b)]Q)P_{(n-[\alpha n])}\varphi\| \\
&\quad + \big(\|a\|_\infty + \|b\|_\infty\big)\|Q_{n-[\alpha n]}\varphi\| + E_{[\alpha n]}(a) + E_{[\alpha n]}(b) \\
&\le \|(aP+bQ)\varphi\| + \|(aP+bQ)Q_{(n-[\alpha n])}\varphi\| \\
&\quad + \big(\|a\|_\infty + \|b\|_\infty\big)\|Q_{n-[\alpha n]}\varphi\| + 2E_{[\alpha n]}(a) + 2E_{[\alpha n]}(b) \\
&\le 2\big(\|a\|_\infty + \|b\|_\infty\big)\|Q_{n-[\alpha n]}\varphi\| + 2E_{[\alpha n]}(a) + 2E_{[\alpha n]}(b),
\end{aligned}$$

since $\|aI\| \le \|a\|_\infty, \|P\| = \|Q\| = 1$, and $\varphi \in \ker(aP+bQ)$.

Using that $W_n^2 = P_n$ and that (see [7, proof of Theorem 7.17]) $W_nL_n(aP + bQ)W_n = L_n(\tilde{a}P + \tilde{b}Q)P_n$ we get $W_nA_{n,r}W_n = L_n(\tilde{a}P + \tilde{b}Q)P_n(P_n - P_{r-1})$. Hence, in the same fashion we find that for all sufficiently large n

$$\|W_nA_{n,r}W_n\psi\| \le 2\big(\|a\|_\infty + \|b\|_\infty\big)\|Q_{(n-[\alpha n])}\psi\| + 2E_{[\alpha n]}(a) + 2E_{[\alpha n]}(b).$$

Here we used that $\|a\|_\infty = \|\tilde{a}\|_\infty$ and $E_n(a) = E_n(\tilde{a})$ for all $a \in C_{N\times N}$ and all $n \in \mathbb{Z}_+$.

Thus, we have proved the following theorem.

Theorem 4.2 (cf. Theorem 2.2)**.** *Let $a, b \in C_{N\times N}$ and let $\alpha \in (0,1)$. If the singular integral operator $aP + bQ$ is Fredholm, then*

$$\begin{aligned}
s_k(A_{n,r}) \le C\max\big(&E_{[\alpha n]}(a), E_{[\alpha n]}(b), \|Q_{n-[\alpha n]}\varphi_1\|, \dots, \|Q_{n-[\alpha n]}\varphi_l\|, \\
&\|Q_{n-[\alpha n]}\psi_1\|, \dots, \|Q_{n-[\alpha n]}\psi_m\|\big)
\end{aligned}$$

with $k = \dim\ker(aP + bQ) + \dim\ker(\tilde{a}P + \tilde{b}Q)Q_{r-1}$, *where the constant* C *does not depend on* n, *and* $\{\varphi_i\}_{i=1}^{l}$ *and* $\{\psi_i\}_{i=1}^{m}$ *are some orthonormal bases of* $\ker(aP + bQ)$ *and* $\ker(\tilde{a}P + \tilde{b}Q)Q_{r-1}$, *respectively.*

This inequality can be used to estimate the convergence speed by the smoothness of the functions a and b. Here are two sample results.

Corollary 4.3. *Let* $a, b \in C_{N\times N}$ *and let the singular integral operator* $aP + bQ$ *be Fredholm. If the functions* a, b *belong to* $\mathcal{H}^s_{N\times N}$ *for some* $s > 0$, *then*

$$s_k(A_{n,r}) = O\left(n^{-s}\right) \quad \text{as } n \to \infty. \tag{4}$$

If, in addition, the functions a, b *belong to* $\mathcal{R}_{N\times N}$, *then there is a* $\rho > 0$ *such that*

$$s_k(A_{n,r}) = O\left(e^{-\rho n}\right) \quad \text{as } n \to \infty. \tag{5}$$

Proof. First we consider the case when the functions a and b belong to the Hölder-Zygmund space $\mathcal{H}^s_{N\times N}$ for some $s > 0$. It it well-known (see e.g. [4, Proposition VII.2.1, Theorem VII.3.1, Theorem II.6.1]) that if the operator $aP+bQ$ is Fredholm, then the functions a, b are invertible and the function $c := b^{-1}a$ admits a factorization $c(t) = c_-(t)d(t)c_+(t)$, $t \in \mathbb{T}$, where d is a diagonal matrix function of the form $d(t) = \operatorname{diag} t^{\kappa_1}, \ldots, t^{\kappa_N}$ with certain integers $\chi_1, \ldots, \chi_N$ and $c_-^{\pm 1} \in \mathcal{H}^{s\,-}_{N\times N}, c_+^{\pm 1} \in \mathcal{H}^{s\,+}_{N\times N}$. Here we denote by $\mathcal{H}^{s\,-}_{N\times N}(\mathcal{H}^{s\,+}_{N\times N})$ the subspace of $\mathcal{H}^s_{N\times N}$ consisting of all functions f the Fourier coefficients f_n of which vanish for $n > 0 (n < 0)$.

One can easily check that this representation of the function c allows us to write the operator $aP + bQ$ in the form $aP + bQ = bc_-(dP + Q)(c_+P + c_-^{-1}Q)$, where the operators bc_-I and $c_+P + c_-^{-1}Q$ are invertible in $\mathcal{L}(L^2_N)$ (their inverses are $c_-^{-1}b^{-1}I$ and $c_+^{-1}P + c_-Q$, respectively).

Further it is well-known that if $a, b \in \mathcal{H}^s_{N\times N}$, then the singular integral operator $aP + bQ$ is bounded on $\mathcal{H}^s_N$. Here $\mathcal{H}^s_N = \mathcal{H}^s_{N\times 1}$ refers to the Hölder-Zygmund space of all column-vectors of length N with components from $\mathcal{H}^s_{1\times 1}$. In particular, the operators bc_-I and $c_+P + c_-^{-1}Q$ and their inverses are bounded on $\mathcal{H}^s_N$, too. This implies that the kernel of the operator $aP + bQ \in \mathcal{L}(L^2_N)$ coincides with kernel of the operator $aP + bQ \in \mathcal{L}(\mathcal{H}^s_N)$. In other words the kernel of the singular integral operator $aP + bQ \in \mathcal{L}(L^2_N)$ is included in the Hölder-Zygmund space $\mathcal{H}^s_N$ whenever the functions a and b belong to $\mathcal{H}^s_{N\times N}$.

Thus, in view of Theorem 4.2 it remains to estimate the order of the approximation for functions from the Hölder-Zygmund spaces. It is well-known that a function f belongs to $\mathcal{H}^s_{N\times N}$ if and only if (see, e.g., [7, §2.34]) $E_n(f) = O(n^{-s})$ as $n \to \infty$. Moreover, from the Parseval's equality we conclude that if $f \in \mathcal{H}^s_N$, then (recall that $E_n(f) = \|f - p_n(f)\|_\infty$)

$$\|Q_n f\| \le \|f - p_n(f)\| \le \text{ const } \|f - p_n(f)\|_\infty = O(n^{-s}) \quad \text{as } n \to \infty.$$

Hence in the case when $a, b \in \mathcal{H}^s_{N\times N}$ we arrive at the estimate (4).

To treat the case of rational functions a and b we introduce the spaces

$$L^2_N(\rho) := \left\{ f \in L^2_N : \sum_{k=-\infty}^{\infty} e^{2\rho|k|} \|f_k\|^2 < \infty \right\},$$

where $\{f_k\}$ is the sequence of the Fourier coefficients of f.

Since the Fourier coefficients of rational functions decay exponentially, we deduce that if $a, b \in \mathcal{R}_{N\times N}$, then the singular integral operator $aP + bQ$ is bounded on $L^2_N(\rho)$ whenever $\rho > 0$ is small enough. Moreover, in this case the function c admits such a factorization that the factors c_-, c_+ are rational functions (see e.g. [4, Theorem I.2.1]).

The estimate (5) can be proved in just the same way as the estimate (4). □

5. Numerical examples

First we consider the matrix representation of the operators

$$A_{n,r} := L_n(aP + bQ)P_n(P_n - W_n P_{r-1} W_n).$$

Let F_{2n+1} be the following $(2n+1)N \times (2n+1)N$ matrices (I_N is the identity $N \times N$ matrix) and let F^{-1}_{2n+1} be their inverses:

$$F_{2n+1} = \left(\frac{1}{\sqrt{2n+1}} e^{\frac{i2\pi jk}{2n+1}} I_N \right)_{j,k=0}^{2n}, \quad F^{-1}_{2n+1} = \left(\frac{1}{\sqrt{2n+1}} e^{-\frac{i2\pi jk}{2n+1}} I_N \right)_{j,k=0}^{2n}.$$

With respect to the standard basis of $\operatorname{im} P_n$ we get

$$A_{n,r} = F^{-1}_{2n+1} \left(a(t_j)\delta_{j,k}\right)_{j,k=0}^{2n} F_{2n+1} P_{n,r} + F^{-1}_{2n+1} \left(b(t_j)\delta_{j,k}\right)_{j,k=0}^{2n} F_{2n+1} Q_{n,r},$$

where $\delta_{j,k}$ is the Kronecker symbol, and

$$\begin{aligned} P_{n,r} &= \operatorname{diag}(\underbrace{0I_N, \dots, 0I_N}_{r}, \underbrace{I_N, \dots, I_N}_{n+1-r}, \underbrace{0I_N, \dots, 0I_N}_{n}) \\ Q_{n,r} &= \operatorname{diag}(\underbrace{0I_N, \dots, 0I_N}_{n+1}, \underbrace{I_N, \dots, I_N}_{n-\max(0,r-1)}, \underbrace{0I_N, \dots, 0I_N}_{\max(0,r-1)}). \end{aligned}$$

Further, taking into account that the singular values can be defined as approximation numbers (see (1)), we find that $s_k(A_{n,r}) = s_k(B_{n,r})$, for all k, n, r, where the matrices $B_{n,r}$ are defined by

$$\begin{aligned} B_{n,r} &= F_{2n+1} A_{n,r} F^{-1}_{2n+1} \\ &= \left(a(t_j)\delta_{j,k}\right)_{j,k=0}^{2n} F_{2n+1} P_{n,r} F^{-1}_{2n+1} + \left(b(t_j)\delta_{j,k}\right)_{j,k=0}^{2n} F_{2n+1} Q_{n,r} F^{-1}_{2n+1}. \end{aligned}$$

Here we used that $\|F_{2n+1}\| = \|F_{2n+1}^{-1}\| = 1$. The advantage of matrices $B_{n,r}$ over $A_{n,r}$ is the ability to compute the matrices $F_{2n+1}P_{n,r}F_{2n+1}^{-1}$, $F_{2n+1}Q_{n,r}F_{2n+1}^{-1}$ in advance, that is independent of the functions a, b. After that one has only to multiply this matrices by the block diagonal matrices $(a(t_j)\delta_{j,k})_{j,k=0}^{2n}$, $(b(t_j)\delta_{j,k})_{j,k=0}^{2n}$.

Now we present three examples, where we compare the applicability of the finite section method and of the collocation method to compute the kernel dimension of the Toeplitz operator $T(a)$ and/or the singular integral operator $aP + Q$.

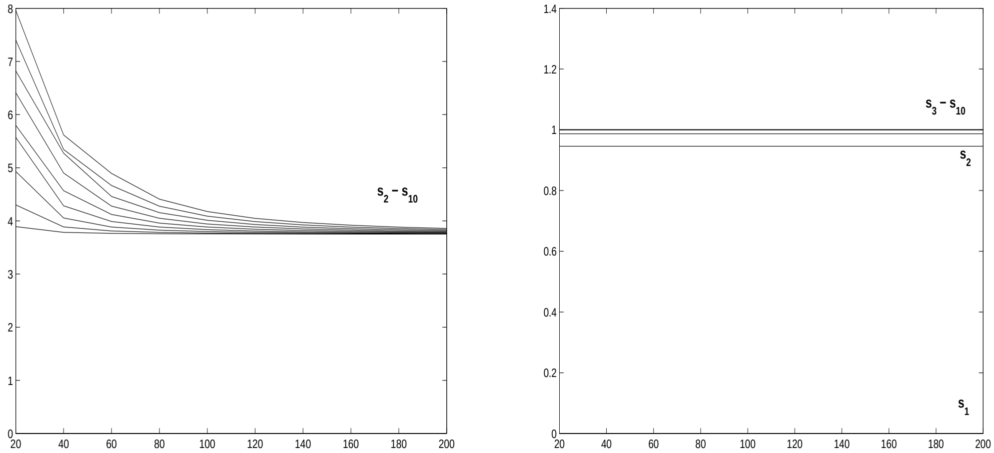

Figure 1: Finite section method for $T(a_1)$ and collocation method for $a_1P + Q$

Example 1. First we consider the scalar function

$$a_1(t) = -2.5t + 2.25 + 7t^{-1} - 3t^{-2} = (t^{-1} - 2.5)t^{-1}(t - 2)(t + 1.5).$$

Using the Wiener-Hopf factorization theory and elementary properties of Toeplitz operators we obtain that $\dim\ker T(a_1) = \dim\ker(a_1P + Q) = 1$ and $\dim\ker T(\tilde{a}_1) = \dim\ker(\tilde{a}_1P + Q) = 0$. In the Figure 1 we plotted the first ten singular values for $T_{n,0}(a_1)$ and $A_{n,0}$. For both methods (finite section and collocation) the first singular value converges very quickly to zero, and the second singular value stays away from zero, that is (see Theorems 2.1 and 3.5) $k = \dim\ker T(a_1) + \dim\ker T(\tilde{a}_1) = \dim\ker(a_1P + Q) + \dim\ker(\tilde{a}_1P + Q) = 1$.

Example 2. Next we consider the 2×2 matrix function

$$\begin{aligned} a_2(t) &= \begin{pmatrix} \frac{10-9t}{10t-9} & \frac{t^3}{10t-9} + t^2 \sin t^{-1} \\ 0 & 1 - 4t \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{10-9t^{-1}} & t \sin t^{-1} \\ 0 & t^{-1} - 4 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 10 - 9t & t^3 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Again using the Wiener-Hopf factorization theory we find that $\dim\ker T(a_2) = \dim\ker(a_2P + Q) = 1$.

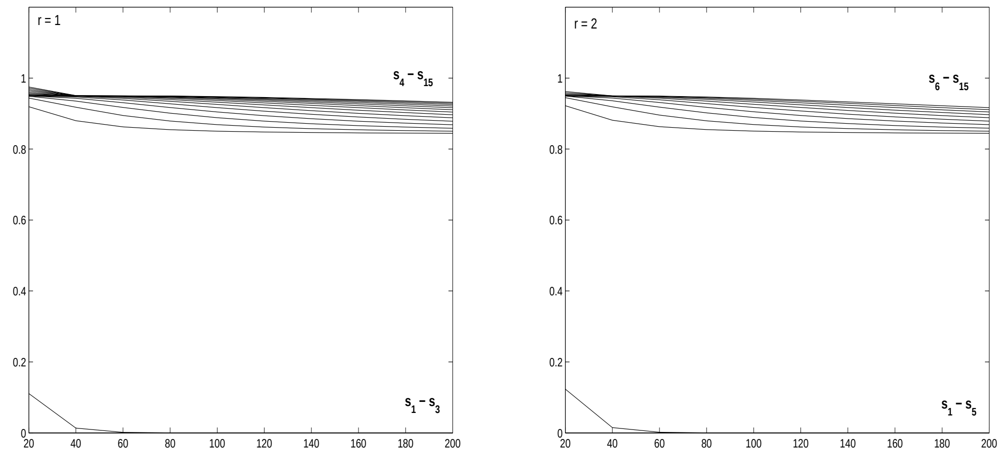

Figure 2: Finite section method for $T(a_2)$

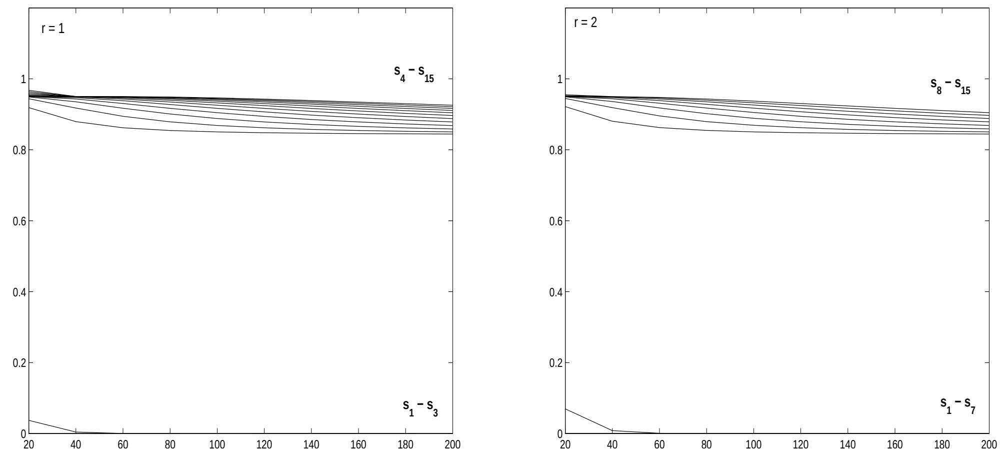

Figure 3: Collocation method for $a_2P + Q$

In the Figures 2 and 3 we plotted the first fifteen singular values for $T_{n,r}(a_2)$ and $A_{n,r}$, respectively, with $r = 1$ and $r = 2$. The pictures show that we have

$$\text{for finite section method} \quad - \quad \begin{cases} \text{3-splitting property}, & r = 1 \\ \text{5-splitting property}, & r = 2 \end{cases}$$

$$\text{for collocation method} \quad - \quad \begin{cases} \text{3-splitting property}, & r = 1 \\ \text{7-splitting property}, & r = 2. \end{cases}$$

Thus, taking into account the remarks after Theorems 2.1 and 3.5, we obtain that

$$\dim\ker T(a_2) = \begin{cases} 3 - 2 * 1, & r = 1 \\ 5 - 2 * 2, & r = 2 \end{cases} = 1$$

$$\dim\ker(a_2 P + Q) = \begin{cases} 3 - 2 * (2 - 1), & r = 1 \\ 7 - 2 * (4 - 1), & r = 2 \end{cases} = 1.$$

Example 3. Now we consider the following piecewise continuous function

$$a_3(e^{i\theta}) = \psi_{0.55}(e^{i\theta}) = \frac{\pi}{\sin 0.55\pi} e^{i0.55\pi} e^{-i0.55\theta}, \qquad 0 \le \theta < 2\pi.$$

It is clear that the function a_3 has only one point of discontinuity at $e^{i\theta} = 1$. Moreover it is known (see e.g. [3, §1.24]) that the operators $T(a_3)$ and a_3P+Q are Fredholm and $\dim\ker T(a_3) = \dim\ker(a_3P + Q) = 1$, $\dim\ker T(\tilde{a}_3) = \dim\ker(\tilde{a}_3P + Q) = 0$. In the Figure 4 we plotted the first ten singular values

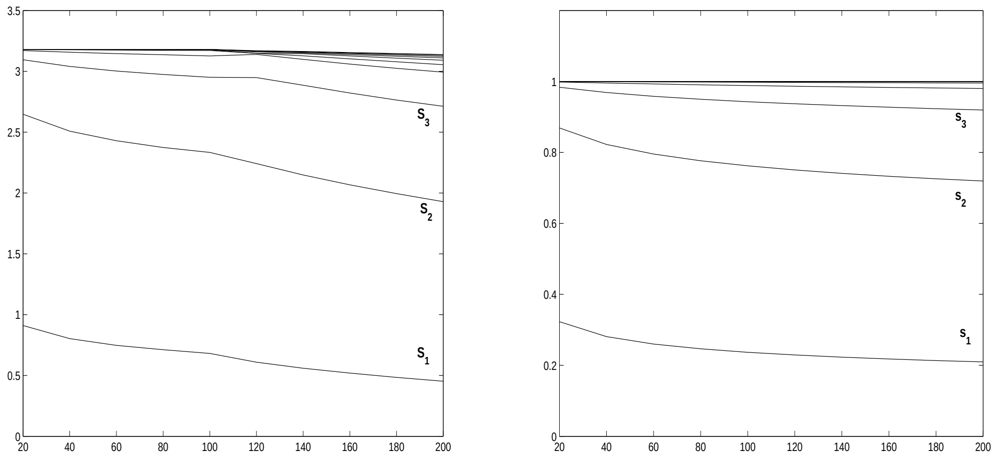

Figure 4: Finite section method for $T(a_3)$ and collocation method for $a_3P + Q$

for $T_{n,0}(a_3)$ and $A_{n,0}$. Looking at the pictures one can not see whether the singular values have the k-splitting property, although Theorems 2.1 and 3.5 imply that for both methods (finite section and collocation) we have the 1-splitting property.

Thus we can conclude that the finite section method and the collocation method are equivalently applicable to compute the kernel dimension of the Toeplitz operator $T(a)$ and/or the singular integral operator $aP + Q$. More precisely, if the k-splitting property is clearly observed for the finite section method, then it is clearly observed for the collocation method too. And if one can not see the k-splitting property for the collocation method, then the finite section method is also useless.

Acknowledgement. This work was supported by the Deutsche Forschungsgemeinschaft, DFG project SI 474110-1.

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Received April 3, 2006; revised January 12, 2007