

Transition to Instability of Planar Viscous Shock Fronts: the Refined Stability Condition

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Abstract. Classical inviscid stability analysis determines stability of shock waves only up to a region of neutral stability occupying an open set of physical parameters. To locate a precise transition point within this region, it has been variously suggested that nonlinear and or viscous effects should be taken into account. Recently, Zumbrun and Serre showed that transition under localized ($L^1 \cap H^s$) perturbations is in fact entirely decided by viscous effects, and gave an abstract criterion for transition in terms of an effective viscosity coefficient β determined by second derivatives of the Evans function associated with the linearized operator about the wave. Here, generalizing earlier results of Kapitula, Bertozzi et al, and Benzoni-Gavage et al., we develop a simplified perturbation formula for β , applicable to general shock waves, that is convenient for numerical and analytical investigation.

Keywords. Evans function, viscous conservation laws, spectral stability

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1. Introduction

Consider a planar shock solution

$$u(x, t) = \bar{u}\left(\frac{x_1 - st}{\nu}\right), \quad \lim_{z \rightarrow \pm\infty} \bar{u}(z) = u_{\pm}, \tag{1.1}$$

of a general system of viscous conservation laws

$$u_t + \sum_j f^j(u)_{x_j} = \nu \sum_{j,k} (B^{jk}(u)u_{x_k})_{x_j}, \tag{1.2}$$

$u, f^j \in \mathbb{R}^n, B \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^d, \nu > 0$, satisfying the traveling-wave ODE

$$B^{11}(\bar{u})\bar{u}' = f^1(\bar{u}) - f^1(u_-) - s(\bar{u} - u_-). \tag{1.3}$$

Solution (1.1) converges as $\nu \rightarrow 0$ to an ideal shock solution

$$u(x, t) = \begin{cases} u_- & x < st \\ u_+ & x \geq st, \end{cases} \tag{1.4}$$

$$f^1(u_+) - f^1(u_-) - s(u_+ - u_-) = 0, \tag{1.5}$$

of the corresponding inviscid system

$$u_t + \sum_j f^j(u)_{x_j} = 0. \tag{1.6}$$

For definiteness, we restrict to the case of a classical Lax-type shock (defined Section 2). Equivalently (see [62, 67] for strictly parabolic systems, [48, 63] for real viscosity systems), we assume that \bar{u} is a unique, transverse orbit solving the connection problem (1.3), connecting nonhyperbolic rest points u_+ and u_- . Nonclassical under- or overcompressive shocks may be treated similarly; see, e.g., the treatment of undercompressive phase transitions in [3].

We have in mind the physical example in which (1.1) represents a gas-dynamical shock wave and (1.2) and (1.6) the compressible Navier–Stokes and Euler equations, respectively. Gas dynamical shocks are under normal circumstances quite stable; however, in extreme parameter ranges they are known to become unstable [2]. Natural questions, therefore, are the stability under perturbation of viscous vs. ideal shocks as solutions of their respective systems, and the predictive value of these mathematical considerations in determining physically observed transitions to instability.

Inviscid stability analysis centers about the Lopatinski determinant

$$\Delta(\tilde{\xi}, \lambda) := \det \left(\mathcal{R}_1^- \cdots \mathcal{R}_{p-1}^- \mathcal{R}_{p+1}^+ \cdots \mathcal{R}_n^+ \lambda[u] + i[f^{\tilde{\xi}}] \right), \tag{1.7}$$

$\tilde{\xi} = (\xi_2, \dots, \xi_d) \in \mathbb{R}^{d-1}$, $\lambda = \gamma + i\tau \in \mathbb{C}$, $\tau > 0$, a spectral determinant whose zeroes correspond to normal modes $e^{\lambda t} e^{i\tilde{\xi} \cdot \tilde{x}} w(x_1)$, $\tilde{x} = (x_2, \dots, x_d)$, of the constant-coefficient linearized equations about (1.4), or spectra of the linearized operator about the wave. Here $f^{\tilde{\xi}} := \sum_{j=2}^d \xi_j f^j$ and $\{\mathcal{R}_{p+1}^+, \dots, \mathcal{R}_n^+\}$ and $\{\mathcal{R}_1^-, \dots, \mathcal{R}_{p-1}^-\}$ denote bases for the unstable/resp. stable subspaces of

$$\mathcal{A}_+(\tilde{\xi}, \lambda) := (\lambda I + idf^{\tilde{\xi}}(u_{\pm})) (df^1(u_{\pm}))^{-1}. \quad (1.8)$$

Weak stability $|\Delta| > 0$ is clearly necessary for linearized stability, while strong, or uniform stability, $\frac{|\Delta|}{|(\tilde{\xi}, \lambda)|} \geq c_0 > 0$, is sufficient for nonlinear stability. Between strong instability, or failure of weak stability, and strong stability, there lies a region of neutral stability corresponding to the appearance of surface waves propagating along the shock front. For details, see, e.g., [4, 42–44, 49, 56–58, 62–64, 67], and references therein¹; for generalizations of (1.7) to the nonclassical and or nonconservative case, see [11, 18, 27, 39, 52, 61, 67].

The region of neutral inviscid stability typically occupies an open set in physical parameter space [4, 42–44, 62, 63], and thus the basic inviscid theory we have just described does not identify a precise transition point from stability to instability as shock parameters are varied. It has been suggested [2] that accounting of additional nonlinear and or second-order transport effects might resolve this issue. On the other hand, it is noted also that, at the high energy levels needed to observe instability, experiments are somewhat inconclusive, with onset of instability sometimes appearing to occur earlier, within the (mathematical) strongly stable region. Barmin and Egorushkin [2] speculate that the latter might be a result of incomplete modeling of phase-transitional effects associated with ionization.

Weakly nonlinear analysis within the inviscid framework² has been pursued by Majda and Rosales in the related context of detonation waves, with interesting asymptotic and numerical results; see [46, 47]. We focus here on a different approach initiated by Zumbrun and Serre [22, 62–64, 67] within the context of the viscous equations (1.2), incorporating nonlinear and viscous effects, concerning long-time stability with respect to localized $(L^1 \cap H^s)$ perturbations.

Remark 1.1. As discussed in [23, 63], inviscid stability theory concerns short-time stability, or well-posedness (automatic for parabolic equations) with respect to H^s perturbations, whereas standard viscous stability theory concerns long-time asymptotic stability with respect to the more restrictive class of $L^1 \cap H^s$ perturbations. These notions are loosely related by rescaling, but in

¹See especially the seminal work [13].

²More precisely, for the Zeldovich–von Neumann–Doering (ZND) model including reaction but not transport effects; as discussed in [31, 62], this model is intermediate to the Chapman–Jouget and reactive Navier–Stokes models analogous to (1.6) and (1.2).

general give complementary information; see discussion [23]. For a short-time viscous analysis in the spirit of [42–44], see the treatment of the inviscid limit in [23–25].

Viscous stability analysis centers about the *Evans function* $D(\tilde{\xi}, \lambda)$, $\tilde{\xi} = (\xi_2, \dots, \xi_d) \in \mathbb{R}^{d-1}$, $\lambda = \gamma + i\tau \in \mathbb{C}$, $\tau > 0$, a spectral determinant analogous to the Lopatinski determinant of the inviscid theory, whose zeroes correspond to normal modes $e^{\lambda t} e^{i\tilde{\xi} \cdot \tilde{x}} w(x_1)$, $\tilde{x} = (x_2, \dots, x_d)$, of the linearized equations about (1.1) (now variable-coefficient), or spectra of the linearized operator about the wave. The main result of [67], establishing a rigorous relation between viscous and inviscid stability, was the asymptotic expansion

$$D(\tilde{\xi}, \lambda) = \gamma \Delta(\tilde{\xi}, \lambda) + o(|(\tilde{\xi}, \lambda)|) \tag{1.9}$$

of D about the origin $(\tilde{\xi}, \lambda) = (0, 0)$, where γ is a constant measuring transversality of (1.1) as a connecting orbit of (1.3). Equivalently, considering $D(\xi, \lambda) = D(\rho \xi_0, \rho \lambda_0)$ as a function of polar coordinates (ρ, ξ_0, λ_0) , we have

$$D|_{\rho=0} = 0 \quad \text{and} \quad \left. \frac{\partial}{\partial \rho} \right|_{\rho=0} D = \gamma \Delta(\xi_0, \lambda_0). \tag{1.10}$$

An important consequence of (1.9) is that *weak inviscid stability*, $|\Delta| > 0$, is necessary for *weak viscous stability*, $|D| > 0$ (an evident necessary condition for linearized viscous stability). For, (1.9) implies that the zero set of D is tangent at the origin to the cone $\{\Delta = 0\}$ (recall, (1.7), that Δ is homogeneous, degree one), hence enters $\{\tau > 0\}$ if $\{\Delta = 0\}$ does. Moreover, in case of *neutral inviscid stability* $\Delta(\xi_0, i\tau_0) = 0$, $(\xi_0, i\tau_0) \neq (0, 0)$, one may extract a further, *refined stability condition*

$$\beta := - \left. \frac{D_{\rho\rho}}{D_{\rho\lambda}} \right|_{\rho=0} \geq 0 \tag{1.11}$$

necessary for weak viscous stability. For, (1.10) then implies

$$D_\rho|_{\rho=0} = \gamma \Delta(\xi_0, i\tau_0) = 0,$$

whence Taylor expansion of D yields that the zero level set of D is concave or convex toward $\tau > 0$ according as the sign of β ; see [67] for details. As discussed in [62, 67], the constant β has a heuristic interpretation as an effective diffusion coefficient for surface waves moving along the front.

Under additional structural assumptions, satisfied in particular for the equations of gas dynamics, converse results have been established in [62–64], showing that $\beta > 0$ is sufficient for “low-frequency linearized stability” in the sense that this condition, together with “high-frequency stability” $|D(\xi, \lambda)| > 0$ for

$|(\xi, \lambda)| \geq c_0 > 0$ and $\tau \geq 0$, is sufficient for linearized and nonlinear stability. This means that, in the context of long-time viscous stability with respect to localized perturbations, *transition to instability is in principle determined* under existing theory.

For, taking a fixed left-state u_- and moving along the Hugoniot curve of right states u_+ satisfying the Rankine–Hugoniot relation (1.5), we find that the associated shock profile transitions from low-frequency stability to low-frequency instability either at the endpoints of the region of neutral inviscid stability (readily computable; see, e.g., [4, 13, 63]) or at a point $\beta = 0$ where β change sign. Likewise, transition from high-frequency stability to high-frequency instability occurs at (generically isolated) points along the Hugoniot curve at which high-frequency eigenvalues cross the imaginary axis. In either case, transition must occur at one of a discrete set of points.

Thus, we have the following situation. Recalling that inviscid theory is basically a low-frequency, or long-wave approximation neglecting inner shock structure, we see that, *insofar as inviscid theory is predictive of stability, a refinement quantifying the transition to instability may be obtained by evaluation of β in (1.11).*

On the other hand, it may well be that, in certain circumstances, stability is determined, rather, by *high-frequency*, viscous effects completely foreign to inviscid analysis; indeed, this, rather than unmodeled phase-transitional effects, might be an explanation for experimentally observed occasional early onset of instability as described in [2]. To detect the latter situation, we see no option other than a fully resolved numerical evaluation of the Evans function as discussed, e.g., in [7–10, 30].

Remark 1.2. In the related context of detonation waves, experimental and theoretical evidence indicates that transition to instability typically occurs at *high frequency*, corresponding to a relative Poincaré–Hopf bifurcation, or complex conjugate pair of eigenvalues crossing the imaginary axis; see, e.g., [31, 40, 41, 59, 60], and references therein³. Whether and when high-frequency transition occurs for shock waves remains an important philosophical question, possibly connected with existence of an increasing thermodynamic entropy; see [63, 1.20.2].

This state of affairs suggests a two-pronged approach to determination of stability transitions, on one hand focusing on low-frequency stability and determination of β , and on the other (in more limited situations) on high-frequency

³In particular, an expansion (1.9) relates the Evans function D for the full, reactive Navier–Stokes equations to the Lopatinski determinant Δ for the simplified Chapman–Jouget model, with Δ strongly stable for an ideal gas despite experimentally observed instability in some regimes [31].

stability and full numerical evaluation of the Evans function, the expectation being that low-frequency stability “usually” determines transition, with infrequent (if any) exceptions in unusual parameter regimes. In this paper, we initiate the first (and expectedly the primary one) of these directions of investigation by deriving a compact, Lyapunov–Schmidt-type formula for β that is convenient for numerical and analytical investigation. We intend to use this formula in followup work to investigate low-frequency transition to instability in various physically and mathematically interesting cases.

Specifically, we establish the following result generalizing (1.9).

Theorem 1.3. *Let \bar{u} be a Lax shock and $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\tau \in \mathbb{R}$ such that $\Delta(\tilde{\xi}, i\tau) = 0$, satisfying (H0)–(H7) (defined below). Then, $\beta := -\frac{D_{\rho\rho}}{D_{\rho\lambda}}(\tilde{\xi}, i\tau)$ is given by*

$$\begin{aligned} \beta &= -\Delta_{\lambda}(\tilde{\xi}, \tau)^{-1} \left[\left\langle l, -2 \left[\frac{\partial}{\partial \rho} L_{\rho} \right]_{|\rho=0} \tilde{y} - \left[\left(\frac{\partial}{\partial \rho} \right)^2 L_{\rho} \right]_{|\rho=0} \bar{u}' \right\rangle + \mathcal{B} \right] \\ &= -\Delta_{\lambda}(\tilde{\xi}, \tau)^{-1} \left[\left\langle l, 2(\lambda I + A^{\tilde{\xi}} - iB^{1\tilde{\xi}}\partial_{x_1} - i\partial_{x_1}B^{\tilde{\xi}1})\tilde{y} + 2B^{\tilde{\xi}\tilde{\xi}}\bar{u}' \right\rangle + \mathcal{B} \right], \end{aligned} \tag{1.12}$$

where $\Delta(\cdot, \cdot)$ is as defined in (1.7), $\langle \cdot, \cdot \rangle$ denotes the standard complex L^2 inner product, l is the constant vector determined by the exterior product

$$l := \mathcal{R}_1^- \wedge \cdots \wedge \mathcal{R}_{p-1}^- \wedge \mathcal{R}_{p+1}^+ \wedge \cdots \wedge \mathcal{R}_n^+, \tag{1.13}$$

\tilde{y} is the (unique, modulo \bar{u}') solution of the (first order) variational equation $L_0\tilde{y} = -\left[\frac{\partial}{\partial \rho}L_{\rho}\right]_{|\rho=0}\bar{u}'$, or

$$\partial_{x_1}(B^{11}\partial_{x_1} - A^1)\tilde{y} = (\lambda I + A^{\tilde{\xi}} - iB^{1\tilde{\xi}}\partial_{x_1} - i\partial_{x_1}B^{\tilde{\xi}1})\bar{u}', \tag{1.14}$$

satisfying boundary conditions

$$\begin{aligned} \tilde{y}(+\infty) &\in (A_+^1)^{-1}\text{Span}\{\mathcal{R}_{p+1}^+, \dots, \mathcal{R}_n^+\} \\ \tilde{y}(-\infty) &\in (A_-^1)^{-1}\text{Span}\{\mathcal{R}_1^-, \dots, \mathcal{R}_{p-1}^-\}, \end{aligned}$$

and \mathcal{B} is a boundary term described in (3.10) of Proposition 3.3 below. (Here, $L_{\rho} = L_{\rho}(\tilde{\xi}, \lambda)$ is the Fourier–Laplace transform of the linearized operator about the wave, written in polar coordinates (ρ, ξ, λ) , ρ the radius; see (3.2).)

Remark 1.4. Noting that Δ can be expressed as

$$\Delta = \langle l, (\lambda I + A^{\tilde{\xi}} - iB^{1\tilde{\xi}}\partial_x - i\partial_x B^{\tilde{\xi}1})\bar{u}' \rangle,$$

(see related calculations (2.26)–(2.29) of [67]) we find that formula (1.12) is indeed independent of the choice of \tilde{y} , since $\Delta = 0$ by assumption.

For gas dynamics, \mathcal{R}_j^\pm and Δ , hence \mathcal{B} and Δ_λ are explicitly computable (see [62, Appendix C], or [4, 13]), hence evaluation of β reduces to solution of (1.14) for \tilde{y} . More generally, \mathcal{B} and Δ_λ may be approximated numerically in well-conditioned fashion. Numerical approximation of \tilde{y} is also a well-conditioned problem, as discussed in Section 4. Thus, Theorem 1.3 *indeed gives a prescription for the desired efficient determination of β* , and thereby of the transition point between stability and instability of viscous shock waves.

Remark 1.5. The vector l can be recognized as an “effective” adjoint eigenfunction dual to \bar{u}' , see discussion [62, 66]. This establishes the connection with Lyapunov–Schmidt decomposition and the generalized Melnikov formulae of [32]. Specifically, our formula may be seen to agree, up to boundary term \mathcal{B} , with those of [32], generalizing one-dimensional results of [5, 6, 65]. As discussed in [5, 65], the appearance of a boundary term is associated with the absence of spectral gap for the linearized operator L_0 . Similar boundary terms have been observed in [33–35] in the context of perturbed nonlinear Schrödinger equations. Related second-derivative calculations may be found in [38].

Plan of the paper. In Section 2, we recall the construction of the Evans and Lopatinski determinants and other background facts. In Section 3, we carry out the proof of (3.1) and Theorem 1.3. Finally, in Section 4, we discuss numerical approximation of the function \tilde{y} appearing in (1.12).

Note. After the completion of this article, we have learned of new inviscid results of Coulombel and Secchi [12], which include among other things the conclusion that inviscid shocks in the “neutral stability” region are in fact nonlinearly stable in the sense of Majda. This important work answers in the positive the longstanding mathematical question of inviscid stability in the neutral regime. However, it leaves open the fundamental physical question whether inviscid well-posedness is a useful criterion to detect transition to instability, lending additional interest to our investigations.

2. Preliminaries

For clarity of exposition, we carry out the analysis in the simpler, strictly parabolic case. It is a routine but tedious exercise to verify that *all calculations carry over to the case of “real”, or physical, partially parabolic viscosity* treated in [62, 64] under the standard hypotheses therein: in particular, to the Navier–Stokes equations of compressible gas- or magnetohydrodynamics (MHD). See [62], Sections 3.1–3.2 for the analog of Proposition 3.1; the calculation of β goes similarly.

2.1. Assumptions. We make the standard hypotheses ([21, 63, 66, 67]):

- (H0) $f^j, B^{jk} \in C^2$.
- (H1) $\Re\sigma(\sum_{j,k} \xi_k \xi_j B^{jk}(\bar{u}(\cdot))) > 0$ for all nonzero $\xi \in \mathbb{R}^d$.
- (H2) $\sigma(\sum_j Df^j(u_{\pm})\xi_j)$ real, semi-simple for all $\xi \in \mathbb{R}^d, s \notin \sigma(Df^1(u_{\pm}))$.
- (H3) $\Re\sigma(-i \sum_j Df^j(u_{\pm})\xi_j - \sum_{j,k} B^{jk}(u_{\pm})\xi_k \xi_j) \leq -\theta|\xi|^2, \theta > 0$.
- (H4) The orbit $\bar{u}(\cdot)$ is a (unique) transverse connection for (1.3) between nonhyperbolic rest points u_{\pm} .

Hypotheses (H0)–(H3) correspond, respectively, to regularity, (local) parabolicity, hyperbolicity of u_{\pm} /nonsonicity of the shock triple (u_{-}, u_{+}, s) , and (strong) L^2 linearized stability of the constant solutions $u \equiv u_{\pm}$. Hypothesis (H4) includes the information that the dimensions of the unstable manifold of u_{-} and the stable manifold of u_{+} , i.e. (see (1.3)), the unstable subspace of $(B^{11})^{-1}(df^1(u_{-}) - sI)$ and the stable subspace of $(B^{11})^{-1}(df^1(u_{+}) - sI)$ sum to $n + 1$. By a lemma of [45] (see [62, 67]), this together with (H2)–(H3) implies that the dimensions i_{-} and i_{+} of the unstable subspace of $df^1(u_{-}) - sI$ and the stable subspace of $df^1(u_{+}) - sI$ sum also to $n + 1$, or, in the language of hyperbolic theory, there are $n + 1$ total characteristics incoming to the shock. The number of incoming characteristics defines (1.4) as a *classical, Lax p-shock*, $p = n - i_{-}$; see [62, 67] for further discussion.

Remark 2.1. Hypothesis (H4) specializes to the Lax case and slightly strengthens the more general “weak transversality” hypothesis of [62, 67] that, local to $\bar{u}(\cdot)$, the set of solutions of (1.1)–(1.3) connecting the same endstates u_{\pm} with same speed s form a smooth manifold $\{\bar{u}^{\delta}\}, \delta \in U \subset \mathbb{R}^{\ell}$. In the present case, $\{\bar{u}^{\delta}\} = \{\bar{u}(\cdot - \delta)\}$ and $\ell = 1$. For more general, nonclassical over- and undercompressive shocks (discussed in detail in [62, 67]), the number of incoming characteristics and the value of ℓ are essentially arbitrary.

2.2. The Evans function. Loosely following [67], we now briefly recall the construction of the Evans function. Without loss of generality, take $s = 0$, so that $\bar{u}(x_1)$ is a standing-wave solution. Linearizing (1.2) about $\bar{u}(\cdot)$, we obtain

$$v_t = Lv := \sum (B^{jk}v_{x_k})_{x_j} - \sum (A^jv)_{x_j}, \tag{2.1}$$

where coefficients

$$B^{jk} := B^{jk}(\bar{u}), \quad A^jv := Df^j(\bar{u})v - DB^{j1}(\bar{u})(v, \bar{u}_{x_1}) \tag{2.2}$$

are smooth functions of x_1 alone.

Seeking normal modes $v(x, t) = e^{\lambda t} e^{i\tilde{\xi} \cdot x} w(x_1)$ of (2.1), $w \in L^2(x_1)$ (equivalently, taking the Fourier–Laplace transform in transverse spatial directions

$\tilde{x} = (x_2, \dots, x_n)$ and time t), we are led to the family of generalized eigenvalue equations

$$\begin{aligned} 0 &= (L_{\tilde{\xi}} - \lambda)w \\ &= (B^{11}w')' - (A^1w)' + i \sum_{j \neq 1} B^{j1} \xi_j w' + i \left(\sum_{k \neq 1} B^{1k} \xi_k w \right)' \\ &\quad - i \sum_{j \neq 1} A^j \xi_j w - \sum_{j \neq 1, k \neq 1} B^{jk} \xi_j \xi_k w - \lambda w, \end{aligned} \tag{2.3}$$

where “ $'$,” denotes $\frac{\partial}{\partial x_1}$. Evidently, existence of solutions corresponds to linear dependency between the unstable manifold at $x_1 \rightarrow -\infty$ and the stable manifold at $x_1 \rightarrow +\infty$ of solutions of (2.3).

Lemma 2.2. *There exist choices $w_j^\pm(\tilde{\xi}, \lambda, x_1)$ of bases*

$$\{w_1^+, \dots, w_2^+\} \quad \text{and} \quad \{w_{n+1}^-, \dots, w_{2n}^-\},$$

spanning the stable/unstable manifolds at $-\infty/+ \infty$ of solutions of (2.3), jointly analytic in $(\tilde{\xi}, \lambda)$ in a neighborhood of any $(\tilde{\xi}, \lambda) \in \mathbb{R}^{d-1} \times \{\Re \lambda > 0\}$.

Proof. It is sufficient to show the corresponding result for the limiting, constant-coefficient equations as $x_1 \rightarrow \pm\infty$, the variable-coefficient result then following by asymptotic ODE theory: specifically, the “conjugation lemma” of [50] (a generalization of the “gap lemma” of [21, 34]), asserting as a consequence of exponential convergence of the coefficients that there is an $2n \times 2n$ bounded invertible change of coordinates $Q(\tilde{\xi}, \lambda, x)$ on $x > 0$ (resp. < 0), locally analytic in $(\tilde{\xi}, \lambda)$, taking solutions (\hat{w}, \hat{w}') of the limiting constant-coefficient equations to solutions $(w, w') = Q(\hat{w}, \hat{w}')$ of the variable-coefficient equations. But, the result for the constant-coefficient equations follows by spectral separation of stable and unstable eigenspaces on $\Re \lambda > 0$, an easy consequence of (H3). For details, see [67]. □

Definition 2.3. The (local) *Evans function* $D(\tilde{\xi}, \lambda)$ associated with (2.3) is defined as the Wronskian

$$D(\tilde{\xi}, \lambda) := \det \begin{pmatrix} w_1^+ & \cdots & w_n^+ & w_{n+1}^- & \cdots & w_{2n}^- \\ w_1^{+'} & \cdots & w_n^{+'} & w_{n+1}^{-'} & \cdots & w_{2n}^{-'} \end{pmatrix} \Big|_{x_1=0}. \tag{2.4}$$

Evidently, $D(\cdot, \cdot)$ is jointly analytic in $(\tilde{\xi}, \lambda)$ on its domain of definition. The Evans function as usual is specified only up to a nonvanishing analytic multiplier, since the functions w_j^\pm are specified only up to analytic change of basis.

Remark 2.4. The Evans function may alternatively be defined globally as a C^∞ function in $(\tilde{\xi}, \lambda)$ [25], and in the one-dimensional case globally analytic in λ [21]. Similarly, using homogeneity to reduce to a single complex argument, the Lopatinski determinant may be chosen globally C^∞ , and globally analytic in the two-dimensional (more generally, the rotationally symmetric) case⁴. Local C^∞ regularity is sufficient for the multidimensional theory of [22–25, 62–64, 67].

We now introduce the main object of our attention, the “radial Evans function”

$$D_{\tilde{\xi}, \lambda}(\rho) := D(\rho\tilde{\xi}, \rho\lambda), \quad \rho \in (0, +\infty).$$

This function is evidently jointly analytic in $(\tilde{\xi}, \lambda, \rho)$ where it is defined. The key point is to understand behavior as $\tau, \rho \rightarrow 0^+$ (recall, $\tau := \Re\lambda$).

2.3. Variable multiplicity and glancing sets. Denote by $a_j^\pm(\xi)$, $j = 1, \dots, n$ the eigenvalues (real by (H2)) of hyperbolic initial-value symbol $\sum_{j=1}^d A_\pm^j \xi_j$, indexed by increasing order, without loss of generality chosen to be (real) homogeneous degree one. Define Ξ_\pm^r to be the sets of ξ at which a_r^\pm changes multiplicity. Away from Ξ_\pm^r , each a_r^\pm is analytic in ξ . Define Θ_\pm^r to be the sets of $\xi \notin (\cup \Xi^r)_\pm$ at which $\frac{\partial a_r^\pm}{\partial \xi_1} = 0$.

Definition 2.5. Setting $\xi = (\xi_1, \dots, \xi_d) =: (\xi_1, \tilde{\xi})$, we define the *variable multiplicity set* \mathcal{V} as the collection of all $(\tilde{\xi}, i\tau)$ such that, for some real ξ_1 and $1 \leq r \leq n$, $\tau = -a_r^\pm(\xi_1, \tilde{\xi})$ and $(\xi_1, \tilde{\xi}) \in \Xi_\pm^r$. Similarly, we define the *glancing set* \mathcal{G} as the collection of all $(\tilde{\xi}, i\tau)$ such that, for some real ξ_1 and $1 \leq r \leq n$, $\tau = -a_r^\pm(\xi_1, \tilde{\xi})$, $(\xi_1, \tilde{\xi}) \in \Theta_\pm^r$. These are codimension one (hence measure zero) in $\mathbb{R}^{d-1} \times \{\Re\lambda = 0\}$, as the union of the projections onto $(\tilde{\xi}, i\tau)$ of the graphs $(\xi, -ia_r^\pm(\xi))$ over the codimension one sets Ξ_\pm^r, Θ_\pm^r .

The glancing set consists of points at which the analytic (temporal) modes $i\tau = -ia_r^\pm(\xi)$ of the hyperbolic initial-value symbol $\sum_j i\xi_j A_\pm^j$ are not analytically invertible as functions $i\xi_1 = \alpha_r(\tilde{\xi}, \tau)$, $r = 1, \dots, n$, describing the (spatial) modes of the hyperbolic initial-boundary-value symbol $-(A_\pm^1)^{-1}(i\tau + \sum_{j=2}^d i\xi_j A_\pm^j)$. The two symbols are linked by the full Fourier transform $(i\tau + \sum_j i\xi_j A_\pm^j)\hat{v} = 0$ of the frozen-coefficient linearized hyperbolic problem.

Remark 2.6. For gas dynamics, \mathcal{V} is empty and \mathcal{G} is readily calculable [49, 63]. Likewise \mathcal{V} and \mathcal{G} are readily calculable for the equations of magnetohydrodynamics (MHD) [37, 51].

Remark 2.7. For a notion of glancing at points of variable multiplicity, see [51].

⁴These cannot so far as we can see be prescribed globally analytically in the general multidimensional case as stated in [67].

2.4. Limiting behavior as $\rho, \tau \rightarrow 0^+$. Define the matrices

$$\mathcal{A}_\pm(\tilde{\xi}, \lambda) := (\lambda I + iA^{\tilde{\xi}})(A^1)_\pm^{-1}$$

conjugate to the hyperbolic initial-boundary-value symbol above, where $A^{\tilde{\xi}} := \sum_{j=2}^d \xi_j A^j$ and $A^j_\pm := A^j(\pm\infty)$. As a consequence of (H2), the stable/unstable subspaces of $\mathcal{A}_\pm(\tilde{\xi}, \lambda)$ remain spectrally separated, hence vary analytically in $(\tilde{\xi}, \lambda)$ on the set $\mathbb{R}^{d-1} \times \{\Re\lambda > 0\}$ [29]; see [49, 67] for further discussion. We have the following extension to the boundary $\Re\lambda = 0$.

Lemma 2.8. *Local to any point $(\tilde{\xi}, \lambda) = (\tilde{\xi}, i\tau) \in \mathbb{R}^{d-1} \times \{\Re\lambda = 0\}$ not belonging to the codimension-one set $\mathcal{V} \cup \mathcal{G}$, there exist analytic choices of bases $\{\mathcal{R}_j^\pm(\tilde{\xi}, \lambda)\}$ spanning the stable (resp. unstable) subspaces of $\mathcal{A}_\pm(\tilde{\xi}, \lambda)$ for $\Re\lambda > 0$.*

Proof. Clearly, the stable (resp. unstable) subspace of \mathcal{A}_\pm at $\Re\lambda = 0$ continues analytically into $\Re\lambda > 0$, by spectral separation, and thus admits a locally analytic basis. It is sufficient, therefore, to consider pure imaginary eigenvalues of \mathcal{A}_\pm .

Defining $i\xi_1 = \alpha_r^\pm(\tilde{\xi}, \tau)$ as above, ξ_1 real, with $\tau = -a_r^\pm(\xi)$ we find that the genuine eigenspace of \mathcal{A} associated to eigenvalue α_r is A^1 times the total eigenspace (recall the semisimplicity assumption of (H2)) of A^ξ associated with a_r , hence varies analytically so long as $(\tilde{\xi}, i\tau) \notin \mathcal{V}$. On the other hand, the condition $(\tilde{\xi}, i\tau) \notin \mathcal{G}$ is readily seen (see, e.g., [49, 62, 67]) to preclude a nontrivial Jordan block for \mathcal{A} , so that the total eigenspace of \mathcal{A} is equal to the genuine eigenspace and thus varies analytically. It therefore admits a locally analytic basis.

Finally, observing that $\Re\alpha_r = \Im\xi_1 = 0$ by (H2) implies a_r is real, hence λ is pure imaginary, we see that $\Re\alpha_r$ is of constant sign on $\Re\lambda > 0$. Thus, for $\Re\lambda > 0$, the stable (resp. unstable) subspace of \mathcal{A}_\pm is the direct sum of eigenspaces associated with all α_r with $\Re\alpha_r < 0$ (resp. > 0), hence by the foregoing discussion admits an analytic bases as claimed. \square

We can now state the following key result describing the structure of bases w_j^\pm in the low-frequency, or “hyperbolic”, limit $\rho \rightarrow 0^+$.

Lemma 2.9. *For any $(\tilde{\xi}_0, \lambda_0) \in \mathbb{R}^{d-1} \times \{\Re\lambda_0 \geq 0\}$ not in $\mathcal{V} \cup \mathcal{G}$, $D_{\tilde{\xi}, \lambda}(\rho)$ and also its component factors w_j^\pm in (2.4) has a unique (jointly) analytic extension onto a neighborhood of $(\tilde{\xi}, \lambda, \rho) = (\tilde{\xi}_0, \lambda_0, 0)$. Moreover, w_j^\pm may be chosen at $(\tilde{\xi}, \lambda, \rho) = (\tilde{\xi}_0, \lambda_0, 0)$, to satisfy the linearized traveling-wave equation*

$$(B^{11}w')' = (A^1w)', \tag{2.5}$$

with boundary conditions

$$\begin{aligned} w_1^+(+\infty) &= \dots = w_{i_+}^+(+\infty) = 0 \\ w_{2n-i_-+1}^-(-\infty) &= \dots = w_{2n}^-(-\infty) = 0 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} w_j^+(+\infty) &= (A_+^1)^{-1} \mathcal{R}_j^+(\tilde{\xi}_0, \lambda_0), \quad j = i_+ + 1, \dots, n, \\ w_{n+j}^-(-\infty) &= (A_-^1)^{-1} \mathcal{R}_j^-(\tilde{\xi}_0, \lambda_0), \quad j = 1, \dots, n - i_-, \end{aligned} \tag{2.7}$$

\mathcal{R}_j^\pm as defined in Lemma 2.8, where i_+ denotes the number of negative eigenvalues of A_+^1 , and i_- the number of positive eigenvalues of A_-^1 . In addition, $w_1^+, \dots, w_{i_+}^+$ and $w_{2n-i_-+1}^-, \dots, w_{2n}^-$ may be chosen independent of $(\tilde{\xi}, \lambda)$.

Proof. Here, again, the essential point is to establish behavior of the limiting constant-coefficient equations as $x_1 \rightarrow \pm\infty$, now written in polar coordinates, for $\rho, \tau \rightarrow 0^+$. At $\rho = 0$, these reduce to $A_\pm^1 w' = B_\pm^{11} w''$, hence have a center subspace of dimension n , consisting of constant solutions, and stable and unstable subspaces of dimensions summing to n . The latter evidently continue analytically, by spectral separation. Projecting onto the (also analytic) continuation of the center subspace, we find after a routine calculation that the former correspond to stable and unstable subspaces of $\rho \mathcal{A}_\pm + O(\rho^2)$, hence continue analytically by an argument quite similar to that of Lemma 2.8. Specifically, fixing $\rho = 0$ and varying $\tau = \Re \lambda$ to the side $\tau > 0$, we find by Lemma 2.8 that each of the eigenvalues $\rho \alpha_j^\pm$ of $\rho \mathcal{A}_\pm + O(\rho^2)$ bifurcating from a single eigenvalue α_j^\pm of \mathcal{A}_\pm must have the same sign for $\tau > 0$, and thus for $\rho \geq 0, \tau \geq 0, (\rho, \tau) \neq (0, 0)$. Noting that the corresponding total eigenspaces vary analytically, by spectral separation, we are done. For details, see [67]. \square

3. Evans function computations

We are now ready to carry out the desired Evans function computations.

3.1. The low-frequency limit. For later use, we first review the proof of expansion (1.9), (1.10) in the simple, analytic case $(\tilde{\xi}, \lambda) \notin \mathcal{V} \cup \mathcal{G}$. Note that the conclusion is somewhat stronger in this case, with quadratic order error term. For discussion of the case $(\tilde{\xi}, \lambda) \notin \mathcal{V}$, see [23–25, 51, 62–64]; for the general case, see [26].

Proposition 3.1 ([62, 67]). *For a Lax shock satisfying (H0)–(H4) and any pair $(\tilde{\xi}, \lambda) \in \mathbb{R}^{d-1} \times \{\Re \lambda \geq 0\}$ not in $\mathcal{V} \cup \mathcal{G}$,*

$$D(\tilde{\xi}, \lambda) = \gamma \Delta(\tilde{\xi}, \lambda) + \mathcal{O}(|(\tilde{\xi}, \lambda)|^2), \tag{3.1}$$

where γ is a constant measuring transversality of the connection $\bar{u}(\cdot)$ (hence nonzero, by (H4)), $\mathcal{O}(\cdot)$ is uniform for $|(\tilde{\xi}, \lambda)|$ bounded, and $\Delta(\cdot, \cdot)$ is the Lopatinski determinant defined in (1.7), or, equivalently, $D_{\tilde{\xi}, \lambda}(0) = 0$ and

$$\frac{\partial}{\partial \rho} D_{\tilde{\xi}, \lambda}(0) = \gamma \Delta(\tilde{\xi}, \lambda).$$

Remark 3.2. Similar, but less explicit formulae hold for nonclassical over- or undercompressive shocks; see [3, 6, 62, 67].

Proof. Rewriting the generalized eigenvalue equation (2.3) in polar coordinates, we have

$$\begin{aligned}
 0 = L_\rho w := & (B^{11}w')' - (A^1w)' + i\rho \sum_{j \neq 1} B^{j1}\xi_j w' + i\rho \left(\sum_{k \neq 1} B^{1k}\xi_k w \right)' \\
 & - i\rho \sum_{j \neq 1} A^j \xi_j w - \rho \lambda w - \rho^2 \sum_{j \neq 1, k \neq 1} B^{jk} \xi_j \xi_k w.
 \end{aligned} \tag{3.2}$$

Recall that the rows of the Evans function determinant are made up of solutions $w_j^\pm(\tilde{\xi}, \lambda, \rho)$, with boundary conditions (2.6)–(2.7) at $\rho = 0$. Note as usual that $L_0 w = (B^{11}w')' - (A^1w)'$ reduces to the one-dimensional linearized operator about the wave.

The variation \bar{u}' satisfies (2.5) and decays exponentially at both $\pm\infty$, hence belongs to both of the fast-decaying manifolds

$$\text{span}\{w_1^+, \dots, w_{i_+}^+\} \quad \text{and} \quad \text{span}\{w_{2n-i_-+1}^-, \dots, w_{2n}^-\}$$

for $\rho = 0$, all $(\tilde{\xi}, \lambda)$ (and indeed lies in the kernel of $L_{\tilde{\xi}}$ for all $\tilde{\xi}$). Without loss of generality, therefore, normalize $w_1^+ = w_{2n}^- = \bar{u}'$ at $\rho = 0$, for all $(\tilde{\xi}, \lambda)$.

With this choice, $D_{\tilde{\xi}, \lambda}(0) = 0$ follows immediately from definition (2.4). Likewise, we find, applying the Leibnitz rule and performing elementary matrix manipulations, that

$$\begin{aligned}
 \frac{\partial}{\partial \rho} D(0) = & \det \begin{pmatrix} \frac{\partial}{\partial \rho} w_1^+ & \cdots & w_{2n}^- \\ \frac{\partial}{\partial \rho} w_1^{+'} & \cdots & w_{2n}^{-'} \end{pmatrix} \Big|_{x_1=0} \\
 & + \det \begin{pmatrix} w_1^+ & \frac{\partial}{\partial \rho} w_2^+ & \cdots & w_{2n}^- \\ w_1^{+'} & \frac{\partial}{\partial \rho} w_2^{+'} & \cdots & w_{2n}^{-'} \end{pmatrix} \Big|_{x_1=0} + \cdots \\
 & + \det \begin{pmatrix} w_1^+ & \cdots & \frac{\partial}{\partial \rho} w_{2n}^- \\ w_1^{+'} & \cdots & \frac{\partial}{\partial \rho} w_{2n}^{-'} \end{pmatrix} \Big|_{x_1=0} \\
 = & \det \begin{pmatrix} \frac{\partial}{\partial \rho} w_1^+ & \cdots & w_{2n}^- \\ \frac{\partial}{\partial \rho} w_1^{+'} & \cdots & w_{2n}^{-'} \end{pmatrix} \Big|_{x_1=0} + 0 + \cdots + 0 \\
 & + \det \begin{pmatrix} w_1^+ & \cdots & \frac{\partial}{\partial \rho} w_{2n}^- \\ w_1^{+'} & \cdots & \frac{\partial}{\partial \rho} w_{2n}^{-'} \end{pmatrix} \Big|_{x_1=0} \tag{3.3} \\
 = & \det \begin{pmatrix} w_1^+ & \cdots & w_{2n-1}^- & (y^- - y^+) \\ w_1^{+'} & \cdots & w_{2n-1}^{-'} & (y^- - y^+)' \end{pmatrix} \Big|_{x_1=0},
 \end{aligned}$$

where $y^+ := \frac{\partial}{\partial \rho} w_1^+$ and $y^- := \frac{\partial}{\partial \rho} w_{2n}^-$ at $\rho = 0$.

To evaluate this quantity, recall that $w = w_j^\pm$ satisfy (2.5) at $\rho = 0$, with boundary conditions (2.6) prescribed at $+\infty, -\infty$ respectively. Integrating (2.5) from $+\infty / -\infty$ to x_1 , we thus obtain the relation

$$B^{11}w' = A^1w - A^1w|_{\pm\infty}, \quad \text{or} \quad B^{11}w' - A^1w = -A^1w|_{\pm\infty}.$$

Referring to (2.6)–(2.7), we find that

$$B^{11}w_j^{\pm'} - A^1w_j^\pm = 0 \tag{3.4}$$

for decaying modes $w_1^+, \dots, w_{i_+}^+$ and $w_{2n-i_-+1}^-, \dots, w_{2n}^-$, while, for the asymptotically constant modes,

$$\begin{aligned} B^{11}w_j^{+'} - A^1w_j^+ &= \mathcal{R}_j^+(\tilde{\xi}, \lambda), \quad j = i_+ + 1, \dots, n \\ B^{11}w_{n+j}^{-'} - A^1w_{n+j}^- &= \mathcal{R}_j^-(\tilde{\xi}, \lambda), \quad j = 1, \dots, n - i_- \end{aligned} \tag{3.5}$$

Differentiating (3.2), we find, further, that derivatives $y_j^\pm := \frac{\partial}{\partial \rho} w_j^\pm$ satisfy the first order variational equations

$$\begin{aligned} L_\rho y &= - \left[\frac{\partial}{\partial \rho} L_\rho \right] w \\ &= -i \sum_{j \neq 1} B^{j1} \xi_j w^{\pm'} - i \left(\sum_{k \neq 1} B^{1k} \xi_k w^\pm \right)' \\ &\quad + i \sum_{j \neq 1} A^j \xi_j w^\pm + \lambda w^\pm + 2\rho i \sum_{j \neq 1, k \neq 1} B^{jk} \xi_j \xi_k w, \end{aligned} \tag{3.6}$$

with boundary conditions

$$y^+(+\infty) = y^-(-\infty) = 0.$$

Here, w^\pm denote w_1^+, w_{2n}^- respectively. Recalling that, for $\rho = 0$, $w_1^+ = w_{2n}^- = \bar{u}'$, and using the definition of A^j in (2.2) to express $A^j \bar{u}' - B^{j1} \xi_j \bar{u}''$ as a perfect derivative, $f^j(\bar{u})' - (B^{j1}(\bar{u})\bar{u}')'$, we obtain, integrating (3.6) from $+\infty / -\infty$, respectively, to x_1 , the relations

$$\begin{aligned} B^{11}y^{\pm'} - A^1y^\pm &= i f^{\tilde{\xi}}(\bar{u}) - i B^{1\tilde{\xi}}(\bar{u})\bar{u}' - i B^{\tilde{\xi}1}(\bar{u})\bar{u}' + \lambda \bar{u} \\ &\quad - (i f^{\tilde{\xi}}(u_\pm) + \lambda u_\pm), \end{aligned} \tag{3.7}$$

and thus

$$B^{11}(y^- - y^+) - A^1(y^- - y^+) = i[f^{\tilde{\xi}}(u)] + \lambda[u]. \tag{3.8}$$

Combining (3.4)–(3.5) with (3.8), and recalling (3.3), we obtain by the row

operation $B^{11}w' - A^1w$ that

$$\frac{\partial D}{\partial \rho}(0) = \frac{1}{\det(B^{11})} \times \begin{vmatrix} w_1^+ & \cdots & w_{i_+}^+ & * & \cdots & * & * & \cdots & * & w_{2n-i_-+1}^- & \cdots & w_{2n-1}^- & * \\ 0 & \cdots & 0 & \mathcal{R}_{i_++1}^+ & \cdots & \mathcal{R}_n^+ & \mathcal{R}_1^- & \cdots & \mathcal{R}_{n-i_-}^- & 0 & \cdots & 0 & \diamond \end{vmatrix}_{x_1=0}$$

where $\diamond = i[f^{\tilde{\xi}}(u)] + \lambda[u]$. Performing $(n - i_+)(n - i_-) + (2n - i_- - i_+)(i_- - 1) \equiv n - i_- \pmod{2}$ column transpositions (exchanging second and third column blocks, then moving both past the fourth), and recalling $p = n - i_- + 1$, we obtain relation (3.1), with

$$\gamma := \det(B^{11})^{-1}(-1)^p \det(w_1^+ \cdots w_{i_+}^+ w_{2n-i_-+1}^- \cdots w_{2n-1}^-)_{|x_1=0}. \tag{3.9}$$

Noting that $w_1^+, \dots, w_{i_+}^+$ and $w_{2n-i_-+1}^-, \dots, w_{2n}^-$ satisfy the (reduced, i.e. first order) linearized traveling-wave ODE

$$B^{11}w' = A^1w,$$

and vanish at $+\infty, -\infty$ respectively, we find that they span the tangent manifold along $\bar{u}(\cdot)$ of the stable/unstable manifolds at u_+/u_- respectively, and, by the discussion in §2, that (3.9) is indeed a transversality coefficient for the traveling-wave ODE. □

3.2. Computing β . Finally, we carry out our main computation. Let $\Delta(\tilde{\xi}, i\tau)$ vanish for some real, nontrivial $\tilde{\xi}, \tau$. Along with (H0)–(H4), assume also

- (H5) $\mathcal{R}_1^-, \dots, \mathcal{R}_{p-1}^-, \mathcal{R}_{p+1}^+, \dots, \mathcal{R}_n^+$ are independent at $(\tilde{\xi}, i\tau)$.
- (H6) $(\tilde{\xi}, i\tau) \notin \mathcal{V} \cup \mathcal{G}$ (defined Section 2.3).
- (H7) Each α_j is pure imaginary.

Hypothesis (H5), made for definiteness, holds always for extreme shocks ($p = n$ or 1 , or equivalently i_- or $i_+ = n$), and holds generically for intermediate shocks ($1 < p < n$). In general, it may be checked in the course of evaluating Δ, Δ_λ , at no extra computational cost. As mentioned earlier, $\mathcal{V} \cup \mathcal{G}$ is codimension one in $\{\Re \lambda = 0\}$, so is also generically satisfied. In particular, since codimension one, these do not interfere with the problem of determining stability transitions as we move along a one-dimensional Hugoniot curve. Finally, as discussed in [4], condition (H7) ensures that the zero $(\tilde{\xi}, i\tau)$ lies in the “hyperbolic domain” for which the bases \mathcal{R}_j may be chosen real at $\rho = 0$, so that the instability is indeed of the “weakly real” type that persists on an open set in parameter space. This is always satisfied for $|\tilde{\xi}|$ sufficiently small with respect to $|\tau|$ (see [56]), and holds in particular for weak instabilities occurring in gas dynamics [42–44, 62–64].

We establish the following result, of which Theorem 1.3 is a corollary.

Proposition 3.3. *Under the assumptions of Theorem 1.3, and taking without loss of generality \mathcal{R}_j^\pm to be genuine eigenvectors of \mathcal{A}_\pm , $\mathcal{A}_\pm \mathcal{R}_j^\pm = \alpha_j^\pm \mathcal{R}_j^\pm$, with \mathcal{L}_j^\pm dual left eigenvectors, $\mathcal{L}_j \mathcal{A}_\pm = \mathcal{L}_j \alpha_j^\pm$, the conclusions of the Theorem hold with*

$$\begin{aligned} \mathcal{B} = & \sum_{j < p} c_j \ell \cdot \left(B_-^{11} \alpha_j^- \mathcal{R}_j^- + \sum_{k \geq p} \frac{\mathcal{L}_k^- B_-(\tilde{\xi}, \alpha_j^-) \mathcal{R}_j^-}{\alpha_k^-} \mathcal{R}_k^- \right) \\ & + \sum_{j > p} c_j \ell \cdot \left(B_+^{11} \alpha_j^+ \mathcal{R}_j^+ + \sum_{k \leq p} \frac{\mathcal{L}_k^+ B_+(\tilde{\xi}, \alpha_j^+) \mathcal{R}_j^+}{\alpha_k^+} \mathcal{R}_k^+ \right), \end{aligned} \tag{3.10}$$

where $B_\pm(\tilde{\xi}, \alpha) := \left(B^{11} \alpha^2 + \sum_{j \neq 1} (B^{j1} + B^{1j}) \xi_j \alpha + \sum_{j,k \neq 1} B^{jk} \xi_j \xi_k \right)_\pm$.

Proof. By Proposition 3.1, $\frac{\partial^2}{\partial \lambda \partial \rho} D(0) = \gamma \Delta_\lambda$. Thus, it suffices to establish

$$\begin{aligned} \left(\frac{\partial}{\partial \rho} \right)^2 D(0) = & \gamma \left[\left\langle l, -2 \left[\frac{\partial}{\partial \rho} L_\rho \right] \Big|_{\rho=0} \tilde{y} - \left[\left(\frac{\partial}{\partial \rho} \right)^2 L_\rho \right] \Big|_{\rho=0} \bar{u}' \right\rangle + \mathcal{B} \right] \\ = & \gamma \left[\left\langle l, 2(\lambda I + A^{\tilde{\xi}} - i B^{1\tilde{\xi}} \partial_{x_1} - i \partial_{x_1} B^{\tilde{\xi}1}) \tilde{y} + 2 B^{\tilde{\xi}\tilde{\xi}} \bar{u}' \right\rangle + \mathcal{B} \right], \end{aligned} \tag{3.11}$$

with \mathcal{B} as in (3.10). We compute this second derivative by a calculation generalizing that of [5] in the one-dimensional case.

Namely, following the general approach of [19, 20, 32, 66], we choose a convenient basis for the computation, consisting of a generalized Jordan chain. Specifically, we first choose again the normalization $w_1^+ = w_{2n}^- = \bar{u}'$ at $\rho = 0$. Next, we observe that, by the matrix manipulations carried out in the proof of Proposition 3.1 and the facts (by transversality assumption (H4)) that $\gamma \neq 0$ and the kernel of $B^{11} \partial_x - A^1$ is spanned by \bar{u}' , the assumed relation $\frac{\partial}{\partial \rho} D^{\tilde{\xi}, i\tau}(0) = 0$ implies a linear dependence modulo \bar{u}' in the functions $w_1^-, \dots, w_{p-1}^-, w_{p+1}^+, \dots, w_n^+$ and $\frac{\partial}{\partial \rho} w_{2n}^- - \frac{\partial}{\partial \rho} w_1^+$, or

$$\sum_{j=1}^{p-1} c_j w_j^- + \sum_{j=p+1}^n c_j w_j^+ + c_p \bar{u}' + \left(\frac{\partial}{\partial \rho} w_{2n}^- - \frac{\partial}{\partial \rho} w_1^+ \right) = 0,$$

where

$$\sum_{j=1}^{p-1} c_j r_j^- + \sum_{j=p+1}^n c_j r_j^+ + (i[f^{\tilde{\xi}}(u)] + \lambda[u]) = 0. \tag{3.12}$$

Note that we have used (H6) in setting the coefficient of $(i[f^{\tilde{\xi}}(u)] + \lambda[u])$ to unity in (3.12). Defining

$$\tilde{w}_{2n}^- := w_{2n}^- + \rho \left(c_p w_{2n}^- + \sum_{j=1}^{p-1} c_j w_j^- \right), \quad \tilde{w}_1^+ := w_1^+ - \rho \sum_{j=p+1}^n c_j w_j^+, \tag{3.13}$$

we thus have the generalized Jordan chain

$$\tilde{w}_1^+ = \tilde{w}_{2n}^- = \bar{u}', \quad \frac{\partial}{\partial \rho} \tilde{w}_1^+ = \frac{\partial}{\partial \rho} \tilde{w}_{2n}^-. \quad (3.14)$$

We may substitute \tilde{w}_1^+ and \tilde{w}_{2n}^- for w_1^+ and w_{2n}^- without changing the value of D . With this choice of basis, we find readily, applying the Leibnitz rule and performing elementary matrix manipulations, similarly as in the proof of Proposition 3.1, that $\frac{\partial}{\partial \rho} D(0) = 0$, and

$$\left(\frac{\partial}{\partial \rho}\right)^2 D(0) = \det \left(\begin{array}{ccc} w_1^+ & \cdots & w_{2n-1}^- \\ w_1^{+'} & \cdots & w_{2n-1}^{-'} \end{array} \begin{array}{c} (\tilde{z}^- - \tilde{z}^+) \\ (\tilde{z}^- - \tilde{z}^+)' \end{array} \right)_{|x_1=0},$$

where $\tilde{z}^+ := (\frac{\partial}{\partial \rho})^2 \tilde{w}_1^+$ and $\tilde{z}^- := (\frac{\partial}{\partial \rho})^2 \tilde{w}_{2n}^-$ at $\rho = 0$.

Now, second derivatives $\tilde{z}_j^\pm := (\frac{\partial}{\partial \rho})^2 \tilde{w}_j^\pm$ satisfy the second order variational equations

$$\begin{aligned} L_\rho \tilde{z} &= -2 \left[\frac{\partial}{\partial \rho} L_\rho \right] \tilde{y} - \left[\left(\frac{\partial}{\partial \rho} \right)^2 L_\rho \right] \tilde{w} \\ &= -2i \sum_{j \neq 1} B^{j1} \xi_j \tilde{y}' - 2i \left(\sum_{k \neq 1} B^{1k} \xi_k \tilde{y} \right)' \\ &\quad + 2i \sum_{j \neq 1} A^j \xi_j \tilde{y} + 2\lambda \tilde{y} + 2\rho \sum_{j \neq 1, k \neq 1} B^{jk} \xi_j \xi_k \tilde{y} \\ &\quad + 2 \sum_{j \neq 1, k \neq 1} B^{jk} \xi_j \xi_k \tilde{w}, \end{aligned} \quad (3.15)$$

where $\tilde{y} := \frac{\partial}{\partial \rho} \tilde{w}$, hence, setting $\rho = 0$ and recalling (3.14), $(B^{11}(\tilde{z}^- - \tilde{z}^+) - A^1(\tilde{z}^- - \tilde{z}^+))' = L_0(\tilde{z}^- - \tilde{z}^+) = 0$, or $B^{11}(\tilde{z}^- - \tilde{z}^+) - A^1(\tilde{z}^- - \tilde{z}^+) \equiv I$ for some constant I .

Recalling again (3.4), (3.5), and performing the same row operation $B^{11}w' - A^1w$ used in the proof of Proposition 3.1, we thus obtain

$$\left(\frac{\partial}{\partial \rho}\right)^2 D(0) = \gamma \det (\mathcal{R}_1^- \cdots \mathcal{R}_{p-1}^- \mathcal{R}_{p+1}^+ \cdots \mathcal{R}_n^+ I) = \gamma l \cdot I,$$

where l is defined as in (1.13). To evaluate $l \cdot I$, observe, from (3.15) and (3.14), that

$$\begin{aligned} L_0 \tilde{z}^\pm &= \left(B^{11}(\tilde{z}^\pm)' - A^1(\tilde{z}^\pm) \right)' \\ &= -2i \sum_{j \neq 1} B^{j1} \xi_j \tilde{y}' - 2i \left(\sum_{k \neq 1} B^{1k} \xi_k \tilde{y} \right)' \\ &\quad + 2(\lambda + iA^{\tilde{\xi}}) \tilde{y} + 2 \sum_{j \neq 1, k \neq 1} B^{jk} \xi_j \xi_k \bar{u}'. \end{aligned}$$

By the Fundamental Theorem of Calculus, therefore,

$$\begin{aligned}
 l \cdot I &= l \cdot (M_0 \tilde{z}^- - M_0 \tilde{z}^+) |_{x=0} \\
 &= \int_{-\infty}^{+\infty} l \cdot L_0 \tilde{z}^\pm dx_1 + l \cdot L_0 \tilde{z}^+ |_{+\infty} - l \cdot L_0 \tilde{z}^- |_{-\infty} \\
 &= \int_{-\infty}^{+\infty} l \cdot \left(-2i \sum_{j \neq 1} B^{j1} \xi_j \tilde{y}' - 2i \left(\sum_{k \neq 1} B^{1k} \xi_k \tilde{y} \right)' \right. \\
 &\quad \left. + 2(\lambda + iA^{\tilde{\xi}}) \tilde{y} + 2 \sum_{j \neq 1, k \neq 1} B^{jk} \xi_j \xi_k \tilde{u}' \right) dx_1 \\
 &\quad + l \cdot \left(B^{11}(\tilde{z}^+) - A^1 \tilde{z}^+ \right) |_{+\infty} - l \cdot \left(B^{11}(\tilde{z}^-) - A^1 \tilde{z}^- \right) |_{-\infty},
 \end{aligned} \tag{3.16}$$

so long as the righthand side converges, where $L_0 =: \partial_{x_1} M_0$.

Differentiating (3.13) with respect to ρ , we obtain

$$\tilde{y}_{2n}^- = y_{2n}^- + \left(c_p w_{2n}^- + \sum_{j=1}^{p-1} c_j w_j^- \right), \quad \tilde{y}_1^+ := y_1^+ - \sum_{j=p+1}^n c_j w_j^+, \tag{3.17}$$

and

$$\tilde{z}_{2n}^- = z_{2n}^- + \left(c_p y_{2n}^- + \sum_{j=1}^{p-1} c_j y_j^- \right), \quad \tilde{z}_1^+ := z_1^+ - \sum_{j=p+1}^n c_j y_j^+, \tag{3.18}$$

where y_j^\pm and z_j^\pm decay exponentially as $x_1 \rightarrow +\infty$ (resp. $-\infty$), as do $\tilde{w}_{2n}^- = \tilde{w}_1^+ = \tilde{u}'$ and $(\tilde{y}_j^\pm)'$. Recalling (2.6)–(2.7), the definition of r_j^\pm as eigenvectors of $(\lambda + iA^{\tilde{\xi}})(A^1)^{-1}$, and the fact that l was chosen orthogonal to all r_j^\pm appearing in (3.17), we find that the integral on the righthand side of (3.16) indeed converges, though

$$\int_{-\infty}^{+\infty} \left(-2i \sum_{j \neq 1} B^{j1} \xi_j \tilde{y}' - 2i \left(\sum_{k \neq 1} B^{1k} \xi_k \tilde{y} \right)' + 2(\lambda + iA^{\tilde{\xi}}) \tilde{y} + 2 \sum_{j \neq 1, k \neq 1} B^{jk} \xi_j \xi_k \tilde{w} \right) dx_1$$

may not.

A more delicate issue is the convergence of boundary terms in \tilde{z}^\pm , which amounts, by (3.18), to the assignment of boundary values for y_j^\pm at $\rho = 0$ for the slow-decaying modes w_j^\pm appearing in formulae (3.13), (3.17). As it will play an important role in the calculations, we carry this out separately, in some detail.

Lemma 3.4. *Assume (H0)–(H7). For $\rho = 0$, let $w_j^+ \sim (A_+^1)^{-1} \mathcal{R}_j^+$ as $x_1 \rightarrow +\infty$, with*

$$\mathcal{A}_+ \mathcal{R}_j^+ = \alpha_j \mathcal{R}_j^+, \quad \mathcal{L}_j^+ \mathcal{A}_+ = \mathcal{L}_j^+ \alpha_j, \quad \mathcal{L}_j^+ \mathcal{R}_j^+ = \delta_j^k$$

at $(\tilde{\xi}, \lambda) \notin \mathcal{V} \cup \mathcal{C}$, where $\mathcal{A}_+ = (\lambda + iA\tilde{\xi})(A^1)_+^{-1}$ as in (1.8) and δ_j^k is the Kronecker delta function. Then, the corresponding variational boundary conditions at $\rho=0$ are

$$\begin{aligned} y_j^+ &\sim -x\alpha_j(A_+^1)^{-1}\mathcal{R}_j^+ + (A_+^1)^{-1}\frac{\partial}{\partial\rho}(\mathcal{R}_j^+), \\ (y_j^+)' &\sim -\alpha_j(A_+^1)^{-1}\mathcal{R}_j^+, \end{aligned} \tag{3.19}$$

where all error terms are exponentially decaying and

$$\begin{aligned} \frac{\partial}{\partial\rho}\mathcal{R}_j^+ &= \sum_{\alpha_k^+ \neq \alpha_j^+} \frac{\mathcal{L}_k^+ B(\tilde{\xi}, \alpha_j)\mathcal{R}_j^+}{\alpha_k} \mathcal{R}_k^+ + \sum_{\alpha_k^+ = \alpha_j^+} d_k \mathcal{R}_k^+, \\ B(\tilde{\xi}, \alpha) &:= B_+^{11}\alpha^2 + \sum_{j \neq 1} (B_+^{j1} + B_+^{1j})\xi_j\alpha + \sum_{j,k \neq 1} B_+^{jk}\xi_j\xi_k. \end{aligned} \tag{3.20}$$

A symmetric result holds for modes $w_j^- \sim (A_-^1)^{-1}\mathcal{R}_j^-$ as $x_1 \rightarrow -\infty$.

Proof. Again, it is sufficient to show the corresponding result for the limiting, constant-coefficient equations as $x_1 \rightarrow \pm\infty$, the variable-coefficient result then following by asymptotic ODE theory (the ‘‘conjugation lemma’’ [50]).

Let us first consider the simpler, *strictly hyperbolic* case, for which, away from \mathcal{G} , the eigenvalues of \mathcal{A}_\pm are simple, and we can choose slow modes analytically as pure exponential modes $w(\tilde{\xi}, \lambda, \rho, x_1) = e^{\mu x_1}v(\tilde{\xi}, \lambda, \rho)$. Substituting this Ansatz into the frozen-coefficient version of (3.2), we obtain

$$\begin{aligned} 0 &= \mu^2 B_+^{11}v - \mu A_+^1 v + i\mu\rho \sum_{j \neq 1} B_+^{j1}\xi_j v + i\mu\rho \sum_{k \neq 1} B_+^{1k}\xi_k v \\ &\quad - i\rho \sum_{j \neq 1} A_+^j \xi_j v - \rho\lambda v - \rho^2 \sum_{j \neq 1, k \neq 1} B_+^{jk}\xi_j \xi_k v, \end{aligned} \tag{3.21}$$

from which we readily find that slow modes satisfy $\mu \sim -\rho\alpha$ as $\rho \rightarrow 0$, so that $\mu_\rho = -\alpha$ at $\rho = 0$ for all modes associated with a given α_j , whereupon (3.19) follow by the product rule. In the general (nonstrictly hyperbolic) case, away from $\mathcal{V} \cup \mathcal{G}$, the eigenvalues α of \mathcal{A}_+ occur in constant-multiplicity blocks, and $w(\tilde{\xi}, \lambda, \rho, x_1) = e^{m x_1}v(\tilde{\xi}, \lambda, \rho)$, with $m = -\rho\alpha I_K + O(\rho^2)$ where K is the multiplicity of the block, from which we obtain the same result.

To make the calculation (3.20), we first note that, modulo the arbitrary term $\sum_{\alpha_k = \alpha_j} d_k \mathcal{R}_k^+$, $\frac{\partial}{\partial\rho}|_{\rho=0}$ is independent under C^1 coordinate changes, being completely determined by the values of \mathcal{R}_j^+ at $\rho = 0$. Thus, we are free to choose modes w_j^+ as pure exponentials $w(\tilde{\xi}, \lambda, \rho, x_1) = e^{\mu x_1}v(\tilde{\xi}, \lambda, \rho)$ so long as the eigenvectors $\mathcal{R} := A^1 v$ are C^1 with respect to ρ : in particular, in the strictly hyperbolic case.

More generally, substituting $\mu_j^+ =: -\rho\alpha_j^+$, $A^1v_j^+ =: \mathcal{R}_j^+$ into (3.21) and dividing out a factor of ρ , we obtain the reduced (slow) equation

$$\left(\mathcal{A}_+ + \rho B(\tilde{\xi}, \alpha_j^+) - \alpha_j^+\right) \mathcal{R}_j^+ = 0 \tag{3.22}$$

with $B(\tilde{\xi}, \alpha)$ as in (3.20), from which we see that α_j^+ , \mathcal{R}_j^+ are generically C^1 as the values of B_+^{jk} are varied. Thus, it is sufficient to establish (3.20) in this case, with the general result following by continuity. Assume, therefore, that α_j^+ , \mathcal{R}_j^+ are C^1 solutions of (3.22). Differentiating (3.22) with respect to ρ at $\rho = 0$, taking products with \mathcal{L}_k^+ for all $\alpha_k^+ \neq \alpha_j^+$, and rearranging using the normalization $\mathcal{L}_k^+ \mathcal{R}_j^+ = \delta_j^k$, we readily obtain (3.20).

See [5] for a similar calculation in the one-dimensional case. □

From (3.19), and the fact that ℓ is orthogonal to all \mathcal{R}_j^\pm under consideration (i.e., the ones occurring as limits of the slow decaying modes appearing in the Evans function), we readily obtain

$$\begin{aligned} \mathcal{B} &:= l \cdot \left(B^{11}(\tilde{z}^+) - A^1\tilde{z}^+ \right) |_{+\infty} - l \cdot \left(B^{11}(\tilde{z}^-) - A^1\tilde{z}^- \right) |_{-\infty} \\ &= \sum_{j \neq p} c_j \ell \cdot \left(B^{11} \alpha_j \mathcal{R}_j^\pm + \frac{\partial}{\partial \rho} (\mathcal{R}_j^\pm) \right), \end{aligned}$$

whence (3.16) yields (3.11), (3.10) as claimed, where c_j are as in (3.12), (3.19), and $\frac{\partial}{\partial \rho} \mathcal{R}_j^\pm$ as in (3.20). □

Remark 3.5. As noted in the one-dimensional case [5], the calculations somewhat simplify in the case of an extreme shock, $p = n$ (resp. $p = 1$), for which $\ell = \mathcal{L}_1^+$ (resp. $\ell = \mathcal{L}_n^-$); in particular, this induces considerable cancellation in the expression (3.10) for \mathcal{B} . In the one-dimensional case, $B(\tilde{\xi}, \alpha) = B^{11}\alpha^2$, leading to still further cancellation.

4. Numerical approximation of \tilde{y}

The computation of \mathcal{B} or Δ amounts (see (3.10), (1.7)) to calculation of eigenvectors and eigenvalues of \mathcal{A}_\pm , a standard numerical problem. In the case of gas dynamics, they may be computed in closed form. Thus, the computation of β reduces (see (1.12)) to the solution of (1.14) for \tilde{y} , a rather nonstandard problem due to the absence of spectral gap for operator L_0 , reflected at this level by the presence of nondecaying solutions as $x_1 \rightarrow \pm\infty$. We may easily remove this degeneracy, however, by integrating the equations from $x_1 = -\infty$

to x_1 , and taking advantage of the facts (see (3.17), Section 3.2, and (2.6)) that

$$\begin{aligned} \lim_{z \rightarrow -\infty} \tilde{y}(z) &= (A_-^1)^{-1} \sum_{j=1}^{p-1} c_j \mathcal{R}_j^-, & \lim_{z \rightarrow -\infty} \tilde{y}'(z) &= 0 \\ \lim_{z \rightarrow +\infty} \tilde{y}(z) &= (A_+^1)^{-1} \sum_{j=p+1}^n c_j \mathcal{R}_j^+, & \lim_{z \rightarrow +\infty} \tilde{y}'(z) &= 0, \end{aligned} \quad (4.1)$$

c_j determined by (3.12), hence

$$\lim_{z \rightarrow -\infty} (B^{11} \tilde{y}' - A^1 \tilde{y})(z) = - \sum_{j=1}^{p-1} c_j \mathcal{R}_j^-,$$

and (see (3.7)) that the righthand side of (1.14) is a perfect derivative, to obtain the first-order system

$$\begin{aligned} B^{11} \tilde{y}' - A^1 \tilde{y} = F &:= i f^{\tilde{\xi}}(\bar{u}) - i B^{1\tilde{\xi}}(\bar{u}) \bar{u}' - i B^{\tilde{\xi}1}(\bar{u}) \bar{u}' + \lambda \bar{u} \\ &+ \sum_{j=1}^{p-1} c_j \mathcal{R}_j^- - (i f^{\tilde{\xi}}(u_-) + \lambda u_-) \end{aligned} \quad (4.2)$$

with boundary conditions (4.1). Defining $\hat{y} := \tilde{y} - y_0$, where

$$y_0 := \left(\chi(z) (A_-^1)^{-1} \sum_{j=1}^{p-1} c_j \mathcal{R}_j^- + (1 - \chi(z)) (A_+^1)^{-1} \sum_{j=p+1}^n c_j \mathcal{R}_j^+ \right),$$

and

$$\chi(z) := \frac{1 + \tanh(z)}{2} \rightarrow \begin{cases} 0, & z \rightarrow +\infty \\ 1, & z \rightarrow -\infty, \end{cases}$$

so that $\lim_{z \rightarrow \pm\infty} \hat{y}(z) = 0$, we may convert (4.2), (4.1) to a boundary-value problem of the standard form discussed in, e.g. [54, 55], namely

$$B^{11} \hat{y}' - A^1 \hat{y} = \hat{F} := F + (B^{11} y_0' - A^1 y_0),$$

with “projective boundary conditions” at infinity,

$$\lim_{z \rightarrow \pm\infty} \Pi_{\pm} \hat{y}(z) = 0,$$

where Π_{\pm} are any projections that are isomorphisms on the subspaces of exponentially growing (resp. decaying) solutions of the limiting constant-coefficient equations $B_+^{11} y' - A_+^1 y = 0$ (resp. $B_-^{11} y' - A_-^1 y = 0$), augmented with a single boundary condition at $z = 0$,

$$\ell \cdot y = 0,$$

to fix a unique solution (recall that $(B^{11}\partial_x - A^1)$ has one-dimensional kernel $\text{span}\{\bar{u}'\}$). As operator $B^{11}\partial_{x_1} - A^1$ has exponential dichotomies on $[0, +\infty)$ and $(-\infty, 0]$, the problem is numerically well-conditioned by standard considerations (see [54,55] for details), and may be solved by any reasonable collocation method on a truncated domain $[-M, M]$, with $M > 0$ sufficiently large. This completes our description of the numerical approximation of β .

Remark 4.1. In practice, one may solve (4.2) with inhomogeneous projective boundary conditions $\lim_{z \rightarrow \pm\infty} \Pi_{\pm}(\tilde{y}(z) - \lim_{z \rightarrow \pm\infty} \tilde{y}(z)) = 0$ at infinity and modified condition $\ell \cdot \tilde{y}(0) = 0$ at $z = 0$, so long as M need not be too large. (Unlike \hat{F} , F is nondecaying as $z \rightarrow \pm\infty$, and so there is a great deal of cancellation, with associated loss of significance, in the righthand side of (4.2) for large values of $|z|$.)

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