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Basic Topological and Geometric Properties of Orlicz Spaces over an Arbitrary Set of Atoms

Henryk Hudzik and Lucjan Szymaszkiewicz

Abstract. Orlicz spaces $l^\varphi(\Gamma)$ over an arbitrary set Γ, being a natural generalizations of Orlicz sequence spaces are studied. The following problems in these spaces are considered: relationships between the Luxemburg norm and the modular, Fatou property, relationships between the Luxemburg norm and the Orlicz norm, equality of the Orlicz norm and the Amemiya norm, order continuous elements, a formula for the norm in the quotient space $l^\varphi(\Gamma)/h^\varphi(\Gamma)$ in terms of the modular I_φ for both the Luxemburg and the Orlicz norm, the problem when the equality of the space $l^\varphi(\Gamma)$ and its subspace $h^\varphi(\Gamma)$ holds, isometric representation of the dual spaces $(h^\varphi(\Gamma))^*$, $(l^\varphi(\Gamma))^*$, $(h^\varphi_o(\Gamma))^*$ and $(l^\varphi_o(\Gamma))^*$, representation of support functionals, criteria for smooth points and extreme points of $S(l^\varphi(\Gamma))$ and problem of the existence of such points. It is worthy noticing that the problem of the existence of smooth points on $S(l^\varphi(\Gamma))$ depends essentially on the assumption if Γ is countable or not.

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1. Introduction

Let X be a vector lattice. A subspace Y of the vector lattice X is *an ideal* if for any two elements $x, y \in X$ the following implication holds:

$$(|x| \leq |y| \wedge y \in Y) \;\Rightarrow\; (x \in Y).$$

If A denotes some property, then by (A) we will denote the class of all Banach spaces satisfying property A.

We say that a vector lattice X is *Dedekind complete* (σ*-Dedekind complete*) if for any subset (countable subset) $A \subset X$ orderly bounded from above there exists a supremum. We write then $X \in$ (DC) ($X \in (\sigma$- DC)).

H. Hudzik: Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland; hudzik@amu.edu.pl
L. Szymaszkiewicz: Institute of Mathematics, Szczecin University, Wielkopolska 15, 70-451 Szczecin, Poland; lucjansz@wmf.univ.szczecin.pl

It is known that in the above definition we can take only sets $A \subset X^+$ (see [19, Theorem 12.1] or [14, Theorem 15.11] and remarks). On the base of this fact the following lemma is quite obvious.

Lemma 1.1. *If* $X \in (\mathrm{DC})$ $(X \in (\sigma\text{-}\mathrm{DC}))$ *and* $Y \subset X$ *is an ideal in* X*, then* $Y \in (\mathrm{DC})$ $(Y \in (\sigma\text{-}\mathrm{DC}))$.

A vector lattice which is simultaneously a normed space is called *a normed lattice* if its norm $\|\cdot\|$ is monotone, i.e., it satisfies the following condition:

$$(|x| \le |y|) \;\Rightarrow\; (\|x\| \le \|y\|).$$

If a normed lattice is complete, then it is called *a Banach lattice*.

We say that a normed lattice X is *order continuous* (more precisely a norm $\|\cdot\|$ on X is order continuous) if for any net (x_α) in X such that $x_\alpha \downarrow 0$ (i.e., $x_\alpha \ge x_\beta$ for $\alpha \le \beta$ and $\inf_\alpha x_\alpha = 0$) we have $\inf_\alpha \|x_\alpha\| = 0$. We write then $X \in (\mathrm{OC})$.

We say that a normed lattice X is *σ-order continuous* if for any sequence (x_n) in X such that $x_n \downarrow 0$ (i.e., $x_1 \ge x_2 \ge \dots$ and $\inf_n x_n = 0$) we have $\|x_n\| \downarrow 0$. We write then $X \in (\sigma\text{-}\mathrm{OC})$.

The following theorem holds (see, for example, [18, Theorem 1.1]).

Theorem 1.2. *For any Banach lattice* X *the following conditions are equivalent:*

1. $X \in (\mathrm{OC})$*;*
2. $X \in (\mathrm{DC})$ *and* $X \in (\mathrm{OC})$*;*
3. $X \in (\sigma\text{-}\mathrm{DC})$ *and* $X \in (\sigma\text{-}\mathrm{OC})$.

For any Banach lattice X define

$$X_a = \left\{ x \in X : \forall_{(x_\alpha)}\, |x| \ge x_\alpha \downarrow 0 \Rightarrow \inf_\alpha \|x_\alpha\| = 0 \right\}.$$

It can be shown (see for example [18, p. 60]) that:

Theorem 1.3. *For any Banach lattice* X *we have:*

1. X_a *is an ideal in* X*;*
2. $X_a \in (\mathrm{OC})$*;*
3. *if* $B \subset X$ *is an ideal such that* $B \in (\mathrm{OC})$ *then* $B \subset X_a$*;*
4. *the ideal* X_a *is closed in* X.

Note that a Banach lattice $X \in (\mathrm{OC})$ if and only if $X = X_a$. The most important class of Banach lattices form Köthe spaces called also Banach function lattices.

Let μ be a measure defined on some σ-algebra Σ of subsets of a nonempty set S. We denote by $L^0(S, \Sigma, \mu)$ the space of all equivalence classes of real measurable functions. There is a natural order in this space $(x \leq y) \iff (x(s) \leq y(s)\ \mu\text{-a.e. in } S)$. It is easy to see that $L^0(S, \Sigma, \mu)$ is a σ-Dedekind complete vector lattice.

We say that a Banach lattice X is a Köthe space if it is an ideal in some $L^0(S, \Sigma, \mu)$. Note that by Lemma 1.1 Köthe spaces are σ-Dedekind complete Banach lattices. For a reference to Köthe spaces see, for example, [12].

Let Γ be an arbitrary nonempty set. Define on Γ the counting measure μ acting on the σ-algebra of all subsets of Γ in the following way: $\mu(A) = \text{card}(A)$ if A is finite and $\mu(A) = \infty$ otherwise. Consequently $l^0(\Gamma) := L^0(\Gamma, 2^\Gamma, \mu)$ is the set of all functions defined on Γ with values in $\mathbb{R}$. For any function $x : \Gamma \to \mathbb{R}$ and a set $A \subset \Gamma$ the symbol $\sum_{i \in A} x(i)$ or the shorter one $\sum_A x$ will denote the integral $\int_A x d\mu$. Notice that for $x \geq 0$ it means that

$$\sum_{i \in A} x(i) = \sup_{\substack{I \subset A \\ \text{card}(I) < \infty}} \sum_{i \in I} x(i).$$

Denote, for the convenience, by Σ_1 the family of all finite subsets of Γ and for any set $A \subset \Gamma$ by x^A the element $x \cdot \chi_A$ (χ_A is the characteristic function of a set A).

Let φ be an Orlicz function. Remind that a function $\varphi : \mathbb{R} \to [0, +\infty]$ is called *an Orlicz function* if it is convex, even, $\varphi(0) = 0$, φ is left-hand side continuous on $[0, +\infty)$ and not being equal to zero identically. By ψ we denote the function conjugate to φ in the sense of Young, defined by

$$\psi(u) = \sup_{v > 0} (|u|v - \varphi(v)).$$

These two functions are connected by the Young inequality

$$|uv| \leq \varphi(u) + \psi(v) \quad (u, v \in \mathbb{R}).$$

Define on the space $l^0(\Gamma)$ the modular I_φ by

$$I_\varphi(x) = \sum_{i \in \Gamma} \varphi(x(i)) \quad (x \in l^0(\Gamma)).$$

From the Young inequality we get

$$\sum_{i \in \Gamma} |x(i)\, y(i)| \leq I_\varphi(x) + I_\psi(y) \quad (\forall\, x, y \in l^0(\Gamma)).$$

Define the Orlicz space $l^\varphi(\Gamma) = \{x \in l^0(\Gamma) : \exists_{\lambda > 0}\, I_\varphi(\lambda x) < \infty\}$ and equip it with the Luxemburg norm

$$\|x\|_\varphi = \inf\left\{k > 0 : I_\varphi\left(\frac{x}{k}\right) \leq 1\right\}.$$

For a reference to Orlicz functions and Orlicz spaces see [13,15–17]. If we define the partial order $x \le y \iff \forall_{i\in\Gamma}\, x(i) \le y(i)$, then

$$|x| \le |y| \Rightarrow I_\varphi(x) \le I_\varphi(y) \quad \text{and} \quad |x| \le |y| \Rightarrow \|x\|_\varphi \le \|y\|_\varphi ,$$

which means that the modular I_φ and the norm $\|\cdot\|_\varphi$ are monotonous. The space $l^\varphi(\Gamma)$ with the partial order $\le$ is a vector lattice and $l^\varphi(\Gamma)$ is an ideal in $l^0(\Gamma)$. To conclude that $l^\varphi(\Gamma)$ is a Köthe space we have to show that space $(l^\varphi(\Gamma), \|\cdot\|_\varphi)$ is complete. With this end in view let us notice the following

Lemma 1.4. *For any sequence $(x_n)_{n\in\mathbb{N}}$ in $l^0(\Gamma)$ we have:*

1. $\|x_n\|_\varphi \to 0$ *implies that* $\forall_{i\in\Gamma}\, x_n(i) \to 0$;
2. *if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $l^\varphi(\Gamma)$, then $\big(x_n(i)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for any $i \in \Gamma$.*

Proof. We will show only the first implication. Let $\|x_n\|_\varphi \to 0$ and take any $i \in \Gamma$. We have $I_\varphi(kx_n) \to 0$ as $n \to \infty$ for any $k > 0$, so we have $\varphi(kx_n(i)) \to 0$ as $n \to \infty$ for any $k > 0$.

Assume that $x_n(i) \not\to 0$ for some $i \in \Gamma$, i.e., there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers and $\varepsilon > 0$ such that $|x_{n_k}(i)| \ge \varepsilon$. Taking then $k := \frac{a_\varphi+1}{\varepsilon}$, where $a_\varphi := \sup\{u \ge 0 : \varphi(u) = 0\}$, we get

$$\varphi(k\, x_{n_k}(i)) = \varphi(k|x_{n_k}(i)|) \ge \varphi(k\varepsilon) = \varphi\left(\frac{a_\varphi+1}{\varepsilon}\varepsilon\right) = \varphi(a_\varphi+1) > 0,$$

which means that $\varphi(kx_n(i)) \not\to 0$. □

Theorem 1.5. *The space $(l^\varphi(\Gamma), \|\cdot\|_\varphi)$ is a Banach space.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(l^\varphi(\Gamma), \|\cdot\|_\varphi)$. By Lemma 1.4 the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit x in $l^0(\Gamma)$. It is enough to show that $\|x_n - x\|_\varphi \xrightarrow{n\to\infty} 0$.

Let us fix two numbers $\varepsilon > 0$ and $\lambda > 0$. We will show that there exists a number $N \in \mathbb{N}$ such that $I_\varphi(\lambda(x_n - x)) \le \varepsilon$ for all $n \ge N$, which will finish the proof.

Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, by definition there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\|_\varphi \le \frac{\varepsilon}{\lambda}$ for $n, m \ge N$. Next we get that $\|\lambda(x_n - x_m)\|_\varphi \le \varepsilon$ and consequently

$$I_\varphi(\lambda(x_n - x_m)) \le \varepsilon. \tag{1}$$

By inequality (1) we get $\varphi\big(\lambda(x_n(i) - x_m(i))\big) \le \varepsilon$ for $n, m \ge N$ and $i \in \Gamma$. Note that

$$\varphi\big(\lambda(x_n(i) - x_m(i))\big) \xrightarrow{m\to\infty} \varphi\big(\lambda(x_n(i) - x(i))\big),$$

for any fixed $n \ge N$ and $i \in \Gamma$. By the Fatou Lemma and inequality (1) we get $I_\varphi(\lambda(x_n - x))) \le \liminf_{m\to\infty} I_\varphi(\lambda(x_n - x_m)) \le \varepsilon$ for $n \ge N$. □

Next lemma will be used repeatedly in the remaining part of this article.

Lemma 1.6. *It holds for any $x \in l^0(\Gamma)$: If $\sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi < \infty$, then $\|x\|_\varphi = \sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi$ and $x \in l^\varphi(\Gamma)$.*

Proof. It is enough to show the inequality $\|x\|_\varphi \le \sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi$. Let $c := \sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi$. We have $I_\varphi\big(\frac{x^I}{c}\big) \le I_\varphi\big(\frac{x^I}{\|x^I\|_\varphi}\big) \le 1$ for any $I \in \Sigma_1$. Therefore $I_\varphi\big(\frac{x}{c}\big) = \sup_{I\in\Sigma_1} I_\varphi\big(\frac{x^I}{c}\big) \le 1$, whence $\|x\|_\varphi \le c$. □

Define now on the space $l^\varphi(\Gamma)$ the Orlicz norm

$$\|x\|_\varphi^o = \sup_{I_\psi(y)\le 1} \sum_\Gamma xy.$$

It is easy to show that it is a norm on $l^\varphi(\Gamma)$ indeed (see [17]). We can also consider $\|x\|_\varphi^o$ for any element $x \in l^0(\Gamma)$ obtaining so called function norm which can take value ∞ on some elements of the space $l^0(\Gamma)$.

Lemma 1.7. *The equality*

$$\|x\|_\varphi^o = \sup\left\{\sum_\Gamma xy : I_\psi\left(y\right) \le 1, \forall_{i\in\Gamma}\, x(i)\cdot y(i) \ge 0, \mathrm{supp}(y) \in \Sigma_1\right\}$$

holds for any $x \in l^0(\Gamma)$.

Proof. It is enough to show the inequality „≤”. Assume first that $\|x\|_\varphi^o < \infty$ and take any $\varepsilon > 0$. From the definition of the Orlicz norm there exists $y \in l^0(\Gamma)$ such that $I_\psi\left(y\right) \le 1$ and $\|x\|_\varphi^o - \varepsilon < \sum_\Gamma xy \le \|x\|_\varphi^o$. We can assume that $\forall_{i\in\Gamma}\, x(i)\cdot y(i) \ge 0$ (if it is not true then we can modify y so that the inequalities are true). Take now $I \in \Sigma_1$ such that $\|x\|_\varphi^o - \varepsilon < \sum_I xy$. Define $z = y^I$. We have $I_\psi\left(z\right) \le I_\psi\left(y\right) \le 1$, $\mathrm{supp}(z) \in \Sigma_1$ and

$$\|x\|_\varphi^o - \varepsilon < \sum_\Gamma xz \le \sup\left\{\sum_\Gamma xy : I_\psi\left(y\right) \le 1, \forall_{i\in\Gamma}\, x(i)\cdot y(i) \ge 0, \mathrm{supp}(y) \in \Sigma_1\right\}.$$

The case when $\|x\|_\varphi^o = \infty$ can be proved analogously. □

Lemma 1.8. *For any $x \in l^0(\Gamma)$ there holds the equality $\|x\|_\varphi^o = \sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi^o$.*

Proof. By

$$\|x\|_\varphi^o = \sup_{\substack{I_\psi(y)\le 1\\ xy\ge 0}} \sum_\Gamma xy = \sup_{\substack{I_\psi(y)\le 1\\ xy\ge 0}} \sup_{I\in\Sigma_1} \sum_I xy = \sup_{\substack{I_\psi(y)\le 1\\ xy\ge 0}} \sup_{I\in\Sigma_1} \sum_\Gamma x^I y = \sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi^o$$

the assertion follows. □

The next two lemmas show relationships between Luxemburg norm and Orlicz norm.

Lemma 1.9. *For any $x \in l^0(\Gamma)$ we have the inequality $\|x\|_\varphi^o \leq 2\|x\|_\varphi$.*

Proof. If $\|x\|_\varphi = \infty$ or $\|x\|_\varphi = 0$, then the inequality is satisfied. Therefore we can assume that $0 < \|x\|_\varphi < \infty$. If now $I_\psi(y) \leq 1$, then by the Young inequality we have

$$\sum_\Gamma \frac{x}{\|x\|_\varphi} y \leq I_\varphi\left(\frac{x}{\|x\|_\varphi}\right) + I_\psi(y) \leq 2.$$

Therefore $\sum_\Gamma xy \leq 2\|x\|_\varphi$. From the arbitrariness of y with $I_\psi(y) \leq 1$ we get the thesis. □

Lemma 1.10. *For any $x \in l^0(\Gamma)$ we have the inequality $\|x\|_\varphi \leq \|x\|_\varphi^o$.*

Proof. We can assume that $\|x\|_\varphi^o < \infty$. Assume first that $\mathrm{supp}(x) \in \Sigma_1$. In a standard way we can show that $\|x\|_\varphi \leq \|x\|_\varphi^o$ (see, for example, [1, Theorem 8.14, p. 274]). If now $x \in l^0(\Gamma)$ is arbitrary, then

$$\sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi \leq \sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi^o = \|x\|_\varphi^o < \infty.$$

From Lemma 1.6 we get the thesis. □

Corollary 1.11. *If $x \in l^0(\Gamma)$ and $\|x\|_\varphi^o < \infty$, then $x \in l^\varphi(\Gamma)$.*

Proof. If $\|x\|_\varphi^o < \infty$ then $\|x\|_\varphi < \infty$. Consequently $x \in l^\varphi(\Gamma)$ since for example $I_\varphi\left(\frac{x}{\|x\|_\varphi+1}\right) \leq 1$. □

Corollary 1.11 and Lemma 1.8 imply the analogue of Lemma 1.6 for the Orlicz norm.

Lemma 1.12. *For any $x \in l^0(\Gamma)$ with $\sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi^o < \infty$ we have $\|x\|_\varphi^o = \sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi^o$ and $x \in l^\varphi(\Gamma)$.*

Let $x \in l^0(\Gamma)$. Define $f_x : (0,+\infty) \to (0,+\infty]$ by

$$f_x(k) = \frac{1}{k}(1 + I_\varphi(kx)).$$

If $x \notin l^\varphi(\Gamma)$ then $f_x(k) = +\infty$ for any $k > 0$. If $x \in l^\varphi(\Gamma)$ then let us denote $k_x := \sup\{k > 0 : I_\varphi(kx) < \infty\}$. From the Lebesgue Dominated Convergence Theorem we get that f_x is continuous on $(0, k_x)$. At k_x the function f_x is left-hand side continuous. Define the Amemiya norm on $l^\varphi(\Gamma)$ by

$$\|x\|_\varphi^A = \inf_{k>0} f_x(k).$$

It can be checked that it is a norm indeed. From the above notices about the function f_x, denoting by $\mathbb{Q}$ the set of rational numbers, we get the following

Lemma 1.13. *For any $x \in l^0(\Gamma)$ we have:*

1. $\|x\|_\varphi^A < \infty$ *if and only if* $x \in l^\varphi(\Gamma)$;
2. $\|x\|_\varphi^A = \inf_{k \in (0,k_x) \cap \mathbb{Q}} f_x(k)$.

Let us recall now the following lemma.

Lemma 1.14 ([6, Lemma 1]). *If* $\frac{\varphi(u)}{u} \xrightarrow{u\to\infty} \infty$, *then* $\forall_{x \in l^\varphi(\Gamma)} \exists_{k>0}\ \|x\|_\varphi^A = f_x(k)$.

For every $x \in l^0(\Gamma)$ let us define the set $K(x) := \left\{k > 0 : \|x\|_\varphi^A = f_x(k)\right\}$. It can happen that $K(x) = \emptyset$ but only in the case, when $\frac{\varphi(u)}{u} \to a < \infty$ if $u \to \infty$.

Lemma 1.15. *Let $x \in l^\varphi(\Gamma)$ and $B \subset \Gamma$. Then for any $k \in K(x^B)$ there exists $l \leq k$ such that $l \in K(x)$.*

Proof. Let $x \in l^\varphi(\Gamma)$ and $B \subset \Gamma$. Suppose that $k \in K(x^B)$, i.e., $\left\|x^B\right\|_\varphi^A = \frac{1}{k}\left(1 + I_\varphi\left(kx^B\right)\right)$. Take any $l > k$. Then we have

$$\begin{aligned}
\frac{1}{l}(1 + I_\varphi(lx)) &= \frac{1}{l}\left(1 + I_\varphi\left(lx^B\right)\right) + \frac{1}{l} I_\varphi\left(lx^{\Gamma\setminus B}\right) \\
&\geq \frac{1}{k}\left(1 + I_\varphi\left(kx^B\right)\right) + \frac{1}{k} I_\varphi\left(kx^{\Gamma\setminus B}\right) \\
&= \frac{1}{k}(1 + I_\varphi(kx)).
\end{aligned}$$

From this inequality and from the fact that $\frac{1}{l}(1 + I_\varphi(lx)) \to \infty$ as $l \to 0$ we get the thesis. □

Lemma 1.16. *If $x \in l^0(\Gamma)$, then $\|x\|_\varphi^o \leq \|x\|_\varphi^A$.*

Proof. Let $x \in l^0(\Gamma)$ and $k > 0$. Then we have

$$\|x\|_\varphi^o = \sup_{I_\psi(y)\leq 1} \frac{1}{k} \sum_\Gamma k\,xy \leq \sup_{I_\psi(y)\leq 1} \frac{1}{k}(I_\varphi(kx) + I_\psi(y)) \leq \frac{1}{k}(I_\varphi(kx) + 1).$$

From arbitrariness of $k > 0$ we get the thesis. □

Theorem 1.17. *For any $x \in l^0(\Gamma)$ we have $\|x\|_\varphi^A = \sup_{B\in\Sigma_1} \left\|x^B\right\|_\varphi^A$.*

Proof. Assume first that $x \notin l^\varphi(\Gamma)$. Then we have to show $\sup_{B\in\Sigma_1} \left\|x^B\right\|_\varphi^A = \infty$. Assume for the contrary that $\sup_{B\in\Sigma_1} \left\|x^B\right\|_\varphi^A < \infty$. By Lemma 1.16, $\left\|x^B\right\|_\varphi^o \leq \left\|x^B\right\|_\varphi^A$ for any $B \in \Sigma_1$. Hence $\sup_{B\in\Sigma_1} \left\|x^B\right\|_\varphi^o \leq \sup_{B\in\Sigma_1} \left\|x^B\right\|_\varphi^A$. Since $\|x\|_\varphi^o = \sup_{B\in\Sigma_1} \left\|x^B\right\|_\varphi^o$ (see Lemma 1.8), so $\|x\|_\varphi^o < \infty$, whence $x \in l^\varphi(\Gamma)$ (Corollary 1.11), a contradiction.

Assume now that $x \in l^\varphi(\Gamma)$. Without lost of generality we can assume that $\|x\|_\varphi^A = 1$. There are two cases to consider:

1. $\forall_{B\in\Sigma_1} K(x^B)=\emptyset$,
2. $\exists_{B\in\Sigma_1} K(x^B)\neq\emptyset$.

Consider them one by one.

Ad 1.: Observe that $A:=\lim_{u\to\infty}\frac{\varphi(u)}{u}<\infty$ by Lemma 1.14 and

$$\left\|x^B\right\|_\varphi^A=\lim_{k\to\infty}\frac{1}{k}\left(1+I_\varphi\left(kx^B\right)\right)=A\left\|x^B\right\|_1,\quad \text{for any } B\in\Sigma_1.$$

We have then

$$\sup_{B\in\Sigma_1}\left\|x^B\right\|_\varphi^A=\sup_{B\in\Sigma_1}A\left\|x^B\right\|_1=A\left\|x\right\|_1=\lim_{k\to\infty}\frac{1}{k}\left(1+I_\varphi\left(kx\right)\right)\geq\left\|x\right\|_\varphi^A.$$

The opposite inequality is obvious.

Ad 2.: Denote by B_0 such a finite set that there exists $l\in K(x^{B_0})$. Define the set $L:=\left\{k>0:\exists_{B\in\Sigma_1}\left(B_0\subset B\wedge k\in K(x^B)\right)\right\}$. We have $l\in L$, so $L\neq\emptyset$. Let $k_0:=\inf L$. We will show now that $k_0\geq 1$. Take any set $B\in\Sigma_1$ such that $B_0\subset B$ and $k\in K(x^B)$. It is enough to prove that $k\geq 1$. Assume for the contrary that $k<1$. From Lemma 1.15 there exists $s\in K(x)$ such that $s\leq k<1$. Therefore, $1=\|x\|_\varphi^A=\frac{1}{s}(1+I_\varphi(sx))\geq\frac{1}{s}>1$, which is a contradiction. Consequently we have shown that $k_0\geq 1$.

Take any $\varepsilon>0$. Let $C\in\Sigma_1$ be such that $B_0\subset C$ and

$$\frac{1}{k_0}\left(1+I_\varphi\left(k_0x^C\right)\right)\geq\frac{1}{k_0}\left(1+I_\varphi\left(k_0x\right)\right)-\frac{\varepsilon}{2}\geq\|x\|_\varphi^A-\frac{\varepsilon}{2}.$$

Then we have for $k_C\in K(x^C)$:

$$\begin{aligned}\|x\|_\varphi^A-\frac{\varepsilon}{2}-\left(\frac{1}{k_0}-\frac{1}{k_C}\right)&\leq\frac{1}{k_0}+\frac{I_\varphi\left(k_0x^C\right)}{k_0}-\frac{1}{k_0}+\frac{1}{k_C}\\&=\frac{1}{k_C}+\frac{I_\varphi\left(k_0x^C\right)}{k_0}\\&\leq\frac{1}{k_C}+\frac{I_\varphi\left(k_Cx^C\right)}{k_C}\\&=\left\|x^C\right\|_\varphi^A\\&\leq\sup_{B\in\Sigma_1}\left\|x^B\right\|_\varphi^A.\end{aligned}$$

The above inequality is satisfied for any set $C\in\Sigma_1$ containing B_0. Therefore, taking C with the above properties and large enough, so that $\frac{1}{k_0}-\frac{1}{k_C}\leq\frac{\varepsilon}{2}$, we get the thesis. □

Theorem 1.18. *For any $x\in l^0(\Gamma)$ we have $\|x\|_\varphi^o=\|x\|_\varphi^A$.*

Proof. We know that $\left\|x^B\right\|_\varphi^o=\left\|x^B\right\|_\varphi^A$ for any $B\in\Sigma_1$ (see [8]). Next taking suprema over all $B\in\Sigma_1$ on both sides of this equality and applying Theorem 1.17 and Lemma 1.8, we get the thesis. □

2. Subspaces $h^\varphi(\Gamma)$ of $l^\varphi(\Gamma)$

Let us define the subspace $h^\varphi(\Gamma)$ of $l^\varphi(\Gamma)$ by

$$h^\varphi(\Gamma) = \Big\{ x \in l^\varphi(\Gamma) : \forall_{\lambda>0} \exists_{I\in\Sigma_1} \sum_{i\in\Gamma\setminus I} \varphi(\lambda x(i)) < \infty \Big\}.$$

Theorem 2.1. *For every $x \in l^0(\Gamma)$ the fact that $x \in h^\varphi(\Gamma)$ is equivalent to the condition*

$$\forall_{\varepsilon>0} \exists_{I\in\Sigma_1} \left\| x - x^I \right\|_\varphi \le \varepsilon.$$

Remark. This means that the subspace $h^\varphi(\Gamma)$ is equal to the closure of the subspace $\{x \in l^0(\Gamma) : \operatorname{supp}(x) \in \Sigma_1\}$ in the norm topology, that is,

$$h^\varphi(\Gamma) = \operatorname{cl} \left\{ x \in l^0(\Gamma) : \operatorname{supp}(x) \in \Sigma_1 \right\}.$$

Proof. Suppose that $x \in l^\varphi(\Gamma)$ and $\forall_{\varepsilon>0} \exists_{I\in\Sigma_1} \left\| x - x^I \right\|_\varphi \le \varepsilon$. Fix $\lambda > 0$. For $\varepsilon = \frac{1}{\lambda}$ there exists a set $I \in \Sigma_1$ such that $\left\| x - x^I \right\|_\varphi \le \frac{1}{\lambda}$. Then $\left\| \lambda(x - x^I) \right\|_\varphi \le 1$ and consequently $I_\varphi\left(\lambda(x - x^I)\right) \le 1$. Hence

$$\sum_{i\in\Gamma\setminus I} \varphi(\lambda x(i)) = I_\varphi\left(\lambda(x - x^I)\right) \le 1 < \infty, \quad \text{that is, } x \in h^\varphi(\Gamma).$$

Now assume that $x \in h^\varphi(\Gamma)$ and $\varepsilon > 0$. For $\lambda = \frac{1}{\varepsilon}$ there exists a set $I \in \Sigma_1$ such that $\sum_{i\in\Gamma\setminus I} \varphi(\frac{x(i)}{\varepsilon}) \le M < \infty$ for some $M > 0$. The above sum can have at most countable number of elements not equal to zero. Hence there exists a finite set $J \supset I$ such that $\sum_{i\in\Gamma\setminus J} \varphi(\frac{x(i)}{\varepsilon}) \le 1$. This means that $I_\varphi\big(\frac{x - x^J}{\varepsilon}\big) \le 1$, whence $\left\| x - x^J \right\|_\varphi \le \varepsilon$. □

The above theorem can be formulated alternatively in the following way:

$$x \in h^\varphi(\Gamma) \iff \inf_{I\in\Sigma_1} \left\| x - x^I \right\|_\varphi = 0$$

or

$$x \in h^\varphi(\Gamma) \iff \text{the net } (x^I)_{I\in\Sigma_1} \text{ converges to } x \text{ in norm.}$$

In the remaining part we will use the following notation:

$$(l^\varphi(\Gamma), \|\cdot\|_\varphi) = l^\varphi(\Gamma), \quad (l^\varphi(\Gamma), \|\cdot\|_\varphi^o) = l_o^\varphi(\Gamma)$$
$$(h^\varphi(\Gamma), \|\cdot\|_\varphi) = h^\varphi(\Gamma), \quad (h^\varphi(\Gamma), \|\cdot\|_\varphi^o) = h_o^\varphi(\Gamma).$$

Sometimes it will be no importance which norm is considered. In such a case we will write simply $l^\varphi(\Gamma)$ and $h^\varphi(\Gamma)$ (i.e., we will use the same notations as for the Luxemburg norm).

Lemma 2.2. *The space $h^\varphi(\Gamma)$ is Dedekind complete.*

Proof. Observe first that the space $l^0(\Gamma)$ is Dedekind complete. By Lemma 1.1 we get that $h^\varphi(\Gamma) \in (\mathrm{DC})$. $\square$

Lemma 2.3. *The space $h^\varphi(\Gamma)$ is σ-order continuous.*

Proof. Let (x_n) be a sequence in $h^\varphi(\Gamma)$ such that $x_n \downarrow 0$. We have to show that $I_\varphi(\lambda x_n) \downarrow 0$ for any $\lambda > 0$. Fix $\lambda > 0$. Since $x_1 \in h^\varphi(\Gamma)$, we can choose a set $I \in \Sigma_1$ such that $I_\varphi\big(\lambda x_1^{\Gamma\setminus I}\big) < \infty$. From the Lebesgue Dominated Convergence Theorem we get $I_\varphi\big(\lambda x_n^{\Gamma\setminus I}\big) \downarrow 0$. Since $x_n \downarrow 0$, so $x_n(i) \downarrow 0$ for any $i \in \Gamma$, in particular for any $i \in I$. Hence $\varphi(\lambda x_n(i)) \downarrow 0$ for any $i \in I$. Finally,

$$\begin{aligned} I_\varphi(\lambda x_n) &= \sum_{i\in\Gamma\setminus I} \varphi(\lambda x_n(i)) + \sum_{i\in I} \varphi(\lambda x_n(i)) \\ &= \Big(I_\varphi\left(\lambda x_n^{\Gamma\setminus I}\right) + \sum_{i\in I} \varphi(\lambda x_n(i))\Big) \downarrow 0. \end{aligned}$$ $\square$

Corollary 2.4. *The space $h^\varphi(\Gamma)$ is order continuous.*

Proof. The proof follows immediately from Theorem 1.2 and from the fact that $h^\varphi(\Gamma)$ is Dedekind complete (see Lemma 2.2) and σ-order continuous (see Lemma 2.3). $\square$

Corollary 2.5. *We have the inclusion $h^\varphi(\Gamma) \subset l^\varphi(\Gamma)_a$.*

Proof. Since $h^\varphi(\Gamma)$ is an ideal in $l^\varphi(\Gamma)$, the thesis follows by Corollary 2.4 and Theorem 1.3. $\square$

Theorem 2.6. *The equality $h^\varphi(\Gamma) = l^\varphi(\Gamma)_a$ holds.*

Proof. It is enough to show that $l^\varphi(\Gamma)_a \subset h^\varphi(\Gamma)$. Take any $x \in l^\varphi(\Gamma)_a$. Since $x \in h^\varphi(\Gamma)$ iff $|x| \in h^\varphi(\Gamma)$, we can assume that $x \geq 0$. We have to show that for every $\lambda > 0$ there exists a set $I \in \Sigma_1$ such that $I_\varphi\left(\lambda x^{\Gamma\setminus I}\right) < \infty$. Fix $\lambda > 0$. The set $\{\Gamma \setminus I : I \in \Sigma_1\}$ equipped with the order

$$\big(\Gamma \setminus I \leq \Gamma \setminus J\big) \iff \big(\Gamma \setminus I \supset \Gamma \setminus J\big)$$

is a directed set. Let us associate with any set $\Gamma \setminus I$ ($I \in \Sigma_1$) the element $x^{\Gamma\setminus I}$. We get a net converging monotonously to zero ($x^{\Gamma\setminus I} \downarrow 0$), i.e., $\big(\Gamma \setminus I \leq \Gamma \setminus J\big)$ yields $\big(x^{\Gamma\setminus I} \geq x^{\Gamma\setminus J}\big)$ and $\inf_{I\in\Sigma_1} x^{\Gamma\setminus I} = 0$. By the assumption $x \in l^\varphi(\Gamma)_a$ and the fact that $x \geq x^{\Gamma\setminus I}$ for any $I \in \Sigma_1$, we get that $\inf_{I\in\Sigma_1} \left\|x^{\Gamma\setminus I}\right\|_\varphi = 0$. Let $I_0 \in \Sigma_1$ be such that $\left\|x^{\Gamma\setminus I_0}\right\|_\varphi \leq \frac{1}{\lambda}$. Then $\left\|\lambda x^{\Gamma\setminus I_0}\right\|_\varphi \leq 1$ and finally $I_\varphi\left(\lambda x^{\Gamma\setminus I_0}\right) \leq 1 < \infty$. $\square$

Theorem 2.7. *The subspace $h^\varphi(\Gamma)$ is also characterized by the equality*

$$h^\varphi(\Gamma) = \left\{x \in l^\varphi(\Gamma) : |x| \geq x_n \downarrow 0 \Rightarrow \|x_n\|_\varphi \downarrow 0\right\}.$$

Proof. By Theorems 2.6 and 1.3 and the fact that every sequence is a net, we have

$$\begin{aligned} h^\varphi(\Gamma) &= \left\{x \in l^\varphi(\Gamma) : |x| \geq x_\alpha \downarrow 0 \Rightarrow \|x_\alpha\|_\varphi \downarrow 0\right\} \\ &\subset \left\{x \in l^\varphi(\Gamma) : |x| \geq x_n \downarrow 0 \Rightarrow \|x_n\|_\varphi \downarrow 0\right\}. \end{aligned}$$

It can be shown that $A := \left\{x \in l^\varphi(\Gamma) : |x| \geq x_n \downarrow 0 \Rightarrow \|x_n\|_\varphi \downarrow 0\right\}$ is an ideal in $l^\varphi(\Gamma)$ and that $A \in (\sigma\text{-}\mathrm{OC})$. Since $l^\varphi(\Gamma) \in (\mathrm{DC})$, so $A \in (\mathrm{DC})$ (see Lemma 1.1) and next $A \in (\mathrm{OC})$ (see Theorem 1.2). By Theorem 1.3, we get $A \subset l^\varphi(\Gamma)_a = h^\varphi(\Gamma)$. □

Theorem 2.8. *If $x \in h^\varphi(\Gamma)$, then* supp(x) *is at most a countable set.*

Proof. Let $x \in h^\varphi(\Gamma)$ and define for any $n \in \mathbb{N}$ the set $A_n = \left\{i \in G : |x(i)| > \frac{1}{n}\right\}$. We will show that $A_n \in \Sigma_1$. Let $I \in \Sigma_1$ be such that $I_\varphi\left(n(a_\varphi + 1)x^{\Gamma\setminus I}\right) < \infty$. Since $A_n = \left(A_n \cap (\Gamma \setminus I)\right) \cup \left(A_n \cap I\right)$, it is enough to show that $A_n \cap (\Gamma \setminus I) \in \Sigma_1$. This is true since

$$\begin{aligned} \varphi(a_\varphi + 1)\mu(A_n \cap (\Gamma \setminus I)) &= \sum_{A_n \cap (\Gamma\setminus I)} \varphi(a_\varphi + 1) \\ &\leq \sum_{i \in \Gamma\setminus I} \varphi(n(a_\varphi + 1)x(i)) \\ &= I_\varphi\left(n(a_\varphi + 1)x^{\Gamma\setminus I}\right) < \infty\,. \end{aligned}$$ □

Lemma 2.9. *For any $x \in l^\varphi(\Gamma)$ the following equality holds:*

$$\inf_{z \in h^\varphi(\Gamma)} \|x - z\|_\varphi = \inf_{\substack{y \in l^\varphi(\Gamma) \\ \mathrm{supp}(y) \in \Sigma_1}} \|x - y\|_\varphi\,.$$

Proof. Let us fix $x \in l^\varphi(\Gamma)$. The inequality „$\leq$" is obvious. We will show the inequality „$\geq$". Take any $\varepsilon > 0$ and $z \in h^\varphi(\Gamma)$. By Theorem 2.1 we have

$$h^\varphi(\Gamma) = \mathrm{cl}\left\{y \in l^\varphi(\Gamma) : \mathrm{supp}(y) \in \Sigma_1\right\}.$$

Hence there exists $y \in l^\varphi(\Gamma)$ such that $\mathrm{supp}(y) \in \Sigma_1$ and $\|z - y\|_\varphi < \varepsilon$. Now we have $\|x - y\|_\varphi \leq \|x - z\|_\varphi + \|z - y\|_\varphi \leq \|x - z\|_\varphi + \varepsilon$. Therefore

$$\inf_{\substack{y \in l^\varphi(\Gamma) \\ \mathrm{supp}(y) \in \Sigma_1}} \|x - y\|_\varphi \leq \|x - z\|_\varphi + \varepsilon.$$

By the arbitrariness of $z \in h^\varphi(\Gamma)$, we get

$$\inf_{\substack{y \in l^\varphi(\Gamma) \\ \operatorname{supp}(y) \in \Sigma_1}} \|x - y\|_\varphi \le \inf_{z \in h^\varphi(\Gamma)} \|x - z\|_\varphi + \varepsilon.$$

As it is true for each $\varepsilon > 0$, we get the thesis. □

Theorem 2.10. *For any $x \in l^\varphi(\Gamma)$ the equalities*

$$d(x) = d_o(x) = \inf_{I \in \Sigma_1} \left\| x - x^I \right\|_\varphi = \inf_{I \in \Sigma_1} \left\| x - x^I \right\|_\varphi^o = \theta(x),$$

hold, where $d(x) = \inf_{y \in h^\varphi(\Gamma)} \|x - y\|_\varphi$, $d_o(x) = \inf_{y \in h^\varphi(\Gamma)} \|x - y\|_\varphi^o$, and $\theta(x) = \inf \left\{ \lambda > 0 : \exists_{I \in \Sigma_1} \sum_{i \in \Gamma \setminus I} \varphi\left(\frac{x(i)}{\lambda}\right) < \infty \right\}$.

Proof. If $x \in h^\varphi(\Gamma)$, then all above numbers are equal to zero. Consequently, we can assume that $x \notin h^\varphi(\Gamma)$. Then $\theta(x) > 0$ since $h^\varphi(\Gamma) = \{x : \theta(x) = 0\}$. First we will show that $\theta(x) \le d(x)$. Take any $\varepsilon > 0$ such that $0 < \varepsilon < \theta(x)$ and $y \in l^\varphi(\Gamma)$ such that $\operatorname{supp}(y) \in \Sigma_1$. Then

$$\sum_{i \in \Gamma \setminus I} \varphi\left(\frac{x(i)}{\theta(x) - \varepsilon}\right) = \infty \quad (\forall\, I \in \Sigma_1),$$

in particular $\sum_{i \in \Gamma \setminus \operatorname{supp}(y)} \varphi\left(\frac{x(i)}{\theta(x) - \varepsilon}\right) = \infty$. Hence

$$\begin{aligned} I_\varphi\left(\frac{x - y}{\theta(x) - \varepsilon}\right) &= \sum_{i \in \Gamma} \varphi\left(\frac{x - y}{\theta(x) - \varepsilon}\right) \\ &= \sum_{i \in \Gamma \setminus \operatorname{supp}(y)} \varphi\left(\frac{x(i)}{\theta(x) - \varepsilon}\right) + \sum_{i \in \operatorname{supp}(y)} \varphi\left(\frac{(x - y)(i)}{\theta(x) - \varepsilon}\right) = \infty. \end{aligned}$$

Hence $\left\| \frac{x - y}{\theta(x) - \varepsilon} \right\|_\varphi > 1$, that is, $\|x - y\|_\varphi > \theta(x) - \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we get $\|x - y\|_\varphi \ge \theta(x)$ and next by the arbitrariness of y with support in Σ_1, we have

$$\inf_{\substack{y \in l^\varphi(\Gamma) \\ \operatorname{supp}(y) \in \Sigma_1}} \|x - y\|_\varphi \ge \theta(x).$$

By Lemma 2.9 we get $d(x) \ge \theta(x)$.

Together with the obvious inequalities we have the following inequalities:

$$\begin{aligned} &\theta(x) \le d(x) \le d_o(x) \le \inf_{I \in \Sigma_1} \left\| x - x^I \right\|_\varphi^o \\ &\theta(x) \le d(x) \le \inf_{I \in \Sigma_1} \left\| x - x^I \right\|_\varphi \le \inf_{I \in \Sigma_1} \left\| x - x^I \right\|_\varphi^o. \end{aligned}$$

Then to finish the proof, it is enough to show that $\inf_{I \in \Sigma_1} \left\| x - x^I \right\|_\varphi^o \le \theta(x)$.

Given any $\varepsilon > 0$, there exists $I \in \Sigma_1$ such that $\sum_{i\in\Gamma\setminus I} \varphi\big(\frac{x(i)}{\theta(x)+\varepsilon}\big) < \infty$. Hence there exists a set $J \in \Sigma_1$ such that $\sum_{i\in\Gamma\setminus J} \varphi\big(\frac{x(i)}{\theta(x)+\varepsilon}\big) < \varepsilon$. For any set $K \in \Sigma_1$ satisfying $K \supset J$ we have

$$\sum_{i\in\Gamma} \varphi\left(\frac{x(i) - x^K(i)}{\theta(x)+\varepsilon}\right) \leq \sum_{i\in\Gamma\setminus J} \varphi\left(\frac{x(i)}{\theta(x)+\varepsilon}\right) < \varepsilon.$$

Next, by Theorem 1.18, we have

$$\left\|x - x^K\right\|_\varphi^o \leq (\theta(x)+\varepsilon)\left(1 + \sum_{i\in\Gamma} \varphi\left(\frac{x(i) - x^K(i)}{\theta(x)+\varepsilon}\right)\right) < (\theta(x)+\varepsilon)(1+\varepsilon),$$

whence

$$\inf_{K\in\Sigma_1} \left\|x - x^K\right\|_\varphi^o = \inf_{\substack{K\in\Sigma_1\\ K\supset J}} \left\|x - x^k\right\|_\varphi^o \leq (\theta(x)+\varepsilon)(1+\varepsilon).$$

By the arbitrariness of $\varepsilon > 0$, we get the thesis. □

Let $L > 1$. We say that the function φ satisfies *condition* δ_L (we will write $\varphi \in \delta_L$) if there exist constants $K > 0, u_0 > 0$ such that $0 < \varphi(u_0) < \infty$ and the inequality $\varphi(Lu) \leq K\varphi(u)$ holds for $|u| \leq u_0$.

Lemma 2.11. *If $\varphi \in \delta_L$ for some $L > 1$, then φ takes value zero only at zero.*

Proof. Suppose that $\varphi \in \delta_L$ for some $L > 1$ and suppose for the contrary that $a_\varphi := \sup\{u \geq 0 : \varphi(u) = 0\} > 0$. We have $\varphi(a_\varphi + \varepsilon) > 0$ since $a_\varphi + \varepsilon > a_\varphi$. On the other hand, since $L > 1$, there exists $\varepsilon > 0$ such that $\frac{a_\varphi+\varepsilon}{L} < a_\varphi$, whence

$$\varphi(a_\varphi + \varepsilon) = \varphi\left(L\frac{a_\varphi+\varepsilon}{L}\right) \leq K\varphi\left(\frac{a_\varphi+\varepsilon}{L}\right) = 0,$$

since $\varphi \in \delta_L$. A contradiction. □

Lemma 2.12. *For any $L > 1$ the fact that $\varphi \in \delta_L$ is equivalent to the fact that $\varphi \in \delta_2$.*

We will omit the standard proof.

Theorem 2.13. *If $\varphi \in \delta_2$, then $h^\varphi(\Gamma) = l^\varphi(\Gamma)$.*

Proof. Let $x \in l^\varphi(\Gamma)$ and $\varphi \in \delta_2$. Then there exists $\lambda_0 > 0$ such that $I_\varphi(\lambda_0 x) < \infty$. Take now any $\lambda > 0$. If $\lambda \leq \lambda_0$, then obviously $I_\varphi(\lambda x) < \infty$. Assume that $\lambda > \lambda_0$, i.e., $\frac{\lambda}{\lambda_0} > 1$. In virtue of Lemma 2.12 we know that $\varphi \in \delta_{\frac{\lambda}{\lambda_0}}$, that is, there exist $K > 0$ and $u_0 > 0$ such that $0 < \varphi(u_0) < \infty$ and $\varphi\big(\frac{\lambda u}{\lambda_0}\big) \leq K\varphi(u)$ for $|u| \leq u_0$.

Define now the set $I = \{i \in \Gamma : |\lambda_0 x(i)| > u_0\}$. Since $0 < \sum_{i\in I} \varphi(u_0) \leq \sum_{i\in I} \varphi(\lambda_0 x(i)) \leq I_\varphi(\lambda_0 x) < \infty$, we get $I \in \Sigma_1$. We have

$$\sum_{i\in\Gamma\setminus I} \varphi(\lambda x(i)) \leq \sum_{i\in\Gamma\setminus I} \varphi\left(\frac{\lambda}{\lambda_0}\lambda_0 x(i)\right) \leq \sum_{i\in\Gamma\setminus I} K\varphi(\lambda_0 x(i)) \leq KI_\varphi(\lambda_0 x) < \infty,$$

which finishes the proof. □

3. Dual spaces of $h^\varphi(\Gamma)$

We start with the following

Lemma 3.1. *For any $x, y \in l^0(\Gamma)$ the following Hölder inequality holds:*

$$\sum_\Gamma |xy| \leq \|x\|_\varphi \|y\|_\psi^o .$$

Proof. If $x = 0$ or $\|x\|_\varphi = \infty$, then the inequality is obvious. Assume that $0 < \|x\|_\varphi < \infty$. Then we have $\sum_\Gamma |xy| = \|x\|_\varphi \sum_\Gamma \frac{|x|}{\|x\|_\varphi}|y| \leq \|x\|_\varphi \|y\|_\psi^o$. □

We denote by X^* the dual space of a normed space $(X, \|\cdot\|)$. For any functional $F \in X^*$ the norm is defined by formula

$$\|F\| := \sup\{|F(x)| : \|x\| \leq 1\}.$$

Theorem 3.2. *For each element $y \in l^\psi(\Gamma)$, the formula $F(x) = \sum_\Gamma xy$, for any $x \in h^\varphi(\Gamma)$, determines a functional $F \in h^\varphi(\Gamma)^*$ with $\|F\| = \|y\|_\psi^o$. Conversely, any functional $G \in h^\varphi(\Gamma)^*$ is determined by some element y from $l^\psi(\Gamma)$, that is, $G(x) = \sum_\Gamma xy$ for any $x \in h^\varphi(\Gamma)$.*

Proof. Let $y \in l^\psi(\Gamma)$. Define $F(x) = \sum_\Gamma xy$. By the Hölder inequality we have $|F(x)| \leq \|x\|_\varphi \|y\|_\psi^o$, i.e., F is a well defined linear functional on $h^\varphi(\Gamma)$ and $\|F\| \leq \|y\|_\psi^o < \infty$, whence F is continuous. Define

$$e_j(i) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \quad (i, j \in \Gamma).$$

Then $y(j) = F(e_j)$ for any $j \in \Gamma$. Then by Lemma 1.8,

$$\begin{aligned} \|y\|_\psi^o = \left\|\left(F(e_i)\right)_{i\in\Gamma}\right\|_\psi^o &= \sup_{I_\varphi(x)\leq 1\, \mathrm{supp}(x)\in\Sigma_1} \sum_{i\in\Gamma} x(i)F(e_i) \\ &= \sup_{I_\varphi(x)\leq 1\, \mathrm{supp}(x)\in\Sigma_1} F(x) \\ &\leq \sup_{I_\varphi(x)\leq 1\, \mathrm{supp}(x)\in\Sigma_1} \|F\| \, \|x\|_\varphi \\ &\leq \|F\|. \end{aligned}$$

Consequently, we have shown the equality $\|F\| = \|y\|_\psi^o$.

The above calculation shows also that if we take any functional $G \in h^\varphi(\Gamma)^*$ and define $y = (G(e_i))_{i\in\Gamma}$, then $\|y\|_\psi^o \le \|G\| < \infty$ and, by Corollary 1.11, we have that $y \in l_o^\psi(\Gamma)$. It remains to show that y determines F equal to G. Indeed, F and G are equal on the set $\{x \in h^\varphi(\Gamma) : \mathrm{supp}(x) \in \Sigma_1\}$ and $h^\varphi(\Gamma)$ is the closure of this set, so by continuity of the functionals F and G, we get $F = G$. □

The above theorem shows that any element $v \in l^\psi(\Gamma)$ determines a linear continuous functional on $h^\varphi(\Gamma)$. We denote it by F_v. Note that we can naturally extend the functional F_v to the functional on $l^\varphi(\Gamma)$ with preserving of the norm. Namely, defining $G(x) = \sum_\Gamma xy$ $(x \in l^\varphi(\Gamma))$, we have of course that $G(x) = F_v(x)$ for $x \in h^\varphi(\Gamma)$. By the Hölder inequality (see Lemma 3.1), we get

$$|G(x)| = \Big|\sum_\Gamma xy\Big| \le \|x\|_\varphi \|y\|_\psi^o \quad (\forall\, x \in l^\varphi(\Gamma)),$$

i.e., G is well defined on $l^\varphi(\Gamma)$ and $\|G\| \le \|y\|_\psi^o$. Since G is an extension of F, so $\|G\| \ge \|F_v\| = \|y\|_\psi^o$. In the following the functional G will be denoted by the same symbol F_v.

Theorem 3.3. *Every functional $x^* \in l^\varphi(\Gamma)^*$ can be written, in the unique manner, as a sum $x^* = F_v + G$, where F_v is determined by some element $v \in l^\psi(\Gamma)$ and G is so-called singular functional, i.e., $G(h^\varphi(\Gamma)) = \{0\}$.*

Proof. Since $x^*|_{h^\varphi(\Gamma)}$ is a functional on $h^\varphi(\Gamma)$, by Theorem 3.2, there exists $v \in l_o^\psi(\Gamma)$ determining this functional. Let F_v denote its extension to $l^\varphi(\Gamma)$ (defined above). Define $G = x^* - F_v$. If $x \in h^\varphi(\Gamma)$, then $G(x) = x^*(x) - F_v(x) = x^*|_{h^\varphi(\Gamma)}(x) - x^*|_{h^\varphi(\Gamma)}(x) = 0$. Assume now that $x^* = F_z + H$, where $z \in l^\psi(\Gamma)$ and $H(h^\varphi(\Gamma)) = \{0\}$. Since $F_v + G = F_z + H$, $F_v - F_z = H - G$. For any $x \in h^\varphi(\Gamma)$ we have $(F_v - F_z)(x) = (H - G)(x) = H(x) - G(x) = 0$, i.e., $F_v(x) = F_z(x)$ for $x \in h^\varphi(\Gamma)$, whence $v = z$, and consequently $G = H$. □

In the proof of the next theorem we will use the following

Lemma 3.4. *Let $x \in l^\varphi(\Gamma)$ and $\big|\sum_{i\in\Gamma} x(i)\big| < \infty$. Then for any $\varepsilon > 0$ there exists a set $I_\varepsilon \in \Sigma_1$ such that $\big|\sum_{i\in\Gamma\setminus I} x(i)\big| < \varepsilon$ for any $I \in \Sigma_1$ satisfying $I \supset I_\varepsilon$.*

Proof. Since $\big|\sum_{i\in\Gamma} x(i)\big| < \infty$, we have $\sum_{i\in\Gamma} x^+(i) < \infty$ and $\sum_{i\in\Gamma} x^-(i) < \infty$. Let $\varepsilon > 0$ and take $K \in \Sigma_1$ such that $\sum_K x^+ > \sum_\Gamma x^+ - \frac{\varepsilon}{2}$. We can assume that if $i \in K$, then $x^+(i) > 0$. Analogously we take $J \in \Sigma_1$ such that $\sum_J x^- > \sum_\Gamma x^- - \frac{\varepsilon}{2}$ and $x^-(i) > 0$ for $i \in J$.

Defining $I_\varepsilon = K \cup J$, we notice that $\sum_{I_\varepsilon} x^+ = \sum_K x^+$ and $\sum_{I_\varepsilon} x^- = \sum_J x^-$. For any $I \in \Sigma_1$ such that $I \supset I_\varepsilon$ we have

$$\begin{aligned} \left|\sum_{\Gamma\setminus I} x\right| &\le \sum_{\Gamma\setminus I} x^+ + \sum_{\Gamma\setminus I} x^- \\ &\le \sum_{\Gamma\setminus I_\varepsilon} x^+ + \sum_{\Gamma\setminus I_\varepsilon} x^- \\ &= \Big(\sum_{\Gamma} x^+ - \sum_{I_\varepsilon} x^+\Big) + \Big(\sum_{\Gamma} x^- - \sum_{I_\varepsilon} x^-\Big) \\ &= \Big(\sum_{\Gamma} x^+ - \sum_{K} x^+\Big) + \Big(\sum_{\Gamma} x^- - \sum_{J} x^-\Big) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Theorem 3.5. *For every functional $x^* = F_v + G \in l^\varphi(\Gamma)^*$, we have*

$$\|x^*\| = \|v\|^o_\psi + \|G\|.$$

Proof. First we will prove the inequality „$\le$”. We have

$$\|x^*\| = \|F_v + G\| \le \|F_v\| + \|G\| = \|v\|^o_\psi + \|G\|.$$

Fix $\varepsilon > 0$ and take $x_1, x_2 \in l^\varphi(\Gamma)$ such that $\|x_1\|_\varphi \le 1$, $\|x_2\|_\varphi \le 1$ and $\|v\|^o_\psi - \varepsilon = \|F_v\| - \varepsilon < F_v(x_1)$, $\|G\| - \varepsilon < G(x_2)$. Since $\|F_v : l^\varphi(\Gamma) \to \mathbb{R}\| = \|F_v : h^\varphi(\Gamma) \to \mathbb{R}\|$, we can assume that $x_1 \in h^\varphi(\Gamma)$ and consequently that $\mathrm{supp}(x_1) \in \Sigma_1$ (since $h^\varphi(\Gamma)$ is the closure of the set $\{x \in l^\varphi(\Gamma) : \mathrm{supp}(x) \in \Sigma_1\}$ and $F_v : h^\varphi(\Gamma) \to \mathbb{R}$ is continuous). Since $\left|\sum_\Gamma v x_2\right| = |F_v(x_2)| < \infty$, by Lemma 3.4, there exists $I_\varepsilon \in \Sigma_1$ such that $\forall_{I \in \Sigma_1, I \supset I_\varepsilon} \left|\sum_{\Gamma\setminus I} v x_2\right| < \varepsilon$. Now take a set $K \in \Sigma_1$ such that $\mathrm{supp}(x_1) \subset K$, $I_\varepsilon \subset K$ and $\sum_{i\in K} \varphi(x_2(i)) > \sum_{i\in\Gamma} \varphi(x_2(i)) - \varepsilon$, and define

$$x(i) = \begin{cases} x_1(i) & \text{for } i \in K, \\ x_2(i) & \text{for } i \in \Gamma \setminus K. \end{cases}$$

Then

$$I_\varphi(x) = \sum_{i\in K} \varphi(x_1(i)) + \sum_{i\in\Gamma\setminus K} \varphi(x_2(i)) < I_\varphi(x_1) + \varepsilon \le 1 + \varepsilon.$$

Therefore, $I_\varphi\left(\frac{x}{1+\varepsilon}\right) \le \frac{1}{1+\varepsilon} I_\varphi(x) \le 1$, that is, $\left\|\frac{x}{1+\varepsilon}\right\|_\varphi \le 1$. Hence $\|x^*\| \ge$

$x^*\left(\frac{x}{1+\varepsilon}\right)$. Next we have

$$\begin{aligned}(1+\varepsilon)\|x^*\| &\geq x^*(x) = x^*(x_1\chi_K) + x^*\left(x_2\chi_{\Gamma\setminus K}\right)\\ &= F_v(x_1\chi_K) + G(x_1\chi_K) + F_v\left(x_2\chi_{\Gamma\setminus K}\right) + G\left(x_2\chi_{\Gamma\setminus K}\right)\\ &= F_v(x_1) + 0 + F_v\left(x_2\chi_{\Gamma\setminus K}\right) + G(x_2\chi_\Gamma - x_2\chi_K)\\ &= F_v(x_1) + F_v\left(x_2\chi_{\Gamma\setminus K}\right) + G(x_2)\\ &= F_v(x_1) + \sum_{i\in\Gamma\setminus K} v(i)x_2(i) + G(x_2)\\ &> \|v\|_\psi^o - \varepsilon + (-\varepsilon) + \|G\| - \varepsilon = \|v\|_\psi^o + \|G\| - 3\varepsilon.\end{aligned}$$

Consequently, we have shown that $(1+\varepsilon)\|x^*\| > \|v\|_\psi^o + \|G\| - 3\varepsilon$. By the arbitrariness of $\varepsilon > 0$, we get the thesis. □

4. Dual spaces of $h_o^\varphi(\Gamma)$

We start with the following

Lemma 4.1. *For any $x \in l^\varphi(\Gamma)$ the following equalities holds:*

$$\|x\|_\varphi = \sup_{\|y\|_\psi^o\leq 1}\sum_\Gamma xy = \sup_{\substack{\|y\|_\psi^o\leq 1\\ \operatorname{supp}(y)\in\Sigma_1}}\sum_\Gamma xy.$$

Proof. Notice that

$$\sup_{\substack{\|y\|_\psi^o\leq 1\\ \operatorname{supp}(y)\in\Sigma_1}}\sum_\Gamma xy \leq \sup_{\|y\|_\psi^o\leq 1}\sum_\Gamma xy \leq \sup_{\|y\|_\psi^o\leq 1}\|x\|_\varphi\|y\|_\psi^o \leq \|x\|_\varphi.$$

It remains to show that

$$\|x\|_\varphi \leq \sup_{\substack{\|y\|_\psi^o\leq 1\\ \operatorname{supp}(y)\in\Sigma_1}}\sum_\Gamma xy. \tag{2}$$

Assume that the inequality (2) holds for the elements with finite supports. Then the equality holds also for all elements. Indeed, let $x \in l^\varphi(\Gamma)$ and y be such that $\|y\|_\psi^o \leq 1$ and $\operatorname{supp}(y) \in \Sigma_1$. Then we have for any $I \in \Sigma_1$,

$$\sum_\Gamma x^I y = \sum_I x^I y \leq \sum_I x^I y + \sum_{i\in\Gamma\setminus I} x(i)y(i)\operatorname{sgn}(x(i)y(i)) = \sum_\Gamma xz,$$

where

$$z(i) = \begin{cases} y(i) & \text{for } i \in I\\ y(i)\operatorname{sgn}(x(i)y(i)) & \text{for } i \in \Gamma\setminus I.\end{cases}$$

Since $\|z\|_\psi^o \le \|y\|_\psi^o \le 1$ and $\operatorname{supp}(z) \subset \operatorname{supp}(y) \in \Sigma_1$, we have

$$\sum_\Gamma x^I y \le \sup_{\substack{\|y\|_\psi^o \le 1 \\ \operatorname{supp}(y)\in\Sigma_1}} \sum_\Gamma xy.$$

Hence

$$\sup_{\substack{\|y\|_\psi^o \le 1 \\ \operatorname{supp}(y)\in\Sigma_1}} \sum_\Gamma x^I y \le \sup_{\substack{\|y\|_\psi^o \le 1 \\ \operatorname{supp}(y)\in\Sigma_1}} \sum_\Gamma xy.$$

Next, by Lemma 1.6 and inequality (2) for elements with finite supports, we get

$$\|x\|_\varphi = \sup_{I\in\Sigma_1} \left\|x^I\right\|_\varphi \le \sup_{I\in\Sigma_1} \sup_{\substack{\|y\|_\psi^o \le 1 \\ \operatorname{supp}(y)\in\Sigma_1}} \sum_\Gamma x^I y \le \sup_{\substack{\|y\|_\psi^o \le 1 \\ \operatorname{supp}(y)\in\Sigma_1}} \sum_\Gamma xy,$$

which is the desired inequality.

Consequently, in order to show that

$$\|x\|_\varphi \le \sup_{\substack{\|y\|_\psi^o \le 1 \\ \operatorname{supp}(y)\in\Sigma_1}} \sum_\Gamma xy,$$

we can assume that $\operatorname{supp}(x) \in \Sigma_1$. Additionally, assume without loss of generality that $\|x\|_\varphi = 1$. Then $I_\varphi(x) \le 1$ and $\forall_{i\in\Gamma}\varphi(x(i)) \le 1$, which gives

$$|x(i)| \le b_\varphi := \sup\{u : \varphi(u) < \infty\} \quad (\forall\, i \in \Gamma).$$

Consider first the case when there exists $i_0 \in \Gamma$ such that $|x(i_0)| = b_\varphi$. Defining $z = \frac{1}{b_\varphi}\chi_{\{i_0\}} \operatorname{sgn}(x(i_0))$, we get $\operatorname{supp}(z) = \{i_0\}$,

$$\|z\|_\psi^o = \sup_{I_\varphi(g)\le 1} \sum_\Gamma zg = \sup_{I_\varphi(g)\le 1} \frac{1}{b_\varphi} g(i_0) \le \frac{1}{b_\varphi} b_\varphi = 1$$

and

$$\sup_{\substack{\|y\|_\psi^o \le 1 \\ \operatorname{supp}(y)\in\Sigma_1}} \sum_\Gamma xy \ge \sum_\Gamma xz = x(i_0)\frac{1}{b_\varphi}\operatorname{sgn}(x(i_0)) = b_\varphi \frac{1}{b_\varphi} = 1.$$

Now assume that $\forall_{i\in\Gamma}|x(i)| < b_\varphi$. Since $\operatorname{supp}(x) \in \Sigma_1$, there exists $\varepsilon_0 > 0$ such that $\forall_{i\in\Gamma}(1+\varepsilon_0)|x(i)| < b_\varphi$. Take any $0 < \varepsilon \le \varepsilon_0$. Then $\|(1+\varepsilon)x\|_\varphi > 1$ and consequently $I_\varphi((1+\varepsilon)x)) > 1$. Define

$$w(i) = \begin{cases} \varphi'((1+\varepsilon)x(i)) & \text{for } i \in \operatorname{supp}(x) \\ 0 & \text{for } i \notin \operatorname{supp}(x), \end{cases}$$

where φ' is the left-hand side derivative of the function φ and $z = \frac{w}{I_\psi(w)+1}$. The elements w and z are well defined, $\operatorname{supp}(z) = \operatorname{supp}(w) \subset \operatorname{supp}(x)$ and $\|z\|_\psi^o \le 1$. Indeed,

$$\begin{aligned}\|z\|_\psi^o = \sup_{I_\varphi(h)\le 1} \sum_\Gamma zh &= \frac{1}{I_\psi(w)+1} \sup_{I_\varphi(h)\le 1} \sum_\Gamma wh \\ &\le \frac{1}{I_\psi(w)+1} \sup_{I_\varphi(h)\le 1} \sum_\Gamma (I_\psi(w) + I_\varphi(h)) \\ &\le \frac{1}{I_\psi(w)+1}(I_\psi(w)+1) = 1.\end{aligned}$$

Now we also have

$$\begin{aligned}\sup_{\substack{\|y\|_\psi^o \le 1 \\ \operatorname{supp}(y)\in\Sigma_1}} \sum_\Gamma xy &\ge \sum_\Gamma xz \\ &= \frac{1}{I_\psi(w)+1}\sum_\Gamma xw \\ &= \frac{1}{1+\varepsilon}\,\frac{1}{I_\psi(w)+1}\sum_\Gamma (1+\varepsilon)xw \\ &= \frac{1}{1+\varepsilon}\,\frac{1}{I_\psi(w)+1}\sum_{i\in\operatorname{supp}(x)} \big((1+\varepsilon)x(i)\varphi'((1+\varepsilon)x(i))\big) \\ &= \frac{1}{1+\varepsilon}\,\frac{1}{I_\psi(w)+1}\sum_{i\in\operatorname{supp}(x)} \Big(\varphi\big((1+\varepsilon)x(i)\big) + \psi\big(\varphi'((1+\varepsilon)x(i))\big)\Big) \\ &= \frac{1}{1+\varepsilon}\,\frac{1}{I_\psi(w)+1}\big(I_\varphi((1+\varepsilon)x) + I_\psi(w)\big) \\ &> \frac{1}{1+\varepsilon}\,\frac{1}{I_\psi(w)+1}\big(1 + I_\psi(w)\big) = \frac{1}{1+\varepsilon}.\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, we get the thesis. □

Theorem 4.2. *For each element $y \in l^\psi(\Gamma)$ the formula $F(x) = \sum_\Gamma xy$ for any $x \in h_o^\varphi(\Gamma)$, determines the functional $F \in h_o^\varphi(\Gamma)^*$ with $\|F\| = \|y\|_\psi$. Conversely, any functional $G \in h_o^\varphi(\Gamma)^*$ is determined by some element y from $l^\psi(\Gamma)$.*

Proof. Let $y \in l^\psi(\Gamma)$. Define $F(x) = \sum_\Gamma xy$. By the Hölder inequality, we get $|F(x)| \le \|x\|_\varphi^o \|y\|_\psi$, i.e., F is a well defined linear functional on $h_o^\varphi(\Gamma)$ and $\|F\| \le \|y\|_\psi$. The opposite inequality follows by Lemma 4.1:

$$\|F\| = \sup_{\substack{\|x\|_\varphi^o\le 1 \\ x\in h_o^\varphi(\Gamma)}} F(x) = \sup_{\substack{\|x\|_\varphi^o\le 1 \\ x\in h_o^\varphi(\Gamma)}} \sum_\Gamma xy \ge \sup_{\substack{\|x\|_\varphi^o\le 1 \\ \operatorname{supp}(x)\in\Sigma_1}} \sum_\Gamma xy = \|y\|_\psi .$$

Now take any $G \in h_o^\varphi(\Gamma)^*$. Since the norms $\|\cdot\|_\varphi^o$ and $\|\cdot\|_\varphi$ are equivalent, $h_o^\varphi(\Gamma)^*$ is equal to $h^\varphi(\Gamma)^*$ as sets and consequently, by Theorem 3.2 there exists $y \in l^\psi(\Gamma)$ such that $G(x) = \sum_\Gamma xy$ for $x \in h^\varphi(\Gamma)$. □

5. Smooth points of $l^\varphi(\Gamma)$

In this section we will give criteria for smooth points of the unit sphere of $l^\varphi(\Gamma)$ equipped with the Luxemburg norm. Next we will point out the differences between the spaces depending on the cardinality of the set Γ. Smooth points in Orlicz spaces or in Musielak-Orlicz spaces corresponding to σ-finite measure spaces were investigated among others in [2–6, 9–11].

Let us define support functionals and smooth points.

Definition 5.1. We say that a functional $x^* \in X^*$ is *a support functional* at a point $x \in \mathrm{S}(X)$ if $\|x^*\| = 1$ and $x^*(x) = 1$. By $\mathrm{Grad}(x)$ we will denote the set of all support functionals at a point $x \in \mathrm{S}(X)$. If we take any $x \neq 0$, then we define that $x^* \in \mathrm{Grad}(x)$ if $\|x^*\| = 1$ and $x^*(x) = \|x\|$.

Definition 5.2. We say that $x \in \mathrm{S}(X)$ is *a smooth point* if $\mathrm{Grad}(x)$ is a singleton.

Now we will give some lemmas. Let us recall that $d(x) := \inf_{y \in h^\varphi(\Gamma)} \|x - y\|_\varphi$.

Lemma 5.3. *If $G \in l^\varphi(\Gamma)^*$ is a singular functional, then $|G(x)| \leq \|G\|\, d(x)$ for any $x \in l^\varphi(\Gamma)$.*

Proof. Take any $x \in l^\varphi(\Gamma)$. For any $y \in h^\varphi(\Gamma)$, we have $|G(x)| = |G(x) - G(y)| = |G(x-y)| \leq \|G\|\, \|x - y\|_\varphi$, so $|G(x)| \leq \|G\| \inf_{y \in h^\varphi(\Gamma)} \|x - y\|_\varphi$. □

Lemma 5.4. *Let $x \in \mathrm{S}(l^\varphi(\Gamma))$. If $d(x) < 1$, then any $x^* \in \mathrm{Grad}(x)$ is a regular functional (i.e., the singular part of x^* is equal to 0).*

Proof. Let $x^* = F_v + G \in \mathrm{Grad}(x)$. Assume for the contrary that $\|G\| > 0$. Then, by Lemma 5.3 and Theorem 3.5, we get

$$1 = x^*(x) = F_v(x) + G(x) \leq \|F_v\|\, \|x\|_\varphi + \|G\|\, d(x) < \|F_v\| + \|G\| = \|x^*\|,$$

whence $\|x^*\| > 1$, a contradiction. □

Lemma 5.5. *Let $x \in l^\varphi(\Gamma)$, $\|x\|_\varphi = 1$ and $d(x) = 1$. Then there exist $y, z \in l^\varphi(\Gamma)$ such that $\|y\|_\varphi = \|z\|_\varphi = 1$, $\mathrm{supp}(y) \cap \mathrm{supp}(z) = \emptyset$ and $x = y + z$.*

Proof. Let $x \in l^\varphi(\Gamma)$ be such that $\|x\|_\varphi = 1$ and $d(x) = 1$. Since $d(x) = \theta(x)$ (see Theorem 2.10), so

$$\inf \left\{ \lambda > 0 : \exists_{I \in \Sigma_1} \sum_{i \in \Gamma \setminus I} \varphi\left(\frac{x(i)}{\lambda}\right) < \infty \right\} = 1.$$

Consequently, for every $0 < \lambda < 1$ and every set $I \in \Sigma_1$, we have

$$\sum_{i \in \Gamma \setminus I} \varphi\left(\frac{x(i)}{\lambda}\right) = \infty. \tag{3}$$

Take now any sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < 1$ and $\lambda_n \xrightarrow{n\to\infty} 1$.

We will define by induction a sequence $(I_n)_{n\in\mathbb{N}}$ of sets such that $I_n \in \Sigma_1$ for $n \in \mathbb{N}$, $I_n \cap I_k = \emptyset$ for $n \neq k$ and

$$\sum_{i\in I_n} \varphi\left(\frac{x(i)}{\lambda_n}\right) > 1. \tag{4}$$

Since $\sum_{i\in\Gamma} \varphi\big(\frac{x(i)}{\lambda_1}\big) = \infty$, there exists $I_1 \in \Sigma_1$ such that $\sum_{i\in I_1} \varphi\big(\frac{x(i)}{\lambda_1}\big) > 1$. Having defined the sets I_k $(k = 1, \ldots, n)$ satisfying the above conditions, let us define the set I_{n+1}. Since $I_1 \cup I_2 \cup \ldots \cup I_n \in \Sigma_1$, by (3), we have

$$\sum_{i\in\Gamma\setminus(I_1\cup I_2\cup\ldots\cup I_n)} \varphi\left(\frac{x(i)}{\lambda_{n+1}}\right) = \infty.$$

Hence there exists a set $I_{n+1} \subset \Gamma \setminus (I_1 \cup I_2 \cup \ldots \cup I_n)$, $I_{n+1} \in \Sigma_1$ such that $\sum_{i\in I_{n+1}} \varphi\big(\frac{x(i)}{\lambda_{n+1}}\big) > 1$. Define now elements y and z by

$$y = x\chi_{I_1\cup I_3\cup I_5\cup I_7\cup\ldots} + x\chi_{\Gamma\setminus(I_1\cup I_2\cup I_3\cup\ldots)} \quad \text{and} \quad z = x\chi_{I_2\cup I_4\cup I_6\cup I_8\cup\ldots}.$$

It is clear that $x = y + z$ and $\operatorname{supp}(y) \cap \operatorname{supp}(z) = \emptyset$. We have to show that $\|y\|_\varphi = \|z\|_\varphi = 1$.

Notice that $I_\varphi(y) \leq I_\varphi(x) \leq 1$ and $I_\varphi(z) \leq I_\varphi(x) \leq 1$. Take any $\lambda < 1$. Then there exists $\lambda_k > \lambda$. We can assume without loss of generality that k is an odd number. By (4) we have

$$I_\varphi\left(\frac{y}{\lambda}\right) \geq I_\varphi\left(\frac{y}{\lambda_k}\right) \geq I_\varphi\left(\frac{1}{\lambda_k} x\chi_{I_k}\right) = \sum_{i\in I_k} \varphi\left(\frac{x(i)}{\lambda_k}\right) > 1$$

and

$$I_\varphi\left(\frac{z}{\lambda}\right) \geq I_\varphi\left(\frac{z}{\lambda_{k+1}}\right) \geq I_\varphi\left(\frac{1}{\lambda_{k+1}} x\chi_{I_{k+1}}\right) = \sum_{i\in I_{k+1}} \varphi\left(\frac{x(i)}{\lambda_{k+1}}\right) > 1.$$

Therefore $\|y\|_\varphi = \|z\|_\varphi = 1$. □

Lemma 5.6. *Let $x \in l^\varphi(\Gamma)$, $\|x\|_\varphi = 1$ and* $\operatorname{card}\{i \in \Gamma : |x(i)| = b_\varphi\} \geq 2$. *Then there exist $y, z \in l^\varphi(\Gamma)$ such that $\|y\|_\varphi = \|z\|_\varphi = 1$,* $\operatorname{supp}(y) \cap \operatorname{supp}(z) = \emptyset$ *and $x = y + z$.*

Proof. The proof is analogous as the proof of Proposition 2 in [10] for $\Gamma = \mathbb{N}$. We will present it here for the sake of completeness.

By the assumption, there exist $j, k \in \Gamma$ such that $j \neq k$ and $|x(j)| = |x(k)| = b_\varphi$. Define $y = x\chi_{\{j\}}$, $z = x\chi_{\Gamma\setminus\{j\}}$. We have $\operatorname{supp}(y) \cap \operatorname{supp}(z) = \emptyset$ and $x = y + z$. It is enough to show that $\|y\|_\varphi = \|z\|_\varphi = 1$. Indeed, $I_\varphi(y) \leq I_\varphi(x) \leq 1$, $I_\varphi(z) \leq I_\varphi(x) \leq 1$ and, for any $\lambda > 1$, we have $I_\varphi(y) = I_\varphi(z) = \infty$. □

We have just proven analogues of Lemmas 0.1, 0.2, 0.3 and 0.4 from [2]. On the base of these results necessary and sufficient conditions for smooth points of the Musielak-Orlicz space were given in that article. Next it can be proved in a similar way, an analogue of Theorem 1.1 from that article. Namely, the following theorem holds.

Theorem 5.7. *A point $x \in \mathrm{S}(l^\varphi(\Gamma))$ is a smooth point if and only if:*

1. $d(x) < 1$ *(let us remind that $d(x)$ denote the distance of x to $h^\varphi(\Gamma)$);*
2. *there exists at most one index $i \in \Gamma$ such that $|x(i)| = b_\varphi$;*
3. *if $|x(i)| < b_\varphi$ for any $i \in \Gamma$ then for any $i \in \Gamma$ such that $\varphi(|x(i)|) < 1$ the equality $\varphi'_-(|x(i)|) = \varphi'_+(|x(i)|)$ holds;*
4. *if $|x(i_0)| = b_\varphi$ for some $i_0 \in \Gamma$, then $I_\varphi(x) < 1$ or $\varphi'_-(b_\varphi) = \infty$ or $\varphi'_+(|x(i)|) = 0$ for any $i \neq i_0$.*

From the above theorem we can deduce the next our theorems.

Theorem 5.8. *If φ is an Orlicz function that is smooth at zero, then there exists smooth points on the unit sphere $\mathrm{S}(l^\varphi(\Gamma))$. In particular, smooth points of $\mathrm{S}(l^\varphi(\Gamma))$ are the elements defined by the formula*

$$x = \sup\{u : \varphi(u) \leq 1\} \cdot \chi_{\{k\}},$$

where k is an arbitrary element of Γ.

Proof. Let us fix $k \in \Gamma$ and notice that x defined in the theorem belongs to $h^\varphi(\Gamma)$, that is, the first claim of Theorem 5.7 is satisfied. From the definition of element x, we get immediately that the second claim of Theorem 5.7 is satisfied too.

Consider now two cases:

`Case 1:` $\varphi(b_\varphi) > 1$. Then $I_\varphi(x) = 1$, that is, $\|x\|_\varphi = 1$. If now $|x(i)| < \varphi^{-1}(1)$, then $i \neq k$ and, consequently, $x(i) = 0$. By the assumption that φ is smooth at zero, we get that conditions 3 and 4 of Theorem 5.7 are satisfied. Therefore x is a smooth point.

`Case 2:` $\varphi(b_\varphi) \leq 1$. Since $I_\varphi(x) \leq 1$ and $I_\varphi(x/\lambda) = \infty$ for every $\lambda < 1$, so $\|x\|_\varphi = 1$. Consequently, we have $x(k) = b_\varphi$ and $x(i) = 0$ for $i \neq k$. Since φ is smooth at zero, so $\varphi'_+(x(i)) = 0$ for $i \neq k$, that is, conditions 3 and 4 of Theorem 5.7 are satisfied, and consequently, x is a smooth point. □

When Γ is an uncountable set and φ is an Orlicz function that takes only finite values, then we can reverse the above theorem.

Theorem 5.9. *Let Γ be an uncountable set and φ be an Orlicz function that takes only finite values. If the unit sphere $\mathrm{S}(l^\varphi(\Gamma))$ has smooth points, then φ is smooth at zero.*

Proof. Let $x \in S(l^\varphi(\Gamma))$ be a smooth point. If $a_\varphi > 0$, then φ is automatically smooth at 0. Assume that $a_\varphi = 0$. Then $x(j) = 0$ for some $j \in \Gamma$ (in fact $x(i) = 0$ for all indices outside a set A which is at most countable). Then by the third condition of Theorem 5.7, we get that φ is smooth at zero. □

We will end our considerations concerning smooth points with an example.

Example 5.10. The unit sphere $S(l^1(\Gamma))$ of $l^1(\Gamma)$ has smooth points if and only if the set Γ is countable.

Namely, if $\Gamma = \mathbb{N}$, then smooth points are all the sequences $x \in S(l^1(\mathbb{N}))$ with $\mathrm{supp}(x) = \mathbb{N}$. The support functional at such a point is just $y \in \left(l^1(\mathbb{N})\right)^* = l^\infty(\mathbb{N})$ defined as $y = (\mathrm{sgn}(x_1), \mathrm{sgn}(x_2), \mathrm{sgn}(x_3), \ldots)$, where sgn is the signum function.

Now assume that Γ is an uncountable set. For any $x \in S(l^1(\Gamma))$ there exists $i_0 \in \Gamma$ such that $x(i_0) = 0$. Notice that for any $\alpha \in [-1, 1]$ the element $y_\alpha \in \left(l^1(\Gamma)\right)^* = l^\infty(\Gamma)$ defined by the formula

$$y_\alpha(i) = \begin{cases} \alpha & \text{for } i = i_0 \\ \mathrm{sgn}(x_i) & \text{for } i \neq i_0. \end{cases}$$

is a support functional at this point. Consequently, at each point of the unit sphere there exists infinitely many support functionals, that is, no point of $x \in S(l^1(\Gamma))$ is a smooth point.

6. Extreme points of $l^\varphi(\Gamma)$

In the last section of this article we will present criteria for extreme points of $S(l^\varphi(\Gamma))$.

Definition 6.1. We call $x \in S(X)$ an *extreme point of the unit ball* of a normed space X if for any $y, z \in S(X)$ the equality $x = \frac{1}{2}(y + z)$ implies that $y = z$.

For an Orlicz function φ we define

$$\mathrm{Ext}(\varphi) := \left\{ u \in \mathbb{R} : \underset{w,v \in \mathbb{R}}{\forall} \left(w \neq v \wedge \frac{w+v}{2} = u \right) \Rightarrow \left(\varphi(u) < \frac{\varphi(w) + \varphi(v)}{2} \right) \right\}.$$

We will use now the following result from [7] giving criteria for extreme points in Orlicz spaces L^φ equipped with the Luxemburg norm.

Theorem 6.2 ([7, Theorem 1]). *Let φ be an Orlicz function and (T, Σ, μ) be* any *measure space without atoms with infinite measure.*

(i) *Assume that φ is a continuous Orlicz function. Then $x \in S(L^\varphi)$ is an extreme point if and only if $I_\varphi(x) = 1$ and either*

(a) $x(t) \in \mathrm{Ext}(\varphi)$ *for* μ*-p.w.* $t \in T$ *or*

(b) *there exist an atom* A *such that* $x(t) \in \mathrm{Ext}(\varphi)$ *for* μ*-a.e.* $t \in T \setminus A$ *and* $x|_A = u_0$*, where* $\varphi(u_0) \neq 0$*.*

(ii) *If* φ *is not continuous and* $I_\varphi(x) = 1$*, then* $x \in \mathrm{S}(L^\varphi)$ *is an extreme point if and only if one of the above conditions (a) or (b) holds.*

(iii) *If* φ *is not continuous and* $I_\varphi(x) < 1$*, then* $x \in \mathrm{S}(L^\varphi)$ *is an extreme point if and only* $|x(t)| = b_\varphi$ *for* μ*-p.w.* $t \in T$*.*

In the above theorem L^φ denotes any Orlicz space, that is, an Orlicz space over arbitrary (not necessarily σ-finite) measure space, which can be purely atomic or mixed, but without atoms with infinite measure. We are interested here mainly in $l^\varphi(\Gamma)$. If we additionally assume that the set Γ is infinite, then the above theorem takes for $l^\varphi(\Gamma)$ a particular simple form.

Theorem 6.3. *For any infinite set* Γ*, we have the following:*

(a) *If* $a_\varphi < b_\varphi$*, then* $x \in \mathrm{S}(l^\varphi(\Gamma))$ *is an extreme point if and only if* $I_\varphi(x) = 1$*,* $|x(i)| \geq a_\varphi$ *for any* $i \in \Gamma$ *and*

$$\mu\{i \in \Gamma : |x(i)| > a_\varphi \wedge |x(i)| \notin \mathrm{Ext}(\varphi)\} \leq 1.$$

(b) *If* $a_\varphi = b_\varphi$*, then* $x \in \mathrm{S}(l^\varphi(\Gamma))$ *is an extreme point if and only if* $|x(i)| = a_\varphi$ *for any* $i \in \Gamma$*.*

Note that if the set Γ is finite and $a_\varphi < b_\varphi$, then the condition $I_\varphi(x) = 1$ is not necessary for $x \in \mathrm{S}(l^\varphi(\Gamma))$ to be an extreme point.

As it is seen from the last theorem, in contrast to the criteria for smooth points, if we consider extreme points of $B(l^\varphi(\Gamma))$ the fact if Γ is countable or not is not so essential as for smooth points.

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