Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 27 (2008), 463–467

Superposition Operator in a Space of Infinitely Differentiable Functions

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Abstract. In this paper we prove a degeneration result for the superposition operator in $V(\mathbb{R}^d)$, a particular space of infinitely differentiable functions which have all derivatives uniformly bounded by a constant that does not depend on the order of derivation.

Keywords. Superposition operator, Roumieu space

Mathematics Subject Classification (2000). 47H30, 26E10

1. Introduction

If f is a function defined on the real line, the (nonlinear) *superposition* operator T_f associated with f is defined by $T_f(u) := f \circ u$. Given a space E of real functions, a natural question consists in finding necessary and sufficient conditions on f (called acting conditions) such that $T_f(E) \subseteq E$ (cf. Appell and Zabrejko [2]).

The superposition operator on various spaces is usually studied in view of applications to partial differential equations. In particular, to study some differential equations of infinite order, Ricceri introduced in [5] a certain space $V(\mathbb{R}^d)$ of infinitely differentiable functions. The question about acting conditions in $V(\mathbb{R}^d)$ is stated in [6] as follows.

Problem. *Let d be a positive integer. Denote by $V(\mathbb{R}^d)$ the space of all functions $u \in C^\infty(\mathbb{R}^d)$ such that, for each bounded subset $\Omega \subset \mathbb{R}^d$, one has*

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \Omega} |D^\alpha u(x)| < +\infty,$$

where $D^\alpha u = \frac{\partial^{\alpha_1 + \cdots + \alpha_d} u}{\partial x_1^{\alpha_1} \cdots x_d^{\alpha_d}}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function such that, for each $u \in V(\mathbb{R}^d)$, the composite function $x \to f(u(x))$ belongs to $V(\mathbb{R}^d)$. Then must f be of the form $f(t) = at + b$?

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The aim of this paper is to give a positive answer to it. Note that the space $V(\mathbb{R}^d)$ can be seen as a particular case of the well studied class of *Roumieu* spaces (see [2, page 217]), but known results for the superposition operator in Roumieu spaces ([1, 4, 9]) do not fit with our problem. The author does not know if the proof presented here could be adapted to improve known results for superposition operator in Roumieu spaces.

2. Result

Let us first remember the following formula for the derivatives of the composite function (for a short proof see [7]). The usual k-order derivative of a function f is denoted by $f^{(k)}$.

Lemma 2.1 (Faà de Bruno Formula)**.** *Suppose that v and f are C^∞ real functions on $\mathbb{R}$. Then the derivatives of the composite function $f \circ v$ are given by*

$$(f \circ v)^{(n)}(x) = \sum \frac{n!}{k_1!k_2!\cdots k_n!} f^{(k)}(v(x)) \left(\frac{v^{(1)}(x)}{1!}\right)^{k_1} \left(\frac{v^{(2)}(x)}{2!}\right)^{k_2} \cdots \left(\frac{v^{(n)}(x)}{n!}\right)^{k_n},$$

where $k = k_1 + k_2 + \cdots + k_n$ and the sum is taken over all $k_1, k_2, \ldots, k_n \in \mathbb{N}_0$ for which $k_1 + 2k_2 + \cdots + nk_n = n$.

We note that if a derivative $v^{(i)}(x)$ is zero, then the corresponding index k_i in the formula above can be assumed to be zero. As a corollary, we have the following result.

Corollary 2.2. *Let f be a C^∞ real function on $\mathbb{R}$ and suppose that v is a real polynomial $v(x) := a_0 + a_1x^{i_1} + a_2x^{i_2} + \cdots + a_rx^{i_r}$, where $r \geq 1$ is an integer, $a_0, \ldots, a_r$ are real numbers and $i_1, \ldots, i_r$ are integers with $0 < i_1 < i_2 < \cdots < i_r$. Then*

$$(f \circ v)^{(n)}(0) = \sum \frac{n!}{h_1!h_2!\cdots h_r!} f^{(k)}(a_0)a_1^{h_1}a_2^{h_2}\cdots a_r^{h_r}$$

where $k = h_1 + h_2 + \cdots + h_r$ and the sum is taken over all $h_1, h_2, \ldots, h_r \in \mathbb{N}_0$ for which $i_1h_1 + i_2h_2 + \cdots + i_rh_r = n$.

The next Lemma will be used in the proof of Theorem 2.4.

Lemma 2.3. *Let f and v be as in the previous corollary. For $i > i_r$ integer and $c \in \mathbb{R}$ define $v_i(x) := v(x) + cx^i$. Then*

$$(f \circ v_i)^{(i_1+i)}(0) = (f \circ v)^{(i_1+i)}(0) + ca_1(i_1 + i)!f^{(2)}(a_0).$$

Proof. Corollary above shows that

$$(f \circ v_i)^{(i_1+i)}(0) = \sum \frac{(i_1+i)!}{h_1!h_2!\cdots h_r!h!} f^{(k)}(a_0) a_1^{h_1} a_2^{h_2} \cdots a_r^{h_r} c^h, \tag{1}$$

where $k = h_1 + h_2 + \cdots + h_r + h$ and the sum is taken over all $h_1, h_2, \ldots, h_r \in \mathbb{N}_0$ and all $h \in \mathbb{N}_0$ for which $i_1 h_1 + i_2 h_2 + \cdots + i_r h_r + ih = i_1 + i$. Since $i_1 < i$, condition above forces index h to be 0 or 1. So the sum in (1) can be split in two terms, regarding if $h = 0$ or $h = 1$. Using again the Corollary above we get that the first one, for $h = 0$, is equal to $(f \circ v)^{(i+i_1)}(0)$. The second one, for $h = 1$, is

$$\sum \frac{(i_1+i)!}{h_1!h_2!\cdots h_r!} f^{(k)}(a_0) a_1^{h_1} a_2^{h_2} \cdots a_r^{h_r} c, \tag{2}$$

where $k = h_1 + h_2 + \cdots + h_r + 1$ and the sum is taken over all $h_1, h_2, \ldots, h_r \in \mathbb{N}_0$ for which $i_1 h_1 + i_2 h_2 + \cdots + i_r h_r = i_1$. Since i_1 is smaller than $i_2, \ldots, i_r$, the sum in (2) reduces to the single term $c a_1 f^{(2)}(a_0)(i_1 + i)!$ and this concludes the proof of the lemma. □

Theorem 2.4. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that, for each $u \in V(\mathbb{R}^d)$, the composite function $x \to f(u(x))$ belongs to $V(\mathbb{R}^d)$. Then f is of the form $f(t) = at + b$.*

Proof. Suppose first that $d = 1$.

Since the identity function $id(x) := x$ belongs to $V(\mathbb{R})$, obviously $f = f \circ id$ is a C^∞ function. The theorem follows if we prove that $f^{(2)}(a_0) = 0$ for every fixed $a_0 \in \mathbb{R}$.

Consider the set $\{0, 1\}$ with the discrete topology, and let Q be its numerable product $\{0, 1\}^{\mathbb{N}}$ endowed with the product topology. For each element $q = (q_k)$ of Q define functions $u_q\colon \mathbb{R} \to \mathbb{R}$ by

$$u_q(x) := a_0 + \sum_{k=1}^{\infty} \frac{q_k}{k!} x^k. \tag{3}$$

It is easy to check that functions u_q belong to $V(\mathbb{R})$. Therefore $f \circ u_q$ belong to $V(\mathbb{R})$ and in particular there exist constants M_q such that

$$\sup_{n \ge 1} \left| (f \circ u_q)^{(n)}(0) \right| \le M_q \qquad \forall q \in Q. \tag{4}$$

We want to use the usual Baire argument (in its version for Hausdorff compact spaces, see for example [3, Theorem 7.12]) to find an upper bound $\bar{M}$ in (4) which does not depend on q. For $M \in \mathbb{N}$ put

$$Q_M := \left\{ q \in Q : \sup_{n \ge 1} \left| (f \circ u_q)^{(n)}(0) \right| \le M \right\}$$

so that by (4) we have $\bigcup_{M\in\mathbb{N}} Q_M = Q$. Moreover Q_M are closed sets in Q: observe that

$$Q_M = \bigcap_{n\in\mathbb{N}} T_n^{-1}([-M,M]),$$

where the functions $T_n\colon Q \to \mathbb{R}$ are defined by

$$T_n(q) := (f \circ u_q)^{(n)}(0).$$

The expression $(f\circ u_q)^{(n)}(0)$ depends on first n derivatives of u_q in 0, that is on $q_1,\dots,q_n$, but does not depend on others components of q. This implies that each T_n is continuous and hence that Q_M are closed subsets of Q.

Since Q is a Hausdorff compact set, by the Baire Lemma there exists $\bar M \in \mathbb{N}$ such that $Q_{\bar M}$ has non–empty interior. Therefore there exist two finite disjoint (non-empty) subsets I, J of $\mathbb{N}$ with the following property: if $q \in Q$ is such that $q_k = 1$ for $k \in I$ and $q_k = 0$ for $k \in J$, then $q \in \operatorname{int}(Q_{\bar M})$ and hence

$$\sup_{n\in\mathbb{N}} \left|(f\circ u_q)^{(n)}(0)\right| \le \bar M. \tag{5}$$

Let $\bar n := \max(I \cup J)$ and suppose $I = \{i_1,\dots,i_r\}$ with $0 < i_1 < i_2 < \cdots < i_r$. For $i > \bar n$ define

$$v(x) := a_0 + \frac{x^{i_1}}{i_1!} + \cdots + \frac{x^{i_r}}{i_r!} \quad \text{and} \quad v_i(x) := a_0 + \frac{x^{i_1}}{i_1!} + \cdots + \frac{x^{i_r}}{i_r!} + \frac{x^i}{i!},$$

so that by (5) it follows that

$$\sup_{n\in\mathbb{N}} \left|(f\circ v)^{(n)}(0)\right| \le \bar M \quad \text{and} \quad \sup_{n\in\mathbb{N}} \left|(f\circ v_i)^{(n)}(0)\right| \le \bar M.$$

In particular, choosing $n = i_1 + i$, we get

$$\left|(f\circ v_i)^{(i_1+i)}(0) - (f\circ v)^{(i_1+i)}(0)\right| \le 2\bar M. \tag{6}$$

We may apply Lemma 2.3, with $c = \frac{1}{i!}$, to find

$$|f^{(2)}(a_0)| \le 2\bar M \frac{i_1!\, i!}{(i_1+i)!},$$

and considering the limit for $i \to +\infty$ we obtain as required that $f^{(2)}(a_0) = 0$.

If $d > 1$ we can repeat the same argument, just considering functions $\tilde u_q(x) = u_q(x_1)$ which depend only on first component x_1 of $x = (x_1,\dots,x_d)$ and using partial derivatives $\frac{\partial^n (f\circ \tilde u_q)}{\partial x_1^n}$ instead of ordinary derivatives $(f\circ u_q)^{(n)}$. $\square$

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Received August 2, 2006