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Uniform Rectifiability from Mean Curvature Bounds

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Abstract. A version of Allard's rectifiability theorem with explicit bounds is given. The condition on the mean curvature would correspond to a bound in Sobolev spaces with fractional negative exponents.

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1. Introduction

The present paper deals with an extension of Allard's famous rectifiability theorem. The way Allard's theorem is stated, it deals with varifolds whose density is bounded from below and whose first variation is a vector valued measure. Recall that

Definition 1. A d-dimensional varifold μ in an open domain $\Omega \subset \mathbb{R}^n$ is a Radon measure on $\Omega \times G_d(n)$, where $G_d(n) = \{P \in \mathbb{R}^{n\times n} \mid P = P^2 = P^t, Tr(P) = d\}$.

And the first variation or mean curvature is given by

Definition 2. The first variation H_μ of a varifold μ is defined by

$$H_\mu(\xi) = \int Tr(PD\xi(x))d\mu(x,P) \quad \text{for } \xi \in C_0^1(\Omega, \mathbb{R}^n),$$

where $D\xi$ denotes the derivative of ξ.

For classical d dimensional surfaces Allard's theorem entails, that from (local) bounds on the d-dimensional measure and the L_1 norm of the mean curvature, compactness of the tangent planes w.r.t. convergence in measure follows. I.e., there exists for arbitrary $\varepsilon > 0$ an abstract modulus of continuity ω_ε of the tangent planes outside of a set of measure ε.

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The present paper gives a short proof of this result using no more geometric measure theory than the Vitali covering lemma, and it extends the result in two ways. It makes the modulus of continuity and the estimate on the exceptional set explicit. And it allows for mean curvature estimates which are weaker than L^1. The interest in the latter comes from free boundary problems with surface tension effects, e.g. the Stefan Gibbs Thompson problem. For these the functional whose first variation is taken, has a bulk and a surface term. From the bulk term one gets terms which still are integrals of $D\xi$ rather than ξ, but these integrals are with respect to a $d+1$ dimensional measure.

The main tool in the proof of the main result of this paper, Theorem 1 below, is the monotonicity formula, just as in Allard's paper [1]. The simplifying trick used here is integration of the monotonicity formula w.r.t. its base point.

Further, we use the following notation:

a) For a d-dimensional varifold μ, we will use the representation

$$\int \psi(x,P)d\mu = \int \left(\int \psi(x,P)d\mu_x(P) \right) d|\mu|(x)$$

for $\psi \in C_0(\Omega \times G_d(n))$, where μ_x is a probability measure on $G_d(n)$ depending on x in a weakly measureable way.

b) δ_p will denote the Dirac measure concentrated at p.

c) H^d denotes the d-dimensional Hausdorff measure.

2. Main result

The main result is an explicit estimate on the measure of the exceptional set, where the logarithmic continuity of the tangent plane is violated.

Theorem 1. *Suppose μ is a d-dimensional varifold in Ω an open domain in $\mathbb{R}^n$. Suppose the following assumptions are satisfied:*

A1. *The density of $|\mu|$ is bounded below, i.e., there exists $\Theta > 0$ such that for $|\mu|$ a.a. $x \in \Omega$*

$$\Theta \leq \limsup_{\rho \to 0} \rho^{-d} |\mu| \left(B_\rho(x) \right).$$

A2. *There exist a Radon measure ν, a vector valued density v, and an $\mathbb{R}^{n\times n}$ valued density A with $|v| \leq 1, |A| \leq 1$ such that the first variation of μ satisfies the identity*

$$H_\mu(\xi) = \int_\Omega \xi \cdot v d\nu + \int_\Omega Tr(AD\xi)d\nu \quad \text{for } \xi \in C_0(\Omega, \mathbb{R}^n). \qquad (1)$$

A3. *The $\mathbb{R}^{n\times n}$ valued density A satisfies the inequality*

$$\rho^{-d-1}\int_{B_\rho(x)}|A|\,d\nu \le \partial_1 F\Big(\rho, \sup_{\rho<R<\mathrm{dist(x,\partial\Omega)}} R^{-d}\nu\big(B_R(x)\big)\Big) \tag{2}$$

for $x\in\Omega$, $0<\rho<\mathrm{dist(x,\partial\Omega)}$, with a function $F:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ satisfying

$$\left.\begin{aligned}&\partial_2 F\ge 0,\ \ \partial_1 F\ge 0,\ \ \partial_1^2 F\le 0\\ &\lim_{\rho\to 0}F(\rho,L)=0 \qquad \text{for all } L\\ &\lim_{L\to\infty}L^{-1}\inf\big\{R^{-d}+F(R,L)\mid R>0\big\}=0\,.\end{aligned}\right\} \tag{3}$$

Then one has for $|\mu|$ a.a. $x\in\Omega$, $\mu_x=\delta_{P_x}$. And for $\epsilon>0$ arbitrary there exist K_ϵ with $|\mu|(K_\epsilon)\le\epsilon$ such that, if $x_1,x_2\notin K_\epsilon, \mathrm{dist(x_i,\partial\Omega)>R_0}$,

$$|P_{x_1}-P_{x_2}|\le\omega_F\big(\epsilon,R_0,\nu(\Omega),|\mu|(\Omega)\big)\,\Big|\ln^{\frac14}|x_1-x_2|\Big|^{-1},$$

where the constant $\omega_F(\epsilon,R_0,\nu(\Omega),|\mu|(\Omega))$ can be given explicitly.

3. Proof

3.1. Structure of the proof. Outside of a set of "exceptionally large mean curvature" $K_{1(L)}$, characterized in terms of the measure ν, the monotonicity formula for $|\mu|$ is used. The formula is integrated with respect to radius and base point. This yields an estimate of the form

$$\int_{\hat\Omega\setminus K_{1(L)}}\int_{\hat\Omega}|x-y|^{-d}\left|(Id-P)\frac{x-y}{|x-y|}\right|^2 d\mu\,d|\mu|, \quad \text{for } \hat\Omega\subset\subset\Omega.$$

Since $|x-y|^{-d}$ is not integrable by the assumption on the density of $|\mu|, \mu_x$ has to be a Dirac measure δ_{P_x} and the continuity estimate for P_x can be derived.

3.2. Proof of Theorem 1.

Step 1: the monotonicity formula. The monotonicity estimate is used for $u(\rho,x)=\rho^{-d}\int\varphi\big(\frac{|x-y|}{\rho}\big)d|\mu|(y)$ where $\varphi'\le 0$; $\varphi(s)=1$ for $s\le\frac12$; $\varphi(s)=0$ for $s\ge 1$ and $|\varphi'|\le 3$. One derives

$$\begin{aligned}\partial_\rho u(\rho,x)&=-d\rho^{-d-1}\int\varphi\left(\frac{|x-y|}{\rho}\right)d|\mu|(y)\\ &\quad-\rho^{-d-1}\int\varphi'\left(\frac{|x-y|}{\rho}\right)\frac{|x-y|}{\rho}\,d|\mu|(y)\end{aligned}$$

$$= - \int Tr\left(PD\left(\frac{x-y}{\rho^{d+1}}\varphi\left(\frac{|x-y|}{\rho}\right)\right)\right) d\mu(P,y)$$
$$- \rho^{-d-1}\int \varphi'\left(\frac{|x-y|}{\rho}\right)\frac{|x-y|}{\rho}\left[1-\left|P\frac{x-y}{|x-y|}\right|^2\right] d\mu(P,y).$$

And using Assumption A1 one gets

$$\partial_\rho u(\rho,x) \geq - \rho^{-d-1}\int \frac{|x-y|}{\rho}\varphi'\left(\frac{|x-y|}{\rho}\right)\left|(Id-P)\frac{x-y}{|x-y|}\right|^2 d\mu(P,y)$$
$$- \rho^{-d-1}\int (x-y)v\varphi\left(\frac{|x-y|}{\rho}\right) d\nu$$
$$- \rho^{-d-1}\int Tr\left(AD\left[(x-y)\varphi\left(\frac{|x-y|}{\rho}\right)\right]\right) d\nu.$$

Now with the Assumption A3 (2) one can estimate, either

$$\partial_\rho\Big(u(\rho,x)+F(\rho,L)(d+3)+\rho L\Big)$$
$$\geq -\rho^{-d-1}\int \frac{|x-y|}{\rho}\varphi'\left(\frac{|x-y|}{\rho}\right)\left|(Id-P)\frac{x-y}{|x-y|}\right|^2 d\mu(P,y)$$

or there exists $\rho \leq \tilde\rho \leq \mathrm{dist(x,\partial\Omega)}$ with $\nu(B_{\tilde\rho}(x)) \geq L\tilde\rho^d$. Define R_L by the equation $(d+3)F(R_L,L)+R_L L = R_L^{-d}$ then, using the monotonicity of F in R:

$$R_L^{-d} \leq \inf\left\{(d+3)F(R,L)+RL+R^{-d}, R>0\right\}$$
$$\leq (d+3)\inf\left\{F(R,L)+R^{-d}, R>0\right\} + \inf\left\{RL+R^{-d}, R>0\right\}$$
$$= (d+3)\inf\left\{F(R,L)+R^{-d}, R>0\right\} + \frac{d+1}{d}\left(\frac{L}{d}\right)^{\frac{d}{d+1}}.$$

So by Assumption A3 (3), $R_L < L^{-1}R_L^{-d} \to 0$ as $L\to\infty$. Define $K_1(L)$ by

$$K_1(L) = \left\{x \,|\, \mathrm{dist(x,\partial\Omega) \geq R_L,\ \limsup_{\rho\to 0} u(\rho,x) < \tfrac{\Theta}{2}}\right\}$$
$$\cup \left\{x \,\middle|\, \mathrm{dist(x,\partial\Omega)\geq R_L,\ \sup_{\Theta<R<dist(x,\partial\Omega)} R^{-d}\nu(B_R(x)) > L}\right\}.$$

Then if $x \notin K_1(L), \rho \leq \mathrm{dist(x,\partial\Omega), dist(x,\partial\Omega) \geq R_L}$,

$$u(\rho,x)+(d+3)F(\rho,L)+\rho L \geq \frac{\Theta}{2}$$
$$\partial_\rho\left(u(\rho,x)+(d+3)F(\rho,L)+\rho L\right) \geq - \rho^{-d-1}\int \frac{|x-y|}{\rho}\varphi'\left(\frac{|x-y|}{\rho}\right)$$
$$\times\left|(Id-P)\frac{x-y}{|x-y|}\right|^2 d\mu(P,y),$$

and for $|\mu|$ a.a. $x \in K_1(L)$ there exists

$$\rho_x = \max\left\{\rho \mid \rho^{-d}\nu(B_\rho(x)) \geq L, \rho \leq \mathrm{dist(x,\partial\Omega)}\right\}.$$

If $10\rho_x \geq R_L$, then $|\mu|(B_{5_{\rho_x}}(x)) \leq \frac{|\mu|(\Omega)}{LR_L^d}\nu(B_{\rho_x}(x))$. If $10\rho_x \leq R_L \leq \mathrm{dist(x,\partial\Omega)}$, then

$$|\mu|(B_{5\rho_x}(x)) \leq (10\rho_x)^d u(10\rho_x,x) \leq (10\rho_x)^d(u(R_L,x)+R_L L+(d+3)F(R_L,L)).$$

In the latter case,

$$|\mu(B_{5\rho_x}(x)) \leq (10\rho_x)^d R_L^{-d}(|\mu|(\Omega)+1) \leq 10^d\frac{|\mu|(\Omega)+1}{LR_L^d}\nu(B_{\rho_x}(x)).$$

To sum up one has the following

Lemma 1. *For L arbitrary, R_L defined by $(d+3)F(R_L,L)+R_L L = R_L^{-d}$ there exists $K_1(L) \subset \Omega$ with*

a) $|\mu|(K_1(L)) \geq 10^d\nu(\Omega)(|\mu|(\Omega)+1)L^{-1}R_L^{-d}$,

and if $\mathrm{dist(x,\partial\Omega)} \leq \mathrm{R_L}$ *for $x \notin K_1(L)$, then for $\rho \leq \mathrm{dist(x,\partial\Omega)}$,*

b) $u(\rho,x)+(d+3)F(\rho,L)+L\rho \geq \dfrac{\Theta}{2}$;

c) $\partial_\rho\big(u(\rho,x)+(d+3)F(\rho,L)+L\rho\big)$
$$\geq -\rho^{-d-1}\int \frac{|x-y|}{\rho}\varphi'\left(\frac{|x-y|}{\rho}\right)\left|(Id-P)\frac{x-y}{|x-y|}\right|^2 d\mu(P,y).$$

Moreover $R_L \leq L^{-1}R_L^{-d} \to 0$ as $L \to \infty$.

Proof. We use Vitali's covering theorem to cover $K_1(L)$ by balls $B_{5\rho_{x_i}}(x_i)$ such that $B_{\rho_{x_i}}(x_i)$ are disjoint. □

With the same methods we can prove another lemma whose use will become apparent later. Rougly speaking it is a bound from below on the right hand side in statement c) in the case that supp$(|\mu|)$ is not locally flat.

Lemma 2. *Define $K_2(L) = \left\{x \mid \exists_{\rho>0}\ |\mu|(B_\rho(x)\cap K_1(L)) > \epsilon(d)|\mu|(B_{5\rho}(x)\right\}$. Then:*

a) $|\mu|(K_2(L)) > \dfrac{1}{\epsilon(d)}|\mu|(K_1(L))$,

and with $\epsilon(d)$ specified below, for any $\Pi \in G_{d-1}^{(n)}$, $x \notin K_2(L)$, $B_{6R_L}(x) \subset \Omega$, $\rho < R < R_L$,

b) $$\int_{(B_R(x)\setminus B_\rho(x))\setminus K_1(L)}\left|(Id-\Pi)\frac{y-x}{|y-x|}\right|^2|y-x|^{-d}d|\mu|(y)$$
$$\geq \Theta c_1(d)\cdot\ln\left(\frac{R}{\rho}\right) - c_2(d,L).$$

Here $\epsilon(d) = \frac{1}{4}5^{-d}$, $c_1(d) = d5^{-2d}2^{-d-9}$, $c_2(d,L) = 2\left(\frac{\Theta}{8r_0}+L\right)R_L + 2^{d+2}|\mu|(\Omega)R_L^{-d}$, *where* $F(r_0, L) = \frac{\Theta}{8(d+3)}$.

Proof. To prove a) cover $K_2(L)$ by balls $B_{\xi\rho_i}(x_i)$ such that $B_{\rho_i}(x_i)$ are disjoint and $|\mu|B_{\rho_i}(x_i) \cap K_1(L_1) \geq \epsilon(d)\mu(B_{5\rho_i}(x_i))$. Then

$$|\mu|\left(K_2(L)\right) \leq \sum |\mu|\left(B_{5\rho_i}(x_i)\right) \leq \frac{1}{\epsilon(d)} \sum |\mu|(B_{\rho_i}(x_i) \cap K_1(L)) \leq \frac{1}{\epsilon(d)} |\mu|(K_1(L)).$$

To prove b) set $\psi(r) = |\mu|((B_r(x)\backslash K_1(L)) \cap \{|(Id - \Pi)(y-x)| \geq \delta(d)|y-x|\}$, then with $\delta(d)$ to be specified later

$$\begin{aligned}\int_{(B_R(x)\backslash B_\rho(x))\backslash K_1(L)} &\left|(Id - \Pi)\frac{y-x}{|y-x|}\right|^2 |y-x|^{-d} d|\mu|y \\ &\geq \delta(d)^2 \int_\rho^R r^{-d} d\psi(r) \\ &\geq d\delta(d)^2 \int_\rho^R r^{-d-1}\psi(r)dr - \delta(d)^2\rho^{-d}\psi(\rho).\end{aligned}$$

In order to estimate $\psi(r)$ from below cover

$$(B_r(x)\backslash K_1(L)) \cap \left\{y|(Id - \Pi)(y-x)| < \delta(d)(y-x)\right\} =: \hat{B}_r(x)$$

by balls $B_{2\sqrt{2}\delta(d)r}(y_i)$ such that $y_i \in \hat{B}_r(x)$ and $B_{\sqrt{2}\delta(d)r}(y_i)$ are disjoint. □

The number of y_i is estimated by $(I + \delta(d))^{d-1}\delta(d)^{1-d}$, and by Lemma 1

$$\begin{aligned}|\mu|\left(B_{2\sqrt{2}\delta(d)r}(y_i)\right) &< \left(4\sqrt{2}\delta(d)r\right)^d u\left(4\sqrt{2}\delta(d)r, y_i\right) \\ &\leq \left(4\sqrt{2}\delta(d)r\right)^d (u(4r), y_i) + (d+3)F(4r, L) + L(4r) \\ &\leq \left(4\sqrt{2}\delta(d)r\right)^d \Big(|\mu|(B_{5r}(x))(4r)^{-d} + (d+3)F(4r,L) + 4rL\Big).\end{aligned}$$

So

$$\begin{aligned}|\mu|\left(\hat{B}_r(x)\right) \leq &\left(\sqrt{2}\right)^d \delta(d)\left(1+\delta(d)\right)^{d-1} |\mu|(B_{5r}(x)) \\ &+ \left(4\sqrt{2}\right)^d \delta(d)\left(1+\delta(d)\right)^{d-1}\left[(d+3)F(4r,L)r^d + 4Lr^{d+1}\right].\end{aligned}$$

Now choose $\delta(d) = \left(\frac{1}{5\sqrt{2}}\right)^d \frac{1}{8}$, so

$$|\mu|\left(\hat{B}_r(x)\right) \leq \frac{1}{4}5^{-d}|\mu|(B_{5r}(x)) + \left(\frac{4}{5}\right)^d \frac{1}{4}\Big[(d+3)F(4r,L)r^d + 4Lr^{d+1}\Big].$$

Using the concavity of F for any r_0 one can estimate further

$$
\begin{aligned}
&|\mu|\left(\hat{B}_r(x)\right)\\
&\leq \frac{1}{4}5^{-d}|\mu|(B_{5r}(x))\left(\frac{4}{5}\right)^d\left((d+3)\frac{F(r_0,L)}{r_0}+L\right)r^{d+1}+\left(\frac{4}{5}\right)^d\frac{d+3}{4}F(r_0,L)r^d.
\end{aligned}
$$

That means:

$$
\begin{aligned}
\int_\rho^R r^{-d-1}\psi(r)dr \geq \int_\rho^R r^{-d-1}\Bigg[&|\mu|(B_r(x))-\left(\epsilon(d)+\frac{1}{4}5^{-d}\right)|\mu|(B_{5r}(x))\\
&-\left(\frac{4}{5}\right)^d\left((d+3)\frac{F(r_0,L)}{r_0}+L\right)-\left(\frac{4}{5}\right)^d\frac{d+3}{4}F(r_{0i}L)r^{-1}\Bigg].
\end{aligned}
$$

Choosing $\epsilon(d)=\frac{1}{4}\,5^{-d}$,

$$
\begin{aligned}
\int_\rho^R r^{-d-1}\psi(r) \geq \frac{1}{2}\int_\rho^R r^{-d-1}|\mu|(B_r(x))-\frac{1}{2}\int_R^{5R} r^{-d-1}|\mu|(B_r(x))\\
-\left(\frac{4}{5}\right)^d\left((d+3)F(r_0,L)\left(\frac{R}{r_0}+\frac{1}{4}\ln R\right)+LR\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
r^{-d-1}|\mu|(B_r(x)) &\geq r^{-1}\frac{\Theta}{2}-(d+3)F(r,L)r^{-1}-L\\
&\geq r^{-1}\left(\frac{\Theta}{2}-(d+3)F(r_0,L)\right)\frac{(d+3)F(r_0,L)+L}{r_0},
\end{aligned}
$$

again using Lemma 1. Finally

$$
\begin{aligned}
\int_\rho^R r^{-d-1}\psi(r) \geq \left[\frac{\Theta}{4}-(d+3)F(r_0,L)\right]\ln\frac{R}{\rho}\\
-2\left((d+3)\frac{F(r_0,L)}{r_0}+L\right)R-52^{d-1}\frac{|\mu|(\Omega)}{|\mathrm{dist(x},\partial\Omega)|^{\mathrm{g}}}
\end{aligned}
$$

and $\rho^{-d}\psi(\rho)\leq 2^d\frac{|\mu|(\Omega)}{(\mathrm{dist(x},\partial\Omega))}d$. Choose $(d+3)F(r_0,L)=\frac{\Theta}{8}$ and the result follows.

Step 2: integration of the monotonicity formula w.r.t. its base point. Take L fixed R_L as in Lemma 1, $R_L < R_0$ define

$$
\Omega_{R_0}=\{x\in\Omega \mid \mathrm{dist(x},\partial\Omega)\geq \mathrm{R_0}\}.
$$

Integrate the monotonicity formula Lemma 1 c) over $x \in \Omega_{R_0} \backslash K_1(L)$ w.r.t. $|\mu|$ and over ρ from 0 to R_0. That gives

$$\begin{aligned}
\Big(R_0^{-d}|\mu|(\Omega) + (d+3)F(R_0,L) + R_0 L\Big)|\mu|(\Omega) \\
\geq -\int_0^{R_0} \Bigg(\int_{\Omega_{R_0}\backslash K_1(L)} \Bigg(\int_\Omega \Bigg(\int_{G_d(n)} \frac{|x-y|}{\rho}\, \varphi' \left(\frac{|x-y|}{\rho}\right) \\
\times \left|(Id-p)\frac{x-y}{|x-y|}\right|^2 d\mu_y(P)\Bigg) d|\mu|(y)\Bigg) d|\mu|(x)\Bigg)\rho^{-d-1} d\rho \\
= \int_{\Omega_{R_0}\backslash K_1(L)} \Bigg(\int_\Omega \Bigg(\int_{\frac{|x-y|}{R_0}}^\infty \sigma^d |\varphi'| \left(\sigma^{-1}\right) d\sigma\Bigg) |x-y|^{-d} \\
\times \left[\int_{G_d(n)} \left|(Id-P)\frac{x-y}{|x-y|}\right|^2 d\mu_y(P)\right] d|\mu|(y)\Bigg) d|\mu|(x).
\end{aligned}$$

Interchanging the order of integration one gets

$$\begin{aligned}
c_\varphi \int_\Omega \Bigg(\int_{G_d(n)} \Bigg(\int_{B_{R_{0/2}}(y)\cap\Omega_{R_0}\backslash K_1(L)} \left|(Id-P)\frac{x-y}{|x-y|}\right|^2 \\
\times |x-y|^{-d} d|\mu|(x)\Bigg) d\mu_y(P)\Bigg) d|\mu|(y) \\
\leq \Big(R_0^{-d}|\mu|(\Omega) + (d+3)F(R_0,L) + R_0 L\Big)|\mu|(\Omega),
\end{aligned}$$

where $c_\varphi = \int_0^\infty \sigma^d |\varphi'|(\sigma^{-1}) d\sigma$. Define

$$\begin{aligned}
K_3(M) \\
= \Bigg\{ y \,\Bigg|\, \int_{G_d(n)} \int_{B_{R_{0\backslash 2}}(y)\cap\Omega_{R_0}\backslash K_1(L)} |x-y|^{-d} \left|(Id-P)\frac{x-y}{|x-y|}\right|^2 d|\mu|(x) d\mu_y(P) > M \Bigg\}.
\end{aligned}$$

Then $|\mu|(K_3(M)) < \frac{1}{M}|\mu|(\Omega)\big(R_0^{-d}|\mu|(\Omega) + (d+3)F(R_0,L) + R_0 L\big)$.

Let us summarise the result of the second step in a lemma.

Lemma 3. *Outside of a set $K_3(M)$ with measure*

$$|\mu|\big(K_3(M)\big) < \frac{1}{M}|\mu|(\Omega)\Big(R_0^{-d}|\mu|(\Omega) + (d+3)F(R_0,L) + R_0 L\Big)$$

one has

$$\int_{G_d(n)} \int_{B_{R_{0\backslash 2}}(x)\backslash K_1(L)} |x-y|^{-d} \left|(Id-P)\frac{x-y}{|x-y|}\right|^2 d|\mu|(y)\, d\mu_x(P) < M.$$

Step 3: proof of the theorem. With the help of a linear algebra lemma it is possible to combine the estimate from below in Lemma 2 with the one from above in Lemma 3. Let us state the lemma first:

Lemma 4. *For $P_1, P_2 \in G_d(n)$ one has a Π in $G_{d-1}(n)$ such that for $x \in \mathbb{R}^n$ arbitrary*

$$\big|(Id-\Pi)x\big|^2 \le \left(1+\frac{2}{|P_1-P_2|^4}\right)\Big[|(Id-P_1)x|^2+|(Id-P_2)x|^2\Big].$$

Proof. Choose $x_1, |x_1| = 1, |(P_1-P_2)x_1| = |P_1-P_2|$. It follows that $(P_1-P_2)x_1 = \pm|P_1-P_2|x_1$ and we may assume $w\log$ that $P_1x_1 \neq 0$. Define Π by $\Pi x = P_1x - \frac{\langle x,P_1x_1\rangle}{|P_1x_1|^2}P_1x_1$, then $\Pi \in G_{d-1}(n)$. We have $|x-\Pi x|^2 = |x-P_1x|^2 + \langle x,P_1x_1\rangle^2|P_1x_1|^{-2}$ and

$$\begin{aligned}
\langle x,P_1x_1\rangle^2 &= \langle P_1x,x_1\rangle^2\\
&= \langle P_1x_1(Id-P_2)x_1\rangle^2|P_1-P_2|^{-2}\\
&= |P_1-P_2|^{-4}\langle P_1x,(Id_2)P_1x_1\rangle^2\\
&= |P_1-P_2|^{-4}\langle (Id-P_2)x-(Id-P_2)(Id-P_1)x,P_1x_1\rangle^2\\
&\le 2\left[\big|\big(Id-P_2\big)x\big|^2+\big|\big(Id-P_1\big)x\big|^2\right]|P_1x_1|^2|P_1-P_2|^{-4}. \qquad \square
\end{aligned}$$

To finish the proof of the theorem, suppose now $x_1, x_2 \notin K_2(L) \cup K_3(M)$, $|x_1-x_2| = \rho$, $\frac{x_1+x_2}{2} = x$. Then

$$\begin{aligned}
M &\ge \int_{G_d(n)} d\mu_{x_i}(P_i)\int_{B_R(x_i)\setminus K_1(L)}\Big|(Id-P_i)(y-x_i)\Big|^2|y-x_i|^{-d-2}d|\mu|\\
&\ge \int_{G_d(n)} d\mu_{x_i}(P_i)\int_{(B_{R-\rho/2}(x)\setminus B_\rho(x))\setminus K_1(L)}\Big|(Id-P_i)(y-x_i)\Big|^2(2|y-x|)^{-d-2}d|\mu|\\
&\ge \int_{G_d(n)} d\mu_{x_i}(P_i)\int_{(B_{R-\rho/2}(x)\setminus B_\rho(x))\setminus K_1(L)}\Big|(Id-P_i)(y-x)\Big|^2(2|y-x|)^{-d-2}d|\mu|\\
&\quad -\int_{B_{R-\rho/2}(x)\setminus B_{\rho(x)}}\left(\frac{\rho}{2}\right)^2(2|y-x|)^{-d-2}d|\mu|.
\end{aligned}$$

The last integral is estimated by

$$\begin{aligned}
&\int_{B_R(x)\setminus B_\rho(x)}\left(\frac{\rho}{2}\right)^2(2|y-x|)^{-d-2}\,d|\mu|\\
&\qquad = 2^{-d}\rho^2\int_\rho^R \tau^{-d-2}d|\mu|(B_\tau(x))\\
&\qquad \le 2^{-d}\rho^2R^{-d-2}|\mu|(B_R(x)) + 2^{-d}\rho^2\int_\rho^R(d+2)\tau^{-d-3}|\mu|(B_\tau(x))d\tau\\
&\qquad \le 2^{-d}\rho^2R^{-d-2}|\mu|(B_R(x)) + 4(d+2)\rho^2\int_{2\rho}^{2R}\tau^{-3}u(\tau,x)d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le 2^{-d}\rho^2 R^{-d-2}|\mu|(B_R(x)) \\
&\quad + 2(d+2)\frac{\rho^2}{(2\rho)^2}\big(u(2R,x) + (d+3)F(R,L) + RL\big) \\
&\le (d+2)\left(\frac{(\mu(\Omega))}{(2R)^d} + (d+3)F(R,L) + RL\right).
\end{aligned}
$$

Combining this with Lemmas 2 and 4 one gets

$$
\begin{aligned}
&\big(2^{d+2}+1\big)M + (d+2)\bigg(\frac{|\mu|(\Omega)}{(2R)^d} + (d+3)F(R,L) + RL\bigg) \\
&\ge \int_{G_d(n)\times G_d(n)} d\mu_{x_1} \times d\mu_{x_2}\left(\frac{|P_1-P_2|^4}{|P_1-P_2|^4+2}\right)\big(\Theta c_1(d)\ln(R/\rho) - c_2(d,L)\big).
\end{aligned}
$$

This implies directly that μ_x is a Dirac measure if $x \notin K_2(L) \cup K_3(M)$.

And secondly,

$$
\begin{aligned}
\Big|P_1 - P_2\Big|^4 \le \left|\ln\frac{1}{|x_1-x_2|}\right|^{-1}\frac{2+2^4}{\Theta c_1(d)}\bigg(&(2^{d+2}+1)M - \Theta c_1(d)\ln(R_L) \\
&+ c_2(d,L) + (d+2)\left(\frac{|\mu|(\Omega)}{2R_L^d} + (d+3)F(R_L,L) + R_L L\right)\bigg).
\end{aligned}
$$

That proves the theorem. □

4. Further conclusions

Rectifiability is now an easy conclusion of the result stated in Theorem 1. First we state a lemma identifying the projections on almost tangent planes, in order to be able to quote standard results from the literature.

Lemma 5. *With the definition of $K_3(M)$ as in Lemma 3 one has: If $x \notin \bigcap_{M\in\mathbb{R}^+} K_3(M)$, then for $\delta > 0$ arbitrary*

$$
\lim_{\rho\to 0}\rho^{-d}\,|\mu|\,\left(\left\{y\;\middle|\;|y-x|<\rho,\ \big|(\mathrm{Id}-P_x)\,(y-x)\big| > \delta\rho\right\}\right) = 0\,.
$$

Proof. This follows from

$$
\begin{aligned}
&\sum (2^{-n})^{-d}\,|\mu|\,\left(\left\{y\;\middle|\;|y-x|<2^{-n},\ \big|(\mathrm{Id}-P_x)\,(y-x)\big| > \delta 2^{-n}\right\}\right) \\
&\le \frac{1}{1-2^{-d}}\,\delta^{-d}\int |x-y|^{-d}\,\big|(\mathrm{Id}-P_x)\,|y-x|\big|^2 d|\mu|\,.
\end{aligned}
$$

□

Remark. So P_x is an almost tangent space for $|\mu|$ a.a. x. Note though that Lemma 5 is not an explicite estimate. As a conclusion one has with standard results from [7, Theorem 3.8.3].

Theorem 2. *Suppose for the d-dimensional varifold μ that its density is positive, i.e.,*

$$0 < \limsup_{\rho \to 0} \rho^{-d} |\mu| \ (B_\rho(x)) \qquad \textit{for } |\mu| \ \textit{a.a. } x,$$

and that the assumptions (1), (2), (3) *from Theorem* 1 *hold. Then μ is rectifiable.*

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