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Global Smooth Solutions of Viscous Compressible Real Flows with Density-Dependent Viscosity

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Abstract. We consider an initial boundary problem of viscous, compressible, heat-conducting real fluids with density-dependent viscosity. More precisely, we assume that the viscosity $\mu(\rho) = \rho^\lambda$, where ρ is the density of flows and λ is a positive constant. The equations of state for the real flows depend nonlinearly upon the temperature and the density unlike the linear dependence for the perfect flows. We prove the global existence (uniqueness) of smooth solutions under the hypotheses: $\lambda \in \left(2(\gamma - 1), \frac{1}{2}\right]$ and $1 \leq \gamma < \frac{5}{4}$, which improves a previous result. In particular, we also show that no vacuum will be developed provided the initial density is far away from vacuum.

Keywords. Viscous, heat-conducting gas, density-dependent viscosity, global existence

Mathematics Subject Classification (2000). Primary 76N10, secondary 76D05, 76N15

1. Introduction

In this paper, we are concerned with viscous, compressible, heat-conducting real fluids confined to a fixed tube with impermeable ends. We will study the global existence of smooth solutions with density-dependent viscosity. The governing equations of one-dimensional gas (under Lagrangian coordinate) are the following three ones for the conservation of mass, the balance of momentum

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and the balance of energy respectively, see [4–6, 8, 10, 11]:

$$u_t - v_x = 0 \tag{1a}$$

$$v_t + P_x(u,\theta) = \left(\frac{\mu(u)v_x}{u}\right)_x \tag{1b}$$

$$\left(e + \frac{v^2}{2}\right)_t + (vP(u,\theta))_x = \left(\frac{\mu(u)vv_x}{u} + \frac{k\theta_x}{u}\right)_x, \quad 0 < x < 1,\ t > 0, \tag{1c}$$

with the initial conditions

$$(u, v, \theta)(x, 0) = (u_0(x), v_0(x), \theta_0(x)), \quad \text{on } [0,1], \tag{2}$$

and the boundary conditions

$$v(d,t) = \theta_x(d,t) = 0, \quad d = 0, 1. \tag{3}$$

Here the unknown functions $u = \frac{1}{\rho}$, v, θ represent the specific volume, the velocity and the temperature of the flows, respectively. $e = e(u,\theta)$, $P = P(u,\theta)$ denote the specific internal energy and the pressure, respectively, and satisfy the relation interrelated by

$$e_u(u,\theta) = -P(u,\theta) + \theta P_\theta(u,\theta) \tag{4}$$

by the second law of thermodynamics. $\mu(u) = u^{-\lambda}$ with $2(\gamma-1) < \lambda \le \frac{1}{2}$ is the viscosity coefficient of the real flows and $k = k(u,\theta)$ is the heat conductivity coefficient depending nonlinearly upon the temperature, refer to (7a). We assume that $e(u,\theta)$ and $P(u,\theta)$ are continuously differentiable and $k(u,\theta)$ is twice continuously differentiable on $0 < u < \infty$ and $0 \le \theta < \infty$. The growth conditions on $e(u,\theta)$, $P(u,\theta)$ and $k(u,\theta)$: There are constants ν, P_1, P_2, k_0, and for any given $C > 0$, there exist positive constants $N(C)$, $k(C)$ and $k_1(C)$, such that for $u \ge C$, $\theta \ge 0$, the following conditions hold:

$$e(u,0) \ge 0, \quad \nu(1+\theta^r) \le e_\theta(u,\theta) \le N(C)(1+\theta^r) \tag{5}$$

$$P(u,\theta) \ge 0, \quad P(u,\theta) \to 0 \ \text{ as } u \to \infty \tag{6a}$$

$$|P_\theta(u,\theta)| \le N(C)\frac{1+\theta^r}{u^\gamma} \tag{6b}$$

$$-\frac{P_2(l + (1-l)\theta + \theta^{1+r})}{u^{\gamma+1}} \le P_u(u,\theta) \le -\frac{P_1(l + (1-l)\theta + \theta^{1+r})}{u^{\gamma+1}} \tag{6c}$$

$$k_0(1+\theta^q) \le k(u,\theta) \le k_1(C)(1+\theta^q) \tag{7a}$$

$$|k_u(u,\theta)| + |k_{uu}(u,\theta)| \le k(C)(1+\theta^q), \tag{7b}$$

where $l = 0$ or 1, $q \ge 2 + 2r$, $r \in [0,1]$ and $\gamma \in [1, \frac{5}{4})$.

Remark 1.1. By integrating (6c) over (u,∞) and noticing (6a), we obtain

$$\frac{P_1[l+(1-l)\theta+\theta^{1+r}]}{\gamma} \le u^\gamma P(u,\theta) \le \frac{P_2[l+(1-l)\theta+\theta^{1+r}]}{\gamma}.$$

The above assumptions are more general than those in [4–6, 8, 10, 11] for the case of $\gamma = 1$, for example, $P(u,\theta) = R\frac{\theta}{u}$, $e(u,\theta) = C_v\theta$ satisfy (5) and (6), where R is the gas constant, $C_v = \frac{R}{\gamma-1}$ is the heat capacity of the gas at constant volume, and γ is the adiabatic exponent. There are many studies on the perfect gas, see [2, 7, 9, 13] and the references therein. However, for gases with high pressure and temperature, the internal energy $e(u,\theta)$ may grow as θ^{1+r} with $r \approx 0.5$ and $k(u,\theta) \propto \theta^q$ with $q \in [4.5, 5.5]$, refer to (5) and (7a). We point out that the perfect gas only corresponds to $r = 0$ and $\gamma = 1$ for which there are more extensive discussions and experimental evidence for the real compressible flows in [1, 12]. From the physical points of view and mathematical analysis it is very important to study the well-posedness of global solutions.

Because of the nonlinearity of state equations, the idea in [9] can not be used to obtain the upper bound and lower bound of the specific volume, for example, $P(u,\theta) = R_1\frac{\theta^{r_1}}{u} + R_2\frac{\theta^{r_2}}{u}$ with constants $R_i > 0$, $r_i > 1$, $i = 1, 2$. In this direction, there are also some results, for instance [3, 10, 11] in which the authors established the global existence of solutions with $\mu(u) \ge C > 0$, whereas the positive lower bound of density does play an important role in obtaining the upper and lower bound of the specific volume. On the other hand, the fact that the viscosity $\mu(u) = u^{-\lambda}$ decreases to zero rapidly as the density tends to vacuum causes another difficulty. Jiang [6] firstly obtained the global existence of smooth solutions for the problem of (1)–(3) with $\lambda \in (0, \frac{1}{3}]$ in the case of $\gamma = 1$. In this paper, we will develop some new novel estimates and efficient methods and extend Jiang's result to $\lambda \in \big(2(\gamma-1), \frac{1}{2}\big]$ with $1 \le \gamma < \frac{5}{4}$.

The main result of this paper is

Theorem 1.2. *Let* $\mu(u) = u^{-\lambda}$. *Assume that* $u_0(x)$, $u_{0x}(x)$, $v_0(x)$, $v_{0x}(x)$, $v_{0xx}(x)$, $\theta_0(x)$, $\theta_{0x}(x)$, $\theta_{0xx}(x)$ *are in* $C^\alpha[0,1]$ *for some* $\alpha \in (0,1)$ *and* $u_0(x)$, $\theta_0(x)$ *are positive on* $[0,1]$ *and satisfy the compatibility conditions. If* $2(\gamma-1) < \lambda \le \frac{1}{2})$ *and* $1 \le \gamma < \frac{5}{4}$, *then there exists a unique solution* (u, v, θ) *to the initial boundary problem of* (1)–(3) *such that*

$$u(x,t) > 0, \quad \theta(x,t) > 0 \quad \text{on } [0,1]\times[0,\infty).$$

Furthermore, for any fixed $T > 0$, *we have*

$$(u, u_x, u_t, u_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}) \in (C^{\alpha,\frac{\alpha}{2}}(Q_T))^{12},$$

and $(u_{tt}, v_{xt}, \theta_{xt}) \in (L^2(Q_T))^3$. *Here, as usual,* $C^\alpha[0,1]$ *denotes the Hölder space on* $[0,1]$ *with exponent* α, $C^{\alpha,\frac{\alpha}{2}}(Q_T)$ *the Hölder space on* $Q_T = [0,1]\times[0,T]$ *with exponents* α *in* x *and* $\frac{\alpha}{2}$ *in* t.

Remark 1.3. If we assume the initial data is in H^1, so are the solutions. By the upper and lower bounds of the density and applying the techniques in [10, 11], we can obtain more higher regularity of the solutions.

2. Proof of Theorem 1.2

Similar to [8], to prove Theorem 1.2, it suffices to show the following results:

$$|||(u, u_x, u_t, u_{xt})|||_\alpha + |||(v, v_x, v_t, v_{xx})|||_\alpha + |||(\theta, \theta_x, \theta_t, \theta_{xx})|||_\alpha \leq C, \tag{8}$$

and

$$0 < C_1 \leq \theta(x,t) \leq C_2, \tag{9}$$

where $|||(f_1, \ldots, f_n|||_\alpha = |||f_1|||_\alpha + \cdots + |||f_n|||_\alpha$ denotes the norm of the space $C^{\alpha,\frac{\alpha}{2}}(Q_T)$, C, C_1 and C_2 are positive constants. To establish (8) and (9), we need some a priori estimates on (u, v, θ). In the sequel, the generic constant $C(C(T))$ will depend on the initial data (the given time T) and the parameters of the system (1) which may be different from line to line.

Firstly, we trivially observe from (1a) and the boundary conditions (3) that $\int_0^1 udx = \int_0^1 u_0(x)dx$. With no loss of generality, we assume

$$\int_0^1 udx = \int_0^1 u_0(x)dx = 1, \tag{10}$$

From (1) and (4), we get

$$e_\theta(u,\theta)\theta_t + \theta P_\theta(u,\theta)v_x - \frac{\mu(u)}{u}v_x^2 = \left(\frac{k(u,\theta)}{u}\theta_x\right)_x. \tag{11}$$

Applying the maximum principle to (11), we obtain $\theta(x,t) > 0$ on $[0,1]\times[0,\infty)$.

As usual, the first lemma is the energy estimate.

Lemma 2.1. *There exists a positive constant C, such that*

$$\int_0^1 \left(\nu(\theta + \theta^{1+r}) + \frac{v^2}{2}\right)(x,t)dx \leq C, \quad \forall 0 \leq t \leq T. \tag{12}$$

Proof. Integrating (1c) over $[0,1] \times [0,t]$ and noticing the boundary conditions (3), we deduce that

$$\int_0^1 (e + \frac{v^2}{2})dx = \int_0^1 (e + \frac{v^2}{2})(x,0)dx,$$

which implies the proof of the lemma together with (5). □

The following lemma embodies the dissipative effects of viscosity and heat diffusion.

Lemma 2.2. *Under the hypotheses of Theorem* 1.2*, we have*

$$\int_0^t\int_0^1\left(\frac{(1+\theta^q)\theta_x^2}{u\theta^2}+\frac{\mu(u)v_x^2}{u\theta}\right)dx\,ds\le C, \tag{13}$$

for $0\le x\le 1$ *and* $0\le t\le T$.

Proof. Define $\eta(u,\theta)$ by $e_\theta=\theta\eta_\theta$, $\eta_u=P_\theta$, and let $\Psi(u,\theta)=e(u,\theta)-\theta\eta(u,\theta)$. Then we deduce that $\Psi_\theta(u,\theta)=-\eta(u,\theta)$, $\Psi_u(u,\theta)=-P(u,\theta)$ and $\theta\Psi_{\theta\theta}=-e_\theta$. Set

$$\begin{aligned}\Phi_1&=\Psi(u,\theta)-\Psi(u,1)-\Psi_\theta(u,\theta)(\theta-1)\\ \Phi_2&=\Psi(u,1)-\Psi(1,1)-\Psi_u(1,1)(u-1),\end{aligned}$$

then we get from (1) and (4),

$$\begin{aligned}&\left(\Phi_1+\Phi_2+\frac{v^2}{2}\right)_t+\frac{k\theta_x^2}{u\theta^2}+\frac{\mu(u)v_x^2}{u\theta}\\ &=\left(\frac{\mu(u)vv_x}{u}+\frac{k\theta_x}{u}+P(1,1)v-vP-\frac{k\theta_x}{u\theta}\right)_x.\end{aligned} \tag{14}$$

Integrating (14) over $[0,1]\times[0,t]$, we arrive at

$$\begin{aligned}&\int_0^1\left(\Phi_1+\Phi_2+\frac{v^2}{2}\right)dx+\int_0^t\int_0^1\left(\frac{k\theta_x^2}{u\theta^2}+\frac{\mu(u)v_x^2}{u\theta}\right)dx\,ds\\ &=\int_0^1\left(\Phi_1+\Phi_2+\frac{v^2}{2}\right)(x,0)dx.\end{aligned}$$

By Taylor's expansion, we obtain

$$\begin{aligned}\Phi_1&=(1-\theta)^2\int_0^1\frac{(1-\tau)e_\theta(u,\theta+\tau(1-\theta))}{\theta+\tau(1-\theta)}d\tau\\ &\ge\nu(1-\theta)^2\int_0^1\frac{(1-\tau)(1+(\theta+\tau(1-\theta))^r)}{\theta+\tau(1-\theta)}d\tau\\ &\ge 0\quad\text{(by (5))}\end{aligned}$$

and $\Phi_2=\Phi_{uu}(\xi,1)(u-1)^2=-P_u(\xi,1)(u-1)^2\ge 0$, which end the proof of the lemma together with (7a). □

Lemma 2.3. *The following inequality holds:*

$$\int_0^t\max_{[0,1]}\theta^{(2+2r)}ds\le C,\quad\forall 0\le t\le T.$$

Proof. In view of the embedding theorem, it yields

$$\theta^{r+1} \le \int_0^1 \theta^{r+1}(x,t)dx + (r+1)\int_0^1 \theta^r|\theta_x|dx.$$

By the Cauchy-Schwarz inequality, (10) and (12), we find

$$\begin{aligned}\theta^{1+r} &\le \int_0^1 \theta^{1+r}dx + C\int_0^1 \theta^r|\theta_x|dx \\ &\le C + C\Big(\int_0^1 \frac{\theta^{(2+2r)}\theta_x^2}{u\theta^2}dx\Big)^{\frac{1}{2}}\Big(\int_0^1 udx\Big)^{\frac{1}{2}} \\ &\le C + C\Big(\int_0^1 \frac{(1+\theta^q)\theta_x^2}{u\theta^2}dx\Big)^{\frac{1}{2}}, \quad \text{for } q \ge 2+2r.\end{aligned}$$

Taking the square and integrating in t, we arrive at

$$\int_0^t \max_{[0,1]} \theta^{2(1+r)}ds \le C\Big(1 + \int_0^t\int_0^1 \frac{(1+\theta^q)\theta_x^2}{u\theta^2}dx\,ds\Big) \le C,$$

where we used (13). □

Lemma 2.4. *Under the hypotheses of Theorem* 1.2*, there holds*

$$u(x,t) \ge C(T) \iff \rho(x,t) \le \frac{1}{C(T)}, \tag{15}$$

for $0 \le x \le 1$ *and* $0 \le t \le T$.

Proof. Motivated by [6, 9], we set

$$\varphi(x,t) = \int_0^t \sigma(x,s)ds + \int_0^x v_0(y)dy, \tag{16}$$

where $\sigma(x,t) = \frac{\mu(u)v_x}{u} - P(u,\theta)$. Thus, we have $\varphi_x = v$ and $\varphi_t = \frac{\mu(u)v_x}{u} - P(u,\theta)$. Hence

$$(u\varphi)_t - (v\varphi)_x = \mu(u)v_x - uP - v^2. \tag{17}$$

Integrating (17) over $[0,1]\times[0,t]$, we derive that

$$\begin{aligned}&\int_0^1 (u\varphi)(x,t)dx \\ &= \int_0^t\int_0^1 (\mu(u)v_x - uP - v^2)(x,s)dx\,ds + \int_0^1 (u\varphi)(x,0)dx = \Phi(t).\end{aligned} \tag{18}$$

From (10), we deduce that there exists a point $x_0(t) \in [0,1]$ such that $\varphi(x_0(t),t) = \int_0^1 (u\varphi)(x,t)dx = \Phi(t)$. This leads to

$$\begin{aligned}\int_0^t \sigma(x_0(t),s)ds &= \varphi(x_0(t),t) - \int_0^{x_0(t)} v_0(y)dy \\ &= \Phi(t) - \int_0^{x_0(t)} v_0(y)dy \\ &= \int_0^t\int_0^1 (\mu(u)v_x - uP - v^2)(x,s)dx\,ds \\ &\quad + \int_0^1 u_0(x)\int_0^x v_0(y)dydx - \int_0^{x_0(t)} v_0(y)dy\end{aligned} \tag{19}$$

together with (16) and (18). Let $M(u) = \int_{\inf_{[0,1]} u_0(x)}^{u} \frac{\mu(\xi)}{\xi} d\xi$. Integrating (1b) over $[x_0(t),x] \times [0,t]$ and using (19), we arrive at

$$\begin{aligned}M(u(x,t)) &= M(u_0(x)) + \int_0^t P(x,s)ds \\ &\quad + \int_{x_0(t)}^x (v(y,t) - v_0(y))dy + \int_0^t \sigma(x_0(t),s)ds \\ &= M(u_0(x)) + \int_0^t P(x,s)ds + \int_{x_0(t)}^x (v(y,t) - v_0(y))dy \\ &\quad + \int_0^t\int_0^1 (\mu(u)v_x - uP - v^2)(x,s)dx\,ds \\ &\quad + \int_0^1 u_0(x)\int_0^x v_0(y)dy\,dx - \int_0^{x_0(t)} v_0(y)dy.\end{aligned}$$

Therefore

$$\begin{aligned}\rho^\lambda = \Big[\inf_{[0,1]} u_0(x)\Big]^{-\lambda} &- \lambda\Big(M(u_0(x)) + \int_0^t P(x,s)ds\Big) \\ &- \lambda\Big(\int_{x_0(t)}^x (v(y,t) - v_0(y))dy + \int_0^t\int_0^1 (\mu(u)v_x - uP)(x,s)dx\,ds\Big) \\ &- \lambda\Big(\int_0^1 u_0(x)\int_0^x v_0(y)dydx - \int_0^{x_0(t)} v_0(y)dy - \int_0^t\int_0^1 v^2 dx\,ds\Big).\end{aligned}$$

To finish up the proof of the lemma, we only need to estimate $-\lambda\int_0^t\int_0^1 (\mu(u)v_x - uP)dx\,ds$. Indeed by Lemma 2.2 and Lemma 2.3, it yields

$$\begin{aligned}\left|\int_0^t\int_0^1 \mu(u)v_x dx\,ds\right| &\le \int_0^t \max_{[0,1]}\theta\Big(\int_0^1 u\mu(u)dx\Big)ds + \int_0^t\int_0^1 \frac{\mu(u)v_x^2}{u\theta}\,dx\,ds \\ &\le \int_0^t \max_{[0,1]}\theta\Big(\int_0^1 u dx\Big)^{1-\lambda} ds + \int_0^t\int_0^1 \frac{\mu(u)v_x^2}{u\theta}\,dx\,ds \le C.\end{aligned} \tag{20}$$

Using (10), we obtain

$$\begin{aligned}\left|\int_0^t\int_0^1 uPdx\,ds\right| &= \left|\int_0^t\int_0^1 u^\gamma P u^{1-\gamma}\right|dx\,ds\\ &\le C\int_0^t \max_{[0,1]}(1+\theta^{1+r})\int_0^1 u^{1-\gamma}dx\,ds\\ &\le C\int_0^t \max_{[0,1]}(1+\theta^{2(1+r)})ds + \int_0^t \max_{[0,1]}\rho^\lambda\Big(\int_0^1 u^{2(1-\gamma)+\lambda}dx\Big)ds.\end{aligned}$$

Since $2(\gamma-1)<\lambda\le\frac{1}{2}$, $1\le\gamma<\frac{5}{4}$, we have

$$\left|\int_0^t\int_0^1 uPdx\,ds\right| \le C(T) + C\int_0^t \max_{[0,1]}\rho^\lambda ds.$$

This together with (20) and the initial conditions yields

$$\max_{[0,1]}\rho^\lambda \le C(T)+C\int_0^t\max_{[0,1]}\rho^\lambda ds + \lambda\int_0^t\int_0^1 v^2dx\,ds \le C(T)+C\int_0^t\max_{[0,1]}\rho^\lambda ds,$$

where we used (12). This implies the proof of the lemma by Gronwall's inequality. □

Lemma 2.5. *There exists a positive constant $C(T)$, such that*

$$\int_0^1 [(u^{-\lambda})_x]^2dx \le C(T), \quad 0\le t\le T.$$

Proof. Substituting (1a) into (1b) and integrating over $[0,t]$, we obtain

$$(u^{-\lambda})_x = (u_0^{-\lambda})_x - \lambda(v-v_0) - \lambda\int_0^t P(u,\theta)_x ds. \tag{21}$$

Multiplying (21) by $(u^{-\lambda})_x$ and integrating it over $[0,1]$, we have

$$\int_0^1 [(u^{-\lambda})_x]^2dx = \int_0^1\left((u_0^{-\lambda})_x - \lambda(v-v_0) - \lambda\int_0^t P(u,\theta)_x ds\right)^2dx.$$

Using the assumptions on the initial data and (12), we deduce

$$\begin{aligned}\int_0^1 [(u^{-\lambda})_x]^2dx &= \int_0^1\left((u_0^{-\lambda})_x - \lambda(v-v_0) - \lambda\int_0^t P(u,\theta)_x ds\right)^2dx\\ &\le C\int_0^1[(u_0^{-\lambda})_x]^2dx + C\int_0^1 v_0^2(x)dx + C\int_0^1 v^2dx\\ &\quad + C\int_0^1\left(\int_0^t P(u,\theta)_x ds\right)^2dx\\ &\le C + C\int_0^1\left(\int_0^t P(u,\theta)_x ds\right)^2dx.\end{aligned} \tag{22}$$

Now we will estimate the integral on the right hand side of (22).
By (6b) and (6c), we obtain

$$\begin{aligned}
&\int_0^1\left(\int_0^t P(u,\theta)_x ds\right)^2 dx\\
&=\int_0^1\left(\int_0^t P_u(u,\theta)u_x+P_\theta(u,\theta)\theta_x ds\right)^2 dx\\
&\le C\int_0^1\left(\int_0^t |P_u(u,\theta)u_x| ds\right)^2 dx + C\int_0^t\int_0^1 (P_\theta(u,\theta)\theta_x)^2 dx\,ds\\
&\le C\int_0^1\left(\int_0^t \frac{(l+(1-l)\theta+\theta^{1+r})|u_x|}{u^{\gamma+1}} ds\right)^2 dx\\
&\quad + C\int_0^t\int_0^1 \frac{(1+\theta^r)^2\theta_x^2}{u^{2\gamma}} dx\,ds\\
&\le C\int_0^1\left(\int_0^t |(u^{-\lambda})_x| u^{\lambda-\gamma}(l+(1-l)\theta+\theta^{1+r}) ds\right)^2 dx\\
&\quad + C\int_0^t\int_0^1 \frac{(1+\theta^r)^2\theta_x^2}{u^{2\gamma}} dx\,ds\\
&\triangleq I_1+I_2.
\end{aligned} \tag{23}$$

We will give the estimates of I_1 and I_2, respectively.

Recalling Hölder's inequality, Lemma 2.3 and Lemma 2.4, we discover

$$\begin{aligned}
I_1 &= C\int_0^1\left(\int_0^t |(u^{-\lambda})_x| u^{\lambda-\gamma}(l+(1-l)\theta+\theta^{1+r}) ds\right)^2 dx\\
&\le C(T)\int_0^1\left(\int_0^t [(u^{-\lambda})_x]^2 ds \times \int_0^t \max_{[0,1]}(1+\theta^{2+2r}) ds\right) dx\\
&\le C(T)\int_0^t\int_0^1 [(u^{-\lambda})_x]^2 dx\,ds,
\end{aligned} \tag{24}$$

where we used Lemma 2.2 in the last line. By Young's inequality, (13) and (15) and noticing $q \ge 2+2r$, one has

$$\begin{aligned}
I_2 &= C\int_0^t\int_0^1 \frac{(1+\theta^r)^2\theta_x^2}{u^{2\gamma}} dx\,ds\\
&= C\int_0^t\int_0^1 \frac{\theta^2(1+\theta^r)^2\theta_x^2}{u^{2\gamma}\theta^2} dx\,ds\\
&\le C(T)\int_0^t\int_0^1 \frac{(1+\theta^{2+2r})\theta_x^2}{u\theta^2} dx\,ds\\
&\le C(T)\int_0^t\int_0^1 \frac{(1+\theta^q)\theta_x^2}{u\theta^2} dx\,ds\\
&\le C(T).
\end{aligned} \tag{25}$$

Collecting (22)–(25), we arrive at

$$\int_0^1 [(u^{-\lambda})_x]^2 dx \le C(T) + C(T)\int_0^t\int_0^1 [(u^{-\lambda})_x]^2 dx\, ds,$$

which implies the lemma by Gronwall's inequality. □

Lemma 2.6. *For* $2(\gamma-1) < \lambda \le \frac{1}{2}$ *and* $1 \le \gamma < \frac{5}{4}$, *there holds*

$$u(x,t) \le C(T) \iff \rho(x,t) \ge \frac{1}{C(T)}, \quad 0 \le x \le 1,\ 0 \le t \le T. \tag{26}$$

Proof. Let $U(t) = \max_{[0,1]\times[0,t]} u(x,s)$. Then it follows from the Sobolev's embedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$ for any $0 < \beta < 1$ that $u^\beta \le \int_0^1 u^\beta dx + \beta \int_0^1 u^{\beta-1}|u_x|\, dx$. From Hölder's inequality, (10) and Lemma 2.5, we have

$$\begin{aligned} u^\beta &\le \int_0^1 u^\beta dx + \beta \int_0^1 u^{\beta-1}|u_x|\, dx \\ &\le \Big(\int_0^1 u dx\Big)^\beta + C\beta\Big(\int_0^1 [(u^{-\lambda})_x]^2 dx\Big)^{\frac{1}{2}}\Big(\int_0^1 u^{2(\beta+\lambda)} dx\Big)^{\frac{1}{2}} \\ &\le 1 + C(T)\beta U^\beta\Big(\int_0^1 u^{2\lambda} dx\Big)^{\frac{1}{2}} \\ &\le 1 + C(T)\beta U^\beta, \qquad \text{since} \quad 2(\gamma-1) < \lambda \le \frac{1}{2}. \end{aligned}$$

This leads to Lemma 2.6 because there exists sufficiently small β such that $C(T)\beta < 1$. □

Based on the estimates (12)–(13), (15) and (26), we can show (8) and (9). We refer reader to [3, 4, 8] for detailed proof. Here we omit it for brevity. With the estimates (8), (15) and (26) in hand, the global existence of smooth solutions can be obtained by extending the local solutions globally in time in view of a priori global estimates, while the existence of local solutions is known from Leray–Schauder fixed point theorem. This completes the proof of Theorem 1.2.

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References

[1] Becker, E., *Gasdynamik.* Stuttgart: Teubner 1966.

[2] Chen, G. Q., Hoff, D. and Trivisa, K., Global solutions of the compressible Navier–Stokes equations with large discontinuous initial data. *Comm. Partial Diff. Eqs.* 25 (2000), 2233 – 2257.

[3] Dafermos, C. M., Global smooth solutions to the initial boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity. *SIAM J. Math. Anal.* 13 (1982), 397 – 408.

[4] Jiang, S., On initial boundary value problems for a viscous, heat-conducting, one-dimensional real gas. *J. Diff. Eqs.* 110 (1994), 157 – 181.

[5] Jiang, S., On the asymptotic behavior of the motion of a viscous, heat-conducting, one-dimensional real gas. *Math. Z.* 216 (1994), 317 – 336.

[6] Jiang, S., Global smooth solutions of the equations of a viscous, heat-conducting one-dimensional gas with density-dependent viscosity. *Math. Nachr.* 190 (1998), 169 – 183.

[7] Kawashima, S. and Nishida, T., Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytropic gases. *J. Math. Kyoto Univ.* 21 (1981), 825 – 837.

[8] Kawohl, B., Global existence of large solutions to initial boundary value problems for a viscous, heat-conducting, one-dimensional real gas. *J. Diff. Eqs.* 58 (1985), 76 – 103.

[9] Kazhikhov, A. V. and Shelukhin, V. V., Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas. *J. Appl. Math. Mech.* 41 (1977), 273 – 282.

[10] Wang, D., Global solutions of the Navier-Stokes equations for viscous compressible flows. *Nonlinear Anal.* 52 (2003), 1867 – 1890.

[11] Wang, D., On the global solution and interface behaviour of viscous compressible real flow with free boundaries. *Nonlinearity* 16 (2003), 719 – 733.

[12] Zel'dovich, Y. B. and Raizer, Y. P., *Physics of Shock Waves and High-temperature Hydrodynamic Phenomena*, Vol 2. NewYork: Academic Press 1967.

[13] Zheng, S. and Shen, W., Global smooth solutions to the Cauchy problem of equations of one-dimensional nonlinear thermoviscoelasticity. *J. Partial Diff. Eqs.* 2 (1989), 26 – 38.

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