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Remarks on $C^{1,\gamma}$-Regularity of Weak Solutions to Elliptic Systems with BMO Gradients

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Abstract. The interior $C^{1,\gamma}$-regularity for a weak solution with BMO-gradient of a nonlinear nonautonomous second order elliptic systems is investigated.

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1. Introduction

In this paper we give conditions guaranteeing that the BMO first derivatives of weak solutions to a nonlinear elliptic system

$$-D_\alpha a_i^\alpha(x, Du) = -D_\alpha f_i^\alpha(x) \quad \text{on } \Omega \subset \mathbb{R}^n,\ i = 1, \dots, N \tag{1}$$

belong to $C^{0,\gamma}_{loc}(\Omega, \mathbb{R}^{nN})$.[1]

The system (1) has been extensively studied. S. Campanato in [2,3] proved that (under suitable assumptions) $Du \in \mathcal{L}^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{nN})$ with $\lambda < n$, and $u \in C^{0,\gamma}_{loc}(\Omega, \mathbb{R}^N)$ for some $\gamma < 1$ if $n = 3$, 4. If Ω has a smooth boundary and a_i^α are differentiable and have controllable growth, then there is a positive ϵ such that $u \in W^{2,2+\epsilon}_{loc}(\Omega, \mathbb{R}^{nN})$ which implies that Du is Hölder continuous on $\overline{\Omega}$ for $n = 2$ (see [8, 12, 13]). For this reason we will concentrate on the case

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[1] Throughout the whole text we use the summation convention over repeated indices.

$n > 2$. From a series of counterexamples starting from the famous De Giorgi work (see [7]) it is well known that Du need not be continuous or even bounded (see [9, 11, 14, 17, 18]) for $n > 2$. Higher smoothness of coefficients does not improve the smoothness of a solution, as there are examples (see [15]) where the coefficients are real analytic while Du is bounded and discontinuous. On the other hand, it follows immediately from so called direct proof of partial regularity (see [6,8]) that if modulus of continuity of $\frac{\partial a_i^\alpha}{\partial p_\beta^j}$ is small enough, then Du is Hölder continuous. For this reason we concentrate on conditions that do not require smallness of the L^∞ norm of the modulus of continuity and they imply that solutions with BMO gradients are $C^{1,\gamma}_{loc}(\Omega, \mathbb{R}^N)$. The condition that Du is in BMO cannot be verified in general (see [18]). On the other hand, for some classes of elliptic systems this assertion is proved in [4–6]. Our result has a local character and the work on the global variant is in progress.

By a weak solution to (1) we understand $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega a_i^\alpha\left(x, Du(x)\right) D_\alpha \varphi^i(x)\, dx = \int_\Omega f_i^\alpha(x) D_\alpha \varphi^i(x)\, dx, \quad \forall\, \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N).$$

Here $\Omega \subset \mathbb{R}^n$ is an open set and, as we are interested in the interior regularity, we do not assume that u solves a boundary value problem nor any smoothness of $\partial\Omega$.

On the coefficients we suppose

(i) (Smoothness) $a_i^\alpha(x,p)$ are differentiable functions in x and p with continuous derivatives. Without loss of generality we suppose that $a_i^\alpha(x,0) = 0$.

(ii) (Growth) For all $(x,p) \in \Omega \times \mathbb{R}^{nN}$ denote $A_{ij}^{\alpha\beta}(x,p) = \frac{\partial a_i^\alpha}{\partial p_\beta^j}(x,p)$ and suppose

$$|a_i^\alpha(x,p)|\,, \quad \left|\frac{\partial a_i^\alpha}{\partial x_s}(x,p)\right| \leq M(1+|p|), \quad \text{and} \quad \left|A_{ij}^{\alpha\beta}(x,p)\right| \leq M,$$

where $M > 0$.

(iii) (Ellipticity) There exists $\nu > 0$ such that for every $x \in \Omega$ and p, $\xi \in \mathbb{R}^{nN}$

$$\nu|\xi|^2 \leq A_{ij}^{\alpha\beta}(x,p)\xi_\alpha^i \xi_\beta^j\,.$$

(iv) (Oscillation of coefficients) There is a real function ω absolutely continuous on $[0,\infty)$, which is bounded, nondecreasing, $\omega(0) = 0$ and such that for all $x \in \Omega$ and p, $q \in \mathbb{R}^{nN}$

$$\left|A_{ij}^{\alpha\beta}(x,p) - A_{ij}^{\alpha\beta}(x,q)\right| \leq \omega\left(|p-q|\right).$$

We set $\omega_\infty = \lim_{t\to\infty}\omega(t) \leq 2M$.

(v) $f_i^\alpha \in W^{1,2}(\Omega)$, $\frac{\partial f_i^\alpha}{\partial x_\beta} \in L^{2,\delta-2}(\Omega)$ for $\delta = n + 2\gamma$, $\gamma \in (0,1)$, α, $\beta = 1, \ldots, n$, $i = 1, \ldots, N$.

It is well known (see [8, p. 169]) that for uniformly continuous $A_{ij}^{\alpha\beta}$ there exists a real function ω satisfying (iv) and, viceversa, (iv) implies uniform continuity of $A_{ij}^{\alpha\beta}$.

In what follows we will understand by pointwise derivative ω' of ω the right derivative which is finite on $(0,\infty)$. For $p \in (1,\infty)$, $\frac{1}{p}+\frac{1}{p'}=1$ denote

$$J_p = \int_0^\infty \frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\,dt, \quad S_p = \sup_{t\in(0,\infty)} \frac{d}{dt}(\omega^{2p'})(t) \quad \text{and} \quad P_p = \min\{J_p, S_p\}.$$

Now we formulate the result.

Theorem 1.1. *Let u be a weak solution to* (1) *such that $Du \in BMO(\Omega,\mathbb{R}^{nN})$ and coefficients a_i^α satisfy the hypotheses* (i), (ii), (iii), (iv) *with the constants M, ν, a modulus of continuity ω and a right hand side f satisfying* (v). *Assume that there is a $p \in \left(1, \frac{n}{n-2}\right]$ such that $P_p < \infty$. Then the inequality*

$$\left(P_p^2 \|Du\|_{BMO(\Omega,\mathbb{R}^{nN})}\right)^{\frac{1}{2p'}} \le \nu^2\, C \tag{2}$$

implies that $Du \in C_{loc}^{0,\gamma}(\Omega,\mathbb{R}^{nN})$.

Remark. Here

$$C = \frac{\kappa_n^{\frac{1}{2p'}}}{12c^\star(\lambda,n)C\left(p,n,\frac{M}{\nu}\right)(8L)^{\frac{n+2}{n+2-\mu}}},$$

$\mu \in (n+2\gamma, n+2)$, L is given in Lemma 2.4, $C(p,n,\frac{M}{\nu})$ is given in (10) and $c^\star(\lambda,n)$ is the embedding constant between BMO and $L^{2,\lambda}$ spaces.

2. Preliminaries and notations

Let $n, N \in \mathbb{N}$, $n \ge 3$. We will consider an open set $\Omega \subset \mathbb{R}^n$ with points $x=(x_1,\dots,x_n)$, $n\ge 3$. For a vector-valued function $u:\Omega\to\mathbb{R}^N$, $u(x) = \left(u^1(x),\dots,u^N(x)\right)$, $N\ge 1$, put $Du=(D_1u,\dots,D_nu)$, $D_\alpha = \frac{\partial}{\partial x_\alpha}$. If $x\in\mathbb{R}^n$ and r is a positive real number, we set $B(x,r)=\{y\in\mathbb{R}^n : |y-x|<r\}$. Denote by $u_{x,r} = (\kappa_n r^n)^{-1}\int_{B(x,r)} u(y)\,dy$ the mean value of the function $u \in L^1(B(x,r),\mathbb{R}^N)$ over the set $B(x,r)$ (κ_n being the volume of unit ball in $\mathbb{R}^n$). Moreover, we set $\phi(x,r) = \int_{B(x,r)} |Du(y)-(Du)_{x,r}|^2\,dy$.

Beside the usually used space $C_0^\infty(\Omega,\mathbb{R}^N)$, the Hölder space $C^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega,\mathbb{R}^N)$, $W_0^{k,p}(\Omega,\mathbb{R}^N)$ we use Morrey spaces $L^{q,\lambda}(\Omega,\mathbb{R}^N)$, Campanato spaces $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$ and the space of functions with bounded mean oscillations $BMO(\Omega,\mathbb{R}^N)$(see, e.g., [10]). By the function space $X_{loc}(\Omega,\mathbb{R}^N)$ we understand the space of all functions which belong to $X(\tilde{\Omega},\mathbb{R}^N)$ for any bounded subdomain $\tilde{\Omega}$ with smooth boundary which is compactly embedded in Ω.

For definitions and more details see [1, 8, 10, 13]. In particular, we will use:

Proposition 2.1. *For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary we have the following*

(a) *For $q \in (1,\infty)$, $0 < \lambda < \mu < \infty$ we have $L^{q,\mu}(\Omega,\mathbb{R}^N) \subset L^{q,\lambda}(\Omega,\mathbb{R}^N)$ and $\mathcal{L}^{q,\mu}(\Omega,\mathbb{R}^N) \subset \mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$;*

(b) *$\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$ is isomorphic to $C^{0,\frac{\lambda-n}{q}}(\overline{\Omega},\mathbb{R}^N)$, for $n < \lambda \le n+q$;*

(c) *$L^{q,n}(\Omega,\mathbb{R}^N)$ is isomorphic to $L^\infty(\Omega,\mathbb{R}^N)$, $\mathcal{L}^{q,n}(\Omega,\mathbb{R}^N)$ is isomorphic to $BMO(\Omega,\mathbb{R}^N)$;*

(d) *$L^{q,\lambda}(\Omega,\mathbb{R}^N)$ is isomorphic to $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$, for $0 < \lambda < n$.*

By means of Nirenberg's difference quotients method we obtain

Lemma 2.2. *Let u be a weak solution to* (1) *and coefficients a_i^α satisfy the hypotheses* (i), (ii), (iii), (iv) *with the constants M, ν and a right hand side $f \in W^{1,2}(\Omega,\mathbb{R}^{nN})$. Then $u \in W^{2,2}_{loc}(\Omega,\mathbb{R}^N)$, and for any $x \in \Omega$ and $R \in (0,\frac{1}{2}\mathrm{dist}(\mathrm{x},\partial\Omega))$ it holds*

$$\int_{B(x,R)} |D^2u|^2\,dx \le C\left(\frac{M}{\nu}\right)\left(\frac{1}{R^2}\int_{B(x,2R)} |Du-(Du)_{x,2R}|^2\,dx + R^n + \int_{B(x,2R)} |Du|^2\,dx + \int_{B(x,2R)} |Df|^2\,dx\right). \tag{3}$$

In what follows we will use an algebraic lemma due to S. Campanato. We start with recalling it.

Lemma 2.3 (see [8], Chapter III., Lemma 2.1)**.** *Let α, d be positive numbers, $A > 0$, $\beta \in [0,\alpha)$. Then there exist ϵ_0, C positive so that for any nonnegative, nondecreasing function ϕ defined on $[0,d]$ and satisfying the inequality*

$$\phi(\sigma) \le \left(A\left(\frac{\sigma}{R}\right)^\alpha + K\right)\phi(R) + BR^\beta, \quad \forall \sigma, R \,:\, 0 < \sigma < R \le d, \tag{4}$$

with $K \in (0,\epsilon_0]$ and $B \in [0,\infty)$ it holds

$$\phi(\sigma) \le C\sigma^\beta\left(d^{-\beta}\phi(d) + B\right), \quad \forall\ \sigma \,:\, 0 < \sigma \le d.$$

For the statement of following Lemma see, e.g., [2,8,13].

Lemma 2.4. *Consider a system of the type* (1) *with $a_i^\alpha(x,p) = A_{ij}^{\alpha\beta}p_\beta^j$, $A_{ij}^{\alpha\beta} \in \mathbb{R}$ (i.e., a linear system with constant coefficients) satisfying* (iii)*. Then there exists a constant $L = L(n,\frac{M}{\nu}) \ge 1$ such that for every weak solution $v \in W^{1,2}(\Omega,\mathbb{R}^N)$, for every $x \in \Omega$ and $0 < \sigma \le R \le \mathrm{dist}(\mathrm{x},\partial\Omega)$ the following estimate holds:*

$$\int_{B(x,\sigma)} |Dv(y)-(Dv)_{x,\sigma}|^2\,dy \le L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B(x,R)} |Dv(y)-(Dv)_{x,R}|^2\,dy\,.$$

Lemma 2.5 ([19, p. 37]). *Let $\psi : [0,\infty] \to [0,\infty]$ be non decreasing function which is absolutely continuous on every closed interval of finite length, $\psi(0) = 0$. If $w \geq 0$ is measurable, $E(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ and μ is the n-dimensional Lebesgue measure, then*

$$\int_{\mathbb{R}^n} \psi \circ w \, dy = \int_0^\infty \mu\left(E(t)\right) \psi'(t) \, dt.$$

Remark. In case of ψ non decreasing and bounded, the assumption of absolute continuity of ψ on every closed interval of finite length is equivalent to the absolute continuity of ψ on $[0,\infty)$.

3. Proof of the main result

Proof of Theorem 1.1. Let x_0 be any fixed point of Ω. We prove that $Du \in \mathcal{L}^{2,\delta}$ on a neighborhood of x_0. Let $R \leq \frac{1}{2}\mathrm{dist}(\mathrm{x}_0, \partial\Omega)$. Where no confusion can result, we will use the notation $B(R)$, $\phi(R)$ and $(Du)_R$ instead of $B(x_0,R)$, $\phi(x_0,R)$ and $(Du)_{x_0,R}$.

Denote $A^{\alpha\beta}_{ij,0} = A^{\alpha\beta}_{ij}\left(x_0, (Du)_R\right)$,

$$\tilde{A}^{\alpha\beta}_{ij}(x) = \int_0^1 A^{\alpha\beta}_{ij}\left(x_0, (Du)_R + t\left(Du(x) - (Du)_R\right)\right) dt.$$

Hence $a^\alpha_i(x_0, Du(x)) - a^\alpha_i(x_0,(Du)_R) = \tilde{A}^{\alpha\beta}_{ij}(x)\left(D_\beta u^j(x) - (D_\beta u^j)_R\right)$. Thus we can rewrite the system (1) as

$$\begin{aligned}-D_\alpha\big(A^{\alpha\beta}_{ij,0} D_\beta u^j\big) &= -D_\alpha\Big(\big(A^{\alpha\beta}_{ij,0} - \tilde{A}^{\alpha\beta}_{ij}\big)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\Big)\\ &\quad - D_\alpha\big(a^\alpha_i(x_0, Du) - a^\alpha_i(x, Du)\big) - D_\alpha\big(f^\alpha_i(x) - (f^\alpha_i)_R\big).\end{aligned}$$

Split u as $v + w$ where v is the solution of the Dirichlet problem

$$-D_\alpha\big(A^{\alpha\beta}_{ij,0} D_\beta v^j\big) = 0 \quad \text{in } B(R), \qquad v - u \in W^{1,2}_0\left(B(R), \mathbb{R}^N\right),$$

and $w \in W^{1,2}_0(B(R), \mathbb{R}^N)$ is the weak solution of the system

$$\begin{aligned}-D_\alpha\big(A^{\alpha\beta}_{ij,0} D_\beta w^j\big) &= -D_\alpha\Big(\big(A^{\alpha\beta}_{ij,0} - \tilde{A}^{\alpha\beta}_{ij}\big)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\Big)\\ &\quad - D_\alpha\big(a^\alpha_i(x_0, Du) - a^\alpha_i(x, Du)\big)\\ &\quad - D_\alpha\big(f^\alpha_i(x) - (f^\alpha_i)_R\big).\end{aligned} \tag{5}$$

For every $0 < \sigma \leq R$ from Lemma 2.4 it follows

$$\int_{B(\sigma)} \left|Dv - (Dv)_\sigma\right|^2 dx \leq L\left(\frac{\sigma}{R}\right)^{n+2} \int_{B(R)} \left|Dv - (Dv)_R\right|^2 dx\,.$$

hence

$$\begin{aligned}
&\int_{B(\sigma)} |Du-(Du)_\sigma|^2\,dx \\
&\le 2L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B(R)}|Dv-(Dv)_R|^2\,dx+4\int_{B(R)}|Dw|^2dx \\
&\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B(R)}|Du-(Du)_R|^2dx+4\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\int_{B(R)}|Dw|^2dx.
\end{aligned} \tag{6}$$

Now as $w \in W_0^{1,2}(B_R,\mathbb{R}^N)$ we can choose a test function $\varphi = w$ in (5) and we get

$$\begin{aligned}
\nu^2\int_{B(R)}|Dw|^2\,dx \le 3\Bigg(&\int_{B(R)}\omega^2\left(|Du-(Du)_R|\right)|Du-(Du)_R|^2\,dx \\
&+\int_{B(R)}|a_i^\alpha(x_0,Du)-a_i^\alpha(x,Du)|^2\,dx \\
&+\int_{B(R)}|f_i^\alpha(x)-(f_i^\alpha)_R|^2\,dx\Bigg).
\end{aligned} \tag{7}$$

From (6), (7) and Poincaré's inequality we have

$$\begin{aligned}
\phi(\sigma) &= \int_{B(\sigma)}|Du-(Du)_\sigma|^2\,dx \\
&\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B(R)}|Du-(Du)_R|^2\,dx \\
&\quad+\frac{12\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2}\Bigg[\int_{B(R)}\omega^2(|Du-(Du)_R|)\,|Du-(Du)_R|^2\,dx \\
&\quad+\int_{B(R)}|a_i^\alpha(x_0,Du)-a_i^\alpha(x,Du)|^2\,dx+c(n)R^2\int_{B(R)}|Df|^2\,dx\Bigg] \\
&\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) \\
&\quad+\frac{12\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2}\left[(I_1+I_2)+c(n)R^\delta\|Df\|^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{nN})}\right]
\end{aligned} \tag{8}$$

where $c(n)$ denotes the constant from Poincaré inequality. Then using Hölder inequality with the exponent p from the assumptions of the Theorem, embedding and Lemma 2.2 we have

$$\begin{aligned}
I_1 &\le \left(\int_{B(R)}|Du-(Du)_R|^{2p}\,dx\right)^{\frac{1}{p}}\left(\int_{B(R)}\omega^{2p'}\left(|Du-(Du)_R|\right)\,dx\right)^{\frac{1}{p'}} \\
&\le C_p^2R^{2-\frac{n}{p'}}\int_{B(R)}|D^2u|^2\,dx\left(\int_{B(R)}\omega^{2p'}\left(|Du-(Du)_R|\right)\,dx\right)^{\frac{1}{p'}}
\end{aligned}$$

$$\leq C\left(p,n,\frac{M}{\nu}\right)\left(\frac{1}{\kappa_n R^n}\int_{B(R)}\omega^{2p'}\left(|Du-(Du)_R|\right)\,dx\right)^{\frac{1}{p'}} \times\left(\phi(2R)+R^{n+2}+R^2\|Du\|^2_{L^2(B(2R),\mathbb{R}^{nN})}+R^\delta\|Df\|^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{nN})}\right) \tag{9}$$

where C_p stands for the embedding constant from $W^{1,2}\left(B(1),\mathbb{R}^{nN}\right)$ into $L^{2p}\left(B(1),\mathbb{R}^{nN}\right)$ and

$$C\left(p,n,\frac{M}{\nu}\right)=C_p^2\times C\left(\frac{M}{\nu}\right)\times(1+c(n)), \tag{10}$$

$C(\frac{M}{\nu})$ is the constant from Lemma 2.2.

Taking in Lemma 2.5 $\psi(t)=\omega^{2p'}(t)$, $w=|Du-(Du)_R|$ on $B(R)$ and $w=0$ otherwise, we have $E_R(t)=\{y\in B(R)\,:\,|Du-(Du)_R|>t\}$ and for the last integral we get

$$\int_{B(R)}\omega^{2p'}\left(|Du-(Du)_R|\right)\,dx=\int_0^\infty\left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(E_R(t)\right)\,dt.$$

Now we can estimate the integral on the right hand side according to assumptions of the theorem. In the first case we assume that $P_p=J_p=\int_0^\infty\frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\,dt<\infty$. As $\mu\left(E_R(t)\right)$ is nonnegative and non-increasing then $\mu\left(E_R(t)\right)\leq\frac{1}{t}\int_0^t\mu\left(E_R(s)\right)ds$ holds, and we have

$$\begin{aligned}\int_0^\infty\left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(E_R(t)\right)\,dt&\leq\int_0^\infty\frac{d}{dt}(\omega^{2p'})(t)\left(\frac{1}{t}\int_0^t\mu\left(E_R(s)\right)\,ds\right)\,dt\\&\leq\int_0^\infty\frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\,dt\int_{B(R)}|Du-(Du)_R|\,dx\\&\leq J_p(\kappa_nR^n)^{\frac12}\phi^{\frac12}(R).\end{aligned} \tag{11}$$

If $P_p=S_p=\sup_{0<t<\infty}\frac{d}{dt}(\omega^{2p'})(t)<\infty$ we have

$$\int_0^\infty\left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(E_R(t)\right)\,dt\leq S_p(\kappa_nR^n)^{\frac12}\phi^{\frac12}(R) \tag{12}$$

Denoting $K^\star=\kappa_n^{-\frac{1}{2p'}}C\left(p,n,\frac{M}{\nu}\right)P_p^{\frac{1}{p'}}\|Du\|^{\frac{1}{2p'}}_{BMO(\Omega,\mathbb{R}^{nN})}$ and using (9), (11), (12) for the estimate of I_1 we get

$$I_1\leq K^\star\phi(2R)+K^\star\left(R^{n+2}+R^2\|Du\|^2_{L^2(B(2R),\mathbb{R}^{nN})}+R^\delta\|Df\|^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{nN})}\right).$$

As we suppose that $Du\in BMO(\Omega,\mathbb{R}^{nN})$ we have from Proposition 2.1 that $Du\in L^{2,\lambda}$ for any $\lambda<n$ and for $R<1$

$$\|Du\|^2_{L^2(B(2R),\mathbb{R}^{nN})}\leq c^\star(\lambda,n)R^\lambda\|Du\|_{BMO(\Omega,\mathbb{R}^{nN})}.$$

Set $\lambda = \delta - 2$ and include $c^\star(\lambda, n)$ into $K^\star$. Hence using (ii)

$$\begin{aligned} I_1 &\le K^\star \phi(2R) + K^\star \left(1 + \|Du\|^2_{BMO(\Omega,\mathbb{R}^{nN})} + \|Df\|^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{nN})}\right) R^\delta, \\ I_2 &\le M^2 R^2 \int_{B(R)} \left(1 + |Du|^2\right) dx \le M^2 \left(\kappa_n R^{n+2} + R^2 \int_{B(R)} |Du|^2 \, dx\right) \\ &\le M^2 \left(\kappa_n + c^\star(\lambda, n)\|Du\|^2_{BMO(\Omega,\mathbb{R}^{nN})}\right) R^\delta. \end{aligned} \tag{13}$$

By means of (13) we get from (8)

$$\begin{aligned} \phi(\sigma) \le & \left[4L\left(\frac{\sigma}{R}\right)^{n+2} + \frac{12\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2} K^\star\right] \phi(2R) + \frac{12\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2} \\ & \times (K^\star + M^2)\left(\kappa_n + c^\star(\lambda, n)\|Du\|^2_{BMO(\Omega,\mathbb{R}^{nN})} + 2\|Df\|^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{nN})}\right) R^\delta. \end{aligned}$$

Thus the inequality (4) is satisfied with

$$\begin{aligned} A &= 4L \\ K &= \frac{12\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2} K^\star \\ B &= \frac{12\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2} (K^\star + M^2) \\ & \quad \times \left(\kappa_n + \|Du\|^2_{BMO(\Omega,\mathbb{R}^{nN})} + 2\|Df\|^2_{L^{2,\delta-2}(\Omega,\mathbb{R}^{nN})}\right). \end{aligned}$$

We take $\alpha = n + 2$, $\beta = n + 2\gamma$. Note that ϵ_0 in Lemma 2.3 can be calculated explicitly (see the proof of Lemma 2.1., Chapter III in [8]). Then assumption (2) implies that $K < \epsilon_0$ and all assumptions of Lemma 2.3 are satisfied. Hence $\phi(\sigma) \le C\sigma^\delta$. The thesis follows from Proposition 2.1, Part (b). □

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