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Growth Estimates for the Gradient of an H-Surface Near Singular Points of the Boundary Configuration

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Abstract. We provide estimates for the gradient growth of surfaces with prescribed mean curvature in $\mathbb{R}^3$ near boundary points, which are mapped onto singular points of the boundary configuration. For corners of a Jordan arc, such estimates were provided by G. Dziuk [Analysis 1 (1981), 63–81]. We consider meeting points of a Jordan arc and a support manifold, as appearing in a partially free boundary problem (see G. Dziuk [Manuscr. Math. 35 (1981), 105–123] for the minimal surface case), and edge-type singularities of a support manifold. In subsequent papers, these results shall be used to derive asymptotic expansions of surfaces with prescribed mean curvature near such singular points.

Keywords. Surfaces of prescribed mean curvature, free boundaries, gradient growth estimates

Mathematics Subject Classification (2000). 53A10, 49Q05, 35B45, 35J55

1. Introduction and main result

In the present paper we discuss the behaviour of surfaces with prescribed mean curvature in $\mathbb{R}^3$ (shortly, H-surfaces) near boundary points, which are mapped onto certain singularities of the free or partially free boundary configuration. We will prove growth estimates for the gradient near such points. Clearly, these estimates are of independent interest, but they provide also a main ingredient for the investigation of the asymptotic behaviour of H-surfaces near those singular boundary points; see [9, 10]. Concerning similar results by G. Dziuk [2, 3], we refer to Remark 2 below. The typical new difficulty in our considerations arises from the fact that an H-surface does not have to meet the support manifold perpendicularly along its free trace.

Let us start with a description of our main result: A *(conformally parametrized) H-surface* $\mathbf{x}(w) : B^+ \to \mathbb{R}^3$ over the upper unit half-disc

$$B^+ := \{w = (u, v) = u + iv \, : \, |w| < 1, \; v > 0\} \subset \mathbb{R}^2 \simeq \mathbb{C}$$

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is a solution of

$$\begin{aligned} &\mathbf{x}(w) \in C^2(\overline{B^+} \setminus \{0\}, \mathbb{R}^3) \cap C^\mu(\overline{B^+}, \mathbb{R}^3) \cap H_2^1(B^+, \mathbb{R}^3) \\ &\Delta \mathbf{x}(w) = 2H(\mathbf{x}(w))\, \mathbf{x}_u \times \mathbf{x}_v(w), \quad w \in B^+ \\ &|\mathbf{x}_u(w)|^2 = |\mathbf{x}_v(w)|^2, \ \mathbf{x}_u \cdot \mathbf{x}_v(w) = 0, \quad w \in B^+, \end{aligned} \tag{1.1}$$

with some $\mu \in (0,1)$ and some prescribed function $H = H(\mathbf{x}) \in C^0(\mathbb{R}^3, \mathbb{R})$. In (1.1), $H_2^1(B^+, \mathbb{R}^3)$ denotes the Sobolev space of componentially measurable mappings $\mathbf{x}(w) : B^+ \to \mathbb{R}^3$, which are quadratically integrable together with their weak first derivatives. Observe that the "surface" $\mathbf{x}(w)$ is not supposed to be immersed.

On the interval $I := (-1,+1) \subset \partial B^+$ we pose one of the following boundary conditions (see Remarks 3–4 below for explanation). Setting $I^- := (-1,0)$, $I^+ := (0,+1)$, we assume:

(B1) $\mathbf{x}(I^-) \subset \Gamma$, $\mathbf{x}(I^+) \subset \mathcal{S}$, $\mathbf{x}(0) = \mathbf{x}_0$. Here $\Gamma \in C^2$ is a closed Jordan arc and $\mathcal{S} \in C^2$ denotes an open regular hypersurface in $\mathbb{R}^3$. Furthermore, $\Gamma \cap \mathcal{S} = \{\mathbf{x}_0\}$ holds true and $\mathbf{x}_0$ is an endpoint of Γ.

(B2) $\mathbf{x}(I^-) \subset \mathcal{S}^-$, $\mathbf{x}(I^+) \subset \mathcal{S}^+$, $\mathbf{x}(0) = \mathbf{x}_0$. Here $\mathcal{S}^\pm \in C^2$ are two open regular hypersurfaces in $\mathbb{R}^3$, which possess a common open boundary arc $\mathcal{C} \subset \partial\mathcal{S}^- \cap \partial S^+$ of class C^2. Finally, $\mathbf{x}_0 \in \mathcal{C}$ holds true.

(B3) $\mathbf{x}(I^-) \subset \mathcal{C}$, $\mathbf{x}(I^+) \subset \mathcal{S}$, $\mathbf{x}(0) = \mathbf{x}_0$. Here $\mathcal{S} \in C^2$ is an open regular hypersurface of $\mathbb{R}^3$ and $\mathcal{C} \subset \partial\mathcal{S}$ denotes an open boundary arc of class C^2, satisfying $\mathbf{x}_0 \in \mathcal{C}$.

(B4) $\mathbf{x}(I^- \cup I^+) \subset \mathcal{S}$, $\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{C}$ with $\mathcal{S}$ and $\mathcal{C}$ as in (B3).

If $I^\pm$ is mapped onto a Jordan arc, it is called the *fixed boundary* of $\mathbf{x}(w)$, whereas it will be named *free boundary*, whenever its image lies on a regular hypersurface.

Next we need the notion of a stationary H-surface $\mathbf{x} = \mathbf{x}(w)$: Let $\mathbf{Q} = \mathbf{Q}(\mathbf{x}) \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ denote a vector-field with $\operatorname{div} \mathbf{Q}(\mathbf{x}) = 2H(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$, and define the associated energy functional

$$E_{\mathbf{Q}}(\mathbf{y}) := \iint\limits_B \left\{ \frac{1}{2} |\nabla \mathbf{y}(w)|^2 + \mathbf{Q}(\mathbf{y}(w)) \cdot \mathbf{y}_u \times \mathbf{y}_v(w) \right\} du\, dv, \quad \mathbf{y} \in \mathcal{A}_{\mathbf{x}}\,.$$

Here $\mathcal{A}_{\mathbf{x}}$ denotes the class of all mappings $\mathbf{y}(w) \in H_2^1(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3)$, which fulfil $\mathbf{y}(w) = \mathbf{x}(w)$ on $\partial B^+ \setminus I$ and the same boundary condition (B1), (B2), (B3), or (B4) as $\mathbf{x}(w)$ on I.

Definition 1 (cf. Definition 2 in [1, Section 5.4])**.** A solution $\mathbf{x} = \mathbf{x}(w)$ of (1.1), which satisfies one of the boundary conditions (B1)–(B4), is called *stationary H-surface* (*w.r.t.* $E_{\mathbf{Q}}$), if the relation

$$\delta E_{\mathbf{Q}}(\mathbf{x}, \boldsymbol{\phi}) := \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \big\{ E_{\mathbf{Q}}(\mathbf{x}_\varepsilon) - E_{\mathbf{Q}}(\mathbf{x}) \big\} \geq 0 \tag{1.2}$$

holds true for all variations $\mathbf{x}_\varepsilon(w) := \mathbf{x}(w) + \varepsilon\boldsymbol{\phi}(w;\varepsilon) \in \mathcal{A}_\mathbf{x}$, $\varepsilon \in [0, \varepsilon_0)$. Here $\varepsilon_0 > 0$ is sufficiently small and $\boldsymbol{\phi} = \boldsymbol{\phi}(\cdot, \varepsilon) \in H_2^1(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3)$ is an admissible family of variations; that means, Dirichlet's integrals of $\boldsymbol{\phi}(\cdot, \varepsilon)$ are uniformly bounded from above in $\varepsilon \in [0, \varepsilon_0]$, and $\boldsymbol{\phi}(w, \varepsilon) \to \boldsymbol{\phi}(w, 0)$ $(\varepsilon \to 0+)$ holds true on $\overline{B^+}$.

Definition 2. A vector-field $\mathbf{Q} = \mathbf{Q}(\mathbf{x}) \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ with $\operatorname{div} \mathbf{Q}(\mathbf{x}) = 2H(\mathbf{x})$ on $\mathbb{R}^3$ is called *admissible* for

– (B1), if the relation

$$|\mathbf{Q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})| < 1 \tag{1.3}$$

holds true on $\mathcal{S}$. Here $\mathbf{n} = \mathbf{n}(\mathbf{x})$ denotes a unit normal field of $\mathcal{S}$.

– (B2), if (1.3) holds true on $\mathcal{S}^\pm \cup \mathcal{C}$ with unit normal fields $\mathbf{n}(\mathbf{x}) = \mathbf{n}^\pm(\mathbf{x})$ of $\mathcal{S}^\pm$.

– (B3) or (B4), if (1.3) holds true on $\mathcal{S} \cup \mathcal{C}$ with a unit normal field $\mathbf{n}(\mathbf{x})$ of $\mathcal{S}$.

Remark 1. The stationarity of $\mathbf{x}(w)$ yields the *natural boundary condition*

$$\mathbf{x}_v(w) + \mathbf{Q}(\mathbf{x}(w)) \times \mathbf{x}_u(w) \perp T_{\mathbf{x}(w)}\mathcal{S} \quad (\text{resp.} \perp T_{\mathbf{x}(w)}\mathcal{S}^\pm), \tag{1.4}$$

where, e.g., $T_\mathbf{x}\mathcal{S}$ denotes the tangential plane of $\mathcal{S}$ at $\mathbf{x} \in \mathcal{S}$; see [8, Theorem 1]. Clearly, (1.4) holds true on I^+ with $T_\mathbf{x}\mathcal{S}$ in cases (B1) and (B3), on $I^\pm$ with $T_\mathbf{x}\mathcal{S}^\pm$ in case (B2), and on $I^- \cup I^+$ with $T_\mathbf{x}\mathcal{S}$ in case (B4).

Formula (1.4) is equivalent to the well known relation

$$\mathbf{N}(w) \cdot \mathbf{n}(\mathbf{x}(w)) = -\mathbf{Q}(\mathbf{x}(w)) \cdot \mathbf{n}(\mathbf{x}(w)) \tag{1.5}$$

along the free trace; here $\mathbf{N}(w) := |\mathbf{x}_u \times \mathbf{x}_v(w)|^{-1}\mathbf{x}_u \times \mathbf{x}_v(w)$ denotes the surface normal of $\mathbf{x}(w)$. Clearly, $\mathbf{N}(w)$ is defined only away from branch points, where the equivalence follows by taking the cross product with $\mathbf{x}_u$. On the other hand, we can extend $\mathbf{N}(w)$ continuously to boundary branch points, in virtue of [8, Theorem 2].

Note that condition (1.3) ensures that the stationary H-surface $\mathbf{x}(w)$ cannot meet the support surface tangentially, according to (1.5). But any positive contact angle, prescribed by the vector field $\mathbf{Q}(\mathbf{x})$, is allowed.

Now we can state our main result:

Theorem 1. *Let* $\mathbf{Q}(\mathbf{x}) \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ *denote an admissible vector-field for one of the boundary conditions* (B1)–(B4)*, and consider a given stationary H-surface* $\mathbf{x} = \mathbf{x}(w) \in C^2(\overline{B^+} \setminus \{0\}) \cap C^\mu(\overline{B^+}) \cap H_2^1(B^+)$ *(w.r.t.* $E_\mathbf{Q}$*), which satisfies the respective boundary condition. In addition, choose* $\nu \in (0, \mu)$*. Then there exists* $\delta \in (0, 1)$ *and a nonnegative constant c such that the estimate*

$$|\nabla \mathbf{x}(w)| \leq c|w|^{\nu - 1}, \quad w \in \overline{S_\delta(0)} \setminus \{0\}, \tag{1.6}$$

holds true with $S_\delta(0) := \{w \in B^+ : |w| < \delta\}$*. The constant* $c \geq 0$ *depends on* ν*,* H*,* $\mathbf{Q}$*, the boundary data, and the modulus of continuity of* $\mathbf{x}$*, but not on the particular point* $w \in \overline{S_\delta(0)} \setminus \{0\}$*.*

Remark 2. Stationary minimal surfaces (that is, the case $\mathbf{Q}(\mathbf{x}) \equiv \mathbf{0}$ and consequently $H(\mathbf{x}) \equiv 0$) satisfying condition (B1) were investigated by G. Dziuk [3]. The growth estimate for the gradient developed there is slightly better then our result in allowing $\nu = \mu$ in (1.6). Observe that the gradient estimate in [3] applies also to stationary minimal surfaces subject to condition (B2), (B3), or (B4), because the estimates were provided near I^- and I^+, independently; compare Section 5 below.

Concerning the behaviour (including gradient growth estimates) of H-surfaces near corners of a Jordan arc Γ, we refer to [2].

Remark 3. Stationary H-surfaces, which satisfy the boundary condition (B1), appear, e.g., in the following *partially free boundary problem*: Construct a surface $\mathbf{x} = \mathbf{x}(w) : B^+ \to \mathbb{R}^3$ with prescribed mean curvature $H = H(\mathbf{x})$ and subject to the boundary conditions

$$\begin{aligned} &\mathbf{x}(w) \in \mathcal{S} \quad \text{for all } w \in I \\ &\mathbf{x}(w) : \partial B^+ \setminus I \to \Gamma \text{ continuously and monotonic} \\ &\mathbf{x}(-1) = \mathbf{p}_1, \quad \mathbf{x}(+1) = \mathbf{p}_2. \end{aligned} \tag{1.7}$$

Here $\Gamma \subset \mathbb{R}^3$ denotes a closed Jordan arc with endpoints $\mathbf{p}_1 \neq \mathbf{p}_2$, and $\mathcal{S} \subset \mathbb{R}^3$ is a two-dimensional submanifold without boundary, such that $\mathcal{S} \cap \Gamma = \{\mathbf{p}_1, \mathbf{p}_2\}$ holds true.

Under certain smallness assumptions on $H = H(\mathbf{x}) \in C^\alpha(\mathbb{R}^3)$ ($\alpha \in (0,1)$) and the vector-field $\mathbf{Q} = \mathbf{Q}(\mathbf{x}) \in C^{1,\alpha}(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{Q} \equiv 2H$, one can show that any minimizer of $E_{\mathbf{Q}}$ (in an appropriate class of surfaces, which satisfy (1.7) in a weak sense) belongs to the class $C^2(B^+) \cap C^\mu(\overline{B^+}) \cap H_2^1(B^+)$, provided the boundary configuration $\{\Gamma, \mathcal{S}\}$ satisfies a *chord-arc condition*; see, e.g., [1, Section 7.5], [7, Section 1]. In addition, if $\Gamma, \mathcal{S} \in C^{2,\alpha}$ holds true, one can prove $\mathbf{x}(w) \in C^2(\overline{B^+} \setminus \{-1,+1\})$; confer [5, 8]. Now, by localizing around $w = \pm 1$ (observe that the problem is conformally invariant), it follows that the resulting mapping $\mathbf{x} = \mathbf{x}(w) : B^+ \to \mathbb{R}^3$ (not renamed) is a stationary H-surface, which fulfils the boundary condition (B1) with $\mathbf{x}_0 = \mathbf{p}_1$ or $\mathbf{x}_0 = \mathbf{p}_2$.

Remark 4. Similarly, one obtains stationary H-surfaces which solve (B2), (B3), or (B4) by localizing minimizers of $E_{\mathbf{Q}}$ in a partially free boundary configuration near *edge-type singularities* of the support manifold $\mathcal{S}$. Here we call $\mathbf{x}_0 \in \mathcal{S}$ edge-type singularity, if we can write $\mathcal{S}$ near $\mathbf{x}_0$ as $\mathcal{S}^- \cup \mathcal{C} \cup \mathcal{S}^+$ with two regular hypersurfaces $\mathcal{S}^\pm \in C^2$ and an open Jordan arc $\mathcal{C} \subset \partial\mathcal{S}^- \cap \partial\mathcal{S}^+$ of class C^2, such that $\mathbf{x}_0 \in \mathcal{C}$ is granted. The conditions (B2), (B3), and (B4) correspond to the particular behaviour of the H-surface near $\mathbf{x}_0$ called transversal, uplifting, or tapping singularity in [10], respectively.

Observe that our result applies also to H-surfaces of higher genus and H-surfaces with completely free boundaries, for example. In addition, *corner-type singularities* of the free support manifold $\mathcal{S}$ can be studied (see again [10]).

The main idea in proving Theorem 1 appears already in [2,3]; see also [1, Section 8.2]: Apply the gradient estimates by E. Heinz (compare, e.g., [12, Chapter XII] or [1, Chapter 7]) near I^- and I^+, independently. To this end, one has to reflect the surface in an appropriate way across $I^\pm$. This turns out to be quite complicated near the free boundary, because H-surfaces will not meet the support manifold perpendicularly, in general: We do not know the "direction", in which we have to reflect. As in [6,8], we overcome this new difficulty by reflecting the surface and its first derivatives, *independently*, and working with a first order system for a certain complex linear combination of these first derivatives. This procedure was inspired by similar arguments in [12, Chapter XII].

We restrict ourselves to the proof of Theorem 1 under the boundary condition (B1). In Section 2 we provide a growth estimate for the area of $\mathbf{x}(w)$ near the free boundary I^+. This estimate is necessary for the gradient estimate near I^+, which will be proved in Section 3. The proof of Theorem 1 for (B1) is completed in Section 4 by sketching how to estimate the gradient growth near the fixed boundary I^-. We finish with a remark on the remaining boundary conditions (B2)–(B4) in Section 5.

2. Estimation of the area growth near the free boundary

The proof of Theorem 1 partially relies on a suitable estimate for a certain Cauchy integral. Therefore, we first have to control the area growth of $\mathbf{x}(w)$ near $w = 0$. For points $w \in B^+ \cup I^+$, which stay away from the fixed boundary I^-, this will be done in Lemma 1 below.

Concerning the notation, we use $B_\varrho(w_0) := \{w \in \mathbb{R}^2 \,:\, |w - w_0| < \varrho\}$ for the disc with radius $\varrho > 0$ around $w_0 \in \mathbb{R}^2$, and we write $B := B_1(0) = B_1(0,0)$. Moreover, we abbreviate $S_\varrho(w_0) := B^+ \cap B_\varrho(w_0)$ and $B^- := B \setminus \overline{B^+}$.

Assume a stationary H-surface $\mathbf{x} = \mathbf{x}(w)$ (i.e., a solution of (1.1), (1.2)) to be given and suppose that $\mathbf{x}$ fulfils the boundary condition (B1). Let us agree with the following normalization, which appears after suitable rotation and translation and will not affect the size of $|\nabla \mathbf{x}(w)|$: There holds $\mathbf{x}_0 = \mathbf{0}$ and there is an open neighbourhood $\mathcal{U} = \mathcal{U}(\mathbf{0}) \subset \mathbb{R}^3$ such that we may represent

$$\mathcal{S}_r := \mathcal{S} \cap \mathcal{U} = \big\{\mathbf{x} = (x^1, x^2, x^3) \in B_r(0,0) \times \mathbb{R} \,:\, x^3 = \psi(x^1, x^2)\big\}$$

with some function $\psi = \psi(x^1, x^2) : \overline{B_r(0,0)} \to \mathbb{R}$ of class $C^2(\overline{B_r(0,0)})$. Furthermore, we suppose

$$\psi(0,0) = \psi_{x^1}(0,0) = \psi_{x^2}(0,0) = 0. \tag{2.1}$$

According to the continuity of $\mathbf{x}(w)$, we find $\delta \in (0,1)$ such that

$$|\mathbf{x}(w)| < r, \quad w \in \overline{S_\delta(0)}, \tag{2.2}$$

holds true. In the following, we will further decrease $\delta \in (0,1)$ several times, always assuming (2.2) to be fulfilled.

Now we introduce the mapping $\mathbf{z}(w) = (z^1(w), z^2(w))$ with

$$\begin{aligned} z^1(w) &:= x^1_w(w) + iq(\mathbf{x}(w))x^2_w(w) + \psi_{x^1}(x^1(w), x^2(w))x^3_w(w) \\ z^2(w) &:= -iq(\mathbf{x}(w))x^1_w(w) + x^2_w(w) + \psi_{x^2}(x^1(w), x^2(w))x^3_w(w) \end{aligned} \tag{2.3}$$

for $w \in \overline{S_\delta(0)} \setminus \{0\}$. Here $x^k_w(w) = \frac{\partial x^k}{\partial w}(w)$ denote one of the Wirtinger derivatives of the components of $\mathbf{x}(w)$:

$$\frac{\partial}{\partial w} := \frac{1}{2}\Big(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\Big), \quad \frac{\partial}{\partial \overline{w}} := \frac{1}{2}\Big(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\Big).$$

And we have abbreviated

$$q(\mathbf{x}) := -\psi_{x^1}(x^1, x^2)Q^1(\mathbf{x}) - \psi_{x^2}(x^1, x^2)Q^2(\mathbf{x}) + Q^3(\mathbf{x}), \quad \mathbf{x} \in B_r(0,0) \times \mathbb{R},$$

with the components $Q^k(\mathbf{x})$ of the admissible vector-field $\mathbf{Q}(\mathbf{x}) \in C^1(\mathbb{R}^3, \mathbb{R}^3)$. Using the relation (1.4), one proves

$$\operatorname{Im} \mathbf{z}(w) = \mathbf{0}, \quad w \in I^+ \cap B_\delta(0); \tag{2.4}$$

see [6, Lemma 2] or [8, Lemma 1]. Next we reflect $\mathbf{x}(w)$ and $\mathbf{z}(w)$ across I:

$$\hat{\mathbf{z}}(w) := \begin{cases} \mathbf{z}(w), & w \in \overline{S_\delta(0)} \setminus \overline{I^-} \\ \overline{\mathbf{z}(\overline{w})}, & w \in \overline{B_\delta(0)} \setminus \overline{S_\delta(0)} \end{cases} \in C^1(B_\delta(0) \setminus I) \cap C^0(B_\delta(0) \setminus \overline{I^-}) \tag{2.5}$$

$$\hat{\mathbf{x}}(w) := \begin{cases} \mathbf{x}(w), & w \in \overline{S_\delta(0)} \\ \mathbf{x}(\overline{w}), & w \in \overline{B_\delta(0)} \setminus \overline{S_\delta(0)} \end{cases} \in C^2(B_\delta(0) \setminus I) \cap C^\mu(B_\delta(0)). \tag{2.6}$$

As in [6, Lemma 3], we find a constant $c > 0$ such that the estimates

$$\begin{aligned} &c^{-1}|\nabla \hat{\mathbf{x}}(w)| \le |\hat{\mathbf{z}}(w)| \le c|\nabla \hat{\mathbf{x}}(w)| \\ &|\hat{\mathbf{z}}_{\overline{w}}(w)| \le c|\hat{\mathbf{z}}(w)|^2, \quad w \in B_\delta(0) \setminus I, \end{aligned} \tag{2.7}$$

are satisfied for sufficiently small $\delta \in (0,1)$. The proof of (2.7) is based on the system (1.1), the normalization (2.1), the continuity of $\hat{\mathbf{x}}(w)$, and the smallness condition (1.3), which yields

$$|\mathbf{q}(\hat{\mathbf{x}}(w))| < 1, \quad w \in \overline{B_\delta(0)}, \tag{2.8}$$

for sufficiently small $\delta \in (0,1)$. The constant $c > 0$ in (2.7) depends on the data and the modulus of continuity of $\mathbf{x}(w)$.

Using (2.7) and the continuity of $\hat{\mathbf{z}}(w)$ in $B_\delta(0) \setminus \overline{I^-}$, the Gaussian integral theorem yields

$$\iint\limits_{B_\delta(0)} \left\{ \hat{\mathbf{z}} \cdot \boldsymbol{\varphi}_{\overline{w}} + |\hat{\mathbf{z}}|^2 \mathbf{h} \cdot \boldsymbol{\varphi} \right\} du\, dv = 0 \tag{2.9}$$

for all $\boldsymbol{\varphi} \in H_2^1(B_\delta(0) \setminus \overline{I^-}, \mathbb{C}^2) \cap C_0^0(B_\delta(0) \setminus \overline{I^-}, \mathbb{C}^2)$. Here the bounded function $\mathbf{h} = \mathbf{h}(w) \in L_\infty(B_\delta(0), \mathbb{C}^2)$ is defined by

$$\mathbf{h}(w) := \begin{cases} |\hat{\mathbf{z}}(w)|^{-2} \hat{\mathbf{z}}_{\overline{w}}(w), & \text{for } w \in B_\delta(0) \setminus \overline{I^-} \text{ with } |\hat{\mathbf{z}}(w)| \neq 0 \\ 0, & \text{otherwise}. \end{cases} \tag{2.10}$$

Now we are able to prove the following

Lemma 1. *Let* $\mathbf{x} = \mathbf{x}(w)$ *be a stationary* H*-surface satisfying* (B1)*, and let* $\mathbf{Q} = \mathbf{Q}(\mathbf{x})$ *be an admissible vector-field. Defining* $\hat{\mathbf{x}}(w)$ *by* (2.6)*, we have the estimate*

$$\iint\limits_{B_\varepsilon(w_0)} |\nabla \hat{\mathbf{x}}(w)|^2\, du\, dv \leq c\Big[1 + \ln \frac{\varepsilon_0}{\varepsilon}\Big] \Big(\frac{\varepsilon}{\varepsilon_0}\Big)^{2\mu} \quad \text{for all } \varepsilon \in (0, \varepsilon_0], \tag{2.11}$$

for all $w_0 \in B_\delta(0)$ *and* $\varepsilon_0 > 0$ *with* $\overline{B_{\varepsilon_0}(w_0)} \subset B_\delta(0) \setminus \overline{I^-}$*. Here* $\delta \in (0,1)$ *is chosen sufficiently small, and the constant* $c > 0$ *depends on the data and the modulus of continuity of* $\mathbf{x}(w)$*, but not on the choice of* w_0*,* ε_0*, and* ε*.*

Proof. For fixed $w_0 \in B_\delta(0) \setminus \overline{I^-}$ and $\varepsilon_0 > 0$ with $\overline{B_{\varepsilon_0}(w_0)} \subset B_\delta(0) \setminus \overline{I^-}$ we choose $\varepsilon \in (0, \varepsilon_0]$ as well as the test function $\boldsymbol{\varphi}(w) = \lambda(w) \boldsymbol{\chi}(w)$ with

$$\lambda(w) := \begin{cases} \frac{\varepsilon}{2}, & 0 \leq |w - w_0| < \frac{\varepsilon}{2} \\ \varepsilon - |w - w_0|, & \frac{\varepsilon}{2} \leq |w - w_0| < \varepsilon \\ 0, & \varepsilon \leq |w - w_0| \end{cases} \tag{2.12}$$

and

$$\boldsymbol{\chi}(w) := \begin{pmatrix} \hat{x}^1(w) - \hat{x}^1(w_0) \\ \hat{x}^2(w) - \hat{x}^2(w_0) \end{pmatrix}, \quad w \in B_\delta(0).$$

Observe that the definition (2.3) and the relation (2.8) imply

$$\begin{pmatrix}\hat{x}^1_{\overline{w}}\\ \hat{x}^2_{\overline{w}}\end{pmatrix} = \frac{1}{1-q(\hat{\mathbf{x}})^2}\left[\begin{pmatrix}\overline{\hat{z}^1}\\ \overline{\hat{z}^2}\end{pmatrix} \pm iq(\hat{\mathbf{x}})\begin{pmatrix}\overline{\hat{z}^2}\\ -\overline{\hat{z}^1}\end{pmatrix}\right] - \frac{\hat{x}^3_{\overline{w}}}{1-q(\hat{\mathbf{x}})^2}\begin{pmatrix}1 & \pm iq(\hat{\mathbf{x}})\\ \mp iq(\hat{\mathbf{x}}) & 1\end{pmatrix}\circ\begin{pmatrix}\psi_{x^1}(\hat{x}^1,\hat{x}^2)\\ \psi_{x^2}(\hat{x}^1,\hat{x}^2)\end{pmatrix} \quad \text{on } B^{\pm}\cap B_\delta(0)\,.$$

Consequently, the equation (2.9) applied to $\boldsymbol{\varphi} = \lambda\boldsymbol{\chi}$ gives

$$\iint\limits_{B_\delta(0)} \frac{1}{1-q(\hat{\mathbf{x}})^2}\lambda\,|\hat{\mathbf{z}}|^2 du\,dv \le \iint\limits_{B_\delta(0)} \frac{|q(\hat{\mathbf{x}})|+c(|\nabla\psi(\hat{x}^1,\hat{x}^2)|+|\boldsymbol{\chi}|)}{1-q(\hat{\mathbf{x}})^2}\lambda\,|\hat{\mathbf{z}}|^2 du\,dv + \iint\limits_{B_\delta(0)} |\boldsymbol{\chi}|\,|\lambda_{\overline{w}}|\,|\hat{\mathbf{z}}|\,du\,dv\,. \tag{2.13}$$

From (2.1) and (2.8) we then infer

$$\iint\limits_{B_\delta(0)} \lambda(w)|\hat{\mathbf{z}}(w)|^2\,du\,dv \le c\iint\limits_{B_\delta(0)} |\lambda_{\overline{w}}(w)|\,|\boldsymbol{\chi}(w)|\,|\hat{\mathbf{z}}(w)|\,du\,dv$$

for sufficiently small $\delta \in (0,1)$ and with some constant $c > 0$ depending on the data and the modulus of continuity of $\mathbf{x}(w)$. According to the definition of $\lambda(w)$, we further obtain

$$\begin{aligned}\frac{\varepsilon}{2}\iint\limits_{|w-w_0|<\frac{\varepsilon}{2}} |\hat{\mathbf{z}}|^2\,du\,dv &\le \iint\limits_{|w-w_0|<\varepsilon} \lambda|\hat{\mathbf{z}}|^2\,du\,dv \\ &\le c\iint\limits_{\frac{\varepsilon}{2}<|w-w_0|<\varepsilon} |\boldsymbol{\chi}|\,|\hat{\mathbf{z}}|\,du\,dv \\ &\le c\,\sigma\iint\limits_{\frac{\varepsilon}{2}<|w-w_0|<\varepsilon} |\hat{\mathbf{z}}|^2\,du\,dv + \frac{c}{\sigma}\iint\limits_{|w-w_0|<\varepsilon} |\boldsymbol{\chi}|^2\,du\,dv\end{aligned} \tag{2.14}$$

with arbitrary $\sigma > 0$ and a constant $c > 0$. Additionally, the definition of $\boldsymbol{\chi}(w)$ and the Hölder-continuity of $\mathbf{x}(w)$ imply

$$\iint\limits_{|w-w_0|<\varepsilon} |\boldsymbol{\chi}(w)|^2\,du\,dv \le c\varepsilon^{2+2\mu}, \quad \varepsilon \in (0,\varepsilon_0]. \tag{2.15}$$

If we write (w_0 is fixed)

$$D(\varepsilon) := \iint\limits_{|w-w_0|<\varepsilon} |\hat{\mathbf{z}}|^2\,du\,dv \quad \text{for } \varepsilon \in (0,\varepsilon_0],$$

the relations (2.14)-(2.15) imply the inequality

$$\frac{\varepsilon}{2} D\Big(\frac{\varepsilon}{2}\Big) \le c\sigma \Big[D(\varepsilon) - D\Big(\frac{\varepsilon}{2}\Big)\Big] + \frac{c}{\sigma}\,\varepsilon^{2+2\mu}, \quad \varepsilon \in (0, \varepsilon_0],$$

with arbitrary $\sigma > 0$. Choosing $\sigma := \frac{\varepsilon}{2c(\mu)(4^\mu - 1)}$, we especially find

$$D\Big(\frac{\varepsilon}{2}\Big) \le \Big(\frac{1}{4}\Big)^{\mu} D(\varepsilon) + \tilde{c}\,\varepsilon^{2\mu}, \quad \varepsilon \in (0, \varepsilon_0], \tag{2.16}$$

with the constant $\tilde{c} := 4^{1-\mu} c^2 (4^\mu - 1)^2 > 0$.

For any $\varepsilon \in (0, \varepsilon_0]$ there exists $k \in \mathbb{N} \cup \{0\}$ such that $\big(\frac{1}{2}\big)^{k+1} \varepsilon_0 < \varepsilon \le \big(\frac{1}{2}\big)^{k} \varepsilon_0$ is valid. Because $D(\varepsilon)$ represents a monotonically increasing function, we may estimate

$$\begin{aligned}
D(\varepsilon) &\le D\Big[\Big(\frac{1}{2}\Big)^{k} \varepsilon_0\Big] \\
&\le \Big(\frac{1}{2}\Big)^{2\mu} D\Big[\Big(\frac{1}{2}\Big)^{k-1} \varepsilon_0\Big] + \tilde{c}\Big[\Big(\frac{1}{2}\Big)^{k-1} \varepsilon_0\Big]^{2\mu} \\
&\le \Big(\frac{1}{2}\Big)^{2\mu} \Big\{\Big(\frac{1}{2}\Big)^{2\mu} D\Big[\Big(\frac{1}{2}\Big)^{k-2} \varepsilon_0\Big] + \tilde{c}\Big[\Big(\frac{1}{2}\Big)^{k-2} \varepsilon_0\Big]^{2\mu}\Big\} + \tilde{c}\Big[\Big(\frac{1}{2}\Big)^{k-1} \varepsilon_0\Big]^{2\mu} \\
&= \Big[\Big(\frac{1}{2}\Big)^{2}\Big]^{2\mu} D\Big[\Big(\frac{1}{2}\Big)^{k-2} \varepsilon_0\Big] + 2 \cdot \tilde{c}\Big[\Big(\frac{1}{2}\Big)^{k-1} \varepsilon_0\Big]^{2\mu} \\
&\ \ \vdots \\
&\le \Big[\Big(\frac{1}{2}\Big)^{k}\Big]^{2\mu} D(\varepsilon_0) + k \cdot \tilde{c}\Big[\Big(\frac{1}{2}\Big)^{k-1} \varepsilon_0\Big]^{2\mu},
\end{aligned}$$

employing (2.16) exactly k times. By virtue of the choice of $k \in \mathbb{N} \cup \{0\}$, we then obtain the inequality

$$D(\varepsilon) \le 4^{\mu} \Big(\frac{\varepsilon}{\varepsilon_0}\Big)^{2\mu} D(\varepsilon_0) + \frac{16^{\mu}\,\tilde{c}}{\ln 2}\, \varepsilon^{2\mu} \ln \frac{\varepsilon_0}{\varepsilon}, \quad \varepsilon \in (0, \varepsilon_0]. \tag{2.17}$$

Thus we arrive at the claimed estimate (2.11), if we note the relation

$$c^{-1} \iint\limits_{B_\varepsilon(w_0)} |\nabla \hat{\mathbf{x}}|^2 \, du\, dv \le D(\varepsilon) \le c \quad \text{for all } \varepsilon \in (0, \varepsilon_0],$$

with a constant $c > 0$ independent of w_0, ε_0, and ε; compare (2.7). □

3. Gradient estimates near the free boundary

Next we use the ingenious technique of E. Heinz[1] to establish an estimate for the gradient of a stationary H-surface $\mathbf{x} = \mathbf{x}(w)$ with the boundary condition (B1) in the sector

$$\Omega_\delta^+ := \Big\{ w = u + iv \in \overline{S_{\frac{\delta}{2}}(0)} \setminus \{0\} \ :\ u \ge 0 \Big\}. \tag{3.1}$$

[1]The way of proving Heinz's gradient estimates used here seems to have been initiated by F. Sauvigny [12, Chapter XII].

Here the number $\delta \in (0,1)$ is chosen as in Lemma 1.

Consider the reflected functions $\hat{\mathbf{z}}(w)$ and $\hat{\mathbf{x}}(w)$ defined in (2.3)–(2.6). We fix a point $w^* \in \Omega_\delta^+$ and set $\varepsilon^* := \frac{1}{2}|w^*|$. And we define the continuous, nonnegative function

$$\phi(w) := \big(\varepsilon^* - |w - w^*|\big)|\hat{\mathbf{z}}(w)|, \quad w \in \overline{B_{\varepsilon^*}(w^*)}.$$

Note the inclusion $\overline{B_{\varepsilon^*}(w^*)} \subset B_\delta(0) \setminus \overline{I^-}$. Because $\phi(w) = 0$ holds true on $\partial B_{\varepsilon^*}(w^*)$, there is a point $w_0 \in B_{\varepsilon^*}(w^*)$ such that $\phi(w_0) = \max_{w \in \overline{B_{\varepsilon^*}(w^*)}} \phi(w)$ is satisfied. Setting $\varepsilon_0 := \varepsilon^* - |w_0 - w^*|$, we intend to apply the representation formula of Pompeiu and Vekua

$$\hat{\mathbf{z}}(w_0) = \frac{1}{2\pi i} \int\limits_{\partial B_\varepsilon(w_0)} \frac{\hat{\mathbf{z}}(w)}{w - w_0}\, dw - \frac{1}{\pi} \iint\limits_{B_\varepsilon(w_0)} \frac{\hat{\mathbf{z}}_{\overline{w}}(w)}{w - w_0}\, du\, dv \tag{3.2}$$

for any $\varepsilon \in (0, \varepsilon_0]$ (compare, e.g., [11, p. 257]); we understand line integrals always to be positively oriented. In order to estimate the line integral in (3.2), we utilize Lemma 1: According to the inclusion $\overline{B_{\varepsilon_0}(w_0)} \subset \overline{B_{\varepsilon^*}(w^*)} \subset B_\delta(0) \setminus \overline{I^-}$, we deduce

$$\iint\limits_{B_\varepsilon(w_0)} |\nabla \hat{\mathbf{x}}(w)|^2\, du\, dv \le c\Big[1 + \ln \frac{\varepsilon_0}{\varepsilon}\Big]\Big(\frac{\varepsilon}{\varepsilon_0}\Big)^{2\mu}$$

for all $\varepsilon \in (0, \varepsilon_0]$. If we set $\varepsilon := \vartheta\varepsilon_0$, we infer

$$\iint\limits_{B_{\vartheta\varepsilon_0}(w_0)} |\nabla \hat{\mathbf{x}}(w)|^2\, du\, dv \le c\big[(1 - \ln\vartheta)\vartheta^{2\alpha}\big]\, \varepsilon^{*2(\mu-\alpha)}$$

for all $\vartheta \in (0, \varepsilon^*]$ and $\alpha \in (0, \mu)$; observe that $\varepsilon^* < 1$ holds true. The Courant–Lebesgue Lemma (see, e.g., [11, p. 50]) now yields

$$\int\limits_{\partial B_{\gamma\vartheta\varepsilon_0}(w_0)} |d\hat{\mathbf{x}}(w)| \le c\sqrt{1 - \ln\vartheta}\, \vartheta^\alpha\, \varepsilon^{*\mu-\alpha} \tag{3.3}$$

for arbitary $\vartheta \in (0, \varepsilon^*]$ and suitable $\gamma = \gamma(\vartheta) \in [\frac{1}{4}, \frac{1}{2}]$. Note that

$$m(\vartheta) := \sqrt{1 - \ln\vartheta}\, \vartheta^\alpha \ \to 0 \ (\vartheta \to 0) \tag{3.4}$$

is fulfilled for any $\alpha \in (0, \mu)$. According to (2.7) and the conformality relations in (1.1), we find

$$|\hat{\mathbf{z}}(w)\, dw| \le c|\hat{\mathbf{x}}_w(w)\, dw| = c|d\hat{\mathbf{x}}(w)| \quad \text{on } \partial B_{\gamma\vartheta\varepsilon_0}(w_0). \tag{3.5}$$

From (3.3)–(3.5) we now conclude

$$\left|\frac{1}{2\pi i}\int\limits_{\partial B_{\gamma\vartheta\varepsilon_0}(w_0)}\frac{\hat{\mathbf{z}}(w)}{w-w_0}\,dw\right|\le\frac{c}{\varepsilon_0}\frac{m(\vartheta)}{\vartheta}\,\varepsilon^{*\mu-\alpha},\quad \vartheta\in(0,\varepsilon^*],\tag{3.6}$$

with a constant $c>0$. The second integral in (3.2) can be estimated as follows:

$$\begin{aligned}\left|\frac{1}{\pi}\iint\limits_{B_{\gamma\vartheta\varepsilon_0}(w_0)}\frac{\hat{\mathbf{z}}_{\overline{w}}(w)}{w-w_0}\,du\,dv\right| &\le \frac{1}{\pi}\iint\limits_{B_{\vartheta\varepsilon_0}(w_0)}\frac{|\hat{\mathbf{z}}_{\overline{w}}(w)|}{|w-w_0|}\,du\,dv\\ &\le c\,\vartheta\varepsilon_0\max_{w\in B_{\vartheta\varepsilon_0}(w_0)}|\hat{\mathbf{z}}(w)|^2\\ &\le c\,\vartheta\varepsilon_0\max_{w\in B_{\vartheta\varepsilon_0}(w_0)}\left\{\frac{\varepsilon^*-|w-w^*|}{(1-\vartheta)\varepsilon_0}|\hat{\mathbf{z}}(w)|\right\}^2\\ &=\frac{c}{\varepsilon_0}\frac{\vartheta}{(1-\vartheta)^2}\max_{w\in B_{\vartheta\varepsilon_0}(w_0)}\phi(w)^2\\ &=\frac{c}{\varepsilon_0}\frac{\vartheta}{(1-\vartheta)^2}\,\phi(w_0)^2\quad\text{for all }\vartheta\in(0,\varepsilon^*],\end{aligned}\tag{3.7}$$

where $c>0$ may increase again. Formulas (3.2), (3.6), and (3.7) are combined to the inequality

$$\phi(w_0)\le c\,\frac{m(\vartheta)}{\vartheta}\,\varepsilon^{*\mu-\alpha}+c\,\frac{\vartheta}{(1-\vartheta)^2}\,\phi(w_0)^2,\tag{3.8}$$

which holds true for any $\vartheta\in(0,\varepsilon^*]$ and any $\alpha\in(0,\mu)$. Abbreviating $a(\vartheta):=c\,\frac{\vartheta}{(1-\vartheta)^2}$, $b(\vartheta):=c\,\frac{m(\vartheta)}{\vartheta}\,\varepsilon^{*\mu-\alpha}$, we rewrite (3.8) into the equivalent form

$$\left[\phi(w_0)-\frac{1}{2a(\vartheta)}\right]^2\ge\frac{1-4a(\vartheta)b(\vartheta)}{4a(\vartheta)^2},\quad \vartheta\in(0,\varepsilon^*].$$

By virtue of $a(\vartheta)b(\vartheta)\to 0$ $(\vartheta\to 0)$, there exists $\vartheta_0\in(0,\varepsilon^*]$ such that $1-4a(\vartheta)b(\vartheta)\ge\frac{1}{4}$ for all $\vartheta\in(0,\vartheta_0]$ is satisfied. Then we have the alternative

$$\phi(w_0)\le m^-(\vartheta)\quad\text{or}\quad\phi(w_0)\ge m^+(\vartheta)\quad\text{for all }\vartheta\in(0,\vartheta_0]\tag{3.9}$$

with the abbrevitions

$$m^{\pm}(\vartheta):=\frac{1}{2a(\vartheta)}\Big\{1\pm\sqrt{1-4a(\vartheta)b(\vartheta)}\Big\}.$$

Note that the quantities $m^{\pm}(\vartheta)$ depend continuously on $\vartheta\in(0,\vartheta_0]$ and that $m^-(\vartheta)<m^+(\vartheta)$ holds true on $(0,\vartheta_0]$. Now the second alternative in (3.9) is

impossible according to the contradiction $+\infty > \phi(w_0) \ge m^+(\vartheta) \to +\infty$ $(\vartheta \to 0)$. Hence the first alternative has to be fulfilled and we especially conclude

$$\phi(w_0) \le m^-(\vartheta_0) = \frac{2b(\vartheta_0)}{1+\sqrt{1-4a(\vartheta_0)b(\vartheta_0)}} \le \frac{4}{3}\, b(\vartheta_0).$$

According to the choice of w_0, we know $\phi(w^*) \le \phi(w_0)$ and the definition of ϕ gives us

$$|\mathbf{z}(w^*)| = |\hat{\mathbf{z}}(w^*)| = \frac{1}{\varepsilon^*}\phi(w^*) \le \frac{1}{\varepsilon^*}\phi(w_0) \le \frac{4}{3}c\frac{m(\vartheta_0)}{\vartheta_0}\varepsilon^{*\mu-\alpha-1} = c|w^*|^{\mu-\alpha-1}$$

for any $\alpha \in (0,\mu)$ and with a constant $c > 0$, which does not depend on $w^* \in \Omega_\delta^+$. Writing $\nu = \mu - \alpha \in (0,\mu)$, formula (2.7) finally implies the following

Lemma 2. *Let* $\mathbf{x} = \mathbf{x}(w)$ *be a stationary* H*-surface satisfying* (B1). *Assume* $\mathbf{Q} = \mathbf{Q}(\mathbf{x})$ *to be an admissible vector-field, and define* Ω_δ^+ *by* (3.1). *For any* $\nu \in (0,\mu)$ *and sufficiently small* $\delta \in (0,1)$ *we then have the estimate*

$$|\nabla\mathbf{x}(w)| \le c|w|^{\nu-1}, \quad w \in \Omega_\delta^+, \tag{3.10}$$

with a constant $c > 0$*, which depends on* ν*, the data, and the modulus of continuity of* $\mathbf{x}(w)$.

4. Behaviour near the fixed boundary and completion of the proof of Theorem 1 for (B1)

We will now discuss the gradient of a stationary H-surface $\mathbf{x} = \mathbf{x}(w)$ subject to (B1) in a neighbourhood of the *fixed boundary* I^+. This can also be achieved with the technique described in [1, Section 8.2] and one would even obtain a slightly better growth estimate for $|\nabla\mathbf{x}(w)|$. However, this will not yield a better result in the mixed boundary problem considered here, according to Lemma 2. So we have included a short description of the proof of estimate (1.6) near I^+, using a variant of the method in Sections 2, 3.

Let the boundary condition (B1) be satisfied and assume $\mathbf{x}_0 = \mathbf{0}$. After a rotation, which will clearly not affect the size of $|\nabla\mathbf{x}(w)|$, we may suppose

$$\Gamma \cap \mathcal{U} = \big\{(x^1,x^2,x^3) \in \mathbb{R}^3 \;:\; x^1 = \gamma_1(x^3),\ x^2 = \gamma_2(x^3),\ x^3 \in [0,r]\big\}$$

with some open neighbourhood $\mathcal{U}$ of $\mathbf{0}$ and with two functions $\gamma_1(t), \gamma_2(t) \in C^2([0,r],\mathbb{R})$ satisfying $\gamma_1(0) = \gamma_2(0) = 0$, $\gamma_1'(0) = \gamma_2'(0) = 0$. We reflect $\gamma_1(t)$, $\gamma_2(t)$ via

$$\hat\gamma_k(t) := \begin{cases} \gamma_k(t), & t \in [0,r] \\ \gamma_k(-t), & t \in [-r,0] \end{cases}, \quad k = 1,2,$$

obtaining functions $\hat{\gamma}_1(t), \hat{\gamma}_2(t) \in C^2([-r,r],\mathbb{R})$. By continuity, there exists $\delta \in (0,1)$ such that

$$|\mathbf{x}(w)| < r \quad \text{for all } w \in \overline{S_\delta(0)}$$

is valid. Now we define the auxiliary function $\mathbf{y}(w) = (y^1(w), y^2(w))$ by virtue of

$$y^1(w) := ix^1_w(w) - i\hat{\gamma}_1'(x^3(w))x^3_w(w)$$
$$y^2(w) := ix^2_w(w) - i\hat{\gamma}_2'(x^3(w))x^3_w(w), \quad w \in \overline{S_\delta(0)} \setminus \{0\},$$

and in analogy to the definition of $\mathbf{z}(w)$ in (2.3). Due to the boundary condition $\mathbf{x} : I^- \to \Gamma$, we then have $\operatorname{Im} \mathbf{y}(w) = 0$ for $w \in I^- \cap B_\delta(0)$. Consequently, the reflected function

$$\hat{\mathbf{y}}(w) := \begin{cases} y(w), & w \in \overline{S_\delta(0)} \setminus \overline{I^+} \\ \overline{y(\overline{w})}, & w \in \overline{B_\delta(0)} \setminus \overline{S_\delta(0)} \end{cases}$$

is of class $C^0(B_\delta(0) \setminus \overline{I^+}, \mathbb{C}^2)$. Moreover, we again extend $\mathbf{x}(w)$ as in (2.6) obtaining $\hat{\mathbf{x}}(w) \in H^1_2(B,\mathbb{R}^3) \cap C^\mu(B,\mathbb{R}^3)$ with $\mu \in (0,1)$. From the system (1.1) and the continuity of $\mathbf{x}(w)$ we then infer

$$\begin{aligned} &c^{-1}|\nabla\hat{\mathbf{x}}(w)| \le |\hat{\mathbf{y}}(w)| \le c|\nabla\hat{\mathbf{x}}(w)| \\ &|\hat{\mathbf{y}}_{\overline{w}}(w)| \le c|\hat{\mathbf{y}}(w)|^2, \quad w \in B_\delta(0) \setminus I, \end{aligned} \tag{4.1}$$

with a constant $c > 0$. From (4.1), the continuity of $\hat{\mathbf{y}}(w)$ in $B_\delta(0) \setminus \overline{I^+}$, and the Gaussian integral theorem we finally deduce the relation

$$\iint\limits_{B_\delta(0)} \left\{\hat{\mathbf{y}} \cdot \boldsymbol{\varphi}_{\overline{w}} + |\hat{\mathbf{y}}|^2 \mathbf{h} \cdot \boldsymbol{\varphi}\right\} du\, dv = 0 \tag{4.2}$$

for all $\boldsymbol{\varphi} \in H^1_2(B_\delta(0) \setminus \overline{I^+}, \mathbb{C}^2) \cap C^0_0(B_\delta(0) \setminus \overline{I^+}, \mathbb{C}^2)$. In (4.2), the bounded function $\mathbf{h} = \mathbf{h}(w) \in L_\infty(B_\delta(0), \mathbb{C}^2)$ is given by (2.10) with $\hat{\mathbf{z}}(w)$ replaced by $\hat{\mathbf{y}}(w)$ and I^- replaced by I^+.

Now we take $w_0 \in B_\delta(0) \setminus \overline{I^+}$ and $\varepsilon_0 > 0$ such that $\overline{B_{\varepsilon_0}(w_0)} \subset B_\delta(0) \setminus \overline{I^+}$. For arbitrary $\varepsilon \in (0,\varepsilon_0]$ we consider the test function $\boldsymbol{\varphi}(w) = \lambda(w)\boldsymbol{\psi}(w)$ with $\lambda = \lambda(w)$ defined by (2.12) and with

$$\boldsymbol{\psi}(w) := -i \begin{pmatrix} [\hat{x}^1(w) - \hat{x}^1(w_0)] - [\hat{\gamma}_1(\hat{x}^3(w)) - \hat{\gamma}_1(\hat{x}^3(w_0))] \\ [\hat{x}^2(w) - \hat{x}^2(w_0)] - [\hat{\gamma}_2(\hat{x}^3(w)) - \hat{\gamma}_2(\hat{x}^3(w_0))] \end{pmatrix}, \quad w \in B_\delta(0).$$

According to $\boldsymbol{\psi}_{\overline{w}}(w) = \overline{\hat{\mathbf{y}}(w)}$ in $B_\delta(0)$ and $|\lambda_{\overline{w}}(w)| = \frac{1}{2}$ in $B_\varepsilon(w_0) \setminus B_{\frac{\varepsilon}{2}}(w_0)$, we derive the estimate

$$\iint\limits_{B_\varepsilon(w_0)} \lambda|\hat{\mathbf{y}}|^2\, du\, dv \le \iint\limits_{B_\varepsilon(w_0)} \lambda|\hat{\mathbf{y}}|^2|\mathbf{h}|\,|\boldsymbol{\psi}|\, du\, dv + \frac{1}{2} \iint\limits_{B_\varepsilon(w_0)\setminus B_{\frac{\varepsilon}{2}}(w_0)} |\hat{\mathbf{y}}|\,|\boldsymbol{\psi}|\, du\, dv$$

from (4.2). We clearly may assume $|\mathbf{h}(w)|\,|\boldsymbol{\psi}(\hat{x}^1(w),\hat{x}^2(w))| \le \frac{1}{2}$ on $B_\delta(0)$, choosing $\delta \in (0,1)$ sufficiently small. For any $\varepsilon \in (0,\varepsilon_0]$ we then find

$$\begin{aligned}\frac{\varepsilon}{2}\iint\limits_{|w-w_0|<\frac{\varepsilon}{2}} |\hat{\mathbf{y}}|^2\,du\,dv &\le \iint\limits_{|w-w_0|<\varepsilon} \lambda|\hat{\mathbf{y}}|^2\,du\,dv \\ &\le c\iint\limits_{\frac{\varepsilon}{2}<|w-w_0|<\varepsilon} |\hat{\mathbf{y}}|\,|\boldsymbol{\psi}|\,du\,dv \\ &\le c\,\sigma\iint\limits_{\frac{\varepsilon}{2}<|w-w_0|<\varepsilon} |\hat{\mathbf{y}}|^2\,du\,dv + \frac{c}{\sigma}\iint\limits_{\frac{\varepsilon}{2}<|w-w_0|<\varepsilon} |\boldsymbol{\psi}|^2\,du\,dv\end{aligned} \tag{4.3}$$

with arbitrary $\sigma > 0$ and a constant $c > 0$, which does not depend on w_0, ε_0, and ε. Inequality (4.3) coincides with the corresponding estimate (2.14) for $\hat{\mathbf{z}}(w)$. Following the considerations below (2.14) closely, we arrive at

Lemma 3. *Let $\mathbf{x} = \mathbf{x}(w)$ be a stationary H-surface with satisfying* (B1)*, and let $\mathbf{Q} = \mathbf{Q}(\mathbf{x})$ denote an admissible vector-field. Then the reflected surface $\hat{\mathbf{x}}(w)$ satisfies the inequality*

$$\iint\limits_{B_\varepsilon(w_0)} |\nabla\hat{\mathbf{x}}|^2\,du\,dv \le c\Big[1+\ln\frac{\varepsilon_0}{\varepsilon}\Big]\Big(\frac{\varepsilon}{\varepsilon_0}\Big)^{2\mu} \quad \text{for any } \varepsilon \in (0,\varepsilon_0],$$

for all $w_0 \in B_\delta(0)$ and $\varepsilon_0 > 0$ with $\overline{B_{\varepsilon_0}(w_0)} \subset B_\delta(0) \setminus \overline{I^+}$. Here $\delta \in (0,1)$ is sufficiently small, and the constant $c > 0$ can be chosen independently of w_0, ε_0, and ε.

Now it is not difficult to transfer the calculations of Section 3 to the function

$$\tilde{\phi}(w) := (\varepsilon^* - |w - w^*|)|\hat{\mathbf{y}}(w)|, \quad w \in \overline{B_{\varepsilon^*}(w^*)},$$

where w^* is a point in the set

$$\Omega_\delta^- := \big\{w = u + iv \in \overline{S_{\frac{\delta}{2}}(0)} \setminus \{0\} \;:\; u \le 0\big\}$$

and $\varepsilon^* := \frac{1}{2}|w^*|$. Thus we find

Lemma 4. *Under the assumptions of Lemma* 3*, any stationary H-surface $\mathbf{x} = \mathbf{x}(w)$ fulfils the estimate*

$$|\nabla\mathbf{x}(w)| \le c|w|^{\nu-1}, \quad w \in \Omega_\delta^-,$$

with sufficiently small $\delta \in (0,1)$ and a constant $c > 0$ depending on ν, the data, and modulus of continuity of $\mathbf{x}(w)$.

Proof of Theorem 1 *for the boundary condition* (B1). The estimates provided by Lemma 2 and Lemma 4 prove (1.6) in the set $\overline{S_{\frac{\delta}{2}}(0)} \setminus \{0\} = \Omega_\delta^- \cup \Omega_\delta^+$ with sufficiently small $\delta \in (0,1)$. Obviously, this is the assertion with δ replaced by $\frac{\delta}{2}$. $\square$

5. Gradient estimates for the boundary conditions (B2)–(B4)

In the preceding sections we have proved Theorem 1 for stationary H-surfaces, which fulfil the boundary condition (B1). The remaining conditions (B2)–(B4), which arise for edge-type singularities of the support manifold in a free or partially free boundary problem (cf. Remark 4), can be examined with exactly the same method. This is because the proof of our theorem was given near the fixed and the free boundary, independently. The only additional ingredient is a well known continuation result, which allows us to extend, for instance, $S^{\pm}$ in the case (B2) smoothly to surfaces, which contain $\mathbf{x}_0$ in their interior; see, e.g., [4, Section 6.9]. This extension does not affect the C^2-norm of the functions $\psi^{\pm} = \psi^{\pm}(x^1, x^2)$, appearing in graph representations of $\mathcal{S}^{\pm}$ as described in Section 2. In the case (B2), the theorem now follows by applying the method of Sections 2, 3 near $I^{\pm}$, independently. The conditions (B3), (B4) can be handled similarly.

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