Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 28 (2009), 119–127

Invertibility of Matrix Wiener-Hopf plus Hankel Operators with Symbols Producing a Positive Numerical Range

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Abstract. We characterize left, right and both-sided invertibility of matrix Wiener–Hopf plus Hankel operators with possibly different Fourier symbols in the Wiener subclass of the almost periodic algebra. This is done when a certain almost periodic matrix-valued function (constructed from the initial Fourier symbols of the Hankel and Wiener–Hopf operators) admits a numerical range bounded away from zero. The invertibility characterization is based on the value of a certain mean motion. At the end, an example of a concrete Wiener–Hopf plus Hankel operator is studied in view of the illustration of the proposed theory.

Keywords. Wiener–Hopf operator, Hankel operator, almost periodic function, invertibility

Mathematics Subject Classification (2000). Primary 47B35, secondary 47A05, 47A12, 47A20, 42A75

1. Introduction

Motivated by the needs of different kinds of applications, there is a growing interest in the study of invertibility properties of the so-called Wiener–Hopf plus Hankel operators (cf., e.g., [4, 7–10, 12, 17, 19, 21, 22, 24, 25]). In fact, these operators occur in a natural manner in many applications. E.g., in the analysis of problems of wave diffraction by wedges (cf. [8, 9, 25]) this is particularly evident due to the use of some symmetrization techniques which generate sums of Hankel and Wiener–Hopf operators. Therefore, an eventual additional knowledge about the invertibility characteristics of these operators is welcome for both theoretical and practical reasons.

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Sponsored by *Fundação para a Ciência e a Tecnologia* (Portugal) under grant number SFRH/BD/38698/2007.

In the present paper, we will consider matrix Wiener–Hopf plus Hankel operators of the form

$$W_{\Phi_1} + H_{\Phi_2} \ : \ [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n \,, \tag{1.1}$$

with W_{Φ_1} and H_{Φ_2} being matrix Wiener–Hopf and Hankel operators defined by

$$W_{\Phi_1} = r_+ \mathcal{F}^{-1} \Phi_1 \cdot \mathcal{F} \quad \text{and} \quad H_{\Phi_2} = r_+ \mathcal{F}^{-1} \Phi_2 \cdot \mathcal{F} J \,,$$

respectively. We use $[L^2_+(\mathbb{R})]^n$ to denote the subspace of $[L^2(\mathbb{R})]^n$ formed by all the vector functions supported on the closure of $\mathbb{R}_+ = (0, +\infty)$, r_+ represents the operator of restriction from $[L^2(\mathbb{R})]^n$ onto $[L^2(\mathbb{R}_+)]^n$, $\mathcal{F}$ denotes the Fourier transformation, J is the reflection operator given by the rule $J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$, and Φ_1 and Φ_2 are $n \times n$ matrix functions with elements belonging to the so-called *APW* algebra. For defining this algebra, let us first consider the algebra of almost periodic functions, usually denoted by *AP*, i.e., the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ ($\lambda \in \mathbb{R}$), where

$$e_\lambda(x) = e^{i\lambda x}, \ x \in \mathbb{R}.$$

APW is the subclass of all functions $\varphi \in AP$ which can be written in the form of an absolutely convergent series, i.e.,

$$APW := \Big\{ \varphi = \sum_j \varphi_j e_{\lambda_j} \ : \ \varphi_j \in \mathbb{C}, \ \lambda_j \in \mathbb{R}, \ \sum_j |\varphi_j| < \infty \Big\}. \tag{1.2}$$

APW becomes a Banach algebra with respect to pointwise addition and multiplication when endowed with the norm $\|\varphi\|_{APW} := \sum_j |\varphi_j|$ (with φ_j in the sense of (1.2)).

We will use the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B. Applying a similar result of *Bohr's Theorem for AP functions*, it holds that for each $\phi \in \mathcal{G}APW$ there exists a real number $\kappa(\phi)$ and a function $\psi \in APW$ such that

$$\phi(x) = e^{i\kappa(\phi)x} e^{\psi(x)} \,, \tag{1.3}$$

for all $x \in \mathbb{R}$ (cf. Theorem 8.11 in [3]). The number $\kappa(\phi)$ is uniquely determined, and is called the *mean motion* of ϕ.

We would like to clarify that in opposition to the Wiener–Hopf plus Hankel case, the properties of Wiener–Hopf operators with almost periodic symbols are already well developed (cf., e.g., [3, 11]). In addition, here we are considering the possibility of $\Phi_1 \neq \Phi_2$ (see (1.1)) in contrast to some previous works that study regularity properties of Wiener–Hopf plus Hankel operators only in the case of $\Phi_1 = \Phi_2$; cf. [4, 5, 20–22].

2. Operator relations for Wiener-Hopf plus Hankel operators with essentially bounded Fourier symbols

The main purpose of this section is to present an explicit operator relation between the above defined Wiener–Hopf plus Hankel operator $W_{\Phi_1} + H_{\Phi_2}$ and a new Wiener–Hopf operator. This will be done in the form of a so-called *delta relation after extension* [6].

For such a purpose, let us first recall some different types of relations between bounded linear operators. Consider two bounded linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, acting between Banach spaces. The operators T and S are said to be *equivalent*, and we will denote this by $T \sim S$, if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that

$$T = E\,S\,F. \tag{2.1}$$

It directly follows from (2.1) that if two operators are equivalent, then they belong to the same *invertibility class* [6,23]. More precisely, one of these operators is invertible, left invertible, right invertible or only generalized invertible, if and only if the other operator enjoys the same property.

The so-called Δ*–relation after extension* was introduced in [6] for bounded linear operators acting between Banach spaces, e.g. $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$. We say that T is Δ–related after extension to S if there is a bounded linear operator acting between Banach spaces $T_\Delta : X_{1\Delta} \to X_{2\Delta}$ and invertible bounded linear operators E and F such that

$$\begin{bmatrix} T & 0 \\ 0 & T_\Delta \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_Z \end{bmatrix} F, \tag{2.2}$$

where Z is an additional Banach space and I_Z represents the identity operator in Z. In the particular case when $T_\Delta : X_{1\Delta} \to X_{2\Delta} = X_{1\Delta}$ is the identity operator, we say that the operators T and S are *equivalent after extension*. It follows from (2.2) that if we have T being Δ–related after extension to S, then the transfer of invertibility properties can only be guaranteed in one direction, that is, from operator S to operator T.

We are now in conditions to present the above announced result which is based on the famous Gohberg–Krupnik–Litvinchuk identity (see [13–16,18], for instance).

Theorem 2.1. *Let $\Phi_1, \Phi_2 \in \mathcal{G}[L^\infty(\mathbb{R})]^{n\times n}$. Then the matrix Wiener–Hopf plus Hankel operator $W_{\Phi_1} + H_{\Phi_2} : [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n$ is Δ–related after extension to the Wiener–Hopf operator $W_\Psi : [L^2_+(\mathbb{R})]^{2n} \to [L^2(\mathbb{R}_+)]^{2n}$ with Fourier symbol*

$$\Psi = \begin{bmatrix} \Phi_1 - \Phi_2 \widetilde{\Phi_1^{-1}} \widetilde{\Phi_2} & -\Phi_2 \widetilde{\Phi_1^{-1}} \\ \widetilde{\Phi_1^{-1}} \widetilde{\Phi_2} & \widetilde{\Phi_1^{-1}} \end{bmatrix}. \tag{2.3}$$

Proof. Let us start by extending $W_{\Phi_1} + H_{\Phi_2}$ on the left by the zero extension operator $\ell_0 : [L^2(\mathbb{R}_+)]^n \to [L^2_+(\mathbb{R})]^n$, which leads to

$$W_{\Phi_1} + H_{\Phi_2} \sim \ell_0(W_{\Phi_1} + H_{\Phi_2}) : [L^2_+(\mathbb{R})]^n \to [L^2_+(\mathbb{R})]^n.$$

Using the notation $P_+ = \ell_0 r_+$ and $P_- = I_{[L^2(\mathbb{R})]^n} - P_+$, we will now extend

$$\ell_0(W_{\Phi_1} + H_{\Phi_2}) = P_+\mathcal{F}^{-1}(\Phi_1 \cdot +\Phi_2 \cdot J)\mathcal{F}_{|P_+[L^2(\mathbb{R})]^n}$$

to the full $[L^2(\mathbb{R})]^n$ space by using the identity in $[L^2_-(\mathbb{R})]^n$. Next we will extend the obtained operator to $[L^2(\mathbb{R})]^{2n}$ with the help of the auxiliary paired operator

$$\mathcal{T} = \mathcal{F}^{-1}(\Phi_1 \cdot -\Phi_2 \cdot J)\mathcal{F}P_+ + P_- : [L^2(\mathbb{R})]^n \to [L^2(\mathbb{R})]^n .$$

Altogether, we have

$$\left[\begin{array}{cc|c} \ell_0(W_{\Phi_1} + H_{\Phi_2}) & 0 & 0 \\ 0 & I_{P_-[L^2(\mathbb{R})]^n} & 0 \\ \hline 0 & 0 & \mathcal{T} \end{array}\right] = E_1 \mathcal{W}_1 F_1$$

with

$$E_1 = \frac{1}{2}\begin{bmatrix} I_{[L^2(\mathbb{R})]^n} & J \\ I_{[L^2(\mathbb{R})]^n} & -J \end{bmatrix}$$

$$F_1 = \begin{bmatrix} I_{[L^2(\mathbb{R})]^n} & I_{[L^2(\mathbb{R})]^n} \\ J & -J \end{bmatrix}\begin{bmatrix} I_{[L^2(\mathbb{R})]^n} - P_-\mathcal{F}^{-1}(\Phi_1 \cdot +\Phi_2 \cdot J)\mathcal{F}P_+ & 0 \\ 0 & I_{[L^2(\mathbb{R})]^n} \end{bmatrix}$$

$$\begin{aligned}
\mathcal{W}_1 &= \begin{bmatrix} \mathcal{F}^{-1}\Phi_1 \cdot \mathcal{F} & 0 \\ \mathcal{F}^{-1}\widetilde{\Phi_2} \cdot \mathcal{F} & 1 \end{bmatrix} P_+ + \begin{bmatrix} 1 & \mathcal{F}^{-1}\Phi_2 \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\Phi_1} \cdot \mathcal{F} \end{bmatrix} P_- \\
&= \begin{bmatrix} 1 & \mathcal{F}^{-1}\Phi_2 \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\Phi_1} \cdot \mathcal{F} \end{bmatrix} (\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ + P_-) \\
&= \begin{bmatrix} 1 & \mathcal{F}^{-1}\Phi_2 \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\Phi_1} \cdot \mathcal{F} \end{bmatrix} (P_+\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ + P_-)(I_{[L^2(\mathbb{R})]^{2n}} + P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+),
\end{aligned}$$

where in the last definition of operator $\mathcal{W}_1$ we are using $P_\pm$ defined in $[L^2(\mathbb{R})]^{2n}$ and Ψ is the same as defined in (2.3). Note that the paired operator

$$I_{[L^2(\mathbb{R})]^{2n}} + P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ : [L^2(\mathbb{R})]^{2n} \to [L^2(\mathbb{R})]^{2n}$$

used above is an invertible operator with inverse given by

$$I_{[L^2(\mathbb{R})]^{2n}} - P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ : [L^2(\mathbb{R})]^{2n} \to [L^2(\mathbb{R})]^{2n}.$$

Therefore, we have just explicitly demonstrated that $W_{\Phi_1} + H_{\Phi_2}$ is Δ–related after extension to the Wiener–Hopf operator

$$W_\Psi = r_+\mathcal{F}^{-1}\Psi \cdot \mathcal{F} : [L^2_+(\mathbb{R})]^{2n} \to [L^2(\mathbb{R}_+)]^{2n} ,$$

and this concludes the proof. □

Corollary 2.2. *Let* $\Phi_1, \Phi_2 \in \mathcal{G}[L^\infty(\mathbb{R})]^{n\times n}$. *If the Wiener–Hopf operator* $W_\Psi : [L^2_+(\mathbb{R})]^{2n} \to [L^2(\mathbb{R}_+)]^{2n}$ *is Fredholm or (left/right/both-sided) invertible, then the Wiener–Hopf plus Hankel operator* $W_{\Phi_1} + H_{\Phi_2} : [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n$ *has the same corresponding property.*

Proof. Due to the Δ-relation after extension between the two operators presented in Theorem 2.1, we derive that:

(i) $\operatorname{im} \begin{bmatrix} W_{\Phi_1} + H_{\Phi_2} & 0 \\ 0 & \mathcal{T} \end{bmatrix}$ is closed if and only if $\operatorname{im} W_\Psi$ is closed;

(ii) $\big([L^2(\mathbb{R}_+)]^n \times [L^2(\mathbb{R})]^n\big) \setminus \overline{\operatorname{im} \begin{bmatrix} W_{\Phi_1} + H_{\Phi_2} & 0 \\ 0 & \mathcal{T} \end{bmatrix}} \simeq [L^2(\mathbb{R}_+)]^{2n} \setminus \overline{\operatorname{im} W_\Psi}$;

(iii) $\ker \begin{bmatrix} W_{\Phi_1} + H_{\Phi_2} & 0 \\ 0 & \mathcal{T} \end{bmatrix} \simeq \ker W_\Psi$.

Then, these properties (i)–(iii) are enough to conclude the desired statement, taking into consideration the definitions of Fredholm property and (lateral/-both-sided) invertibility. □

3. An invertibility characterization based on a mean motion and a numerical range

We recall that the *numerical range* of a complex matrix $\Theta \in \mathbb{C}^{m\times m}$ is defined by

$$\mathcal{H}(\Theta) = \{(\Theta\eta, \eta) : \eta \in \mathbb{C}^m, \|\eta\| = 1\}.$$

If $\Phi \in [APW]^{m\times m}$, then (due to the definition of APW) we also have that $\mathcal{H}(\Phi(x))$ is well-defined for all $x \in \mathbb{R}$. In this way, the numerical range of Φ is said to be *bounded away from zero* if $\inf_{x\in\mathbb{R}} \operatorname{dist}\left(\mathcal{H}\left(\Phi(x)\right),\, 0\right) > 0$ or, equivalently, if there is an $\varepsilon > 0$ such that

$$\left|\big(\Phi(x)\eta, \eta\big)\right| \geq \varepsilon\|\eta\|^2 \quad \text{for all } x \in \mathbb{R} \text{ and all } \eta \in \mathbb{C}^m.$$

Consider $\Phi \in [APW]^{m\times m}$ and $\eta \in \mathbb{C}^m\setminus\{0\}$. If the numerical range of Φ is bounded away from zero, then the function $(\Phi\eta, \eta)$ given by

$$(\Phi\eta, \eta)(x) = (\Phi(x)\eta, \eta) , \quad x \in \mathbb{R}$$

is invertible in APW. Thus, the mean motion of $(\Phi\eta, \eta)$, denoted by $\kappa\big((\Phi\eta, \eta)\big)$, is well-defined for all $\eta \in \mathbb{C}^m \backslash \{0\}$. Moreover, due to a theorem by Babadzhanyan and Rabinovich (see Theorem 9.9 in [3], and cf. also [1], [2]), we have that $\kappa\big((\Phi\eta, \eta)\big)$ is independent of $\eta \in \mathbb{C}^m \backslash \{0\}$.

Theorem 3.1. *Let us consider $\Phi_1, \Phi_2 \in \mathcal{G}[APW]^{n\times n}$ such that the numerical range of Ψ (defined in (2.3)) is bounded away from zero. We have the following characterization of the invertibility properties of the operator $W_{\Phi_1} + H_{\Phi_2}$:*

(a) *If $\kappa\big((\Psi\,\eta, \eta)\big) = 0$, then $W_{\Phi_1} + H_{\Phi_2}$ is invertible.*
(b) *If $\kappa\big((\Psi\,\eta, \eta)\big) > 0$, then $W_{\Phi_1} + H_{\Phi_2}$ is left invertible.*
(c) *If $\kappa\big((\Psi\,\eta, \eta)\big) < 0$, then $W_{\Phi_1} + H_{\Phi_2}$ is right invertible.*

Proof. The assertion is now a consequence of the Δ–relation after extension presented in the last section, and of the corresponding result for Wiener–Hopf operators (cf. Corollary 9.10 in [3]). Indeed, first, the hypotheses in (a), (b), and (c) give us (from [3, Corollary 9.10]) the invertibility, left invertibility, and right invertibility of W_Ψ, respectively. Then, by using Corollary 2.2, these three cases lead us to the final conclusion about the Wiener–Hopf plus Hankel operator. □

It is clear that the condition on the numerical range of Ψ – to be bounded away from zero – is fundamental in the last result. In view of this, it is also clear that not all Fourier symbol matrix functions Φ_1 and Φ_2 in $\mathcal{G}[APW]^{n\times n}$ yield such a property for the corresponding Ψ matrix function. For instance,

$$\Phi_1(x) = \Phi_2(x) = \begin{bmatrix} 2e^{\,e^{ix}} & e^{\,e^{ix}} \\ e^{-e^{ix}} & e^{-e^{ix}} \end{bmatrix}, \quad x \in \mathbb{R} \tag{3.1}$$

is invertible in $[APW]^{2\times 2}$ but produces a matrix function Ψ which does not have a numerical range bounded away from zero. In fact, for Φ_1 and Φ_2 in (3.1), if we take any $\eta = (\eta_1, \eta_2, 0, 0)^\top \in \mathbb{C}^4$ such that $\|\eta\| = 1$ then a direct computation yields $([\Psi(x)]\eta, \eta) = 0$ for all $x \in \mathbb{R}$. Note that in such computation the identity $\Phi_1 = \Phi_2$ plays an important role. Anyway, we would like to point out that such a particular case of equal Fourier symbols $\Phi_1 = \Phi_2$ is also possible to consider by using matrices bounded away from zero, and even in a simpler way than in the present more complex case (cf. [20, Chapter 6]).

4. Example

To illustrate the previous theorem, we will present in this last section a concrete case of an invertible Wiener–Hopf plus Hankel operator $W_{\phi_1} + H_{\phi_2}$ with different APW Fourier symbols ϕ_1 and ϕ_2:

$$\phi_1(x) = 2e^{2e^{-ix}} \quad \text{and} \quad \phi_2(x) = e^{e^{ix}+e^{-ix}}, \quad x \in \mathbb{R}.$$

Note that ϕ_1 and ϕ_2 are invertible and ϕ_1^{-1}, $\phi_2^{-2} \in APW$. This yields that ϕ_1, $\phi_2 \in \mathcal{G}APW$. In this case, for the element Ψ from the last section, we have the particular form

$$\Psi(x) = \begin{bmatrix} \frac{3}{2}e^{2e^{-ix}} & -\frac{1}{2}e^{e^{-ix}-e^{ix}} \\ \frac{1}{2}e^{-e^{ix}+e^{-ix}} & \frac{1}{2}e^{-2e^{ix}} \end{bmatrix}, \quad x \in \mathbb{R}.$$

Let us now analyze that the numerical range of Ψ is bounded away from zero. Considering $\eta = (\eta_1, \eta_2)^\top \in \mathbb{C}^2$, such that $\|\eta\| = 1$, it follows that

$$\begin{aligned} (\Psi(x)\eta, \eta) &= \left(\begin{bmatrix} \frac{3}{2}\eta_1 e^{2e^{-ix}} - \frac{1}{2}\eta_2 e^{e^{-ix}-e^{ix}} \\ \frac{1}{2}\eta_1 e^{-e^{ix}+e^{-ix}} + \frac{1}{2}\eta_2 e^{-2e^{ix}} \end{bmatrix}, \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \right) \\ &= \frac{3}{2}|\eta_1|^2 e^{2e^{-ix}} + \frac{1}{2}|\eta_2|^2 e^{-2e^{ix}} + \Im m(\eta_1\overline{\eta_2}) i e^{-2i\sin x}, \end{aligned}$$

for all $x \in \mathbb{R}$. Therefore,

$$\mathcal{H}\big(\Psi(x)\big) = \left\{ \frac{3}{2}|\eta_1|^2 e^{2e^{-ix}} + \frac{1}{2}|\eta_2|^2 e^{-2e^{ix}} + \Im m(\eta_1\overline{\eta_2})\, i\, e^{-2i\sin x}, \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \mathbb{C}^2, \|\eta\| = 1 \right\} \tag{4.1}$$

for $x \in \mathbb{R}$. Then, we have that

$$\operatorname{dist}\Big(\mathcal{H}\big(\Psi(x)\big),\ 0\Big) = \left[\left| \frac{3}{2}|\eta_1|^2 e^{2e^{-ix}} + \frac{1}{2}|\eta_2|^2 e^{-2e^{ix}} \right|^2 + \left| \Im m(\eta_1\overline{\eta_2}) e^{-2i\sin x} \right|^2 \right]^{\frac{1}{2}}$$

with $\eta = (\eta_1, \eta_2)^\top \in \mathbb{C}^2$ such that $\|\eta\| = 1$.

Let $\|\eta\| = 1$. It being clear that $\Im m(\eta_1\overline{\eta_2})e^{-2i\sin x} = 0$ if and only if $\Im m(\eta_1\overline{\eta_2}) = 0$, we will now verify that $\left| \frac{3}{2}|\eta_1|^2 e^{2e^{-ix}} + \frac{1}{2}|\eta_2|^2 e^{-2e^{ix}} \right| \neq 0$ for all $x \in \mathbb{R}$. We have

$$\begin{aligned} \left| \frac{3}{2}|\eta_1|^2 e^{2e^{-ix}} + \frac{1}{2}|\eta_2|^2 e^{-2e^{ix}} \right| &= \left| \frac{3}{2}|\eta_1|^2 e^{-2i\sin x + 2\cos x} + \frac{1}{2}|\eta_2|^2 e^{-2i\sin x - 2\cos x} \right| \\ &= \left| e^{-2i\sin x} \right| \left| \frac{3}{2}|\eta_1|^2 e^{2\cos x} + \frac{1}{2}|\eta_2|^2 e^{-2\cos x} \right| \\ &= \left| \frac{3}{2}|\eta_1|^2 e^{2\cos x} + \frac{1}{2}|\eta_2|^2 e^{-2\cos x} \right| > 0\,. \end{aligned}$$

Altogether, this leads to

$$\inf_{x\in\mathbb{R}} \operatorname{dist}\Big(\mathcal{H}\big(\Psi(x)\big),\ 0\Big) > 0, \tag{4.2}$$

i.e., the numerical range of Ψ is bounded away from zero.

Now, to compute the corresponding mean motion, we start by considering $\eta = (1,0)^\top$. From (4.1), it follows that $(\Psi(x)\eta,\eta) = \frac{3}{2}e^{2e^{-ix}}$, $x \in \mathbb{R}$. Due to the fact that $e^{2e^{-ix}} \in \mathcal{G}APW$, from the analogue of Bohr's Theorem for AP functions (cf. (1.3)), we have that

$$k\big((\Psi\eta,\eta)\big) = 0 \tag{4.3}$$

for $\eta = (1,0)^\top$. From (4.2) and (4.3), and according to the *Babadzhanyan and Rabinovich Theorem* (mentioned above), it follows that $k\big((\Psi\eta,\eta)\big) = 0$ for all $\eta \in \mathbb{C}^2\backslash\{0\}$. Therefore, applying Theorem 3.1, we conclude that $W_{\phi_1} + H_{\phi_2}$ is an invertible operator.

Acknowledgment. This work was supported in part by *Unidade de Investigação Matemática e Aplicações* of Universidade de Aveiro through the Portuguese Science Foundation (*FCT–Fundação para a Ciência e a Tecnologia*).

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Received July 26, 2007; revised January 3, 2008