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Description of Pointwise Multipliers in Pairs of Besov Spaces $B_1^k(\mathbb{R}^n)$

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Abstract. Necessary and sufficient conditions for a function to be a multiplier mapping the Besov space $B_1^m(\mathbb{R}^n)$ into the Besov space $B_1^l(\mathbb{R}^n)$ with integer l and m, $0 < l \leq m$, are found. It is shown that multipliers between $B_1^m(\mathbb{R}^n)$ and $B_1^l(\mathbb{R}^n)$ form the space of traces of multipliers between the Sobolev classes $W_1^{m+1}(\mathbb{R}_+^{n+1})$ and $W_1^{l+1}(\mathbb{R}_+^{n+1})$.

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1. Introduction

By a multiplier acting from one Banach function space S_1 into another S_2, we mean a function γ such that $\gamma\, u \in S_2$ for any $u \in S_1$. We adopt the notation $M(S_1 \to S_2)$ for the space of multipliers $\gamma : S_1 \to S_2$ with the norm

$$\|\gamma\|_{M(S_1\to S_2)} = \sup\{\|\gamma\, u\|_{S_2} : \ \|u\|_{S_1} \leq 1\}$$

and we write MS instead of $M(S \to S)$.

A theory of multipliers in spaces of differentiable functions was developed in the book [4]. Necessary and sufficient conditions for a function to belong to the space $M(B_p^m(\mathbb{R}^n) \to B_p^l(\mathbb{R}^n))$ with $p \in (1,\infty)$ and integer m and l, subject to $0 < l \leq m$, were given in [4] for $l = m$ and in [5] for $l < m$. The case of $p \in [1,\infty)$ and non-integer l and m such that $0 < l \leq m$ was characterized in [4]. The space $M(B_{p,1}^{1/p}(\mathbb{R}^n) \to B_{p,\infty}^{1/p}(\mathbb{R}^n))$ was described by Gulisashvili [1]. Two results on multipliers preserving a Besov class are due to Sickel and Smirnov [6], who described $MB_{p,q}^s(\mathbb{R}^n)$ for $1 \leq p \leq q \leq \infty$, $s > \frac{n}{p}$, and to Koch and Sickel [2], who characterized the spaces $MB_{\infty,1}^0(\mathbb{R}^n)$ and $MB_{\infty,\infty}^0(\mathbb{R}^n)$.

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A complete description of the space $M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))$ in the case of positive integer m and l, $m \ge l$, is the main result of the present paper. We show also that $M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))$ is the space of traces on $\mathbb{R}^n$ of functions in $M(W_1^{m+1}(\mathbb{R}_+^{n+1}) \to W_1^{l+1}(\mathbb{R}_+^{n+1}))$.

Let $s = k + \alpha$, where $\alpha \in (0,1]$ and k is a nonnegative integer. Further, let $\Delta_h^{(2)} u(x) = u(x+h) - 2u(x) + u(x-h)$ and

$$(C_s u)(x) = \int_{\mathbb{R}^n} |\Delta_h^{(2)} \nabla_k u(x)|\, |h|^{-n-\alpha} dh, \tag{1}$$

where ∇_k stands for the gradient of order k, i.e., $\nabla_k u = \{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}\}$, $\alpha_1 + \dots + \alpha_n = k$. The Besov space $B_1^s(\mathbb{R}^n)$ is introduced as the space of functions defined on $\mathbb{R}^n$, which have the finite norm

$$\|u\|_{B_1^s(\mathbb{R}^n)} = \|C_s u\|_{L_1(\mathbb{R}^n)} + \|u\|_{L_1(\mathbb{R}^n)}. \tag{2}$$

Throughout this paper, the equivalence relation $a \sim b$ means that there exist positive constants c_1, c_2 such that $c_1 b \le a \le c_2 b$. Let $\mathcal{B}_r^{(n)}(x)$ denote the ball $\{y \in \mathbb{R}^n : |y - x| < r\}$ and let $\mathcal{B}_r^{(n)} = \mathcal{B}_r^{(n)}(0)$. Further, we adopt the notation $\mathcal{B}$ for $\mathcal{B}_1^{(n)}$. In what follows, c stands for different constants depending on l, m, and n. We also use the notation $(\Delta^{(2)} u)(x,y) = u(x) - 2u\big(\frac{x+y}{2}\big) + u(y)$.

With any Banach function space S of functions on $\mathbb{R}^n$, one can associate the spaces

$$S_{\rm loc} = \big\{u :\ \eta u \in S \quad \text{for all}\, \eta \in C_0^\infty\big\}$$

and

$$S_{\rm unif} = \Big\{u :\ \sup_{z \in \mathbb{R}^n} \|\eta_z\, u\|_S < \infty\Big\},$$

where $\eta_z(x) = \eta(x - z)$, $\eta \in C_0^\infty$, $\eta = 1$ on $\mathcal{B}$.

The following theorem proved in Section 3 provides a complete characterization of the space of multipliers $M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))$. The integral over $\big(\mathcal{B}_r^{(n)}(z)\big)^2$ stands for the double integral over $\mathcal{B}_r^{(n)}(z)$.

Theorem 1.1. *Let m and l be integers, $m \ge l \ge 1$. The following equivalence relation holds:*

$$\|\gamma\|_{M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))} \sim \sup_{\substack{z \in \mathbb{R}^n \\ r \in (0,1)}} r^{m-n} \bigg(\int_{(\mathcal{B}_r^{(n)}(z))^2} \big|\big(\Delta^{(2)} \nabla_{l-1} \gamma\big)(x,y)\big| \frac{dx\, dy}{|x-y|^{n+1}} + r^{-l} \|\gamma\|_{L_1(\mathcal{B}_r^{(n)}(z))} \bigg). \tag{3}$$

Remark 1.2. It is obvious that for $m = l$ relation (3) can be written as

$$\|\gamma\|_{MB_1^l(\mathbb{R}^n)} \sim \sup_{\substack{z \in \mathbb{R}^n \\ r \in (0,1)}} r^{m-n} \int_{(\mathcal{B}_r^{(n)}(z))^2} \big|\big(\Delta^{(2)} \nabla_{l-1} \gamma\big)(x,y)\big| \frac{dx\, dy}{|x-y|^{n+1}} + \|\gamma\|_{L_\infty(\mathbb{R}^n)}. \tag{4}$$

Another description of the space $M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))$, involving the function $C_l\gamma$ given by (1), is presented in the following assertion.

Theorem 1.3. *Let m and l be an integers, $m \ge l \ge 1$. The following equivalence relation holds:*

$$\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))} \sim \sup_{\substack{z\in\mathbb{R}^n \\ r\in(0,1)}} r^{m-n}\|C_l\gamma\|_{L_1(\mathcal{B}_r^{(n)}(z))} + \|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}. \tag{5}$$

If $m = l$, then

$$\|\gamma\|_{MB_1^l(\mathbb{R}^n)} \sim \sup_{\substack{z\in\mathbb{R}^n \\ r\in(0,1)}} r^{m-n}\|C_l\gamma\|_{L_1(\mathcal{B}_r^{(n)}(z))} + \|\gamma\|_{L_\infty(\mathbb{R}^n)}. \tag{6}$$

For $m \ge n$ and $m > l$,

$$\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))} \sim \sup_{z\in\mathbb{R}^n}\Big(\|C_l\gamma\|_{L_1(\mathcal{B}_1^{(n)}(z))} + \|\gamma\|_{L_1(\mathcal{B}_1^{(n)}(z))}\Big) \tag{7}$$

which, in its turn, is equivalent to $\|\gamma\|_{B^l_{1,\mathrm{unif}}(\mathbb{R}^n)}$.

We use the notation $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$ and $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$. By $W_1^k(\mathbb{R}^{n+1}_+)$ we mean the space of functions defined on $\mathbb{R}^{n+1}_+$ with the finite norm

$$\|U\|_{W_1^k(\mathbb{R}^{n+1}_+)} = \|\nabla_k U\|_{L_1(\mathbb{R}^{n+1}_+)} + \|U\|_{L_1(\mathbb{R}^{n+1}_+)}.$$

The following theorem shows that $M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))$ is the space of traces on $\mathbb{R}^n$ of functions in $M(W_1^{m+1}(\mathbb{R}^{n+1}_+) \to W_1^{l+1}(\mathbb{R}^{n+1}_+))$.

Theorem 1.4. *Let m and l be integers, $m \ge l \ge 1$.*

(i) *Suppose that $\gamma \in M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))$. Then the Dirichlet problem*

$$\Delta\Gamma = 0 \text{ on } \mathbb{R}^{n+1}_+, \quad \Gamma|_{\mathbb{R}^n} = \gamma$$

has a unique solution in $M(W_1^{m+1}(\mathbb{R}^{n+1}_+) \to W_1^{l+1}(\mathbb{R}^{n+1}_+))$ and the estimate

$$\|\Gamma\|_{M(W_1^{m+1}(\mathbb{R}^{n+1}_+)\to W_1^{l+1}(\mathbb{R}^{n+1}_+))} \le c\,\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))} \tag{8}$$

holds.

(ii) *Suppose that $\Gamma \in M(W_1^{m+1}(\mathbb{R}^{n+1}_+) \to W_1^{l+1}(\mathbb{R}^{n+1}_+))$. If γ is the trace of Γ on $\mathbb{R}^n$, then*

$$\gamma \in M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))$$

and the estimate

$$\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))} \le c\,\|\Gamma\|_{M(W_1^{m+1}(\mathbb{R}^{n+1}_+)\to W_1^{l+1}(\mathbb{R}^{n+1}_+))} \tag{9}$$

holds.

2. Auxiliary assertions

2.1. Properties of functions in Besov spaces. We start with the statement of a well-known trace and extension results.

Lemma 2.1 ([8])**.** *Suppose $m \geq 1$.*

(i) *Let U be an arbitrary function in the space $W_1^{l+1}(\mathbb{R}_+^{n+1})$. Then for almost all $x \in \mathbb{R}^n$, there exists the limit $u(x) = \lim_{\rho \to 0} U(x, \rho)$, the function u belongs to the space $B_1^l(\mathbb{R}^n)$ and*

$$\|u\|_{B_1^l(\mathbb{R}^n)} \leq c \, \|U\|_{W_1^{l+1}(\mathbb{R}_+^{n+1})}.$$

(ii) *Let Tu denote the Poisson operator of the function $u \in L_{1,\mathrm{unif}}(\mathbb{R}^n)$ defined by*

$$(Tu)(x, y) = \frac{1}{|S^n|} \int_{\mathbb{R}^n} \frac{y \, u(\xi) d\xi}{(y^2 + |x - \xi|^2)^{\frac{n+1}{2}}}, \quad (x, y) \in \mathbb{R}_+^{n+1}, \tag{10}$$

where $|S^n|$ is the area of the boundary S^n of the $(n+1)$-dimensional unit ball. Then

$$\|Tu\|_{W_1^{l+1}(\mathbb{R}_+^{n+1})} \leq c \, \|u\|_{B_1^l(\mathbb{R}^n)}.$$

The next lemma contains an interpolation inequality for functions in $B_1^l(\mathbb{R}^n)$.

Lemma 2.2. *Let $u \in B_1^l(\mathbb{R}^n)$. Then, for any $j = 0, \dots, l-1$,*

$$\|u\|_{B_1^{l-j}(\mathbb{R}^n)} \leq c \, \|u\|_{B_1^l(\mathbb{R}^n)}^{\frac{l-j}{l}} \, \|u\|_{L_1(\mathbb{R}^n)}^{\frac{j}{l}}. \tag{11}$$

Proof. We introduce the function

$$(C_s^{(q)} u)(x) = \int_{\mathbb{R}^n} \frac{|\Delta_h^{(q)} u(x)|}{|h|^{n+s}} dh \tag{12}$$

with any integer $q > s$, where $\Delta_h^{(q)} u(x)$ is the difference of order q defined by $\Delta_h^{(q)} u(x) = \sum_{i=0}^{q} \binom{q}{i} (-1)^i \, u(x + (q-i)h)$. Given $s > 0$, the equivalence relation

$$\|u\|_{B_1^s(\mathbb{R}^n)} \sim \|C_s^{(q)} u\|_{L_1(\mathbb{R}^n)} + \|u\|_{L_1(\mathbb{R}^n)} \tag{13}$$

holds for all values of q greater than s (see [7, Section 3.5.3]). Let $q > l$. Summing up the two inequalities

$$\int_{\mathbb{R}^n} \int_{\mathcal{B}} \frac{|\Delta_h^{(q)} u(x)|}{|h|^{n+l-j}} dh \, dx \leq \int_{\mathbb{R}^n} \int_{\mathcal{B}} \frac{|\Delta_h^{(q)} u(x)|}{|h|^{n+l}} dh \, dx$$

and

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus \mathcal{B}} \frac{|\Delta_h^{(q)} u(x)|}{|h|^{n+l-j}} dh \, dx \leq c \, \|u\|_{L_1(\mathbb{R}^n)},$$

we find that

$$\|C^{(q)}_{l-j}u\|_{L_1(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\Delta^{(q)}_h u(x)|}{|h|^{n+l-j}} dh\, dx \le c\big(\|C^{(q)}_l u\|_{L_1(\mathbb{R}^n)} + \|u\|_{L_1(\mathbb{R}^n)}\big). \quad (14)$$

By (13), $\|C_{l-j}u\|_{L_1(\mathbb{R}^n)} \le c\big(\|C^{(q)}_{l-j}u\|_{L_1(\mathbb{R}^n)} + \|u\|_{L_1(\mathbb{R}^n)}\big)$. This, together with (14) and (13), leads to

$$\|C_{l-j}u\|_{L_1(\mathbb{R}^n)} \le c\big(\|C_l u\|_{L_1(\mathbb{R}^n)} + \|u\|_{L_1(\mathbb{R}^n)}\big). \quad (15)$$

The interpolation inequality (11) follows by dilation transform in (15). □

We introduce the space $B^l_1(\mathcal{B})$ of functions on the unit ball $\mathcal{B}$ with the finite norm

$$\|u\|_{B^l_1(\mathcal{B})} = \sum_{j=0}^{l-1} \|\nabla_j u\|_{L_1(\mathcal{B})} + \sum_{j=0}^{l-1} \int_{\mathcal{B}}\int_{\mathcal{B}} \big|(\Delta^{(2)}_y \nabla_j u)(x)\big| \frac{dx\, dy}{|x-y|^{n+1}}.$$

A local variant of inequality (11) is contained in the next statement.

Corollary 2.3. *Let $u \in B^l_1(\mathcal{B})$. Then, for any $j = 0, \dots, l-1$,*

$$\|u\|_{B^{l-j}_1(\mathcal{B})} \le c\, \|u\|^{\frac{l-j}{l}}_{B^l_1(\mathcal{B})} \|u\|^{\frac{j}{l}}_{L_1(\mathcal{B})}. \quad (16)$$

Proof. It is standard (see [7, Section 4.5]) that u can be extended onto $\mathbb{R}^n$ so that

$$\|u\|_{B^l_1(\mathbb{R}^n)} \le c\, \|u\|_{B^l_1(\mathcal{B})} \quad (17)$$

$$\|u\|_{L_1(\mathbb{R}^n)} \le c\, \|u\|_{L_1(\mathcal{B})}. \quad (18)$$

These inequalities, combined with Lemma 2.2, give (16). □

We need the following Hardy type inequality.

Lemma 2.4. *Let $u \in B^l_1(\mathbb{R}^n)$, $l < n$. Then*

$$\int_{\mathbb{R}^n} |x|^{-l}\, |u(x)|\, dx \le c\, \|u\|_{B^l_1(\mathbb{R}^n)}. \quad (19)$$

Proof. Let $U \in W^{l+1}_1(\mathbb{R}^{n+1}_+)$ be an arbitrary extension of u. We have

$$\int_{\mathbb{R}^n} |x|^{-l}\, |u(x)|\, dx = \frac{2^l - 1}{l} \int_0^\infty \frac{dr}{r^{l+1}} \int_{\mathcal{B}^{(n)}_{2r}\setminus\mathcal{B}^{(n)}_r} |u(x)|\, dx. \quad (20)$$

To estimate the right-hand side of (20), we use the standard trace inequality

$$\int_{\mathcal{B}^{(n)}_{2r}\setminus\mathcal{B}^{(n)}_r} |u(x)|\, dx \le c \int_{\mathcal{G}_{2r}\setminus\mathcal{G}_r} \big(r^{-1}|U(z)| + |\nabla U(z)|\big) dz,$$

where $\mathcal{G}_r = \mathcal{B}_r^{(n+1)} \cap \mathbb{R}_+^{n+1}$. Together with (20), this implies

$$\int_{\mathbb{R}^n} |x|^{-l}\, |u(x)|\, dx \le c \int_{\mathbb{R}_+^{n+1}} \left(\frac{|U(z)|}{|z|} + |\nabla U(z)|\right) \frac{dz}{|z|^l}. \tag{21}$$

Iterating the Hardy type inequality

$$\int_{\mathbb{R}_+^{n+1}} |\nabla_j U(z)| \frac{dz}{|z|^{l+1-j}} \le c \int_{\mathbb{R}_+^{n+1}} |\nabla_{j+1} U(z)| \frac{dz}{|z|^{l-j}}$$

with $j = 0, \ldots, l-1$, we find that the right-hand side in (21) is dominated by $c \int_{\mathbb{R}_+^{n+1}} |\nabla_{l+1} U(z)| dz$. Taking into account that u is the trace of U on $\mathbb{R}^n$ and using part (i) of Lemma 2.1, we complete the proof. □

The next lemma contains two more inequalities for intermediate derivatives of functions given on the ball $\mathcal{B}_r^{(n)}$.

Lemma 2.5. *Let l be a positive integer and let $j = 0, \ldots, l-1$. Then, for any $r \in (0, 1]$,*

$$\begin{aligned} &r^{j-l} \|\nabla_j u\|_{L_1(\mathcal{B}_r^{(n)})} \\ &\le c \left(\int_{(\mathcal{B}_r^{(n)})^2} \big|(\Delta^{(2)} \nabla_{l-1} u)(x,y)\big| \frac{dx\, dy}{|x-y|^{n+1}} + r^{-l} \|u\|_{L_1(\mathcal{B}_r^{(n)})} \right) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &r^{j+1-l} \int_{(\mathcal{B}_r^{(n)})^2} \big|(\Delta^{(2)} \nabla_j u)(x,y)\big| \frac{dx\, dy}{|x-y|^{n+1}} \\ &\le c \left(\int_{(\mathcal{B}_r^{(n)})^2} \big|(\Delta^{(2)} \nabla_{l-1} u)(x,y)\big| \frac{dx\, dy}{|x-y|^{n+1}} + r^{-l} \|u\|_{L_1(B_r)} \right). \end{aligned} \tag{23}$$

Proof. By dilation, the proof reduces to the case $r = 1$. It is well known that for $j = 1, \ldots, l-2$,

$$\|\nabla_j u\|_{L_1(\mathcal{B})} \le c \left(\|\nabla_{l-1} u\|_{L_1(\mathcal{B})} + \|u\|_{L_1(\mathcal{B})} \right). \tag{24}$$

Hence, it suffices to prove (22) for $j = l-1$. We introduce the function $\varphi \in C_0^\infty(\mathcal{B})$ subject to $\int_{\mathbb{R}^n} \varphi(y) dy = 1$. Integrating by parts, we have

$$\begin{aligned} \nabla_{l-1} u(x) = &\int_{\mathcal{B}} \varphi(y) \big(\Delta^{(2)} \nabla_{l-1} u\big)(x,y) dy \\ &+ (-1)^{l-1} \int_{\mathcal{B}} \left(2^{2-l} u\left(\frac{x+y}{2}\right) - u(y) \right) (\nabla_{l-1} \varphi)(y) dy. \end{aligned}$$

Therefore,

$$\int_{\mathcal{B}} |\nabla_{l-1}u(x)|dx \le \int_{\mathcal{B}} \left| \int_{\mathcal{B}} \varphi(y)\big(\Delta^{(2)}\nabla_{l-1}u\big)(x,y)\, dy \right| dx + c\, \|u\|_{L_1(\mathcal{B})}.$$

Since the right-hand side does not exceed

$$c\Big(\int_{\mathcal{B}} \int_{\mathcal{B}} \big| \big(\Delta^{(2)}\nabla_{l-1}u\big)(x,y)\big| \frac{dydx}{|x-y|^{n+1}} + \|u\|_{L_1(\mathcal{B})} \Big),$$

we arrive at (22) with $j = l-1$. The proof of (22) is complete. Finally, (23) results from the definition of the space $B_1^l(\mathcal{B})$ and inequalities (16) and (22). □

2.2. Auxiliary estimates for the Poisson operator. We deal with the operator T defined by (10).

Lemma 2.6. *Let $\gamma \in W_{1,loc}^{l-1}(\mathbb{R}^n)$. Then*

$$\int_0^\infty \left| \frac{\partial^{l+1}(T\gamma)}{\partial y^{l+1}} \right| dy \le c\, (C_l\gamma)(x), \tag{25}$$

where $(C_l\gamma)(x)$ is defined by (1).

Proof. For any n-dimensional multi-index α with $|\alpha| = 2$,

$$\begin{aligned} D_x^\alpha(T\gamma)(x,y) &= y^{-n-2} \int_{\mathbb{R}^n} (D^\alpha\zeta)\Big(\frac{\xi - x}{y}\Big)\gamma(\xi)d\xi \\ &= y^{-n-2} \int_{\mathbb{R}^n} (D^\alpha\zeta)\Big(\frac{h}{y}\Big)\gamma(x+h)dh \\ &= y^{-n-2} \int_{\mathbb{R}^n} \zeta_{0,\alpha}\Big(\frac{h}{y}\Big)\Delta_h^{(2)}\gamma(x)dh, \end{aligned} \tag{26}$$

where $\zeta_{0,\alpha} = \frac{1}{2}(D^\alpha\zeta)(\xi)$. The last equality in (26) is valid because $D^\alpha\zeta$ is even and satisfies $\int_{\mathbb{R}^n} D^\alpha\zeta(t)dt = 0$. Since

$$(T\gamma)(x,y) = \frac{1}{2}y^{-n} \int_{\mathbb{R}^n} \zeta\Big(\frac{h}{y}\Big)\Delta_h^{(2)}\gamma(x)dh + \gamma(x), \tag{27}$$

it follows for any n-dimensional multi-index β with $|\beta| = 1$ that

$$\begin{aligned} \frac{\partial}{\partial y}D_x^\beta(T\gamma)(x,y) &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\partial}{\partial y}\Big(y^{-n-1}\big(D_y^\beta\zeta\big)\Big(\frac{h}{y}\Big)\Big)\Delta_h^{(2)}\gamma(x)dh \\ &= y^{-n-2}\int_{\mathbb{R}^n} \zeta_{0,\beta}\Big(\frac{h}{y}\Big)\Delta_h^{(2)}\gamma(x)dh, \end{aligned} \tag{28}$$

where $\zeta_{0,\beta} = -\frac{1}{2}\big((n+1+\langle\xi,\nabla\rangle)D^\beta\zeta\big)(\xi)$.

Suppose $l \geq 2$. Let $\tau = \alpha + \delta$, where $|\tau| = l+1, |\alpha| = 2, |\delta| = l-1$. By (26),

$$D_x^\tau (T\gamma)(x,y) = y^{-n-2} \int_{\mathbb{R}^n} \zeta_{0,\alpha}\left(\frac{h}{y}\right) \Delta_h^{(2)} (D^\delta \gamma)(x) dh. \tag{29}$$

Next, let $\tau = \beta + \delta$, where $|\tau| = l$, $|\beta| = 1$, $|\delta| = l-1$. In view of (28),

$$\frac{\partial}{\partial y} D_x^\tau (T\gamma)(x,y) = y^{-n-2} \int_{\mathbb{R}^n} \zeta_{0,\beta}\left(\frac{h}{y}\right) \Delta_h^{(2)} (D^\delta \gamma)(x) dh. \tag{30}$$

Suppose that $l+1$ is even, then the harmonicity of $T\gamma$ implies

$$\frac{\partial^{l+1}}{\partial y^{l+1}} (T\gamma)(x,y) = (-\Delta_x)^{\frac{l+1}{2}} (T\gamma)(x,y).$$

Hence by (29),

$$\left| \frac{\partial^{l+1}}{\partial y^{l+1}} (T\gamma)(x,y) \right| \leq c\, y^{-n-2} \int_{\mathbb{R}^n} \zeta_1 \left(\frac{h}{y}\right) \left| (\Delta_h^{(2)} \nabla_{l-1} \gamma)(x) \right| dh, \tag{31}$$

where

$$0 < \zeta_1(\xi) \leq c\,(1+|\xi|)^{-n-3}. \tag{32}$$

If $l+1$ is odd, then we have by harmonicity of $T\gamma$

$$\frac{\partial^{l+1}}{\partial y^{l+1}} (T\gamma)(x,y) = \frac{\partial}{\partial y} (-\Delta_x)^{\frac{l}{2}} (T\gamma)(x,y).$$

This, together with (30), gives

$$\left| \frac{\partial^{l+1}}{\partial y^{l+1}} (T\gamma)(x,y) \right| \leq c\, y^{-n-2} \int_{\mathbb{R}^n} \zeta_2 \left(\frac{h}{y}\right) \left| (\Delta_h^{(2)} \nabla_{l-1} \gamma)(x) \right| dh, \tag{33}$$

where

$$0 < \zeta_2(\xi) \leq c(1+|\xi|)^{-n-2}. \tag{34}$$

Hence,

$$\begin{aligned} \int_0^\infty \left| \frac{\partial^{l+1}(T\gamma)}{\partial y^{l+1}} \right| dy &\leq c \int_0^\infty dy \int_{\mathbb{R}^n} \frac{|(\Delta_h^{(2)} \nabla_{l-1} \gamma)(x)|}{(y+|h|)^{n+2}} dh \\ &= \frac{c}{n+1} \int_{\mathbb{R}^n} \frac{|(\Delta_h^{(2)} \nabla_{l-1} \gamma)(x)|}{|h|^{n+1}} dh \end{aligned}$$

which completes the proof. $\square$

Lemma 2.7. *Suppose* $\gamma \in W^{l-1}_{1,\mathrm{loc}}(\mathbb{R}^n)$ *and let*

$$N = \sup_{\substack{x\in\mathbb{R}^n \\ r\in(0,1)}} r^{m-n} \|C_l\gamma\|_{L_1(\mathcal{B}^{(n)}_r(x))}. \tag{35}$$

Then, for any $y \in (0,1]$,

$$\left|\frac{\partial^{l+1}(T\gamma)(x,y)}{\partial y^{l+1}}\right| \le c\, N\, y^{-m-1}.$$

Proof. By Lemma 2.6,

$$\int_{\mathcal{B}^{(n)}_r(x)} dz \int_0^\infty \left|\frac{\partial^{l+1}(T\gamma)(z,y)}{\partial y^{l+1}}\right| dy \le c\, N\, r^{n-m} \tag{36}$$

for $r \in (0,1)$. Let $\frac{r}{2} < y \le r$. Applying the mean value theorem for harmonic functions, we find

$$\left|\frac{\partial^{l+1}(T\gamma)(x,y)}{\partial y^{l+1}}\right| \le \frac{c}{r^{n+1}} \int_{\mathcal{B}^{(n)}_r(x)} dz \int_{\frac{r}{2}}^r \left|\frac{\partial^{l+1}(T\gamma)(z,\eta)}{\partial \eta^{l+1}}\right| d\eta.$$

By (36), the right-hand side is dominated by cNr^{-1-m}. The proof is complete. □

The next assertion is based mainly on two previous lemmas.

Corollary 2.8. *Let* $0 < l < m \le n$ *and let* $\gamma \in W^{l-1}_{1,\mathrm{loc}}(\mathbb{R}^n)$. *Then, for all* $x \in \mathbb{R}^n$,

$$|\gamma(x)| \le c\left(N^{\frac{l}{m}}\big((C_l\gamma)(x)\big)^{\frac{m-l}{m}} + \|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}\right).$$

Proof. We use the inequality due to Verbitsky

$$|\gamma(x)| \le c\left(\|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)} + \int_0^1 \left|\frac{\partial^{l+1}(T\gamma)(x,y)}{\partial y^{l+1}}\right| y^l dy\right) \tag{37}$$

(see [4, Section 2.6.1]). Introducing the notation

$$\varphi(y) = \begin{cases} \left|\frac{\partial^{l+1}(T\gamma)(x,y)}{\partial y^{l+1}}\right| & \text{for } 0 < y \le 1 \\ 0 & \text{for } y > 1, \end{cases}$$

for any $R > 0$, we have

$$\int_0^1 \left|\frac{\partial^{l+1}(T\gamma)(x,y)}{\partial y^{l+1}}\right| y^l dy = \int_0^\infty \varphi(y)\, y^l\, dy \le R^l \int_0^R \varphi(y) dy + \int_R^\infty \varphi(y) y^l dy.$$

By Lemma 2.6, the first term in the right-hand side is majorized by $cR^l(C_l\gamma)(x)$ and by Lemma 2.7,

$$\int_R^\infty \varphi(y) y^l dy \le c\, N \int_R^\infty y^{l-m-1} dy = c\, N\, R^{l-m}.$$

Choosing R as $R = N^{\frac{1}{m}}\big((C_l\gamma)(x)\big)^{-\frac{1}{m}}$, we arrive at the inequality

$$\int_0^\infty \varphi(y)y^l dy \le c\, N^{\frac{l}{m}}\big((C_l\gamma)(x)\big)^{\frac{m-l}{m}}$$

which together with (37) completes the proof. □

Corollary 2.9. *Suppose* $\gamma \in W^{l-1}_{1,\mathrm{loc}}(\mathbb{R}^n)$. *For any integer* $l \ge 1$ *and any* $z \in \mathbb{R}^n$

$$r^{m-n-l}\|\gamma\|_{L_1(\mathcal{B}_r^{(n)}(z))} \le c\left(\sup_{\rho\in(0,1)} \rho^{m-n}\|C_l\gamma\|_{L_1(\mathcal{B}_\rho^{(n)}(z))} + \|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}\right). \tag{38}$$

Proof. By Corollary 2.8,

$$\begin{aligned} &r^{m-n-l}\|\gamma\|_{L_1(\mathcal{B}_r^{(n)}(z))} \\ &\le c\left(N^{\frac{l}{m}}\, r^{m-n-l}\int_{\mathcal{B}_r^{(n)}(z)} \big((C_l\gamma)(x)\big)^{\frac{m-l}{m}} dx + r^{m-l}\|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}\right)\end{aligned}$$

which does not exceed

$$c\left(N^{\frac{l}{m}}\left(r^{m-n}\int_{\mathcal{B}_r^{(n)}(z)} (C_l\gamma)(x)dx\right)^{\frac{m-l}{m}} + \|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}\right)$$

by Hölder's inequality. Using (35), we complete the proof. □

In the next two lemmas, we return to the Poisson operator T.

Lemma 2.10. *Let* $r < y < 1$. *Then, for any* $k \ge 1$

$$|\nabla_k(T\gamma)(x,y)| \le c\, r^{l-m-k} \sup_{\substack{z\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-l}\|\gamma\|_{L_1(\mathcal{B}_\rho^{(n)}(z))}. \tag{39}$$

Proof. By (10),

$$|\nabla_k(T\gamma)(x,y)| \le c\int_{\mathbb{R}^n} \frac{|\gamma(\xi)|}{(|x-\xi|+y)^{n+k}} d\xi. \tag{40}$$

We have

$$\begin{aligned}\int_{\mathcal{B}_r^{(n)}(x)} \frac{|\gamma(\xi)|}{(|x-\xi|+y)^{n+k}} d\xi &\le c\, y^{-n-k}\int_{\mathcal{B}_r^{(n)}(x)} |\gamma(\xi)|d\xi \\ &\le c\, r^{l-m-k} \sup_{\substack{z\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-l}\|\gamma\|_{L_1(\mathcal{B}_\rho^{(n)}(z))}.\end{aligned} \tag{41}$$

Besides,

$$\int_{\mathbb{R}^n\setminus\mathcal{B}_r^{(n)}(x)} \frac{|\gamma(\xi)|}{(|x-\xi|+y)^{n+k}}d\xi \le \int_{\mathbb{R}^n\setminus\mathcal{B}_r^{(n)}(x)} \frac{|\gamma(\xi)|}{|x-\xi|^{n+k}}d\xi$$
$$\le c\,r^{-n}\int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}(x)} dz \int_{\mathcal{B}_r^{(n)}(z)} \frac{|\gamma(\xi)|}{|x-\xi|^{n+k}}d\xi.$$

Since $|\xi - x| > \frac{|z-x|}{2}$, it follows that the right-hand side of the last inequality does not exceed

$$c\,r^{l-m}\int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}(x)} \frac{dt}{|t-x|^{n+k}} \sup_{\substack{x\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-l}\|\gamma\|_{L_1(\mathcal{B}_\rho^{(n)}(x))}.$$

Therefore,

$$\int_{\mathbb{R}^n\setminus\mathcal{B}_r^{(n)}(x)} \frac{|\gamma(\xi)|}{(|x-\xi|+y)^{n+k}}\,d\xi \le c\,r^{l-m-k} \sup_{\substack{z\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-l}\|\gamma\|_{L_1(\mathcal{B}_\rho^{(n)}(z))}. \tag{42}$$

Now, (39) results by combining inequalities (41), (42), and (40). □

Lemma 2.11. *Let $y > 1$ and let $k \ge 0$, then*

$$|\nabla_k(T\gamma)(x,y)| \le c\,y^{-k}\|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}. \tag{43}$$

Proof. First, observe that

$$\int_{\mathcal{B}_y^{(n)}(x)} \frac{|\gamma(\xi)|\,d\xi}{(|x-\xi|+y)^{n+k}} \le cy^{-n-k}\int_{\mathcal{B}_y^{(n)}(x)}|\gamma(\xi)|d\xi \le cy^{-k}\|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}. \tag{44}$$

Clearly,

$$\int_{\mathbb{R}^n\setminus\mathcal{B}_y^{(n)}(x)} \frac{|\gamma(\xi)|}{(|x-\xi|+y)^{n+k}}d\xi \le c\int_{\mathbb{R}^n\setminus\mathcal{B}_2^{(n)}(x)} dz \int_{\mathcal{B}_1^{(n)}(z)} \frac{|\gamma(\xi)|}{|x-\xi|^{n+k}}d\xi. \tag{45}$$

Since $|\xi - x| > \frac{|z-x|}{2}$, the right-hand side of (45) is dominated by

$$c\,\|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus\mathcal{B}_2^{(n)}(x)} \frac{dz}{|z-x|^{n+k}}.$$

Therefore,

$$\int_{\mathbb{R}^n\setminus\mathcal{B}_y^{(n)}(x)} \frac{|\gamma(\xi)|}{(|x-\xi|+y)^{n+k}}d\xi \le c\,y^{-k}\|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}. \tag{46}$$

Summing up inequalities (44) and (46), and then using estimate (40), we complete the proof. □

3. Proof of Theorem 1.1

We make use of the norm

$$\|v\|_{B_1^l(\mathcal{B}_r^{(n)})} = \sum_{j=0}^{l-1} r^{j-l}\|\nabla_j v\|_{L_1(\mathcal{B}_r^{(n)})} + \sum_{j=0}^{l-1} r^{j+1-l}\int_{(\mathcal{B}_r^{(n)})^2} \big|\Delta_y^{(2)}\nabla_j v(x)\big|\frac{dx\,dy}{|x-y|^{n+1}}$$

defined for a positive integer l and $r \in (0,1]$. Lemma 2.5 implies

$$\|v\|_{B_1^l(\mathcal{B}_r^{(n)})} \sim \int_{(\mathcal{B}_r^{(n)})^2} \big|(\Delta_y^{(2)}\nabla_{l-1}v)(x)\big|\frac{dx\,dy}{|x-y|^{n+1}} + r^{-l}\|v\|_{L_1(\mathcal{B}_r^{(n)})}. \tag{47}$$

By dilation in (16), we obtain

$$\|v\|_{B_1^{l-j}(\mathcal{B}_r^{(n)})} \leq c\,\|v\|^{1-\frac{j}{l}}_{B_1^l(\mathcal{B}_r^{(n)})}\|v\|^{\frac{j}{l}}_{L_1((\mathcal{B}_r^{(n)})} \tag{48}$$

for any $j = 0,\dots,l-1$.

Owing to (47), the required relation (3) can be written as

$$\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))} \sim \sup_{\substack{z\in\mathbb{R}^n\\ r\in(0,1)}} r^{m-n}\|\gamma\|_{B_1^l(\mathcal{B}_r^{(n)}(z))}. \tag{49}$$

In view of Lemma 2.4 and (47), we obtain

$$\|v\|_{B_1^l(\mathcal{B}_r^{(n)})} \leq c\,\|v\|_{B_1^l(\mathbb{R}^n)} \tag{50}$$

for $l < n$. Let $u(y) = \eta\left(\frac{y-x}{r}\right)$, where $r \in (0,1)$ for $m < n$ and $r = 1$ for $m \geq n$, and $\eta \in C_0^\infty(\mathcal{B}_2^{(n)})$, $\eta = 1$ on $\mathcal{B}_1^{(n)}$. Setting this u into the inequality

$$\|\gamma\,u\|_{B_1^l(\mathbb{R}^n)} \leq \|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))}\|u\|_{B_1^m(\mathbb{R}^n)} \tag{51}$$

and using (50) with $v = \gamma u$, we have

$$\|\gamma\|_{B_1^l(\mathcal{B}_r^{(n)}(x))} \leq c\,r^{n-m}\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))} \tag{52}$$

for any $x \in \mathbb{R}^n$. The required lower estimate for the norm $\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))}$ follows from (47).

Now we obtain the upper estimate for the norm $\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))}$. Let, as before, $T\gamma$ stand for the Poisson integral of γ. For any $U \in W_1^{m+1}(\mathbb{R}_+^{n+1})$, we have by Lemma 2.1 that

$$\|\gamma\,u\|_{B_1^l(\mathbb{R}^n)} \leq c\,\|(T\gamma)U\|_{W_1^{l+1}(\mathbb{R}_+^{n+1})}, \tag{53}$$

where $u(x) = U(x,0)$. Let $X = (x,y) \in \mathbb{R}^{n+1}_+$ and let $\mathcal{G}_r(X) = \mathcal{B}^{(n+1)}_r(X) \cap \mathbb{R}^{n+1}_+$ as in the proof of Lemma 2.4. It is shown in [4, Sect. 6.1], that for any integer $l \in [0,m)$,

$$\|\Gamma\|_{M(W_1^{m+1}(\mathbb{R}^{n+1}_+)\to W_1^{l+1}(\mathbb{R}^{n+1}_+))} \sim \sup_{\substack{X\in\mathbb{R}^{n+1}_+\\ r\in(0,1)}} r^{m-n}\|\nabla_{l+1}\Gamma\|_{L_1(\mathcal{G}_r(X))} + \sup_{X\in\mathbb{R}^{n+1}_+}\|\Gamma\|_{L_1(\mathcal{G}_1(X))}. \tag{54}$$

The first supremum in (54) can be replaced by $\sup_{X\in\mathbb{R}^{n+1}_+}\|\nabla_{l+1}\Gamma\|_{L_1(\mathcal{G}_1(X))}$ in the case $m \geq n$. Furthermore,

$$\|\Gamma\|_{MW_1^{l+1}(\mathbb{R}^{n+1}_+)} \sim \sup_{\substack{X\in\mathbb{R}^{n+1}_+\\ r\in(0,1)}} r^{l-n}\|\nabla_{l+1}\Gamma\|_{L_1(\mathcal{G}_r(X))} + \|\Gamma\|_{L_\infty(\mathbb{R}^{n+1}_+)}. \tag{55}$$

This and (53) give

$$\|\gamma\, u\|_{B_1^l(\mathbb{R}^n)} \leq c\, K_{m,l}\|U\|_{W_1^{m+1}(\mathbb{R}^{n+1}_+)}, \tag{56}$$

where

$$K_{m,l} = \sup_{\substack{X\in\mathbb{R}^{n+1}_+\\ r\in(0,1)}} r^{m-n}\|\nabla_{l+1}(T\gamma)\|_{L_1(\mathcal{G}_r(X))} + \sup_{X\in\mathbb{R}^{n+1}_+}\|T\gamma\|_{L_1(\mathcal{G}_1(X))}. \tag{57}$$

We introduce one more notation

$$k_{m,l} := \sup_{\substack{z\in\mathbb{R}^n\\ r\in(0,1)}} r^{m-n}\int_{(\mathcal{B}^{(n)}_r(z))^2}\left|\Delta^{(2)}\nabla_{l-1}\gamma(x,y)\right|\frac{dx\,dy}{|x-y|^{n+1}} \tag{58}$$

and intend to show that

$$K_{m,l} \leq c\left(k_{m,l} + \sup_{z\in\mathbb{R}^n}\|\gamma\|_{L_1(\mathcal{B}^{(n)}_1(z))}\right). \tag{59}$$

Then the upper estimate for $\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))}$ follows from (56) by Lemma 2.1 and the arbitrariness of U.

Let us justify (59). When estimating $\|\nabla_{l+1}(T\gamma)\|_{L_1(\mathcal{G}_r(X_0))}$, where $X_0 \in \mathbb{R}^{n+1}_+$, it suffices to take $X_0 = (0,y_0)$. Suppose first that $y_0 > 2$. Then by Lemma 2.11, $r^{m-n}\|\nabla_{l+1}(T\gamma)\|_{L_1(\mathcal{G}_r(X_0))} \leq c\,\|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}$. For $2 > y_0 > 2r$, in view of Lemma 2.10, we have

$$r^{m-n}\|\nabla_{l+1}(T\gamma)\|_{L_1(\mathcal{G}_r(X_0))} \leq c \sup_{\substack{x\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-l}\|\gamma\|_{L_1(\mathcal{B}^{(n)}_\rho(x))}.$$

Given any $r \in (0,1)$, it remains to estimate the norm $\|\nabla_{l+1}(T\gamma)\|_{L_1(\mathcal{G}_r(X_0))}$ for $y_0 < 2r$.

For any even $k \geq 2$ and $|\sigma| = l + 1 - k$, the harmonicity of $T\gamma$ in $\mathbb{R}^{n+1}_+$ implies $\frac{\partial^k}{\partial y^k} D^\sigma_x (T\gamma)(x, y) = D^\sigma_x (-\Delta_x)^{\frac{k}{2}} (T\gamma)(x, y)$. Hence by (29),

$$\left| \frac{\partial^k}{\partial y^k} D^\sigma_x (T\gamma)(x, y) \right| \leq c\, y^{-n-2} \int_{\mathbb{R}^n} \zeta_1 \left(\frac{h}{y} \right) \left| (\Delta^{(2)}_h \nabla_{l-1} \gamma)(x) \right| dh, \tag{60}$$

where ζ_1 obeys (32). Similarly, $\frac{\partial^k}{\partial y^k} D^\sigma_x (T\gamma)(x, y) = \frac{\partial}{\partial y} D^\sigma_x (-\Delta_x)^{\frac{k-1}{2}} (T\gamma)(x, y)$ for any odd $k \geq 3$. Using (30), we have

$$\left| \frac{\partial^k}{\partial y^k} D^\sigma_x (T\gamma)(x, y) \right| \leq c\, y^{-n-2} \int_{\mathbb{R}^n} \zeta_2 \left(\frac{h}{y} \right) \left| (\Delta^{(2)}_h \nabla_{l-1} \gamma)(x) \right| dh \tag{61}$$

with ζ_2 satisfying (34).

Introducing the notation

$$J_1 := \int_{\mathcal{G}_r(X_0)} y^{-n-2} dx\, dy \int_{\mathcal{B}^{(n)}_y} \zeta_2 \left(\frac{h}{y} \right) \left| \Delta^{(2)}_h (\nabla_{l-1} \gamma)(x) \right| dh, \tag{62}$$

we deduce from (34) that for $y_0 < 2r$

$$\begin{aligned} J_1 &\leq c \int_0^{3r} y^{-n-2} dy \int_{\mathcal{B}^{(n)}_{2r}} dx \int_{\mathcal{B}^{(n)}_y} \left| \Delta^{(2)}_h \nabla_{l-1} \gamma)(x) \right| dh \\ &= c \int_{\mathcal{B}^{(n)}_{3r}} |h|^{-n-1} dh \int_{\mathcal{B}^{(n)}_{2r}} \left| (\Delta^{(2)}_h \nabla_{l-1} \gamma)(x) \right| dx. \end{aligned}$$

Therefore,

$$J_1 \leq c\, r^{n-m}\, k_{m,l} \tag{63}$$

with $k_{m,l}$ given by (58).

Let

$$J_2 := \int_{\mathcal{G}_r(X_0)} \int_{\mathbb{R}^n \backslash \mathcal{B}^{(n)}_y} \zeta_2 \left(\frac{h}{y} \right) \left| (\Delta^{(2)}_h \nabla_{l-1} \gamma)(x) \right| dh\, y^{-n-2} dx\, dy. \tag{64}$$

In view of (61) and (34), we have for the inner integral over $\mathbb{R}^n \backslash \mathcal{B}^{(n)}_y$

$$\int_{\mathbb{R}^n \backslash \mathcal{B}^{(n)}_y} \zeta_2 \left(\frac{h}{y} \right) \left| (\Delta^{(2)}_h \nabla_{l-1} \gamma)(x) \right| dh \leq c\, y^{n+2} \int_{\mathbb{R}^n \backslash \mathcal{B}^{(n)}_y} \frac{|(\Delta^{(2)}_h \nabla_{l-1} \gamma)(x)| dh}{|h|^{n+2}}.$$

We write the integral in the right-hand side as the sum of two integrals, one taken over $\mathcal{B}^{(n)}_{2r} \backslash \mathcal{B}^{(n)}_y$ and another over $\mathbb{R}^n \backslash \mathcal{B}^{(n)}_{2r}$. We see that

$$\begin{aligned} &\int_0^{3r} dy \int_{\mathcal{B}^{(n)}_r} \left| (\Delta^{(2)}_h \nabla_{l-1} \gamma)(x) \right| dx \int_{\mathcal{B}^{(n)}_{2r} \backslash \mathcal{B}^{(n)}_y} \frac{dh}{|h|^{n+2}} \\ &\qquad \leq \int_{\mathcal{B}^{(n)}_r} dx \int_{\mathcal{B}^{(n)}_{2r}} \frac{|(\Delta^{(2)}_h \nabla_{l-1} \gamma)(x)| dh}{|h|^{n+1}} \\ &\qquad \leq c\, r^{n-m}\, k_{m,l}. \end{aligned} \tag{65}$$

Besides,

$$\int_0^{3r} dy \int_{\mathcal{B}_r^{(n)}} dx \int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}} \frac{|(\Delta_h^{(2)}\nabla_{l-1}\gamma)(x)|dh}{|h|^{n+2}} \leq c(I+I_+ + I_-), \tag{66}$$

where

$$I = \int_0^{3r} dy \int_{\mathcal{B}_r^{(n)}} |\nabla_{l-1}\gamma(x)|\, dx \int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}} \frac{dh}{|h|^{n+2}}$$

and

$$I_\pm = \int_0^{3r} dy \int_{\mathcal{B}_r^{(n)}} \int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}} \frac{|\nabla_{l-1}\gamma(x\pm h)|dh}{|h|^{n+2}} dx.$$

Clearly,

$$I \leq c\, r^{-1} \int_{\mathcal{B}_r^{(n)}} |\nabla_{l-1}\gamma(x)|dx. \tag{67}$$

Hence,

$$I \leq c\, r^{n-m}\Big(\sup_{\substack{z\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-1} \int_{\mathcal{B}_\rho^{(n)}(z)} |\nabla_{l-1}\gamma(x)|dx\Big) \tag{68}$$

Obviously,

$$I_\pm \leq c\, r^{1-n} \int_{\mathcal{B}_r^{(n)}} \int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}} \int_{\mathcal{B}_r^{(n)}(\xi)} \frac{|\nabla_{l-1}\gamma(x\pm h)|dh}{|h|^{n+2}}\, d\xi\, dx.$$

In view of the the estimate $|\xi| \leq r + |h| < \frac{1}{2}|\xi| + |h|$, this implies

$$I_\pm \leq c\, r^{1-n} \int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}} \int_{\mathcal{B}_r^{(n)}(\xi)} \int_{\mathcal{B}_r^{(n)}} |\nabla_{l-1}\gamma(x\pm h)|\, dx\, dh \frac{d\xi}{|\xi|^{n+2}} \tag{69}$$

and therefore,

$$I_\pm \leq c\, r^{n-m}\Bigg(\sup_{\substack{z\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-1} \int_{\mathcal{B}_\rho^{(n)}(z)} |\nabla_{l-1}\gamma(x)|dx\Bigg). \tag{70}$$

Summing up (65), (67), and (70), we conclude that J_2 defined by (64) is subject to the inequality

$$J_2 \leq c\, r^{n-m}\Bigg(k_{l,m} + \sup_{\substack{z\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-1} \int_{\mathcal{B}_\rho^{(n)}(z)} |\nabla_{l-1}\gamma(x)|dx\Bigg). \tag{71}$$

Together with (63), this leads to

$$\begin{aligned} &r^{m-n}\|\nabla_{l+1}(T\gamma)\|_{L_1(\mathcal{G}_r(X_0))} \\ &\leq c\Bigg(k_{l,m} + \sup_{\substack{z\in\mathbb{R}^n\\ \rho\in(0,1)}} \rho^{m-n-1} \int_{\mathcal{B}_\rho^{(n)}(z)} |\nabla_{l-1}\gamma(x)|dx\Bigg). \end{aligned} \tag{72}$$

It remains to show that

$$\sup_{X_0 \in \mathbb{R}^{n+1}_+} \|T\gamma\|_{L_1(\mathcal{G}_1(X_0))} \leq c \sup_{x \in \mathbb{R}^n} \|\gamma\|_{L_1(\mathcal{B}_1^{(n)}(x))}. \tag{73}$$

If $y_0 \geq 2$, this inequality stems directly from (43). Let $y_0 < 2$. Clearly,

$$\begin{aligned} \|T\gamma\|_{L_1(\mathcal{G}_1(X_0))} \leq & \int_0^3 \int_{\mathcal{B}_1^{(n)}} \int_{\mathcal{B}_y^{(n)}(x)} \zeta\Big(\frac{\xi - x}{y}\Big) |\gamma(\xi)| d\xi \, dx \, \frac{dy}{y^n} \\ & + \int_0^3 dy \int_{\mathcal{B}_1^{(n)}} \int_{\mathbb{R}^n \setminus \mathcal{B}_y^{(n)}(x)} \zeta\Big(\frac{\xi - x}{y}\Big) |\gamma(\xi)| d\xi \, dx \frac{dy}{y^n}. \end{aligned} \tag{74}$$

The first term in the right-hand side does not exceed

$$\int_{\mathcal{B}} \zeta(t) \, dt \int_0^3 \int_{\mathcal{B}} |\gamma(x + ty)| \, dx \, dy \leq c \sup_{z \in \mathbb{R}^n} \|\gamma\|_{L_1(\mathcal{B}_1^{(n)}(z))}. \tag{75}$$

Since ζ is the Poisson kernel, the second term in (74) is dominated by

$$\begin{aligned} & c \int_0^3 \int_{\mathcal{B}_1^{(n)}} \int_{\mathbb{R}^n \setminus \mathcal{B}_y^{(n)}} \frac{|\gamma(x+h)| \, dh}{(y + |h|)^{n+1}} \, dx \, y \, dy \\ & \leq c \int_0^3 \int_{\mathcal{B}_1^{(n)}} \int_{\mathbb{R}^n \setminus \mathcal{B}_{2y}^{(n)}} d\xi \int_{\mathcal{B}_y^{(n)}(\xi)} \frac{|\gamma(x+h)| \, dh}{|h|^{n+1}} \, dx \frac{dy}{y^{n-1}}. \end{aligned} \tag{76}$$

In view of the inequality $|h| > \frac{|\xi|}{2}$, the right-hand side in (76) is majorized by

$$c \int_0^3 \int_{\mathbb{R}^n \setminus \mathcal{B}_{2y}^{(n)}} \int_{\mathcal{B}_y^{(n)}(\xi)} \int_{\mathcal{B}_1^{(n)}} |\gamma(x+h)| \, dx \, dh \frac{d\xi}{|\xi|^{n+1}} \frac{dy}{y^{n-1}} \leq c \sup_{z \in \mathbb{R}^n} \|\gamma\|_{L_1(\mathcal{B}_1^{(n)}(z))}.$$

Combining the last estimate with (75) and (76), we arrive at (73).

Now, summing up inequalities (63), (71), and (73), we conclude that the value $K_{l,m}$ defined by (57) satisfies

$$K_{l,m} \leq c\bigg(k_{l,m} + \sup_{\substack{z \in \mathbb{R}^n \\ r \in (0,1)}} r^{m-n-1} \int_{\mathcal{B}_r^{(n)}(z)} |\nabla_{l-1}\gamma(x)| dx + \sup_{z \in \mathbb{R}^n} \|\gamma\|_{L_1(\mathcal{B}_1^{(n)}(z))}\bigg). \tag{77}$$

Estimating the second term in the right-hand side by Lemma 2.5, we arrive at (59). The result follows for $l < m$. For $m = l$, instead of (73), we use the maximum principle $\|T\gamma\|_{L_\infty(\mathbb{R}^{n+1}_+)} \leq \|\gamma\|_{L_\infty(\mathbb{R}^n)}$. The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.3

The desired lower estimate for the norm $\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))}$ follows from (52) and the estimate

$$\|C_l\gamma\|_{L_1(\mathcal{B}_r^{(n)}(z))} \le c \sup_{\xi\in\mathbb{R}^n} \|\gamma\|_{B_1^l(\mathcal{B}_r^{(n)}(\xi))}, \tag{78}$$

which holds for all $z \in \mathbb{R}^n$ and $r \in (0, 1]$. In fact, in order to justify (78), it suffices to check that

$$\int_{\mathcal{B}_r^{(n)}(z)}\int_{\mathbb{R}^n\setminus\mathcal{B}_r^{(n)}} \big|\big(\Delta_h^{(2)}\nabla_{l-1}\gamma\big)(x)\big|\frac{dh}{|h|^{n+1}}dx \le cr^{-1}\sup_{\xi\in\mathbb{R}^n} \|\nabla_{l-1}\gamma\|_{L_1(\mathcal{B}_r^{(n)}(\xi))}. \tag{79}$$

Clearly,

$$\int_{\mathcal{B}_r^{(n)}(z)}\int_{\mathbb{R}^n\setminus\mathcal{B}_r^{(n)}} |\nabla_{l-1}\gamma(x)|\frac{dh}{|h|^{n+1}}\,dx \le c\,r^{-1}\sup_{\xi\in\mathbb{R}^n} \|\nabla_{l-1}\gamma\|_{L_1(\mathcal{B}_r^{(n)}(\xi))}. \tag{80}$$

Besides,

$$\begin{aligned}&\int_{\mathcal{B}_r^{(n)}(z)}\int_{\mathbb{R}^n\setminus\mathcal{B}_r^{(n)}} |\nabla_{l-1}\gamma(x\pm h)|\frac{dh}{|h|^{n+1}}\,dx\\ &\le \frac{c}{r^n}\int_{\mathcal{B}_r^{(n)}(z)}\int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}}\int_{\mathcal{B}_r^{(n)}(\xi)} |\nabla_{l-1}\gamma(x\pm h)|\frac{dh}{|h|^{n+1}}d\xi\,dx.\end{aligned}$$

Since $|\xi| < r + |h|$, it follows that $|h| > \frac{|\xi|}{2}$ and, therefore, the right-hand side of the last inequality is dominated by

$$c\int_{\mathbb{R}^n\setminus\mathcal{B}_{2r}^{(n)}}\int_{\mathcal{B}_r^{(n)}(\xi)} |\nabla_{l-1}\gamma(x\pm h)|dh\frac{d\xi}{|\xi|^{n+1}} \le c\,r^{-1}\sup_{z\in\mathbb{R}^n} \|\nabla_{l-1}\gamma\|_{L_1(\mathcal{B}_r^{(n)}(z))}$$

which together with (80) implies (79).

To get the required upper estimate for $\|\gamma\|_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))}$, we combine (77) with Lemma 2.5 and Corollary 2.9 to conclude

$$K_{m,l} \le c\Bigg(\sup_{\substack{z\in\mathbb{R}^n\\ r\in(0,1)}} r^{m-n}\|C_l\gamma\|_{L_1(\mathcal{B}_r^{(n)}(z))} + \|\gamma\|_{L_{1,\mathrm{unif}}(\mathbb{R}^n)}\Bigg).$$

Using this in (56), the result follows.

For $m \ge n$, the right-hand side in (49) is obviously equivalent to $B^l_{1,\mathrm{unif}}(\mathbb{R}^n)$. The proof is complete. □

5. Proof of Theorem 1.4

(i) Suppose that $\gamma \in M(B_1^m(\mathbb{R}^n) \to B_1^l(\mathbb{R}^n))$. Then by Theorem 1.1, the right-hand side in (3) is finite. Taking into account (59), we conclude that $K_{m,l}$ defined in (57) is finite. The reference to the equivalence relation (54) completes the proof of part (i).

(ii) Let $U \in W_1^{m+1}(\mathbb{R}_+^{n+1})$ and $U(x,0) = u(x)$. Clearly, by part (i) of Lemma 2.1,

$$\|\gamma\, u\|_{B_1^l(\mathbb{R}^n)} \le c\,\|\Gamma U\|_{W_1^{l+1}(\mathbb{R}_+^{n+1})} \le c\,\|\Gamma\|_{M(W_1^{m+1}(\mathbb{R}_+^{n+1})\to W_1^{l+1}(\mathbb{R}_+^{n+1}))}\|U\|_{W_1^{m+1}(\mathbb{R}_+^{n+1})}.$$

Minimizing the right-hand side over all extensions U of u and using part (ii) of Lemma 2.1, we complete the proof. □

6. Interpolation inequality for multipliers

We start with the following known assertion.

Lemma 6.1 ([3]). *Let $s > 0$ and let μ be a measure in $\mathbb{R}^n$. The best constant C in the inequality $\int_{\mathbb{R}^n} |u|\, d\mu \le C\|u\|_{B_1^s(\mathbb{R}^n)}$ is equivalent to*

$$\sup_{\substack{x\in\mathbb{R}^n\\ r\in(0,1)}} r^{s-n}\mu\big(\mathcal{B}_r^{(n)}(x)\big).$$

For $s \ge n$ the last supremum should be replaced by $\sup_{x\in\mathbb{R}^n} \mu\big(\mathcal{B}_1^{(n)}(x)\big)$.

¿From Lemma 6.1, one readily obtains

Corollary 6.2. *Let $0 < s < n$, then*

$$\|\gamma\|_{M(B_1^s(\mathbb{R}^n)\to L_1(\mathbb{R}^n))} \sim \sup_{\substack{x\in\mathbb{R}^n\\ r\in(0,1)}} r^{s-n}\|\gamma\|_{L_1(\mathcal{B}_r^{(n)}(x))}. \tag{81}$$

Let $s \ge n$, then

$$\|\gamma\|_{M(B_1^s(\mathbb{R}^n)\to L_1(\mathbb{R}^n))} \sim \sup_{x\in\mathbb{R}^n} \|\gamma\|_{L_1(\mathcal{B}_1^{(n)}(x))}. \tag{82}$$

Theorem 6.3. *Let m and l be integers, $m \ge l > 0$, and let $j = 0,\dots,l-1$. Then*

$$\|\gamma\|_{M(B_1^{m-j}(\mathbb{R}^n)\to B_1^{l-j}(\mathbb{R}^n))} \le c\,\|\gamma\|^{1-\frac{j}{l}}_{M(B_1^m(\mathbb{R}^n)\to B_1^l(\mathbb{R}^n))}\|\gamma\|^{\frac{j}{l}}_{M(B_1^{m-l}(\mathbb{R}^n)\to L_1(\mathbb{R}^n))}. \tag{83}$$

Proof. By (48), $\|u\|_{B_1^{l-j}(\mathcal{B}_r^{(n)}(x))} \le c\,\|u\|^{1-\frac{j}{l}}_{B_1^l(\mathcal{B}_r^{(n)}(x))}\|\gamma\|^{\frac{j}{l}}_{L_1(\mathcal{B}_r^{(n)}(x))}$. Hence,

$$\sup_{\substack{x\in\mathbb{R}^n\\ r\in(0,1)}} r^{m-j-n}\|u\|_{B_1^{l-j}(\mathcal{B}_r^{(n)}(x))}$$

$$\le c\left(\sup_{\substack{x\in\mathbb{R}^n\\ r\in(0,1)}} r^{m-n}\|u\|_{B_1^{l-j}(\mathcal{B}_r^{(n)}(x))}\right)^{1-\frac{j}{l}}\left(\sup_{\substack{x\in\mathbb{R}^n\\ r\in(0,1)}} r^{m-l-n}\|\gamma\|_{L_1(\mathcal{B}_r^{(n)}(x))}\right)^{\frac{j}{l}}.$$

It remains to apply Theorem 1.1 and Corollary 6.2. □

References

[1] Gulisashvili, A., Multipliers in Besov spaces and the theory of functions (in Russian). *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 135 (1984), 36 – 50.

[2] Koch, H. and Sickel, W., Pointwise multipliers of Besov spaces of smoothness zero and spaces of continuous functions. *Rev. Mat. Iberoamericana* 18 (2002), 587 – 626.

[3] Maz'ya, V., Summability, with respect to an arbitrary measure, of functions from Sobolev-Slobodeckiĭ spaces (in Russian). *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 92 (1979), 192 – 202.

[4] Maz'ya, V. and Shaposhnikova, T., *Theory of Multipliers in Spaces of Differentiable Functions.* Boston: Pitman 1985.

[5] Maz'ya, V. and Shaposhnikova, T., Characterisation of multipliers in pairs of Besov spaces. In: *Operator Theoretical Methods and Appl. to Math. Physics.* Oper. Theory Adv. Appl. 147. Basel: Birkhäuser 2004, pp. 365 – 386.

[6] Sickel, W. and Smirnow, I., Localization properties of Besov spaces and its associated multiplier spaces. *Jenaer Schriften Math./Inf.* 21/99; Jena 1999.

[7] Triebel, H., *Theory of Function Spaces.* Basel: Birkhäuser 1992.

[8] Uspenskii, S. V., Imbedding theorems for classes with weights. *Trudy Mat. Inst. Steklova* 60 (1961), 282 – 303 (in Russian); Engl. transl.: *Amer. Math. Soc. Transl.* 87 (1970), 121 – 145.

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