Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 28 (2009), 165–182

Wong-Zakai Type Approximations for Stochastic Differential Equations Driven by a Fractional Brownian Motion

Constantin Tudor

Abstract. We consider Wong–Zakai type approximations for a class of Itô–Volterra equations related to the fractional Brownian motion. The quadratic mean convergence, uniformly on compact time intervals, of the approximations to the solution of an Itô–Volterra equation with a modified drift is obtained.

Keywords. Fractional Brownian motion, Itô–Volterra equations, quadratic mean convergence, Wong–Zakai approximations

Mathematics Subject Classification (2000). 60H07

1. Introduction

The approximation of SDE's of Itô type (a.s. or in mean square) by ordinary Riemann–Stieltjes equations is considered by Ikeda–Watanabe [6], Karatzas–Shreve [7], Wong–Zakai [9,10]. It is known that if we replace the Brownian motion in the stochastic differential by some smooth approximation (such as linear interpolation, mollifier, etc.), then the solution of the approximating equation converges (a.s. or in mean square) to the Stratonovich form of the original equation.

In the present paper we consider a class of Itô–Volterra equations of the form

$$X_t = \xi + \int_0^t K_H(t,s) b(X_s) ds + \int_0^t K_H(t,s) \sigma(X_s) dW_s, \quad t \in [0,1], \tag{1}$$

where $\xi \in R, b, \sigma : R \longrightarrow R$ are measurable functions, W is a Brownian motion and $K_H : [0,1]^2 \longrightarrow R$ is a deterministic kernel (here $H \in \left(\frac{1}{2}, 1\right)$) such that the process

$$B_t^H = \int_0^t K_H(t,s) dW_s, \tag{2}$$

C. Tudor: Faculty of Mathematics and Computer Science, Str. Academiei 14, 010014 Bucharest, Romania; ctudor@fmi.unibuc.ro

is a fractional Brownian motion with Hurst parameter H, i.e., B^H is a continuous Gaussian process with:

(i) $B_0^H = 0$;

(ii) for every $s, t \in [0,1]$, $E(B_s^H B_t^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.

Such class of equations is introduced by Coutin–Decreusefond [1–3] and named stochastic differential equations driven by a fractional Brownian motion (the name comes from the fact that if $\sigma \equiv 1$, then the stochastic term is exactly the fractional Brownian motion B^H).

This kind of stochastic equations are used as models for signal and observation processes in the filtering theory in the presence of fractional Brownian motion [2]. Approximation results for related SDE's are given in Grecksch and Anh [5].

We consider for (1) the approximation obtained by linear interpolation of the Brownian motion (Wong–Zakai approximations) and we prove that if $b \in C_b^1, \sigma \in C_b^2$, then the approximations converge, in quadratic mean and uniformly on $[0,1]$, to the solution of the limiting SDE with corrected drift (which for Itô equations is the Stratonovich form):

$$\begin{aligned} X_t = \xi &+ \int_0^t K_H(t,s)\left[b(X_s) + a_H s^{H-\frac{1}{2}}(\sigma\sigma')(X_s)\right] ds \\ &+ \int_0^t K_H(t,s)\sigma(X_s)dW_s, \quad t \in [0,1], \end{aligned} \tag{3}$$

$$a_H = \frac{2H-1}{2H+1}\left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{\frac{1}{2}} B\left(\frac{3}{2}-H, H-\frac{1}{2}\right),$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

2. Preliminaries

In what follows we fix $H \in (\frac{1}{2}, 1)$ and a Brownian motion $(W_t)_{0\le t\le 1}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. We consider the deterministic kernel K_H defined by

$$K_H(t,s) = \begin{cases} c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, & 0 < s \le t \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

where

$$c_H = \left(H - \frac{1}{2}\right)\left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

In the next proposition we summarize the main properties of K_H (see [1,2,4]).

Proposition 2.1.

(i_1) *The mapping* $s \longrightarrow K_H(t,s)$ *is continuous on the set* $0 < s \leq t$ *and there exists a positive constant* θ_H *such that*

$$K_H(t,s) \leq \theta_H s^{\frac{1}{2}-H}, \quad 0 < s \leq t \leq 1. \tag{4}$$

(i_2) *For every* $0 < s \leq t$

$$\int_0^t |K_H(t,r) - K_H(s,r)|^2 \, dr = (t-s)^{2H}. \tag{5}$$

(i_3) *The mapping* $t \longrightarrow K_H(t,s)$ *is differentiable on the set* $0 < s < t$ *and*

$$\frac{\partial}{\partial t} K_H(t,s) = c_H \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}.$$

(i_4) *For each* $1 \leq p < \frac{2}{2H-1}$,

$$\sup_{0 \leq t \leq 1} \|K_H(t,\cdot)\|_{L^p([0,1])} < \infty. \tag{6}$$

(i_5) *For each* $f \in L^\infty([0,1])$ *the function* $g(t) = \int_0^t K_H(t,s) f(s) ds$, $0 \leq t \leq 1$, *is derivable and*

$$g'(t) = \int_0^t \frac{\partial}{\partial t} K_H(t,s) f(s) ds. \tag{7}$$

Remark 2.2 ([4,8])**.** The process B^H defined by (2) is a fractional Brownian motion.

3. Wong-Zakai approximations

We consider the measurable mappings $b, \sigma : R \longrightarrow R$ and we introduce the following assumption.

Assumption (H): $b \in C_b^1$, $\sigma \in C_b^2$.

Proposition 3.1 ([1–3])**.** *Under Assumption* (H) *the Itô–Volterra equation* (3) *has a pathwise unique continuous solution* $(X_t)_{0 \leq t \leq 1}$ *such that*

$$\sup_{0 \leq t \leq 1} E\left(|X_t|^2\right) < \infty. \tag{8}$$

We consider the particular partition $\Delta_n : 0 < \frac{1}{n} < \cdots < \frac{j}{n} < \frac{j+1}{n} < \cdots < 1$, $\Delta_{n,j} = (\frac{j}{n}, \frac{j+1}{n}]$. and the linear interpolation $W_t^n = (nt-j)W(\Delta_{n,j}) + W_{\frac{j}{n}}$, $\frac{j}{n} \leq t \leq \frac{j+1}{n}$, where $W(\Delta_{n,j}) = W_{\frac{j+1}{n}} - W_{\frac{j}{n}}$.

The *Wong–Zakai approximations* $(X_t^n)_{0\le t\le 1}$ associated with (3) are defined by

$$X_t^n = \xi + \int_0^t K_H(t,s) b(X_s^n)\, ds + \int_0^t K_H(t,s)\sigma(X_s^n)\, dW_s^n, \quad t\in[0,1], \tag{9}$$

where the second integral is to be understood in the Lebesgue-Stieltjes sense.

Next C is a positive constant which may vary from line to line and is independent of n and $0 \le t < 1$. The following proposition shows that the Wong–Zakai approximations are well defined.

Proposition 3.2. *Under Assumption* (H) *the Volterra equation* (9) *has a pathwise unique continuous solution* $(X_t)_{0\le t\le 1}$ *which satisfies*

$$\sup_n \sup_{0\le t\le 1} E\left(|X_t^n|^2\right) < \infty. \tag{10}$$

Proof. The equation (9) is a Volterra equation for every fixed $\omega\in\Omega$. The proof of existence and uniqueness uses the standard Picard approximations (successive approximations) (see [1, 2] for the stochastic case).

We write

$$\int_0^t K_H(t,s)\sigma(X_s^n)\, dW_s^n = I_n(t) + J_n(t), \tag{11}$$

where $I_n(t) = \int_0^{\frac{[nt]}{n}} K_H(t,s)\sigma(X_s^n)\, dW_s^n$ and $J_n(t) = \int_{\frac{[nt]}{n}}^t K_H(t,s)\sigma(X_s^n)\, dW_s^n$. Applying (7) we can write

$$\begin{aligned} I_n(t) &= n\sum_{j=0}^{[nt]-1} W(\Delta_{n,j}) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\sigma(X_s^n)\, ds \\ &= n\sum_{j=0}^{[nt]-1} W(\Delta_{n,j}) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\Big\{\sigma(X_{\frac{j}{n}}^n) + \int_{\frac{j}{n}}^s \sigma'(X_u^n) \\ &\quad \times \left[\int_0^u \frac{\partial K_H(u,r)}{\partial u}\sigma(X_r^n)\, dW_r^n + \int_0^u \frac{\partial K_H(u,r)}{\partial u} b(X_r^n) dr\right] du\Big\} ds. \end{aligned}$$

Therefore the following equality holds:

$$I_n(t) = I_{n,1}(t) + I_{n,2}(t) + I_{n,3}(t), \tag{12}$$

where

$$\begin{aligned} I_{n,1}(t) &= n\sum_{j=0}^{[nt]-1} W(\Delta_{n,j})\, \sigma\Big(X_{\frac{j}{n}}^n\Big) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\, ds \\ I_{n,2}(t) &= n\sum_{j=0}^{[nt]-1} W(\Delta_{n,j}) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \\ &\quad \times \left[\int_{\frac{j}{n}}^s \sigma'(X_u^n)\left(\int_0^u \frac{\partial K_H(u,r)}{\partial u}\sigma(X_r^n) dW_r^n\right) du\right] ds \end{aligned}$$

and

$$I_{n,3}(t) = n \sum_{j=0}^{[nt]-1} W(\Delta_{n,j}) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \times \left[\int_{\frac{j}{n}}^{s} \sigma'(X_u^n) \left(\int_0^u \frac{\partial K_H(u,r)}{\partial u} b(X_r^n)\, dr \right) du \right] ds.$$

The independence of the increments of the Brownian motion, Schwartz's inequality with respect to s, Assumption (H) and (6) imply

$$\begin{aligned} E\left(|I_{n,1}(t)|^2\right) &= n^2 \sum_{j=0}^{[nt]-1} E\Big(\left| W(\Delta_{n,j})\, \sigma(X^n_{\frac{j}{n}}) \right|^2 \Big) \left| \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\, ds \right|^2 \\ &\leq C \sum_{j=0}^{[nt]-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)\, ds \\ &\leq C \sup_{0\leq u\leq 1} \int_0^1 K_H^2(u,s)\, ds < \infty. \end{aligned} \tag{13}$$

In order to estimate $E\left(|I_{n,2}(t)|^2\right)$ it is convenient to write the equality

$$I_{n,2}(t) = I_{n,2}^{(1)}(t) + I_{n,2}^{(2)}(t), \tag{14}$$

where

$$I_{n,2}^{(1)}(t) = n \sum_{j=0}^{[nt]-1} W(\Delta_{n,j}) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \times \left[\int_{\frac{j}{n}}^{s} \sigma'(X_u^n) \Big(\int_0^{\frac{j}{n}} \frac{\partial K_H(u,r)}{\partial u} \sigma(X_r^n) dW_r^n \Big) du \right] ds$$

$$I_{n,2}^{(2)}(t) = n \sum_{j=0}^{[nt]-1} W(\Delta_{n,j}) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \times \left[\int_{\frac{j}{n}}^{s} \sigma'(X_u^n) \Big(\int_{\frac{j}{n}}^{u} \frac{\partial K_H(u,r)}{\partial u} \sigma(X_r^n) dW_r^n \Big) du \right] ds.$$

Now we estimate $E\big(|I_{n,2}^{(i)}(t)|^2\big)$, $i = 1, 2$. By using Schwartz's inequality with respect to s, Assumption (H) and the independence of the increments of the Brownian motion, we obtain

$$\begin{aligned} E\Big(\left| I_{n,2}^{(1)}(t) \right|^2 \Big) &\leq C n^3 \sum_{j=0}^{[nt]-1} \Big(\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)\, ds \Big) E\left[W^2(\Delta_{n,j}) \right] \\ &\quad \times \int_{\frac{j}{n}}^{\frac{j+1}{n}} \Big(s - \frac{j}{n} \Big) \int_{\frac{j}{n}}^{s} E\left(\left| \int_0^{\frac{j}{n}} \frac{\partial K_H(u,r)}{\partial u} \left| \frac{dW_r^n}{dr} \right| dr \right|^2 \right) du\, ds, \end{aligned} \tag{15}$$

and, from Fubini's theorem, Schwartz's inequality and the equality

$$\int_0^s \frac{\partial K_H(s,r)\,dr}{\partial s} = \frac{2a_H}{2H+1}\; s^{H-\frac{1}{2}}, \tag{16}$$

we deduce

$$\begin{aligned}
&E\left(\left|\left(\int_0^{\frac{j}{n}} \frac{\partial K_H(u,r)}{\partial u}\left|\frac{dW^n_r}{dr}\right|dr\right)\right|^2\right) \\
&\qquad = \int_0^u\int_0^u \prod_{j=1}^2 \frac{\partial K_H(u,r_j)}{\partial u} E\left(\prod_{j=1}^2\left|\frac{dW^n_{r_j}}{dr_j}\right|\right)dr_1dr_2 \\
&\qquad \le \int_0^u\int_0^u \prod_{j=1}^2 \frac{\partial K_H(u,r_j)}{\partial u}\left[E\left(\left|\frac{dW^n_{r_j}}{dr_j}\right|^2\right)\right]^{\frac{1}{2}}dr_j \\
&\qquad = \left|\int_0^u \frac{\partial K_H(u,r)}{\partial u}\left[E\left(\left|\frac{dW^n_r}{dr}\right|^2\right)\right]^{\frac{1}{2}}dr\right|^2 \\
&\qquad \le \sup_{0\le r\le 1} E\left(\left|\frac{dW^n_r}{dr}\right|^2\right)\left[\int_0^u \frac{\partial K_H(u,r)}{\partial u}dr\right]^2 \\
&\qquad = C_H B\left(\frac{3}{2}-H, H-\frac{1}{2}\right)\sup_{0\le r\le 1} E\left(\left|\frac{dW^n_r}{dr}\right|^2\right)u^{2H-1} \\
&\qquad \le Cn.
\end{aligned}$$

Now, utilizing the above inequality in (15) and taking into acoount (6), we get

$$\begin{aligned}
E\left(\left|I^{(1)}_{n,2}(t)\right|^2\right) &\le Cn^3 \sum_{j=0}^{[nt]-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)\,ds\right]\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(s-\frac{j}{n}\right)^2 ds \\
&\le C\sup_{0\le u\le 1}\int_0^1 K_H^2(u,s)\,ds < \infty.
\end{aligned} \tag{17}$$

For the second term $I^{(2)}_{n,2}(t)$, we have by Fubini's theorem

$$\begin{aligned}
I^{(2)}_{n,2}(t) = n^2 &\sum_{j=0}^{[nt]-1} W^2\left(\Delta_{n,j}\right)\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \\
&\times\left[\int_{\frac{j}{n}}^{s}\sigma'(X^n_u)\left(\int_{\frac{j}{n}}^{u}\frac{\partial K_H(u,r)}{\partial u}\sigma(X^n_r)dr\right)du\right]ds \\
&\le Cn^2\sum_{j=0}^{[nt]-1} W^2\left(\Delta_{n,j}\right)\int_{\frac{j}{n}}^{\frac{j+1}{n}}\int_{\frac{j}{n}}^{s} K_H(t,s)\int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u}\frac{\partial K_H(u,r)}{\partial u}dr\,du\,ds \\
&= Cn^2\sum_{j=0}^{[nt]-1} W^2\left(\Delta_{n,j}\right)\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\int_{\frac{j}{n}}^{s} K_H(s,r)\,dr\,ds,
\end{aligned}$$

and then, using Schwartz's inequality and (6),

$$\begin{aligned}
E\left(\left|I_{n,2}^{(2)}(t)\right|^2\right) &\le Cn^3 \sum_{j=0}^{[nt]-1} E\left[W^4\left(\Delta_{n,j}\right)\right] \left|\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\int_{\frac{j}{n}}^{s} K_H(s,r)dr\, ds\right|^2 \\
&= Cn \sum_{j=0}^{[nt]-1} \left|\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\int_{\frac{j}{n}}^{s} K_H(s,r)\, dr\, ds\right|^2 \\
&\le C \sum_{j=0}^{[nt]-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)\left|\int_{\frac{j}{n}}^{s} K_H(s,r)\, dr\right|^2 ds \qquad (18)\\
&\le C \sup_{0\le s\le 1}\left|\int_0^1 K_H(s,r)\, dr\right|^2 \sum_{j=0}^{[nt]-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)\, ds \\
&\le C \sup_{0\le s\le 1}\left|\int_0^1 K_H^2(s,r)dr\right|^3 < \infty\,.
\end{aligned}$$

Utilizing (17), (18) in (14) we deduce

$$E\left(|I_{n,2}(t)|^2\right) \le C, \quad 0\le t\le 1. \qquad (19)$$

Also, it is easily seen that

$$E\left(|I_{n,3}(t)|^2\right) \le C, \quad 0\le t\le 1. \qquad (20)$$

Now from (13), (19), (20) and (12) we get

$$E\left(|I_{n}(t)|^2\right) \le C, \quad 0\le t\le 1. \qquad (21)$$

Also, by Schwartz's inequality and (6), we have for every $0 \le t \le 1$,

$$E(|J_n(t)|^2) \le C\int_{\frac{[nt]}{n}}^{t} K_H^2(t,s)\, ds \le C \sup_{0\le t\le 1}\int_0^1 K_H^2(t,s)\, ds < \infty, \qquad (22)$$

$$E\left(\left|\int_0^t K_H(t,s)b(X_s^n)\, ds\right|^2\right) \le C \sup_{0\le t\le 1}\int_0^1 K_H^2(t,s)\, ds < \infty. \qquad (23)$$

Finally, by (21)–(23) and (11) it follows (10). □

The main result is given by the following theorem.

Theorem 3.3. *Under Assumption* (H) *we have the convergence*

$$\lim_{n\to\infty} \sup_{0\le t\le 1} E\left(|X_t^n - X_t|^2\right) = 0.$$

The proof of Theorem 3.3 requires a few lemmas which we give below.

Lemma 3.4. *For each $k \geq 1$ there exist $C_k, \alpha_k > 0$ such that*

$$\sup_{0\leq s\leq 1} E\left(\left|X^n_s - X^n_{\frac{[ns]}{n}}\right|^{2k}\right) \leq \frac{C_k}{n^{\alpha_k}}, \quad n \geq 1 \tag{24}$$

$$\sup_{0\leq s\leq 1} E\left(\left|X_s - X_{\frac{[ns]}{n}}\right|^{2k}\right) \leq \frac{C_k}{n^{\alpha_k}}, \quad n \geq 1. \tag{25}$$

Proof. We can write

$$\begin{aligned} X^n_s - X^n_{\frac{[ns]}{n}} &= \int_0^{\frac{[ns]}{n}} \left[K_H(s,r) - K_H\left(\frac{[ns]}{n}, r\right)\right] b(X^n_r)\, dr \\ &+ \int_0^{\frac{[ns]}{n}} \left[K_H(s,r) - K_H\left(\frac{[ns]}{n}, r\right)\right] \sigma(X^n_r)\, dW^n_r \\ &+ \int_{\frac{[ns]}{n}}^s K_H(s,r)\, b(X^n_r) dr + \int_{\frac{[ns]}{n}}^s K_H(s,r)\, \sigma(X^n_r)\, dW^n_r, \end{aligned}$$

and then, by using Proposition 2.1 and Hölder's inequality, we obtain for $2 < p < \frac{2}{2H-1}$,

$$\begin{aligned} &E\left(\left|X^n_s - X^n_{\frac{[ns]}{n}}\right|^{2k}\right) \\ &\leq C\Bigg\{\left[\int_0^s \left|K_H(s,r) - K_H\left(\frac{[ns]}{n}, r\right)\right|^2 dr\right]^k \\ &\quad + \left[\int_0^s \left|K_H(s,r) - K_H\left(\frac{[ns]}{n}, r\right)\right|^2 dr\right]^k \int_0^1 E\left(\left|\frac{dW^n_r}{dr}\right|^{2k}\right) dr \\ &\quad + \frac{1}{n^k}\left(\int_{\frac{[ns]}{n}}^s |K_H(s,r)|^2\, dr\right)^k \\ &\quad + n^k \max_j E\left(|W(\Delta_{n,j})|^{2k}\right)\left(\int_{\frac{[ns]}{n}}^s |K_H(s,r)|^2\, dr\right)^k\Bigg\} \\ &\leq C\left[\frac{1}{n^{2Hk}} + \frac{1}{n^{(2H-1)k}} + \left(\frac{1}{n^{\frac{pk}{2}}} + \frac{1}{n^{\frac{(p-2)k}{2}}}\right)\left(\int_0^1 |K_H(s,r)|^p\, dr\right)^{\frac{2k}{p}}\right] \\ &\leq \frac{C_k}{n^{\alpha_k}}, \end{aligned}$$

with $\alpha_k = \min\left((2H-1)\,k, \frac{(p-2)k}{2}\right)$, and therefore (24) is proved.

Denote $b_1(r) = b(X_r) + a_H r^{H-\frac{1}{2}} (\sigma\sigma')(X_r)$. We have

$$\begin{aligned} X_s - X_{\frac{[ns]}{n}} = & \int_0^{\frac{[ns]}{n}} \left[K_H(s,r) - K_H\left(\frac{[ns]}{n}, r\right)\right] b_1(r)\, dr \\ & + \int_0^{\frac{[ns]}{n}} \left[K_H(s,r) - K_H\left(\frac{[ns]}{n}, r\right)\right] \sigma(X_r)\, dW_r \\ & + \int_{\frac{[ns]}{n}}^s K_H(s,r)\, b_1(r)\, dr + \int_{\frac{[ns]}{n}}^s K_H(s,r)\, \sigma(X_r)\, dW_r, \end{aligned}$$

and reasoning as above we have the estimates

$$\begin{aligned} & E\left(\left| \int_0^{\frac{[ns]}{n}} \left[K_H(s,r) - K_H\left(\frac{[ns]}{n}, r\right)\right] \sigma(X_r)\, dW_r \right|^{2k} \right) \\ & \leq C \left[\int_0^s \left| K_H(s,r) - K_H\left(\frac{[ns]}{n}, r\right)\right|^2 dr \right]^k = \frac{C}{n^{2Hk}}, \end{aligned}$$

and

$$E\left(\left| \int_{\frac{[ns]}{n}}^s K_H(s,r)\, \sigma(X_r)\, dW_r \right|^{2k} \right) \leq C \left[\int_{\frac{[ns]}{n}}^s |K_H(s,r)|^2\, dr \right]^k \leq \frac{C}{n^{\frac{(p-2)k}{2}}},$$

where from it is easily to deduce (25). □

Lemma 3.5. *We have the convergence*

$$\beta_n := \sup_{0\leq t\leq 1} \int_0^t \left| n \int_{\frac{[ns]}{n}}^{\frac{[ns]+1}{n}} K_H(t,r)\, dr - K_H(t,s) \right|^2 ds \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. For $\varepsilon > 0$ denote

$$\beta_{n,\varepsilon} = \sup \left\{ |K_H(t,r) - K_H(t,s)| : \varepsilon \leq r, s \leq t \leq 1, |r-s| \leq \frac{1}{n} \right\}.$$

Since for $r < s$ we have

$$\begin{aligned} \left| K_H(t,r) - K_H(t,s) \right| \leq & \ c_H \left| r^{\frac{1}{2}-H} - s^{\frac{1}{2}-H} \right| \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du \\ & + c_H r^{\frac{1}{2}-H} \int_r^s (u-r)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du \\ \leq & \ C \left| r^{\frac{1}{2}-H} - s^{\frac{1}{2}-H} \right| + C(s-r)^{H-\frac{1}{2}}, \end{aligned}$$

it follows that

$$\beta_{n,\varepsilon} \leq C \sup_{\varepsilon\leq r,s\leq 1, |r-s|\leq\frac{1}{n}} \left| r^{\frac{1}{2}-H} - s^{\frac{1}{2}-H} \right| + \frac{C}{n^{H-\frac{1}{2}}} \overset{n\to\infty}{\longrightarrow} 0. \tag{26}$$

Next, by using (4), Schwartz's inequality and (26), we obtain

$$\begin{aligned}
\beta_n &\le 2 \sup_{0\le t\le 1}\left[\int_0^{\varepsilon}\left|n\int_{\frac{[ns]}{n}}^{\frac{[ns]+1}{n}} K_H(t,r)\,dr\,ds\right|^2 + 2\int_0^{\varepsilon}|K_H(t,s)|^2\,ds\right] \\
&\quad + \int_{\varepsilon}^{1}\sup_{\varepsilon\le t\le 1}\left|n\int_{\frac{[ns]}{n}}^{\frac{[ns]+1}{n}} K_H(t,r)\,dr - K_H(t,s)\right|^2 ds \\
&\le Cn\int_0^{\varepsilon}\int_{\frac{[ns]}{n}}^{\frac{[ns]+1}{n}} r^{1-2H}dr\,ds + C\int_0^{\varepsilon} s^{1-2H}ds + C\beta_{n,\frac{\varepsilon}{2}} \\
&\le C\left(\varepsilon^{2(1-H)} + \beta_{n,\frac{\varepsilon}{2}}\right) \longrightarrow 0 \quad \text{as } n\to\infty, \varepsilon\longrightarrow 0. \qquad \square
\end{aligned}$$

Lemma 3.6. *Let $\varphi_n : [0,1] \longrightarrow R$ be the function defined by*

$$\varphi_n(s) = n\left[\int_0^s K_H(s,r)\,dr - \int_0^{\frac{[ns]}{n}} K_H\left(\frac{[ns]}{n}, r\right) dr\right].$$

Then

$$\lim_{n\to\infty}\sup_{0\le t\le 1}\sum_{j=0}^{[nt]-1}\left|\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\left(\varphi_n(s) - a_H s^{H-\frac{1}{2}}\right) ds\right|^2 = 0.$$

Proof. The relation $\int_0^s K_H(s,r)\,dr = a_H s^{H+\frac{1}{2}}$, the mean value theorem and (4) imply

$$\begin{aligned}
A_n(t) &:= \sum_{j=0}^{[nt]-1}\left|\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\left(\varphi_n(s) - a_H s^{H-\frac{1}{2}}\right) ds\right|^2 \\
&= a_H^2\sum_{j=0}^{[nt]-1}\left|\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\left[n\left(s^{H+\frac{1}{2}} - \left(\frac{j}{n}\right)^{H+\frac{1}{2}}\right) - s^{H-\frac{1}{2}}\right] ds\right|^2 \\
&\le \left(H-\frac{1}{2}\right)^2 a_H^2\sum_{j=0}^{[nt]-1}\left|\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)s^{H-\frac{1}{2}}ds\right|^2 A_n(t) \\
&\le \frac{C}{n}. \qquad \square
\end{aligned}$$

Proof of Theorem 3.3. Next we use the notation from the proof of Proposition 3.2. The following equality holds:

$$X_t^n - X_t = \alpha_n(t) + \beta_n(t) + \gamma_n(t), \tag{27}$$

where

$$\alpha_n(t) = \int_0^t K_H(t,s)\,[b(X_s^n) - b(X_s)]\,ds$$

$$\beta_n(t) = I_n(t) - \int_0^{\frac{[nt]}{n}} K_H(t,s)\sigma(X_s)\,dW_s - a_H \int_0^{\frac{[nt]}{n}} K_H(t,s)s^{H-\frac{1}{2}}\left(\sigma\sigma'\right)(X_s)\,ds$$

$$\gamma_n(t) = J_n(t) - \int_{\frac{[nt]}{n}}^t K_H(t,s)\sigma(X_s)\,dW_s - a_H \int_{\frac{[nt]}{n}}^t K_H(t,s)s^{H-\frac{1}{2}}\left(\sigma\sigma'\right)(X_s)\,ds.$$

From (4) and Assumption (H) it follows

$$E\big(|\alpha_n(t)|^2\big) \le C\int_0^t K_H^2(t,s)E\big(|X_s^n - X_s|^2\big)ds \le C\int_0^t s^{1-2H}E\big(|X_s^n - X_s|^2\big)ds. \quad (28)$$

and, if $2 < p < \frac{2}{2H-1}$, we have by (4) and (6)

$$\begin{aligned} E\left(|\gamma_n(t)|^2\right) &\le C\bigg(\int_{\frac{[nt]}{n}}^t K_H^2(t,s)ds + \frac{1}{n^2}\bigg) \\ &\le C\left[\frac{1}{n^{\frac{p-2}{p}}}\left(\int_0^1 K_H^p(t,s)ds\right)^{\frac{2}{p}} + \frac{1}{n^2}\right] \\ &\le C\left(\frac{1}{n^{\frac{p-2}{p}}} + \frac{1}{n^2}\right). \end{aligned} \quad (29)$$

Taking into account (12) we obtain the equality

$$\beta_n(t) = \beta_{n,1}(t) + \beta_{n,2}(t) + I_{n,3}(t), \quad (30)$$

where

$$\beta_{n,1}(t) = I_{n,1}(t) - \int_0^{\frac{[nt]}{n}} K_H(t,s)\sigma(X_s)\,dW_s$$

$$\beta_{n,2}(t) = I_{n,3}(t) - a_H \int_0^{\frac{[nt]}{n}} K_H(t,s)s^{H-\frac{1}{2}}\left(\sigma\sigma'\right)(X_s)\,ds.$$

We write $\beta_{n,1}(t)$ in the form

$$\begin{aligned} \beta_{n,1}(t) &= \int_0^{\frac{[nt]}{n}} \left[\left(n\int_{\frac{[ns]}{n}}^{\frac{[ns]+1}{n}} K_H(t,r)\,dr\right)\sigma\Big(X^n_{\frac{[ns]}{n}}\Big) - K_H(t,s)\sigma(X_s)\right]dW_s \\ &= \int_0^{\frac{[nt]}{n}} \left[\left(n\int_{\frac{[ns]}{n}}^{\frac{[ns]+1}{n}} K_H(t,r)\,dr - K_H(t,s)\right)\sigma\Big(X^n_{\frac{[ns]}{n}}\Big)\right]dW_s \\ &\quad + \int_0^{\frac{[nt]}{n}} \left[K_H(t,s)\left(\sigma\Big(X^n_{\frac{[ns]}{n}}\Big) - \sigma(X_s)\right)\right]dW_s, \end{aligned}$$

and then we have

$$
\begin{aligned}
E\left(|\beta_{n,1}(t)|^2\right) &\le 2C\int_0^{\frac{[nt]}{n}}\left|n\int_{\frac{[ns]}{n}}^{\frac{[ns]+1}{n}} K_H(t,r)dr - K_H(t,s)\right|^2 ds\\
&\quad + 2C\int_0^{\frac{[nt]}{n}} K_H^2(t,s)E\Big(\Big|X^n_{\frac{[ns]}{n}} - X_s\Big|^2\Big)ds\\
&\le C\int_0^{\frac{[nt]}{n}}\left|n\int_{\frac{[ns]}{n}}^{\frac{[ns]+1}{n}} K_H(t,r)dr - K_H(t,s)\right|^2 ds\\
&\quad + C\int_0^{\frac{[nt]}{n}} K_H^2(t,s)E\Big(\Big|X^n_{\frac{[ns]}{n}} - X^n_s\Big|^2\Big)ds\\
&\quad + C\int_0^{\frac{[nt]}{n}} K_H^2(t,s)E\left(|X^n_s - X_s|^2\right)ds,
\end{aligned}
$$

and, by Lemma 3.4, Lemma 3.5, (4) and (6), we get

$$
\begin{aligned}
E\left(|\beta_{n,1}(t)|^2\right) &\le C\int_0^t K_H^2(t,s)E\left(|X^n_s - X_s|^2\right)ds + \delta_n\\
&\le C\int_0^t s^{1-2H}E\left(|X^n_s - X_s|^2\right)ds + \delta_n,\ \ \delta_n \longrightarrow 0.
\end{aligned}
\tag{31}
$$

The term $\beta_{n,2}(t)$ can be writen as

$$
\beta_{n,2}(t) = \sum_{j=1}^{7}\beta^{(j)}_{n,2}(t), \tag{32}
$$

where

$$
\begin{aligned}
\beta^{(1)}_{n,2}(t) &= n\sum_{j=0}^{[nt]-1} W\left(\Delta_{n,j}\right)\sigma'\big(X^n_{\frac{j}{n}}\big)\\
&\quad\times\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\int_{\frac{j}{n}}^{s}\bigg(\int_0^{\frac{j}{n}}\frac{\partial K_H(u,r)}{\partial u}\sigma(X^n_r)\,dW^n_r\bigg)du\,ds\\
\beta^{(2)}_{n,2}(t) &= n\sum_{j=0}^{[nt]-1} W\left(\Delta_{n,j}\right)\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\bigg[\int_{\frac{j}{n}}^{s}\Big[\sigma'(X^n_u) - \sigma'\big(X^n_{\frac{j}{n}}\big)\Big]\\
&\quad\times\bigg(\int_0^{\frac{j}{n}}\frac{\partial K_H(u,r)}{\partial u}\sigma(X^n_r)dW^n_r\bigg)du\bigg]ds\\
\beta^{(3)}_{n,2}(t) &= n^2\sum_{j=0}^{[nt]-1} W^2\left(\Delta_{n,j}\right)\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\int_{\frac{j}{n}}^{s}\sigma'(X^n_u)\\
&\quad\times\int_{\frac{j}{n}}^{u}\frac{\partial K_H(u,r)}{\partial u}\big(\sigma(X^n_r) - \sigma(X_r)\big)dr\,du\,ds
\end{aligned}
$$

$$\beta_{n,2}^{(4)}(t) = n^2 \sum_{j=0}^{[nt]-1} W^2(\Delta_{n,j}) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)$$

$$\times \int_{\frac{j}{n}}^{s} \left(\sigma'(X_u^n) - \sigma'(X_u)\right) \int_{\frac{j}{n}}^{u} \frac{\partial K_H(u,r)}{\partial u} \sigma(X_r)\, dr\, du\, ds$$

$$\beta_{n,2}^{(5)}(t) = n^2 \sum_{j=0}^{[nt]-1} W^2(\Delta_{n,j}) (\sigma\sigma')\left(X_{\frac{j}{n}}\right) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)$$

$$\times \int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u} \frac{\partial K_H(u,r)}{\partial u}\, dr\, du\, ds - a_H \int_0^{\frac{[nt]}{n}} K_H(t,s) s^{H-\frac{1}{2}} (\sigma\sigma')(X_s)\, ds$$

$$= n^2 \sum_{j=0}^{[nt]-1} W^2(\Delta_{n,j}) (\sigma\sigma')\left(X_{\frac{j}{n}}\right) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \int_{\frac{j}{n}}^{s} K_H(s,r)\, dr\, ds$$

$$- a_H \int_0^{\frac{[nt]}{n}} K_H(t,s) s^{H-\frac{1}{2}} (\sigma\sigma')(X_s)\, ds$$

$$\beta_{n,2}^{(6)}(t) = n^2 \sum_{j=0}^{[nt]-1} W^2(\Delta_{n,j}) \sigma'\left(X_{\frac{j}{n}}\right) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)$$

$$\times \int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u} \frac{\partial K_H(u,r)}{\partial u} \left[\sigma(X_r) - \sigma\left(X_{\frac{j}{n}}\right)\right] dr\, du\, ds$$

$$\beta_{n,2}^{(7)}(t) = n^2 \sum_{j=0}^{[nt]-1} W^2(\Delta_{n,j}) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)$$

$$\times \int_{\frac{j}{n}}^{s} \left(\sigma'(X_u^n) - \sigma'\left(X_{\frac{j}{n}}\right)\right) \int_{\frac{j}{n}}^{u} \frac{\partial K_H(u,r)}{\partial u} \sigma(X_r)\, dr\, du\, ds.$$

From independence of the increments of the Brownian motion, Schwartz's inequality, Fubini's theorem and the equality (16) we obtain

$$E\left(\left|\beta_{n,2}^{(1)}(t)\right|^2\right) = n^2 \sum_{j=0}^{[nt]-1} E\left(\left| W(\Delta_{n,j}) \sigma'\left(X_{\frac{j}{n}}^n\right) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\right.\right.$$

$$\left.\left. \times \int_{\frac{j}{n}}^{s} \left(\int_0^{\frac{j}{n}} \frac{\partial K_H(u,r)}{\partial u} \sigma(X_r^n) dW_r^n\right) du\, ds\right|^2\right)$$

$$\leq C n^2 \sum_{j=0}^{[nt]-1} E\left(W^2(\Delta_{n,j})\right)$$

$$\times E\left(\left| \int_{\frac{j}{n}}^{\frac{j+1}{n}} |K_H(t,s)| \int_{\frac{j}{n}}^{s}\int_0^{\frac{j}{n}} \frac{\partial K_H(u,r)}{\partial u} \left|\frac{dW_r^n}{dr}\right| dr\, du\, ds\right|^2\right)$$

and further

$$
\begin{aligned}
&E\left(\left|\beta_{n,2}^{(1)}(t)\right|^2\right)\\
&\leq Cn\sum_{j=0}^{[nt]-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)\,ds\right]\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(s-\frac{j}{n}\right)\\
&\quad\times\int_{\frac{j}{n}}^{s} E\left(\left|\int_0^u \frac{\partial K_H(u,r)}{\partial y}\left|\frac{dW_r^n}{dr}\right| dr\right|^2\right) du\,ds\\
&= Cn\sum_{j=0}^{[nt]-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)ds\right]\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(s-\frac{j}{n}\right)\\
&\quad\times\int_{\frac{j}{n}}^{s}\int_0^u\int_0^u \frac{\partial K_H(u,r_1)}{\partial u}\frac{\partial K_H(u,r_2)}{\partial u} E\left(\left|\frac{dW_{r_1}^n}{dr_1}\right|\left|\frac{dW_{r_1}^n}{dr_1}\right|\right) dr_1\,dr_2\,du\,ds\\
&\leq Cn\sum_{j=0}^{[nt]-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)ds\right]\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(s-\frac{j}{n}\right)\int_{\frac{j}{n}}^{s}\int_0^u\int_0^u \frac{\partial K_H(u,r_1)}{\partial u}\frac{\partial K_H(u,r_2)}{\partial u}\\
&\quad\times\left[E\left(\left|\frac{dW_{r_1}^n}{dr_1}\right|^2\right)\right]^{\frac{1}{2}}\left[E\left(\left|\frac{dW_{r_1}^n}{dr_1}\right|^2\right)\right]^{\frac{1}{2}} dr_1\,dr_2\,du\,ds\\
&\leq Cn\sum_{j=0}^{[nt]-1}\left[\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)ds\right]\sup_{0\leq r\leq 1} E\left(\left|\frac{dW_r^n}{dr}\right|^2\right)\int_{\frac{j}{n}}^{\frac{j+1}{n}}\left(s-\frac{j}{n}\right)^2 ds,
\end{aligned}
$$

and thus

$$
E\left(\left|\beta_{n,2}^{(1)}(t)\right|^2\right)\leq\frac{C}{n}\sum_{j=0}^{[nt]-1}\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H^2(t,s)\,ds\leq\frac{C}{n}\sup_{0\leq u\leq 1}\int_0^1 K_H^2(u,s)\,ds. \qquad (33)
$$

Next, reasoning as above, we deduce

$$
\begin{aligned}
&E\left(\left|\beta_{n,2}^{(2)}(t)\right|^2\right)\\
&\leq Cn^3\sum_{j=0}^{[nt]-1} E\left[W^2\left(\Delta_{n,j}\right)\left|\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\int_{\frac{j}{n}}^{s}\int_0^{\frac{j}{n}}\frac{\partial K_H(u,r)}{\partial u}\left|\frac{dW_r^n}{dr}\right| dr\,du\,ds\right|^2\right] \qquad (34)\\
&\leq\frac{C}{n}\sup_{0\leq u\leq 1}\int_0^1 K_H^2(u,s)\,ds.
\end{aligned}
$$

An application of Assumption (H) and Fubini's theorem imply

$$
\left|\beta_{n,2}^{(3)}(t)\right|\leq Cn^2\sum_{j=0}^{[nt]-1} W^2\left(\Delta_{n,j}\right)\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s)\int_{\frac{j}{n}}^{s} K_H(s,r)\,dr\,ds,
$$

and, using the independence of the increments of the Brownian motion, we get

$$E\left(\left|\beta_{n,2}^{(3)}(t)\right|^2\right) \leq Cn^2 \left[\sum_{j=0}^{[nt]-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \int_{\frac{j}{n}}^{s} K_H(s,r)\, dr\, ds\right]^2,$$

which, by Schwartz's inequality (in s) and (4), yields

$$E\left(\left|\beta_{n,2}^{(3)}(t)\right|^2\right) \leq \frac{C}{n}. \tag{35}$$

Applying Fubini's theorem again, we obtain

$$\begin{aligned}\left|\beta_{n,2}^{(4)}(t)\right| &\leq Cn^2 \sum_{j=0}^{[nt]-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \int_{\frac{j}{n}}^{s}\int_{\frac{j}{n}}^{u} \frac{\partial K_H(u,r)}{\partial u}\, dr\, ds \\ &= Cn^2 \sum_{j=0}^{[nt]-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \int_{\frac{j}{n}}^{s} K_H(s,r)\, dr\, ds,\end{aligned}$$

and similarly for $\beta_{n,2}^{(6)}(t)$, $\beta_{n,2}^{(7)}(t)$. Therefore (see (35))

$$E\left(\left|\beta_{n,2}^{(i)}(t)\right|^2\right) \leq \frac{C}{n}, \quad i = 4, 6, 7. \tag{36}$$

Next, we write the term $\beta_{n,2}^{(5)}(t)$ in the form

$$\beta_{n,2}^{(5)}(t) = a_{n,1}(t) + a_{n,2}(t) + a_{n,3}(t) + a_{n,4}(t),$$

where

$$\begin{aligned}a_{n,1}(t) &= a_H \sum_{j=0}^{[nt]-1} \left[\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) s^{H-\frac{1}{2}} \Big((\sigma\sigma')\left(X_{\frac{j}{n}}\right) - (\sigma\sigma')\left(X_s\right)\Big)\, ds\right] \\ a_{n,2}(t) &= n \sum_{j=0}^{[nt]-1} (\sigma\sigma')\left(X_{\frac{j}{n}}\right) w^2\left(\Delta_{n,j}\right) \Bigg[n \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \\ &\quad \times \left(\int_0^s K_H(s,r)\, dr - \int_0^{\frac{j}{n}} K_H\Big(\frac{j}{n}, r\Big) dr\right) ds - a_H \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) s^{H-\frac{1}{2}}\, ds\Bigg] \\ a_{n,3}(t) &= a_H \sum_{j=0}^{[nt]-1} (\sigma\sigma')\left(X_{\frac{j}{n}}\right)\left(n w^2\left(\Delta_{n,j}\right) - 1\right) \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) s^{H-\frac{1}{2}}\, ds \\ a_{n,4}(t) &= n^2 \sum_{j=0}^{[nt]-1} (\sigma\sigma')\left(X_{\frac{j}{n}}\right) w^2\left(\Delta_{n,j}\right) \\ &\quad \times \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \int_0^{\frac{j}{n}} \left(K_H\Big(\frac{j}{n}, r\Big) - K_H(s,r)\right) dr\, ds.\end{aligned}$$

From (4) and Assumption (H), we deduce

$$|a_{n,1}(t)| \le C \sum_{j=0}^{[nt]-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} \left| X_s - X_{\frac{j}{n}} \right| ds \le C \int_0^1 \left| X_s - X_{\frac{j}{n}} \right| ds,$$

and then

$$E\left(|a_{n,1}(t)|^2\right) \le C^2 \int_0^1 E\left(\left| X_s - X_{\frac{j}{n}} \right|^2 \right) ds \le \frac{C}{n^\alpha}, \tag{37}$$

for some $\alpha > 0$ (by Lemma 3.4).

Next, by Assumption (H) and Lemma 3.6,

$$\begin{aligned} E\Big(\big|a_{n,2}(t)\big|^2\Big) \le C \sum_{j=0}^{[nt]-1} \Bigg| \int_{\frac{j}{n}}^{\frac{j+1}{n}} & K_H(t,s) \\ & \times \left[n \left(\int_0^s K_H(s,r)\, dr - \int_0^{\frac{j}{n}} K_H\Big(\frac{j}{n}, r\Big) dr \right) - a_H s^{H-\frac{1}{2}} \right] ds \Bigg|^2 \\ \longrightarrow 0\,, \end{aligned} \tag{38}$$

and from independence, Assumption (H) and (4) it follows that

$$E\left(|a_{n,3}(t)|^2\right) \le C \sum_{j=0}^{[nt]-1} E\left(\left| n w^2(\Delta_{n,j}) - 1 \right|^2 \right) \left| \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) s^{H-\frac{1}{2}}\, ds \right|^2,$$

which implies the estimate

$$E\left(|a_{n,3}(t)|^2\right) \le \frac{C}{n}\,. \tag{39}$$

Next, arguing with independence again, it follows that

$$E\left(|a_{n,4}(t)|^2\right) \le C n^2 \left[\sum_{j=0}^{[nt]-1} \left| \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_H(t,s) \int_0^{\frac{j}{n}} \Big(K_H\Big(\frac{j}{n}, r\Big) - K_H(s,r) \Big) dr\, ds \right| \right]^2.$$

It is easily seen that by Schwartz's inequality and (5),

$$E\left(|a_{n,4}(t)|^2\right) \le \frac{C}{n^{2H-1}}. \tag{40}$$

From (37)–(40) we obtain, for some $\alpha > 0$,

$$E\left(\left| \beta_{n,2}^{(5)}(t) \right|^2 \right) \le \frac{C}{n^\alpha}. \tag{41}$$

Now, utilizing (33)–(36) and (41) in (32) we obtain

$$E\left(|\beta_{n,2}(t)|^2\right) \leq \frac{C}{n^\alpha}. \tag{42}$$

Finally, by using (28)–(32) and (42) in (27), we arrive to an estimate of the form

$$\sup_{t\leq u} E\left(|X_t^n - X_t|^2\right) \leq \varepsilon_n + C\int_0^u s^{1-2H} \sup_{r\leq s} E\left(|X_r^n - X_r|^2\right) ds, \tag{43}$$

where $\varepsilon_n \longrightarrow 0$. By (43) and Gronwall's lemma it follows that

$$\sup_{0\leq t\leq 1} E\left(|X_t^n - X_t|^2\right) \longrightarrow 0,$$

and the proof is complete. □

3.1. Acknowledgement. The author was supported by the CNCSIS-ESF research contract Nr. 3-RNP, 2007–2009. The author thanks the referees for careful reading and suggestions.

References

[1] Coutin, L. and Decreusefond, L., Stochastic differential equations driven by a fractional Brownian motion. Unpublished manuscript 1997.

[2] Coutin, L. and Decreusefond, L., Abstract nonlinear filtering theory in the presence of fractional Brownian motion. *Ann. Appl. Probab.* 9 (1999)(4), 1058 – 1090.

[3] Coutin, L. and Decreusefond, L., Stochastic Volterra equations with singular kernel. In: *Stochastic Analysis and Mathematical Physics.* Progress Prob. 50. Boston (MA): Birkhäuser 2001, pp. 39 – 50.

[4] Decreusefond, L. and Üstünel, S., Stochastic analysis of the fractional Brownian motion. *Potential Anal.* 10 (1999), 177 – 214.

[5] Grecksch, W. and Anh, V. V., Approximation of stochastic differential equations with modified fractional Brownian motion. *Z. Anal. Anwendungen* 17 (1998), 715 – 727.

[6] Ikeda, N. and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes.* Amsterdam: North-Holland: Kodansha 1981.

[7] Karatzas, I. and Shreve, S. E., *Brownian Motion and Stochastic Calculus* (sec. ed.). New York: Springer 1991.

[8] Pipiras, V. and Taqqu, M. S., Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli* 7 (2001)(6), 873 – 897.

[9] Wong, E. and Zakai, M., On the convergence of ordinary integrals to stochastic integrals. *Ann. Math. Statist.* 36 (1965), 1560 – 1564.

[10] Wong, E. and Zakai, M., On the relation between ordinary and stochastic differential equations. *Internat. J. Engin. Sci.* 3 (1965), 213 – 229.

Received December 5, 2006; revised January 28, 2008