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Source Representation Strategy for Optimal Boundary Control Problems with State Constraints

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Abstract. A state-constrained optimal boundary control problem governed by a linear elliptic equation is considered. In order to obtain the optimality conditions for the solutions to the model problem, a Slater assumption has to be made that restricts the theory to the two-dimensional case. This difficulty is overcome by a source representation of the control and combined with a Lavrentiev type regularization. Optimality conditions for the regularized problem are derived, where the corresponding Lagrange multipliers have L^2-regularity. By the spectral theorem for compact and normal operators, the convergence result of Tröltzsch and Yousept in [*Comput. Optim. Appl.* 42 (2009), 43–66] is extended to a higher dimensional case. Moreover, the convergence for vanishing regularization parameter of the adjoint state associated with the regularized problem is shown. Finally, the uniform boundedness of the regularized Lagrange multipliers in $L^1(\Omega)$ is verified by a maximum principle argument.

Keywords. Optimal boundary control problem, pointwise state constraints, elliptic equations, source representation, Lavrentiev type regularization, spectral theorem

Mathematics Subject Classification (2000). 49J20, 49M05, 65K10

1. Introduction

The study of optimal control problems with pointwise state-constraints has become much more of a challenge in recent years. One of the major difficulties when dealing with such problems is the lack of regularity of the corresponding Lagrange multipliers. Mainly, due to the presence of pointwise state-constraints, Lagrange multipliers are in general nonregular and might have measure type components, see Casas [5, 6], Alibert and Raymond [1] and Bergounioux and

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Kunisch [3]. Therefore, a direct application of semismooth Newton methods, or equivalently primal-dual active set strategies (cf. [8, 13]) to infinite dimensional optimal control problems with state-constraints is not possible.

To overcome this difficulty, two regularization concepts were proposed in recent years. First, Ito and Kunisch [12] suggested the use of a "Moreau-Yosida" type regularization approach, which removes the pointwise state inequality constraints by adding a penalty term to the objective functional. Hereafter, the penalized problems are solved in an efficient way. We also refer to [2, 4, 9, 11]. Secondly, a "Lavrentiev" type regularization (cf. [16]) to the pointwise state inequality constraints was introduced by Meyer, Rösch and Tröltzsch in [19]. In contrast to the first method, it preserves, in some sense, the structure of the state-constrained problem. In the case of distributed control, the Lavrentiev type regularization has been successfully applied, see [10, 18, 19]. However, it cannot be directly used in case of boundary control since the domain of the control and that of the state do not fit together. This obstacle was overcome in [22] by a generalization of the Lavrentiev type regularization concept. We also refer to [15], where a virtual control concept is introduced.

Let us state the model problem that we focus on in this paper:

$$\text{minimize } J(u,y) := \frac{1}{2}\int_\Omega (y(x) - y_d(x))^2\, dx + \frac{\kappa}{2}\int_\Gamma u(x)^2\, ds \tag{P}$$

subject to the elliptic boundary value problem

$$\begin{aligned} -\Delta y &= 0 \quad \text{in } \Omega \\ \partial_\nu y + y &= u \quad \text{on } \Gamma \end{aligned} \tag{1}$$

and to the pointwise state constraint

$$y(x) \le \psi(x) \quad \text{for almost all } x \text{ in } \Omega. \tag{2}$$

Here, the domain $\Omega \subset \mathbb{R}^N$, $N \ge 2$, is assumed to be bounded with a $\mathcal{C}^{0,1}$ boundary Γ. The functions y_d, ψ and the cost parameter $\kappa > 0$ are assumed to be given data. The first difficulty involved in this problem is basically the derivation of the Karush–Kuhn–Tucker (KKT) type optimality conditions. To obtain them, a Slater assumption has to be made. This is natural, since even the proof of Fritz–John type theorems is based on the requirement that the cone of non-negative functions of the space of constraints has a non-empty interior, see Luenberger [17]. This restricts, however, the theory to the two-dimensional case, since the control-to-state mapping $u \mapsto y$ is defined from $L^2(\Omega)$ to $\mathcal{C}(\overline{\Omega}) \cap H^1(\Omega)$ only for $N = 2$, see [1, 6]. If $N = 2$, then the KKT type optimality conditions for (P) follow from [1, 6]. As elements of the dual space $\mathcal{C}(\overline{\Omega})^*$, the Lagrange multipliers associated with the state constraint (2) are in general only Borel measures. Our method to overcome these two di! fficulties consists of a source representation of the boundary control as the image of a "distributed" control $v \in L^2(\Omega)$, by means of the adjoint control-to-state operator:

$$u = S^* v.$$

Here, we denote by $S : L^2(\Gamma) \to L^2(\Omega)$ the control-to-state operator $S : u \mapsto y$ with range in $L^2(\Omega)$. Hereafter, we approximate the pointwise state constraint (2) by a mixed-control-state constraint, i.e., we consider

$$\lambda v + y \leq \psi \quad \text{a.e. in } \Omega,$$

where we used the new auxiliary control v instead of u.

The necessary optimality conditions of the regularized problem can be stated without the Slater assumption and we obtain them for every dimension $N \geq 2$. Additionally, the Lagrange multipliers associated with the regularized problem have a L^2-regularity. However, the convergence of the regularized solutions to the optimal solution of (P) was shown in [22] under a Slater assumption. In other words, the restriction to $N = 2$ was needed. The main goal of the present paper is to extend the convergence result from [22] to dimensions $N > 2$. This extension is possible by an application of the spectral theorem for compact and normal operators. We also show the convergence of the adjoint state associated with the regularized problem. Moreover, by a maximum principle argument, the uniform boundedness of regularized Lagrange multipliers in $L^1(\Omega)$ is verified. The paper is organized as follows: First of all, we introduce the general assumptions as well as the notation used in this paper. Afterwards, the regularization of the model problem is presented in Section 3. We show the convergence of the regularized control towards the optimal solution of the original problem in Section 4. Hereafter, the investigation of the adjoint state and Lagrange multiplier associated with the regularized problem is carried out.

2. Problem formulation

Let us first introduce the general assumptions for the model problem (P) including the notation used troughout the paper. We consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with a $\mathcal{C}^{0,1}$-boundary Γ. The upper bound $\psi \in \mathcal{C}(\overline{\Omega})$ and the desired state $y_d \in L^2(\Omega)$ are assumed to be fixed. We denote further by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. If V is a linear normed function space, then we use the notation $\|\cdot\|_V$ for the standard norm used in V. Further, let us define the solution operator to the elliptic equation (1) by $G : L^2(\Gamma) \to H^1(\Omega)$ that assigns to each $u \in L^2(\Gamma)$ the weak solution $y = y(u) \in H^1(\Omega)$ of

$$\begin{aligned} -\Delta y &= 0 \quad \text{in } \Omega \\ \partial_\nu y + y &= u \quad \text{on } \Gamma, \end{aligned}$$

where ∂_ν denotes the normal derivative with respect to the outward unit normal vector. The solution operator G with range in $L^2(\Omega)$ is denoted by $S : L^2(\Gamma) \to L^2(\Omega)$, i.e., we set $S = i_0 G$ where i_0 is the compact embedding operator from $H^1(\Omega)$ to $L^2(\Omega)$. Hereafter, the control problem (P) can be expressed as follows:

$$\begin{cases} \text{minimize } f(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{over } u \in L^2(\Gamma) \\ \text{subject to } Gu \le \psi \quad \text{a.e. in } \Omega. \end{cases} \tag{P}$$

One can show that, independently of the dimension $N \ge 2$, the problem (P) admits a unique solution $\bar{u} \in L^2(\Gamma)$ provided that the admissible set $\{u \in L^2(\Gamma) \mid G(u) \le \psi \text{ a.e. in } \Omega\}$ is not empty. In the rest of the paper, we assume hence that the admissible set for (P) is not empty and denote the optimal solution to (P) by $\bar{u}$ with the associated state $\bar{y} = G\bar{u}$. As pointed out in the introduction, to obtain the optimality conditions for (P), we require the following assumption:

Assumption 2.1. Assume that $N = 2$. Then, we say that the Slater assumption is satisfied, if there exists a function $u_0 \in L^2(\Gamma)$ (a so-called Slater point) such that

$$G(u_0)(x) < \psi(x) \quad \forall\, x \in \overline{\Omega}.$$

This assumption makes only sense in the two-dimensional case, since the solution operator G is defined from $L^2(\Omega)$ to $H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ only for $N = 2$, cf [1, 6]. In the following, we present the optimality system for (P) in an appropriate sense defined in [1, 6].

Theorem 2.1 (First-order optimality conditions for (P))**.** *Let Assumption* 2.1 *be satisfied. Then, $\bar{u}$ is optimal for* (P) *if and only if there exist an adjoint state $p \in W^{1,s}(\Omega)$ for all $1 \le s < \frac{N}{N-1}$ and a Lagrange multiplier $\mu \in \mathcal{C}^*(\overline{\Omega})$ such that*

$$\begin{aligned} -\Delta \bar{y} &= 0 \quad \text{in } \Omega, & -\Delta p &= \bar{y} - y_d + \mu_{|\Omega} \quad \text{in } \Omega \\ \partial_\nu \bar{y} + \bar{y} &= \bar{u} \quad \text{on } \Gamma, & \partial_\nu p + p &= \mu_{|\Gamma} \quad \text{on } \Gamma \\ p + \kappa \bar{u} &= 0 \quad \text{on } \Gamma \\ \langle \mu, \bar{y} - \psi \rangle_{\mathcal{C}^*,\mathcal{C}} &= 0, & \langle \mu, w \rangle_{\mathcal{C}^*,\mathcal{C}} &\ge 0 \quad \forall w \in \mathcal{C}(\overline{\Omega}) \text{ with } w \ge 0. \end{aligned}$$

Notice that $\langle \cdot, \cdot \rangle_{\mathcal{C}^*,\mathcal{C}}$ stands for the duality pairing between $\mathcal{C}^*(\overline{\Omega})$ and $\mathcal{C}(\overline{\Omega})$. In the case of $N > 2$, the state $y = y(u)$ associated to the boundary data $u \in L^2(\Omega)$ is in general not continuous. Therefore, we do not know how to derive the KKT type optimality condition for the solution to (P) for $N > 2$ under reasonable assumptions.

3. Source representation and regularization

We overcome the difficulties mentioned above by using the source representation:

$$u = S^*v,$$

with a new distributed control $v \in L^2(\Omega)$. The adjoint operator S^* is defined from $L^2(\Gamma) \to L^2(\Omega)$, which is represented by $S^*v = w_{|\Gamma}$ where $w(v) = w \in H^1(\Omega)$ is defined as the solution of the following problem:

$$-\Delta w = v \text{ in } \Omega, \qquad \partial_\nu w + w = 0 \text{ on } \Gamma. \tag{3}$$

Hence for each $v \in L^2(\Omega)$, the state $y(v) = y \in H^1(\Omega)$ is given by the solution of

$$\begin{aligned} -\Delta y &= 0 \text{ in } \Omega, \qquad \partial_\nu y + y = w \text{ on } \Gamma \\ -\Delta w &= v \text{ in } \Omega, \qquad \partial_\nu w + w = 0 \text{ on } \Gamma. \end{aligned}$$

Hereafter, we approximate the pointwise state constraint in (P) by the following mixed pointwise control-state constraint:

$$\lambda v + SS^*v \le \psi \quad \text{a.e. in } \Omega.$$

Finally, as introduced in [22], the complete regularization of the boundary control problem (P) is given by

$$\begin{cases} \text{minimize } \tilde{g}(v) := \dfrac{1}{2}\|SS^*v - y_d\|^2_{L^2(\Omega)} + \dfrac{\kappa}{2}\|S^*v\|^2_{L^2(\Gamma)} + \dfrac{\varepsilon(\lambda)}{2}\|v\|^2_{L^2(\Omega)} \\ \text{over } v \in L^2(\Omega) \\ \text{subject to } \lambda v + SS^*v \le \psi \text{ a.e. in } \Omega, \end{cases} \tag{P_λ}$$

with positive regularization parameters $\varepsilon(\lambda), \lambda$. Here, the objective functional $\tilde{g}$ can also be written as $\tilde{g}(v) = f(S^*v) + \frac{\varepsilon(\lambda)}{2}\|v\|^2_{L^2(\Omega)}$. Recall that $f : L^2(\Gamma) \to \mathbb{R}$ is the control reduced objective functional of (P), which is given by $f(u) = \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|u\|^2_{L^2(\Gamma)}$.

Approaching the control problem (P) in a such way, we obtain the optimality conditions for (P_λ) without any restriction on the dimension. Moreover, the associated Lagrange multipliers of the regularized problem have a better regularity then those of (P). As demonstrated in [22], for each $\lambda > 0$ and $\varepsilon(\lambda) > 0$, there exists a unique solution to (P_λ) which satisfies the following optimality conditions:

Theorem 3.1 (First-order optimality conditions)**.** *Let $\lambda > 0$ and $\varepsilon(\lambda) > 0$ be arbitrarily fixed and let v_λ be the optimal solution to* (P$_\lambda$)*. Then, there exist Lagrange multiplier $\mu_\lambda \in L^2(\Omega)$ and adjoint states $p_\lambda, q_\lambda \in H^1(\Omega)$ such that the following optimality system is satisfied:*

$$\begin{aligned} -\Delta y_\lambda &= 0 \ \ in\ \Omega, & -\Delta w_\lambda &= v_\lambda \ in\ \Omega \\ \partial_\nu y_\lambda + y_\lambda &= w \ \ on\ \Gamma, & \partial_\nu w_\lambda + w_\lambda &= 0 \quad on\ \Gamma \end{aligned} \tag{4}$$

$$\begin{aligned} -\Delta p_\lambda &= y_\lambda - y_d + \mu_\lambda \ in\ \Omega, & -\Delta q_\lambda &= 0 \qquad in\ \Omega \\ \partial_\nu p_\lambda + p_\lambda &= 0 \qquad\qquad on\ \Gamma, & \partial_\nu q_\lambda + q_\lambda &= \kappa w_\lambda + p_\lambda \ on\ \Gamma \end{aligned} \tag{5}$$

$$\varepsilon(\lambda) v_\lambda + q_\lambda + \lambda \mu_\lambda = 0 \quad a.e.\ in\ \Omega \tag{6}$$

$$\lambda v_\lambda + y_\lambda \le \psi \quad a.e.\ in\ \Omega \tag{7}$$

$$(\mu_\lambda, \lambda v_\lambda + y_\lambda - \psi)_{L^2(\Omega)} = 0,\ \ \mu_\lambda \ge 0\,. \tag{8}$$

4. Convergence analysis

For vanishing regularization parameter $\lambda \to 0$, the convergence of the regularized solutions was studied in [22]. The strong convergence of $(S^*\bar{v}_\lambda)_{\lambda>0}$ to the optimal solution $\bar{u}$ of (P) in $L^2(\Gamma)$ was shown under the Slater assumption and the assumption that the parameter $\varepsilon(\lambda)$ satisfies $\varepsilon(\lambda) = c_0 \lambda^{1+c_1}$ with some constants $c_0 > 0$ and $0 \le c_1 < 1$. However, the result was restricted to the two dimensional case, since the continuity of the state was needed.

Our goal now is to extend the convergence result from [22] to dimensions $N > 2$. In fact, we are able to show the convergence without the Slater condition, which leads to the fact that the convergence result is true for every choice of dimension $N \ge 2$.

Throughout this section, we denote by I_{Ω} and I_{Γ} the identity operators on $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. Notice that the operator $SS^* : L^2(\Omega) \to L^2(\Omega)$ is positive semidefinite and compact. For this reason, Fredholm's theorem implies that the operator $(\lambda I_{\Omega} + SS^*) : L^2(\Omega) \to L^2(\Omega)$ is continuously invertible for all $\lambda > 0$. By the same argumentation, the operator $(\lambda I_{\Gamma} + S^*S) : L^2(\Gamma) \to L^2(\Gamma)$ is continuously invertible for all $\lambda > 0$. We start with the following auxiliary result.

Lemma 4.1. *Let $\lambda > 0$. Then, the identity $(\lambda I_{\Omega} + SS^*)^{-1} S = S(\lambda I_{\Gamma} + S^*S)^{-1}$ holds true.*

Proof. Let $\lambda > 0$ and $u \in L^2(\Gamma)$. We define $\tilde{u} = (\lambda I_{\Gamma} + S^*S)^{-1} u$ and write

$$(\lambda I_{\Omega} + SS^*) S\tilde{u} = \lambda S\tilde{u} + SS^*S\tilde{u} = S(\lambda I_{\Gamma} + S^*S)\tilde{u}.$$

Consequently, we have $S\tilde{u} = (\lambda I_{\Omega} + SS^*)^{-1}S(\lambda I_{\Gamma} + S^*S)\tilde{u}$, which implies, due to the definition of $\tilde{u}$, that $S(\lambda I_{\Gamma} + S^*S)^{-1}u = (\lambda I_{\Omega} + SS^*)^{-1}Su$. □

Henceforth, let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers converging to zero. Since the operator $S^*S : L^2(\Gamma) \to L^2(\Gamma)$ is self-adjoint and compact, the spectral theorem for normal compact operators implies the existence of an orthonormal system $\{e_i\}_{i=1}^{\infty}$ of $L^2(\Gamma)$ consisting of eigenvectors of S^*S. We denote here by σ_i the corresponding eigenvalue of the eigenvector e_i, i.e.,

$$S^*Se_i = \sigma_i e_i \quad \forall i \in \mathbb{N}.$$

In the following, we prove an auxiliary result that is essential for our convergence analysis. It should be underlined that the particular construction of the auxiliary sequence below (9) is adapted from [18], Lemma 3.1.

Theorem 4.1. *Let* $\{\hat{v}_n\}_{n=1}^{\infty}$ *be the sequence in* $L^2(\Omega)$ *defined by*

$$\hat{v}_n := (\lambda_n I_{\Omega} + SS^*)^{-1}\bar{y}, \tag{9}$$

where $\bar{y} = S\bar{u} \in H^1(\Omega)$ *denotes the optimal state of* (P). *Then, for each* $n \in \mathbb{N}$, $\hat{v}_n$ *is a feasible control of* (P_{λ_n}) *and* $\{S^*\hat{v}_n\}_{n=1}^{\infty}$ *converges strongly to* $\bar{u}$ *in* $L^2(\Gamma)$ *as* $n \to \infty$.

Proof. Since $\bar{y}$ satisfies the state constraints in (P), $\hat{v}_n$ is obviously a feasible control of (P_{λ_n}) for every $n \in \mathbb{N}$. Now, let us verify that $\{S^*\hat{v}_n\}_{n=1}^{\infty}$ converges strongly to $\bar{u}$ as $n \to \infty$. To this aim, we write first

$$S^*\hat{v}_n - \bar{u} = S^*(\lambda_n I_{\Omega} + SS^*)^{-1}\bar{y} - \bar{u} = S^*(\lambda_n I_{\Omega} + SS^*)^{-1}S\bar{u} - \bar{u}.$$

Thus, by Lemma 4.1, one finds that

$$\begin{aligned} S^*\hat{v}_n - \bar{u} &= S^*S(\lambda_n I_{\Gamma} + S^*S)^{-1}\bar{u} - \bar{u} \\ &= S^*S(\lambda_n I_{\Gamma} + S^*S)^{-1}\bar{u} - (\lambda_n I_{\Gamma} + S^*S)(\lambda_n I_{\Gamma} + S^*S)^{-1}\bar{u} \\ &= (S^*S - \lambda_n I_{\Gamma} - S^*S)(\lambda_n I_{\Gamma} + S^*S)^{-1}\bar{u} \\ &= -\lambda_n(\lambda_n I_{\Gamma} + S^*S)^{-1}\bar{u}. \end{aligned} \tag{10}$$

By Parseval's identity, we infer from (10) that

$$\begin{aligned} \|S^*\hat{v}_n - \bar{u}\|^2_{L^2(\Gamma)} &= \|\lambda_n(\lambda_n I_{\Gamma} + S^*S)^{-1}\bar{u}\|^2_{L^2(\Gamma)} \\ &= \sum_{k=1}^{\infty} \left(\lambda_n(\lambda_n I_{\Gamma} + S^*S)^{-1}\bar{u}, e_k\right)^2_{\Gamma} \\ &= \sum_{k=1}^{\infty} \left(\bar{u}, \frac{\lambda_n}{\lambda_n + \sigma_k}(\lambda_n I_{\Gamma} + S^*S)^{-1}(\lambda_n + \sigma_k)e_k\right)^2_{\Gamma} \\ &= \sum_{k=1}^{\infty} (\bar{u}, e_k)^2_{\Gamma} \left(\frac{\lambda_n}{\lambda_n + \sigma_k}\right)^2. \end{aligned}$$

Hence, we obtain

$$\|S^*\hat{v}_n - \bar{u}\|^2_{L^2(\Gamma)} = \sum_{k=1}^{\infty} (\bar{u}, e_k)^2_\Gamma \left(\frac{\lambda_n}{\lambda_n + \sigma_k} \right)^2. \tag{11}$$

For each $k \in \mathbb{N}$, we define now a continuous and non-negative function $f_k : \mathbb{R} \to \mathbb{R}^+_0$ by $f_k(x) = (\bar{u}, e_k)^2_\Gamma \left(\frac{x}{x+\sigma_k}\right)^2$. Notice that $\sigma_k \geq 0$ holds true for all $k \in \mathbb{N}$, since the operator S^*S is positive semidefinite. For this reason, $\sup_{x\in\mathbb{R}} |f_k(x)| \leq (\bar{u}, e_k)^2_\Gamma$ holds true for all k. Further, Parseval's identity implies that $\sum_{k=1}^{\infty}(\bar{u}, e_k)^2_\Gamma = \|\bar{u}\|^2_{L^2(\Gamma)}$, and hence the function series $\sum_{k=1}^{\infty} f_k$ converges uniformly. Therefore, we obtain finally from (11):

$$\lim_{n\to\infty} \|S^*\hat{v}_n - \bar{u}\|^2_{L^2(\Gamma)} = \lim_{n\to\infty} \sum_{k=1}^{\infty} f_k(\lambda_n) = \sum_{k=1}^{\infty} \lim_{n\to\infty} f_k(\lambda_n) = \sum_{k=1}^{\infty} f_k(0) = 0. \qquad \square$$

Lemma 4.2. *The sequence* $\{\lambda_n \|\hat{v}_n\|^2_{L^2(\Omega)}\}_{n=1}^{\infty}$ *converges to zero, as* $n \to \infty$.

Proof. Lemma 4.1 implies that

$$\begin{aligned}\lambda_n \|\hat{v}_n\|^2_{L^2(\Omega)} = \lambda_n(\hat{v}_n, \hat{v}_n)_\Omega &= (\lambda_n(\lambda_n I_\Omega + SS^*)^{-1} S\bar{u}, \hat{v}_n)_\Omega \\ &= (\lambda_n S(\lambda_n I_\Gamma + S^*S)^{-1}\bar{u}, \hat{v}_n)_\Omega \\ &= (\lambda_n(\lambda_n I_\Gamma + S^*S)^{-1}\bar{u}, S^*\hat{v}_n)_\Gamma.\end{aligned}$$

Hence by (10), we have

$$\lambda_n \|\hat{v}_n\|^2_{L^2(\Omega)} = (\bar{u} - S^*\hat{v}_n, S^*\hat{v}_n)_\Gamma \leq \|S^*\hat{v}_n - \bar{u}\|_{L^2(\Gamma)} \|S^*\hat{v}_n\|_{L^2(\Gamma)}.$$

Therefore, the lemma is verified by Theorem 4.1. $\square$

For the parameter $\varepsilon(\lambda)$, we make the following assumption:

Assumption 4.1. The regularization parameter $\varepsilon = \varepsilon(\lambda)$ satisfies

$$\varepsilon = c_0 \lambda^{1+c_1}$$

with some constants $c_0 > 0$ and $0 \leq c_1 < 1$.

Let now $\{v_n\}_{n=1}^{\infty}$ be the sequence of the optimal solutions of (P_{λ_n}). Further, we define the corresponding sequence of optimal states to (P_{λ_n}) by $\{y_n\}_{n=1}^{\infty}$.

Lemma 4.3. *Let Assumption* 4.1 *be satisfied. Then, the sequence* $\{S^*v_n\}_{n=1}^{\infty}$ *is uniformly bounded in* $L^2(\Gamma)$.

Proof. Since $\hat{v}_n$ is a feasible control of (P_{λ_n}), it holds that

$$f(S^*v_n) + \frac{\varepsilon(\lambda_n)}{2}\|v_n\|^2_{L^2(\Omega)} \leq f(S^*\hat{v}_n) + \frac{\varepsilon(\lambda_n)}{2}\|\hat{v}_n\|^2_{L^2(\Omega)}.$$

By Theorem 4.1, we also know that $S^*\hat{v}_n$ converges strongly to $\bar{u}$. For this reason, due to Assumption 4.1, Lemma 4.2 implies that there exists a constant $\hat{c} > 0$ independent of n such that

$$f(S^*v_n) + \frac{\varepsilon(\lambda_n)}{2}\|v_n\|^2_{L^2(\Omega)} \leq f(S^*\hat{v}_n) + \frac{\varepsilon(\lambda_n)}{2}\|\hat{v}_n\|^2_{L^2(\Omega)} \leq \hat{c} \quad \forall n \in \mathbb{N}. \tag{12}$$

Therefore, in a view of the presence of the Tikhonov parameter $\kappa > 0$ in the objective functional f, the assertion is verified. □

By the preceding lemma, we can find a subsequence of $\{S^*v_n\}_{n=1}^\infty$, denoted w.l.o.g. again by the same symbol $\{S^*v_n\}_{n=1}^\infty$, converging weakly to some $\tilde{u} \in L^2(\Gamma)$. Our goal now is to show that $\tilde{u}$ minimizes the original unregularized problem. For this purpose, we show first the feasibility of $\tilde{u}$ for (P), i.e., $S\tilde{u} \leq \psi$ a.e. in Ω.

Lemma 4.4. *Let Assumption* 4.1 *be satisfied. Then every weak limit of any subsequence* $\{S^*v_{n_k}\}$ *is feasible for* (P).

Proof. We know that it holds $\lambda_n v_n + SS^*v_n \leq \psi$ for all n. Therefore, it suffices to show that $\lambda_n v_n$ converges to zero. By (12), we have $\frac{\varepsilon(\lambda_n)}{2}\|v_n\|^2_{L^2(\Omega)} \leq \hat{c}$ for all n, and hence $\frac{\varepsilon(\lambda_n)}{2\lambda_n^2}\|\lambda_n v_n\|^2_{L^2(\Omega)} \leq \hat{c}$ for all n. ¿From Assumption 4.1, we infer then

$$\|\lambda_n v_n\|^2_{L^2(\Omega)} \leq \frac{2\hat{c}\lambda_n^2}{\varepsilon(\lambda_n)} \leq \lambda_n^{1-c_1}\frac{2\hat{c}}{c_0}.$$

This implies $\lambda_n v_n \to 0$ in $L^2(\Omega)$ as $n \to \infty$ and hence the Lemma is shown. □

Now, we are able to prove our main result that shows convergence without Slater condition, independently of the dimension N.

Theorem 4.2. *Let Assumption* 4.1 *be satisfied, then the sequence* $\{S^*v_n\}_{n=1}^\infty$ *converges strongly in* $L^2(\Gamma)$ *to the (unique) optimal control* $\bar{u} \in L^2(\Gamma)$ *of the unregularized problem* (P).

Proof. Since v_n is the optimal solution to (P_{λ_n}) and $\hat{v}_n$ is feasible for (P_{λ_n}), we have

$$f(S^*v_n) \leq f(S^*v_n) + \frac{\varepsilon(\lambda_n)}{2}\|v_n\|^2_{L^2(\Omega)} \leq f(S^*\hat{v}_n) + \frac{\varepsilon(\lambda_n)}{2}\|\hat{v}_n\|^2_{L^2(\Omega)}. \tag{13}$$

By Theorem 4.1, $S^*\hat{v}_n$ converges strongly to $\bar{u}$. Consequently, thanks to Assumption 4.1, Lemma 4.2 implies that

$$\lim_{n\to\infty}\left\{f(S^*\hat{v}_n)+\frac{\varepsilon(\lambda_n)}{2}\|\hat{v}_n\|^2_{L^2(\Omega)}\right\}=f(\bar{u}). \tag{14}$$

Finally applying the lower semicontinuity of f, we infer from (13) and (14):

$$f(\tilde{u})\le\liminf_{n\to\infty}f(S^*v_n)\le\limsup_{n\to\infty}\left\{f(S^*\hat{v}_n)+\frac{\varepsilon(\lambda_n)}{2}\|\hat{v}_n\|^2_{L^2(\Omega)}\right\}=f(\bar{u}).$$

Hence, since $\tilde{u}$ is a feasible control of (P), see Lemma 4.4, and the optimal solution of (P) is unique, one obtains $\tilde{u}=\bar{u}$. Notice that the latter equality $\lim_{n\to\infty}f(S^*v_n)=f(\bar{u})$ implies the convergence of $\{S^*v_n\}_{n=1}^\infty$ in norm and hence, together with the weak convergence, the strong convergence of $\{S^*v_n\}_{n=1}^\infty$ to $\bar{u}$ in $L^2(\Omega)$ is verified. □

5. Discussion on adjoint states and Lagrange multipliers

In the following, we study the behavior of the adjoint state and Lagrange multiplier associated with (P_λ) for $\lambda\to 0$. Basically, the result below follows again from the spectral theorem for compact normal operators. Let us define the sequences of corresponding adjoint states of (P_{λ_n}) by $\{q_n\}_{n=1}^\infty$ and $\{p_n\}_{n=1}^\infty$. Moreover, $\{\mu_n\}_{n=1}^\infty$ denotes the sequence of associated Lagrange multipliers associated with the mixed control-state constraint of (P_{λ_n}). We point out again that all the results in this section hold true without the Slater assumption. Therefore, the results are again true without any restriction on the dimension N.

Theorem 5.1. *Let Assumption* 4.1 *be satisfied. Then, the sequence* $\{q_n\}_{n=1}^\infty$ *converges strongly in* $L^2(\Omega)$ *to zero.*

Proof. According to the optimality conditions for (P_{λ_n}), see Theorem 3.1, the functions q_n, p_n are given by the solutions of

$$\begin{aligned} -\Delta q_n &= 0 && \text{in } \Omega, & -\Delta p_n &= y_n - y_d + \mu_n && \text{in } \Omega\\ \partial_\nu q_n + q_n &= \kappa S^* v_n + p_n && \text{on } \Gamma, & \partial_\nu p_n + p_n &= 0 && \text{on } \Gamma. \end{aligned}$$

Applying the solution operator S and the adjoint operator S^* (see (3)), we obtain from the equations above:

$$q_n = S(\kappa S^* v_n + p_{n|_\Gamma}), \qquad p_{n|_\Gamma} = S^*(y_n - y_d + \mu_n),$$

and hence

$$q_n = S(\kappa S^* v_n + S^*(y_n - y_d + \mu_n)). \tag{15}$$

By the optimality condition, (6), this is equivalent to

$$\begin{aligned} q_n &= S\left(\kappa S^* v_n + S^*\left(y_n - y_d - \frac{1}{\lambda_n} q_n - \frac{\varepsilon(\lambda_n)}{\lambda_n} v_n\right)\right) \\ &= SS^*(y_n - y_d) + \left(\kappa - \frac{\varepsilon(\lambda_n)}{\lambda_n}\right) y_n - \frac{1}{\lambda} SS^* q_n, \end{aligned}$$

which implies that $(\lambda_n I_\Omega + SS^*)q_n = \lambda_n\{SS^*(y_n - y_d) + (\kappa - \frac{\varepsilon(\lambda_n)}{\lambda_n})y_n\}$. Notice that we have used the relation $y_n = SS^* v_n$. Let us define now an auxiliary sequence $\{z_n\}_{n=1}^\infty$ by

$$z_n = SS^*(y_n - y_d) + \left(\kappa - \frac{\varepsilon(\lambda_n)}{\lambda_n}\right) y_n.$$

Since $\{y_n\}_{n=1}^\infty$ converges strongly to $\bar{y}$ in $L^2(\Omega)$ (Theorem 4.2), Assumption 4.1 implies that

$$\lim_{n\to\infty} z_n = SS^*(\bar{y} - y_d) + \kappa \bar{y} := \bar{z} \quad \text{in } L^2(\Omega). \tag{16}$$

Now, q_n can equivalently be written as

$$q_n = \lambda_n(\lambda_n I_\Omega + SS^*)^{-1} z_n = \lambda_n(\lambda_n I_\Omega + SS^*)^{-1}(z_n - \bar{z}) + \lambda_n(\lambda_n I_\Omega + SS^*)^{-1}\bar{z}. \tag{17}$$

Since the operator $SS^* : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint and compact, the spectral theorem for normal compact operators implies the existence of an orthonormal system $\{\tilde{e}_i\}_{i=1}^\infty$ of $L^2(\Omega)$ consisting of eigenvectors of SS^*. Here, we denote by τ_i the corresponding eigenvalue of the eigenvector $\tilde{e}_i$. Notice that $\tau_k \geq 0$ for all k, since SS^* is positive semidefinite. Now, by the completeness relation of $\{\tilde{e}_i\}_{i=1}^\infty$, we have

$$\begin{aligned} \|\lambda_n(\lambda_n I_\Omega + SS^*)^{-1}(z_n - \bar{z})\|_{L^2(\Omega)}^2 &= \sum_{k=1}^\infty \left(\lambda_n(\lambda_n I_\Omega + SS^*)^{-1}(z_n - \bar{z}), \tilde{e}_k\right)_\Omega^2 \\ &= \sum_{k=1}^\infty \left(z_n - \bar{z}, \frac{\lambda_n}{\lambda_n + \tau_k}(\lambda_n I_\Omega + SS^*)^{-1}(\lambda_n + \tau_k)\tilde{e}_k\right)_\Omega^2 \\ &= \sum_{k=1}^\infty (z_n - \bar{z}, \tilde{e}_k)_\Omega^2 \left(\frac{\lambda_n}{\lambda_n + \tau_k}\right)^2 \\ &\leq \sum_{k=1}^\infty (z_n - \bar{z}, \tilde{e}_k)_\Omega^2 \\ &= \|z_n - \bar{z}\|_{L^2(\Omega)}^2. \end{aligned}$$

Thus by (16), it holds that $\lim_{n\to\infty} \lambda_n(\lambda_n I_\Omega + SS^*)^{-1}(z_n - \bar{z}) = 0$ in $L^2(\Omega)$. On the other hand, analogously to the proof of Theorem 4.1, we have

$$\lim_{n\to\infty} \lambda_n(\lambda_n I_\Omega + SS^*)^{-1}\bar{z} = 0 \quad \text{in } L^2(\Omega).$$

Therefore, thanks to (17), the assertion of the theorem is justified. □

Our next goal is to show the uniform boundedness of $\{\mu_n\}_{n=1}^\infty$ in $L^1(\Omega)$. In fact, assuming higher regularity on the boundary Γ, we are able to show the boundedness. It should be underlined here that the result is derived without the Slater assumption. Let us start with the following maximum principle. For the proof, we refer the reader, e.g., to [14].

Lemma 5.1 (Maximum principle)**.** *Let $\xi \in \mathbb{R}^+ \setminus \{0\}$ and $u \in L^2(\Gamma)$ with $u \geq \xi$ a.e. on Γ. Then, the weak solution $y = y(u)$ of*

$$\begin{aligned} -\Delta y &= 0 \qquad \text{in } \Omega \\ \partial_\nu y + y &= u \qquad \text{on } \Gamma \end{aligned}$$

satisfies $y \geq \xi$ a.e. in Ω.

Theorem 5.2. *Let Assumption 4.1 be satisfied and suppose that the boundary Γ is a class of $\mathcal{C}^{1,1}$. Then, $\{\mu_n\}_{n=1}^\infty \subset L^2(\Omega)$ is uniformly bounded in $L^1(\Omega)$.*

Proof. First, we define by $\mathbf{1} \in \mathcal{C}^\infty(\overline{\Omega})$ the constant function: $z(x) = 1$ for all $x \in \overline{\Omega}$. By the assumption, the boundary Γ is a class of $\mathcal{C}^{1,1}$ and hence for $s \in (1, \frac{N}{N-1})$ and $r = \frac{s}{s-1}$, the map

$$T : W^{2,r}(\Omega) \to W^{1+\frac{1}{s},r}(\Gamma) \times W^{\frac{1}{s},r}(\Gamma), \quad z \mapsto (z_{|\Gamma}, \partial_\nu z)$$

is surjective, see [6], Lemma 4.4, which is based on [7] and [20]. Hence, there exists $\hat{w} \in W^{2,r}(\Omega)$ such that

$$\hat{w} = \mathbf{1} \;\text{ on } \Gamma, \quad \partial_\nu \hat{w} = -\mathbf{1} \;\text{ on } \Gamma.$$

Notice that since $s < \frac{N}{N-1}$, it holds that $r > N \geq 2$. For this reason, there exists $\hat{v} \in L^2(\Omega)$ with

$$-\Delta \hat{w} = \hat{v} \;\text{ in } \Omega, \quad \partial_\nu \hat{w} + \hat{w} = 0 \;\text{ on } \Gamma.$$

Consequently, see definition of S^* in (3), $S^*\hat{v} = \hat{w}_{|\Gamma} = \mathbf{1}_{|\Gamma}$ and hence by the maximum principle, Lemma 5.1, it holds that

$$1 \leq SS^*\hat{v} \quad \text{a.e. in } \Omega. \tag{18}$$

Recall from (15) that q_n is given by $q_n = S(\kappa S^* v_n + S^*(y_n - y_d + \mu_n)) = \kappa y_n + SS^*(y_n - y_d) + SS^*\mu_n$. Hence, taking account of Theorem 4.2 and Theorem 5.1, we have

$$\lim_{n\to\infty} SS^*\mu_n = -\kappa\bar{y} - SS^*(\bar{y} - y_d) \quad \text{in } L^2(\Omega). \tag{19}$$

Since $\mu_n \geq 0$ holds true for all $n \in \mathbb{N}$, one finds from (18) that

$$\|\mu_n\|_{L^1(\Omega)} = (\mu_n, \mathbf{1})_{L^2(\Omega)} \leq (\mu_n, SS^*\hat{v})_{L^2(\Omega)} \leq \|SS^*\mu_n\|_{L^2(\Omega)}\|\hat{v}\|_{L^2(\Omega)}.$$

In view of (19), we finally arrive at the conclusion that there exists a constant $c > 0$ independent of n such that $\|\mu_n\|_{L^1(\Omega)} \leq c$ for all $n \in \mathbb{N}$. Thus, the assertion of the theorem is justified. □

Remark 5.1. We point out that, based on the preceding theorem, one can show the existence of a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ converging weakly* in $\mathcal{C}(\overline{\Omega})^*$ to some $\mu^* \in \mathcal{C}(\overline{\Omega})^*$. However, since y_n converges strongly to $\bar{y}$ in general only in $L^2(\Omega)$, we cannot expect that a weak limit of the subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ is a Lagrange multiplier associated with the state constraint in (P).

In the last part of this section, we show the boundedness of $\{p_n\}_{n=1}^{\infty}$ in $W^{1,s}(\Omega)$ with $1 \leq s < \frac{N}{N-1}$. For this purpose, we make use of some results on elliptic problems with measure-valued data which has been studied in [1, 5, 6]. We also refer to [21]. Let us consider the following elliptic problem with measure data $\pi \in \mathcal{C}(\overline{\Omega})^*$:

$$\begin{cases} -\Delta p = \pi_{|\Omega} & \text{in } \Omega \\ \partial_\nu p + p = \pi_{|\Gamma} & \text{on } \Gamma. \end{cases} \tag{20}$$

A function $p \in W^{1,s}(\Omega)$ with $s \geq 1$ is called a (weak) solution of (20) if

$$\int_\Omega \nabla p \nabla z \, dx + \int_\Gamma pz \, ds = \langle \pi, z \rangle_{\mathcal{C}^*,\mathcal{C}} \quad \forall z \in \mathcal{C}^1(\overline{\Omega}).$$

In the following, we provide some important results concerning the solvability of (20), see for instance [21, p. 274–275].

Lemma 5.2. *Let* $\pi \in \mathcal{C}(\overline{\Omega})^*$. *Then,* (20) *admits a unique solution* $p \in W^{1,s}(\Omega)$ *with* $1 \leq s < \frac{N}{N-1}$. *Furthermore there exists a constant* $c(\Omega) > 0$ *depending only on* Ω *such that*

$$\|p\|_{W^{1,s}(\Omega)} \leq c(\Omega)\|\pi\|_{\mathcal{C}(\overline{\Omega})^*}.$$

Based on this result and by Theorem 5.2, we state in the following the uniform boundedness of $\{p_n\}_{n=1}^{\infty}$ in $W^{1,s}(\Omega)$ with $1 \leq s < \frac{N}{N-1}$.

Corollary 5.1. *Let Assumption* 4.1 *be satisfied and suppose that the boundary* Γ *is a class of* $\mathcal{C}^{1,1}$. *Then, the sequence* $\{p_n\}_{n=1}^{\infty}$ *is uniformly bounded in* $W^{1,s}(\Omega)$ *with* $1 \leq s < \frac{N}{N-1}$.

Proof. We recall that for each $n \in \mathbb{N}$, p_n is given by the solution of

$$\begin{cases} -\Delta p_n = y_n - y_d + \mu_n & \text{in } \Omega \\ \partial_\nu p_n + p_n = 0 & \text{on } \Gamma. \end{cases}$$

By Theorem 4.2, we know that y_n converges strongly in $L^2(\Omega)$ to $\bar{y}$. In particular, $\{y_n\}_{n=1}^{\infty}$ is uniformly bounded in $L^2(\Omega)$. For each $n \in \mathbb{N}$, we consider $\mu_n \in L^2(\Omega)$ as an element of $\mathcal{C}(\overline{\Omega})^*$. More precisely, for each $z \in \mathcal{C}(\overline{\Omega})$: $\langle \mu_n, z\rangle_{\mathcal{C}^*,\mathcal{C}} = (\mu_n, z)_{L^2(\Omega)}$ Hence, utilizing Lemma 5.2, it suffices to show that $\{\mu_n\}_{n=1}^{\infty}$ is uniformly bounded in $\mathcal{C}(\overline{\Omega})^*$:

$$\|\mu_n\|_{\mathcal{C}^*(\overline{\Omega})} = \sup_{\|y\|_{\mathcal{C}(\overline{\Omega})=1}} |\langle \mu_n, y\rangle_{\mathcal{C}^*,\mathcal{C}}| = \sup_{\|y\|_{\mathcal{C}(\overline{\Omega})=1}} |(\mu_n, y)_\Omega| \leq \|\mu_n\|_{L^1(\Omega)}.$$

Therefore, by Theorem 5.2, we have just verified that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is uniformly bounded in $\mathcal{C}(\overline{\Omega})^*$ and hence the assertion of the theorem is justified by Lemma 5.2. □

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