Eigenvalue Distribution
of Semi-Elliptic Operators
in Anisotropic Sobolev Spaces

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Abstract. We study the spectral properties of the compact non-negative self-adjoint operator $T = A^{-1} \circ \mathrm{tr}^\Gamma$ acting in the anisotropic Sobolev space $H_2^{s,a}(\mathbb{R}^n)$ and give two-sided estimates for the asymptotic behaviour of its eigenvalues $\lambda_k(T)$, where A is a semi-elliptic differential operator of type

$$Au(x) = (-1)^{s_1} \frac{\partial^{2s_1} u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n} \frac{\partial^{2s_n} u(x)}{\partial x_n^{2s_n}} + u(x),$$

and tr^Γ a special trace operator on an anisotropic d-set Γ.

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1. Introduction

Let us consider a differential expression with real coefficients $A(\mathrm{D}) = \sum a_\alpha \mathrm{D}^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, and $|\alpha| = \sum_{i=1}^n \alpha_i$. Let $l = (l_1, \ldots, l_n)$, $(l_k > 0, 1 \le k \le n)$ be a fixed multi-index. We write $(\alpha : 2l) = \sum_{k=1}^n \frac{\alpha_k}{2l_k}$. We study the following differential operator:

$$A(\mathrm{D})u = \sum_{(\alpha:2l)=1} a_\alpha \mathrm{D}^\alpha u.$$

$A(\mathrm{D})$ is said to be semi-elliptic if the corresponding polynomial is positive,

$$A(\xi) = \sum_{(\alpha:2l)=1} a_\alpha \xi^\alpha > 0, \qquad \xi \in \mathbb{R}^n \setminus \{0\}.$$

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First we explain the physical background of the interest in such operators and start with a classical situation. Let Ω be a bounded domain in the plane $\mathbb{R}^2$ with C^∞ boundary $\partial\Omega$, interpreted as a membrane fixed at its boundary. Vibrations of such a membrane in $\mathbb{R}^3$ are measured by the deflection $v(x,t)$, where $x = (x_1, x_2) \in \Omega$, and $t \geq 0$ stands for the time. In other words, the point $(x_1, x_2, 0)$ in $\mathbb{R}^3$ with $(x_1, x_2) \in \Omega$ of the membrane at rest, is deflected to $(x_1, x_2, v(x,t))$ at time $t > 0$. Up to constants the usual physical description is given by

$$\Delta v(x,t) = m(x)\frac{\partial^2 v(x,t)}{\partial t^2}, \quad x \in \Omega,\ t \geq 0, \tag{1.1}$$

and

$$v(y,t) = 0 \quad \text{if } y \in \partial\Omega,\ t \geq 0,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian and the right-hand side of (1.1) is Newton's law with the mass density $m(x)$. To find the eigenfrequencies one inserts $v(x,t) = u(x)e^{i\lambda t}$ with $\lambda \in \mathbb{R}$ in (1.1) and obtains

$$-\Delta u(x) = \lambda^2 m(x)u(x), \quad x \in \Omega; \quad u(y) = 0 \ \text{if} \ y \in \partial\Omega,$$

where one is interested in non-trivial solutions $u(x)$. Hence one asks for eigenfunctions and eigenvalues of the operator

$$B = (-\Delta)^{-1} \circ m(\cdot),$$

where $(-\Delta)^{-1}$ is the inverse of the *Dirichlet Laplacian* $-\Delta$. We use the notation *Dirichlet Laplacian* always with the understanding that vanishing boundary data at $\partial\Omega$ are incorporated into domains of definition for $-\Delta$ in the function spaces considered, preferably $B_{pq}^s(\Omega)$ and $H_p^s(\Omega)$ with $1 < p \leq \infty$ and $s > \frac{1}{p}$ (this will be specified in greater detail in the next subsection). If ϱ is a positive eigenvalue of B, then $\lambda = \varrho^{-\frac{1}{2}}$ is the related eigenfrequency. Of special interest is the problem what happens when the mass density $m(x)$ shrinks to a fractal set Γ and a related Radon measure μ with

$$\text{supp } \mu = \Gamma \subset \Omega.$$

This refers to eigenfrequencies and eigenfunctions of drums with a fractal membrane. This is what we call fractal drums and fractal Laplacians (extending this notation to $n \in \mathbb{N}$, where Ω is a bounded domain in $\mathbb{R}^n$).

We want to mention that the notion of fractal drums has several meanings. As for the study of fractal membranes in smooth domains, there are the papers by Fujita [7], Naimark and Solomyak [11,12], Solomyak and Verbitsky [15], and by Edmunds and Triebel [2]. Further results on the vibration of "fractal drums" are obtained in different settings. Maybe the best known version is connected with the study of the Laplacian on a fractal, as it is done for example in the

works of Kigami and Lapidus, see [8,9]. A detailed discussion on these different aspects concerning fractal drums can de found in [23, Sections 26.2, 30.1–30.5].

Our motivation in this paper is Triebel's (isotropic) result in [24] for the fractal elliptic operator of type

$$B_s = (-\Delta + \mathrm{id})^{-s} \circ \mathrm{tr}^\Gamma, \quad s > 0. \tag{1.2}$$

Then B_s is a compact, non-negative, self-adjoint operator in $W_2^s(\mathbb{R}^n)$, where

$$\mathrm{tr}^\Gamma = \mathrm{id}_\Gamma \circ \mathrm{tr}_\Gamma,$$

and $\mathrm{tr}_\Gamma : W_2^s(\mathbb{R}^n) \to L_2(\Gamma)$ is the trace operator, and id_Γ is the dual of the trace operator. If we restrict the outcome to the classical example of a compact d-set with $0 < d < n$ and $n - d < 2s \le n$, we get that

$$\lambda_k(B_s) \sim k^{-\frac{1}{d}(d+2s-n)}, \tag{1.3}$$

see [24, Theorem 3, Remark 10]. We look for an anisotropic counterpart of (1.2), (1.3).

An important first step in this context was made by Farkas in the papers [6] and [4]. He studied the operator

$$A^{-1} \circ \mathrm{tr}^\Gamma, \tag{1.4}$$

where

$$\mathrm{tr}^\Gamma : \; B_{p1}^{\frac{2-d}{p},a}(\mathbb{R}^2) \to B_{p\infty}^{-\frac{2-d}{p'},a}(\mathbb{R}^2), \tag{1.5}$$

A^{-1} is the inverse of

$$Au(x) = (-1)^{t_1} \frac{\partial^{2t_1} u(x)}{\partial x_1^{2t_1}} + (-1)^{t_2} \frac{\partial^{2t_2} u(x)}{\partial x_n^{2t_2}} + u(x),$$

and he proved that the operator $A^{-1} \circ \mathrm{tr}^\Gamma$ is compact, non-negative, and self-adjoint in $W_2^{t,a}(\mathbb{R}^2)$ and its positive eigenvalues can be estimated by

$$\lambda_k(A^{-1} \circ \mathrm{tr}^\Gamma) \sim ck^{-\frac{1}{d}(d+2t-2)}. \tag{1.6}$$

We extend this result to the case $\mathbb{R}^n$, $n \ge 2$, related to a generalised notion of anisotropic d-sets introduced in [18], that is, a set $\Gamma \subset \mathbb{R}^n$ satisfying that $\mu(B^a(\gamma, r)) \sim r^d$, $0 < r < 1$, where $B^a(\gamma, r) = \{y \in \mathbb{R}^n : \; |y - \gamma|_a \le r\}$, $\gamma \in \Gamma$, $0 < d < n$, and μ a positive Radon measure in $\mathbb{R}^n$ with compact support $\Gamma = \mathrm{supp}\, \mu$, $0 < \mu(\mathbb{R}^n) < \infty$, and $|\Gamma| = 0$. Our main aim is to study operators of type (1.4) in the case $\mathbb{R}^n$, and to prove counterparts of (1.6). We shall apply approximation number results from [18] (instead of related ones for entropy numbers as in [6]), following thus ideas in [24].

The plan of the paper is the following. First we recall some basic notation and concepts in anisotropic spaces. Then we give the definitions and some important properties of anisotropic function spaces of Besov and Sobolev type. In Section 4 we deal with the concepts of anisotropic d-sets, approximation numbers of related embeddings and trace operators, and recall some results from [18] that will be applied afterwards. Finally, in the last section we formulate and prove our main result and briefly discuss it.

2. Preliminaries

2.1. General notation. As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ stands for the complex numbers, and $\mathbb{Z}^n$ means the lattice of all points in $\mathbb{R}^n$ with integer-valued components. We use the equivalence "$\sim$", in $\varphi(x) \sim \psi(x)$, in the sense that there are two positive numbers c_1 and c_2 such that

$$c_1 \, \varphi(x) \le \psi(x) \le c_2 \, \varphi(x)$$

for all admitted values of x, where φ, ψ are non-negative functions. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index, then $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $\alpha \in \mathbb{N}_0^n$, the derivatives D^α have the usual meaning, x^α means $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $\alpha\gamma = \alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n$, $\gamma \in \mathbb{R}^n$, stands for the scalar product in $\mathbb{R}^n$.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. All unimportant positive constants will be denoted by c, occasionally with additional subscripts within the same formula.

2.2. Anisotropic distance function. Let $a = (a_1, \ldots, a_n)$ be a fixed n-tuple of positive numbers with $a_1 + \cdots + a_n = n$, then we call a an *anisotropy*. If $a = (1, \ldots, 1)$ we speak about the "isotropic case".

The action of $t \in [0, \infty)$ on $x \in \mathbb{R}^n$ is defined by the formula

$$t^a x = (t^{a_1} x_1, \ldots, t^{a_n} x_n).$$

For $t > 0$ and $s \in \mathbb{R}$ we put $t^{sa}x = (t^s)^a x$. In particular, we write $t^{-a}x = (t^{-1})^a x$ and $2^{-ja}x = (2^{-j})^a x$.

Definition 2.1. An *anisotropic distance function* is a continuous function $u : \mathbb{R}^n \to \mathbb{R}$ with the properties $u(x) > 0$ if $x \ne 0$ and $u(t^a x) = tu(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$.

Remark 2.2. It is easy to see that $u_\lambda : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u_\lambda(x) = \left(\sum_{i=1}^{n} |x_i|^{\frac{\lambda}{a_i}} \right)^{\frac{1}{\lambda}} \tag{2.1}$$

is an anisotropic distance function for every $0 < \lambda < \infty$, u_2 is usually called the anisotropic distance of x to the origin, see [13, 4.2.1]. It is well known, see [1, 1.2.3] and [26, 1.4], that any two anisotropic distance functions u and u' are equivalent (in the sense that there exist constants $c, c' > 0$ such that $cu(x) \leq u'(x) \leq c'u(x)$ for all $x \in \mathbb{R}^n$) and that if u is an anisotropic distance function, there exists a constant $c > 0$ such that $u(x + y) \leq c(u(x) + u(y))$ for all $x, y \in \mathbb{R}^n$. We want to use smooth anisotropic distance functions. Note that for appropriate values of λ one can obtain arbitrary (finite) smoothness of the function u_λ from (2.1), cf. [1, 1.2.4]. A standard method concerning the construction of anisotropic distance functions in $C^\infty(\mathbb{R}^n \backslash \{0\})$ was given in [16].

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x \neq 0$, let $|x|_a$ be the unique positive number t such that

$$\frac{x_1^2}{t^{2a_1}} + \cdots + \frac{x_n^2}{t^{2a_n}} = 1$$

and let $|0|_a = 0$; then $|\cdot|_a$ is an anisotropic distance function in $C^\infty(\mathbb{R}^n \backslash \{0\})$, see [26, 1.4/3,8]. Plainly, $|x|_a$ is in the isotropic case the Euclidean distance of x to the origin.

3. Anisotropic function spaces

Before introducing the function spaces under consideration we need to recall some notation. By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \leq \infty$, stands for the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\| f \mid L_p(\mathbb{R}^n) \| := \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

with the usual modification if $p = \infty$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\widehat{\varphi}(\xi) \equiv (\mathcal{F}\varphi)(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) \, \mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{3.1}$$

denotes the Fourier transform of φ. As usual, $\mathcal{F}^{-1}\varphi$ or $\varphi^\vee$, stands for the inverse Fourier transform, given by the right-hand side of (3.1) with i in place of $-i$.

Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi(x) = 1 \quad \text{if} \quad |x|_a \leq 1 \quad \text{and} \quad \text{supp } \varphi \subset \{x \in \mathbb{R}^n : |x|_a \leq 2\},$$

and for each $j \in \mathbb{N}$ let

$$\varphi_j^a(x) := \varphi(2^{-ja}x) - \varphi(2^{(-j+1)a}x), \quad x \in \mathbb{R}^n.$$

Then the sequence $(\varphi_j^a)_{j=0}^{\infty}$, with $\varphi_0 = \varphi$, forms a smooth anisotropic dyadic resolution of unity, cf. [13, Section 4.2]. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the compact support of $\varphi_j^a \widehat{f}$ implies by the Paley–Wiener–Schwartz theorem that $(\varphi_j^a \widehat{f})^{\vee}$ is an entire analytic function on $\mathbb{R}^n$.

Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $a = (a_1, \ldots, a_n)$ an anisotropy, and $(\varphi_j^a)_{j=0}^{\infty}$ a smooth anisotropic dyadic resolution of unity. Then $B_{pq}^{s,a}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f \mid B_{pq}^{s,a}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j^a \widehat{f})^{\vee} | L_p(\mathbb{R}^n)\|^q \right)^{\frac{1}{q}} \tag{3.2}$$

(with the usual modification if $q = \infty$) is finite.

Note that there is a parallel definition for spaces of type $F_{pq}^{s,a}(\mathbb{R}^n)$, $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $a = (a_1, \ldots, a_n)$ an anisotropy, when interchanging the order of ℓ_q- and L_p- quasi-norms in (3.2). It is obvious, that the quasi-norm (3.2) depends on the chosen system $(\varphi_j^a)_{j \in \mathbb{N}_0}$, but not the space $B_{pq}^{s,a}(\mathbb{R}^n)$ (in the sense of equivalent quasi-norms); therefore we omit in our notation the subscript φ in the sequel. It is well-known that $B_{pq}^{s,a}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$), and, as in the isotropic case, $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{pq}^{s,a}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ for all admissible values of p, q, s, see [21, Section 2.3.3]. If $s \in \mathbb{R}$ and $0 < p < \infty$, $0 < q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $B_{pq}^{s,a}(\mathbb{R}^n)$, see [26, Section 3.5] and [1, Section 1.2.10]. Note that we indicated the only (formal) difference to the isotropic counterparts of (3.2) by the additional superscript at the smooth anisotropic dyadic resolution of unity $(\varphi_j^a)_{j=0}^{\infty}$.

We want to point out that if $0 < p < \infty$ and $s \in \mathbb{R}$, then

$$B_{pp}^{s,a}(\mathbb{R}^n) = F_{pp}^{s,a}(\mathbb{R}^n). \tag{3.3}$$

If $1 < p < \infty$ and $s \in \mathbb{R}$, then (in the sense of equivalent quasi-norms)

$$F_{p2}^{s,a}(\mathbb{R}^n) = H_p^{s,a}(\mathbb{R}^n), \tag{3.4}$$

where

$$H_p^{s,a}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left\| \left(\sum_{k=1}^{n} (1 + \xi_k^2)^{\frac{s}{2a_k}} \widehat{f} \right)^{\vee} | L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

are the anisotropic Bessel potential spaces (see [17,Remark 11], [19,Section 2.5.2] and [26, Section 3.11]).

Furthermore, if $1 < p < \infty$, $s > 0$ and if $s_i = \frac{s}{a_i} \in \mathbb{N}$, $i = 1, \ldots, n$, then (in the sense of equivalent quasi-norms)

$$F_{p2}^{s,a}(\mathbb{R}^n) = W_p^{s,a}(\mathbb{R}^n), \tag{3.5}$$

where

$$W_p^{s,a}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \ \|f|L_p(\mathbb{R}^n)\| + \sum_{k=1}^n \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \, |L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

are the classical anisotropic Sobolev spaces on $\mathbb{R}^n$. As a consequence of (3.3), (3.4) and (3.5) we have

$$B_{22}^{s,a}(\mathbb{R}^n) = F_{22}^{s,a}(\mathbb{R}^n) = H_2^{s,a}(\mathbb{R}^n) = W_2^{s,a}(\mathbb{R}^n), \tag{3.6}$$

for $s > 0$ and $s_i = \frac{s}{a_i} \in \mathbb{N}$, $i = 1, \ldots, n$.

4. Traces and approximation numbers

4.1. General measures. Let μ be a positive Radon measure in $\mathbb{R}^n$ with compact support

$$\Gamma = \operatorname{supp} \mu, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0,$$

where $|\Gamma|$ is the Lebesgue measure of Γ. For $1 \le p < \infty$ we denote by $L_p(\Gamma) = L_p(\Gamma, \mu)$ the usual complex Banach space, normed by

$$\|f|L_p(\Gamma, \mu)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \mu(\,\mathrm{d}x) \right)^{\frac{1}{p}} = \left(\int_\Gamma |f(\gamma)|^p \mu(\,\mathrm{d}\gamma) \right)^{\frac{1}{p}}.$$

Since μ is Radon, $\mathcal{S}(\mathbb{R}^n)|\Gamma$ is dense in $L_p(\Gamma)$, for details see [23, p. 7]. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\operatorname{tr}_\Gamma \varphi = \varphi|\Gamma$ makes sense pointwise. If $1 < p < \infty$ and $s > 0$, then the embedding $\operatorname{tr}_\Gamma B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$ must be understood as follows: we ask whether there is a positive number $c > 0$ such that for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\operatorname{tr}_\Gamma \varphi|L_p(\Gamma)\| \le c\|\varphi|B_{pp}^{s,a}(\mathbb{R}^n)\|. \tag{4.1}$$

If this is the case, we use that $\mathcal{S}(\mathbb{R}^n)$ is dense in $B_{pp}^{s,a}(\mathbb{R}^n)$ for $0 < p < \infty$; hence this inequality can be extended by completion to any $f \in B_{pp}^{s,a}(\mathbb{R}^n)$ and the resulting function is denoted by $\operatorname{tr}_\Gamma f$,

$$\operatorname{tr}_\Gamma : B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma). \tag{4.2}$$

The independence of $\operatorname{tr}_\Gamma f$ from the approximating sequence is shown in the standard way. On the other hand, if $f \in L_p(\Gamma)$ is given, then f can be interpreted in the usual way as a tempered distribution $\operatorname{id}_\Gamma f$, given by

$$(\operatorname{id}_\Gamma f)(\varphi) = \int_\Gamma f(\gamma)\varphi(\gamma)\mu(\,\mathrm{d}\gamma) = \int_\Gamma f(\gamma)(\operatorname{tr}_\Gamma\varphi)(\gamma)\mu(\,\mathrm{d}\gamma), \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \quad (4.3)$$

We call id_Γ the identification operator. Let again $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have for the dual spaces,

$$(L_p(\Gamma))' = L_{p'}(\Gamma) \quad \text{and} \quad (B_{pp}^{\sigma,a}(\mathbb{R}^n))' = B_{p'p'}^{-\sigma,a}(\mathbb{R}^n)$$

for any $\sigma \in \mathbb{R}$. The first assertion is well known, the second is a consequence of [25, Section 5.1.7]. In particular, all $B_{pp}^{s,a}(\mathbb{R}^n)$ and also $L_p(\Gamma)$ with $1 < p < \infty$ are reflexive. By (4.3), the operators tr_Γ and id_Γ are dual to each other. Hence (4.2) is equivalent to

$$\operatorname{id}_\Gamma : \ L_{p'}(\Gamma) \hookrightarrow B_{p'p'}^{-s,a}(\mathbb{R}^n), \quad (4.4)$$

and

$$(\operatorname{tr}_\Gamma)' = \operatorname{id}_\Gamma, \qquad (\operatorname{id}_\Gamma)' = \operatorname{tr}_\Gamma. \quad (4.5)$$

In the following we study the existence of the trace operator. Let Q_{jm}^a be rectangles in $\mathbb{R}^n$ with side lengths $2^{-ja_1}, \ldots, 2^{-ja_n}$ and centered at $2^{-ja}m$, where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. Let

$$\mu_j = \sup_{m\in\mathbb{Z}^n} \mu(Q_{jm}^a), \quad j \in \mathbb{N}_0.$$

Proposition 4.1. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $s > 0$. Let μ be the Radon measure in $\mathbb{R}^n$ with*

$$\Gamma = \operatorname{supp}\mu \ \ \text{compact}, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0, \quad (4.6)$$

and

$$\sum_{j\in\mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})}\mu_j^{p'-1} < \infty, \quad \text{where} \ \ \mu_j = \sup_{m\in\mathbb{Z}^n} \mu(Q_{jm}^a). \quad (4.7)$$

Then the operator tr_Γ,

$$\operatorname{tr}_\Gamma : B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma) \quad (4.8)$$

exists and is compact. Furthermore, there is a constant c (depending on p and s) such that for all measures μ with (4.6), (4.7),

$$\|\operatorname{tr}_\Gamma\| \le c \left(\sum_{j\in\mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})}\mu_j^{p'-1} \right)^{\frac{1}{p'}}.$$

The proof can be found in [18, 4.5].

Next we recall the concept of approximation numbers. Let A and B be two Banach spaces and let $T \in L(A, B)$. Then for any $k \in \mathbb{N}$ the kth approximation number $a_k(T)$ of T is given by

$$a_k(T) = \inf \left\{ \|T - L\| \, : \, L \in L(A, B), \, \text{rank}\, L < k \right\},$$

where rank L is the dimension of the range of L. These numbers have various properties, where we collected some of them in the following lemma for convenience.

Lemma 4.2. *Let A and B be two Banach spaces and let T, $S \in L(A, B)$.*

(i) $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq 0$;

(ii) *for all n, $m \in \mathbb{N}$, $a_{m+n-1}(S + T) \leq a_m(S) + a_n(T)$;*

(iii) *for all n, $m \in \mathbb{N}$, and $R \in L(B, C)$, $a_{m+n-1}(RT) \leq a_m(R)a_n(T)$;*

(iv) $a_n(T) = 0$ *if and only if* rank $T < n$.

These formulations coincide essentially with [3, II. Proposition 2.2], where one finds also a short proof. Further properties, comments and references may be found in [18, p. 11–18], [3, Chapter II.]. We restricted ourselves to those assertions which we shall need later on.

Now we consider some connections between approximation numbers and spectral assertions of compact operators. Let A be a complex quasi-Banach space and $T \in L(A)$ a compact map. We know from [18, Theorem 1.2] that the spectrum of T, apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity: let $\{\lambda_k(T) : k \in \mathbb{N}\}$ be the sequence of all non-zero eigenvalues of T, repeated according to their algebraic multiplicity and ordered so that

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq 0. \tag{4.9}$$

If T has only $m(< \infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities, we put $\lambda_k(T) = 0$ for $k > M$.

Proposition 4.3.

(i) *Let A and B two Banach spaces and $T \in L(A, B)$ be compact with dual operator $T' \in L(A', B')$, then*

$$a_k(T) = a_k(T') \quad \text{for all} \ \ k \in \mathbb{N}.$$

(ii) *Let H be a Hilbert space and let $T \in L(H)$ be a compact, non-negative and self-adjoint operator. Then the approximation numbers $a_k(T)$ of T coincide with its eigenvalues (ordered as in (4.9)).*

Remark 4.4. Proofs of these well-known assertions may be found in [3, Proposition 2.5, p. 55] for (i), and [3, Theorem 5.10, p. 91] for (ii).

Let $T = \mathrm{tr}_\Gamma$ according to Proposition 4.1. We strengthen (4.7) by

$$\sum_{j \geq J} 2^{-jp'(s-\frac{n}{p})}\mu_j^{p'-1} \sim 2^{-Jp'(s-\frac{n}{p})}\mu_J^{p'-1}, \quad J \in \mathbb{N}_0, \tag{4.10}$$

where only the cases $s \leq \frac{n}{p}$ are of interest, otherwise (4.10) is always satisfied.

Proposition 4.5. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $s > 0$. Let μ be a Radon measure in $\mathbb{R}^n$ with (4.6) and (4.10). Let $a_k = a_k(\mathrm{tr}_\Gamma)$ be the approximation numbers of the compact operator tr_Γ in (4.8). There are two positive numbers c and c' such that*

$$a_{c2^{nJ}} \leq c' 2^{-J(s-\frac{n}{p})}\mu_J^{\frac{1}{p}}, \quad J \in \mathbb{N}_0,$$

where $c2^{nJ}$ is always assumed to be a natural number.

The proof can be found in [18, 4.6].

4.2. Anisotropic d-sets in $\mathbb{R}^n$.

We assume that μ is a positive Radon measure in $\mathbb{R}^n$ with compact support

$$\Gamma = \mathrm{supp}\,\mu, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0,$$

where $|\Gamma|$ denotes the Lebesgue measure of Γ. Let again $a = (a_1, \ldots, a_n)$ be a given anisotropy.

Definition 4.6. Let $0 < d < n$. Then $\Gamma \subset \mathbb{R}^n$ is called an *anisotropic d-set*, if

$$\mu(B^a(\gamma, r)) \sim r^d, \quad 0 < r < 1, \tag{4.11}$$

where $B^a(\gamma, r) = \{y \in \mathbb{R}^n : |y - \gamma|_a \leq r\}$ and $\gamma \in \Gamma$.

The existence of such anisotropic d-sets as well as some examples are shown in [18, 4.7].

Now we are prepared to formulate our main result an approximation numbers of related compact trace operators in [18].

Theorem 4.7. *Let the anisotropic d-set Γ and μ be given according to (4.11), and $0 < d < n$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{n}{p} \geq s > \frac{n-d}{p}$. Let $a_k = a_k(\mathrm{tr}_\Gamma)$ be the approximation numbers of the compact operator tr_Γ. Then there exist numbers $c, c' > 0$ so that for all $k \in \mathbb{N}$,*

$$ck^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}} \leq a_k\left(\mathrm{tr}_\Gamma : B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)\right) \leq c'k^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}}.$$

The proof can be found in [18, 4.8]

5. Main results

In this section we restrict ourselves to the case $p = 2$ since we have Proposition 4.3(ii) for Hilbert spaces only.

Again let Γ be an anisotropic d-set with respect to the anisotropy $a = (a_1, \ldots, a_n)$, then by (4.2) and (3.6), $\mathrm{tr}_\Gamma : H_2^{s,a}(\mathbb{R}^n) \hookrightarrow L_2(\Gamma)$, $s > 0$. By (3.6), (4.4) and (4.5) we have that $\mathrm{id}_\Gamma : L_2(\Gamma) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n)$, and consequently,

$$\mathrm{tr}^\Gamma = \mathrm{id}_\Gamma \circ \mathrm{tr}_\Gamma : H_2^{s,a}(\mathbb{R}^n) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n). \tag{5.1}$$

Let $s_1, \ldots, s_n \in \mathbb{N}$ and let $s \in \mathbb{R}$ be defined by

$$\frac{1}{s} = \frac{1}{n}\left(\frac{1}{s_1} + \cdots + \frac{1}{s_n}\right). \tag{5.2}$$

Let A be the operator defined by

$$Au(x) = (-1)^{s_1}\frac{\partial^{2s_1}u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n}\frac{\partial^{2s_n}u(x)}{\partial x_n^{2s_n}} + u(x), \tag{5.3}$$

where $x \in \mathbb{R}^n$. Using elementary properties of the Fourier transform we have $Au = \left((1 + \xi_1^{2s_1} + \cdots + \xi_n^{2s_n})\hat{u}\right)^\vee$ for any $u \in \mathcal{S}'(\mathbb{R}^n)$.

It is well known, see for example [10], that A is a lift operator for the scale $B_{pq}^{t,a}(\mathbb{R}^n)$, $t \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. More precisely, A maps any space $B_{pq}^{t,a}(\mathbb{R}^n)$ onto $B_{pq}^{t-2s,a}(\mathbb{R}^n)$ and $\|A(\cdot)|B_{pq}^{t-2s,a}(\mathbb{R}^n)\|$ is an equivalent quasi-norm on $B_{pq}^{t,a}(\mathbb{R}^n)$, the inverse A^{-1} of A has to be understood in this way.

Theorem 5.1. *Let $\Gamma \subset \mathbb{R}^n$ be an anisotropic d-set according to Definition 4.6 with respect to the anisotropy a. Let tr^Γ be the operator given by (5.1), $s_i \in \mathbb{N}$, $i = 1, \ldots, n$, $\frac{1}{s} = \frac{1}{n}\left(\frac{1}{s_1} + \cdots + \frac{1}{s_n}\right)$, A given by*

$$Au(x) = (-1)^{s_1}\frac{\partial^{2s_1}u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n}\frac{\partial^{2s_n}u(x)}{\partial x_n^{2s_n}} + u(x)$$

with $0 < d < n$, and $\frac{n}{2} \geq s > \frac{n-d}{2}$. Then

$$T = A^{-1} \circ \mathrm{tr}^\Gamma \tag{5.4}$$

is a compact, non-negative self-adjoint operator in $H_2^{s,a}(\mathbb{R}^n)$ and with null space

$$N(T) = \left\{f \in H_2^{s,a}(\mathbb{R}^n) : \mathrm{tr}_\Gamma f = 0\right\}.$$

Let $(\lambda_k)_{k \in \mathbb{N}}$ be the sequence of all positive eigenvalues of T, repeated according to multiplicity and ordered by their magnitude. Then

$$\lambda_k \sim k^{-\frac{1}{d}(2s-n+d)}, \quad k \in \mathbb{N}.$$

We begin the proof of Theorem 5.1 with some preparation.

Lemma 5.2. *Let s be given by* (5.2) *and A the operator from* (5.3).

1. *There exists a constant $c > 0$ such that $\langle Au, u\rangle_{L_2(\mathbb{R}^n)} \geq c\|u|L_2(\mathbb{R}^n)\|$ for any $u \in L_2(\mathbb{R}^n)$.*
2. *There exist two constants $c_1, c_2 > 0$ such that*

$$c_1\|u|H_2^{s,a}(\mathbb{R}^n)\|^2 \leq \langle Au, u\rangle_{L_2(\mathbb{R}^n)} \leq c_2\|u|H_2^{s,a}(\mathbb{R}^n)\|^2.$$

Proof of Lemma 5.2. We closely follow the ideas in [6, Section 6.2, Lemma 6.3] where Farkas proved the lemma for the case if $n = 2$. This can be extended to the case $n \in \mathbb{N}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\langle A\varphi, \varphi\rangle_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(A\varphi)(x)\overline{\varphi(x)}\,\mathrm{d}x$. After integration by parts we have

$$\langle A\varphi, \varphi\rangle_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(\left|\frac{\partial^{s_1}\varphi(x)}{\partial x_1^{s_1}}\right|^2 + \cdots + \left|\frac{\partial^{s_n}\varphi(x)}{\partial x_n^{s_n}}\right|^2 + |\varphi(x)|^2\right)\mathrm{d}x\,,$$

and the conclusion of the lemma follows immediately using the density of $C_0^\infty(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$ and in $W_2^{s,a}(\mathbb{R}^n)$, together with (3.6). $\square$

Finally we can prove Theorem 5.1.

Proof of Theorem 5.1.

`Step 1:` We first show that T given by (5.4) is a compact, non-negative self-adjoint operator in $H_2^{s,a}(\mathbb{R}^n)$. In view of (5.4) and $\mathrm{tr}^\Gamma = \mathrm{id}_\Gamma \circ \mathrm{tr}_\Gamma$ the operator $T = A^{-1} \circ \mathrm{tr}^\Gamma$ can be decomposed into

$$
\begin{aligned}
\mathrm{tr}_\Gamma : \quad & H_2^{s,a}(\mathbb{R}^n) \hookrightarrow L_2(\Gamma) \\
\mathrm{id}_\Gamma : \quad & L_2(\Gamma) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n) \\
A^{-1} : \ & H_2^{-s,a}(\mathbb{R}^n) \hookrightarrow H_2^{s,a}(\mathbb{R}^n).
\end{aligned}
\tag{5.5}
$$

By Lemma 5.2 we have that the operator A is positive-definite as an operator acting in $L_2(\mathbb{R}^n)$ and we may fix the norm in $H_2^{s,a}(\mathbb{R}^n)$ by $\|A^{\frac{1}{2}}(\cdot)|L_2(\mathbb{R}^n)\|$ and a corresponding scalar product. By Proposition 4.1, (3.6) and (4.1) there exists a constant $c > 0$ such that $\|\mathrm{tr}_\Gamma\varphi|L_2(\Gamma)\| \leq c\|\varphi|H_2^{s,a}(\mathbb{R}^n)\|$ for all $\varphi \in H_2^{s,a}(\mathbb{R}^n)$. Defining

$$q(f, g) = \int_\Gamma f(\gamma)\overline{g(\gamma)}\mu(\mathrm{d}\gamma) \quad \text{for any} \ \ f, g \in H_2^{s,a}(\mathbb{R}^n),$$

it is clear that $q(\cdot, \cdot)$ is a non-negative quadratic form in $H_2^{s,a}(\mathbb{R}^n)$. Then there exists a non-negative and self-adjoint operator $\widetilde{T}$ uniquely determined such that

$q(f,g) = \langle \widetilde{T}f, g \rangle_{H_2^{s,a}(\mathbb{R}^n)}$ for any $f, g \in H_2^{s,a}(\mathbb{R}^n)$, see for example [22, p. 91]. Furthermore,

$$\|\mathrm{tr}_\Gamma f | L_2(\Gamma)\| = \|\sqrt{\widetilde{T}}f | H_2^{s,a}(\mathbb{R}^n)\|, \tag{5.6}$$

where $\sqrt{\widetilde{T}} = \widetilde{T}^{\frac{1}{2}}$. So it remains to prove that the above operator is the same as in (5.4). Let $f \in H_2^{s,a}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then by (4.3), (4.5) and (5.1),

$$
\begin{aligned}
\langle \mathrm{tr}^\Gamma f, \varphi \rangle_{L_2(\mathbb{R}^n)} = \int_\Gamma f(\gamma)\overline{\varphi(\gamma)}\, \mathrm{d}\mu(\gamma) &= \langle \widetilde{T}f, \varphi \rangle_{H_2^{s,a}(\mathbb{R}^n)} \\
&= \langle A^{\frac{1}{2}}\widetilde{T}f, A^{\frac{1}{2}}\varphi \rangle_{L_2(\mathbb{R}^n)} \\
&= \langle A\widetilde{T}f, \varphi \rangle_{L_2(\mathbb{R}^n)},
\end{aligned}
\tag{5.7}
$$

where the second equality in (5.7) is justified by the fact that we fixed the scalar product in $H_2^{s,a}(\mathbb{R}^n)$ by $\langle u, v \rangle_{H_2^{s,a}(\mathbb{R}^n)} = \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v \rangle_{L_2(\mathbb{R}^n)}$. Considered as a dual pairing in $(\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^n))$ we obtain $A\widetilde{T}f = \mathrm{tr}^\Gamma f$, and we have that $\widetilde{T} = T$ by (5.4). The compactness is a consequence of Theorem 4.7 and (5.5).

Step 2: We prove that there is a number $c > 0$ such that

$$\lambda_k \le ck^{-\frac{1}{d}(2s-n+d)}, \quad k \in \mathbb{N}. \tag{5.8}$$

Recall that the identification operator id_Γ is the dual of the trace operator tr_Γ by (4.5). Thus standard reasoning for dual operators, Proposition 4.3(i), and Theorem 4.7 imply that

$$a_k(\mathrm{id}_\Gamma) = a_k(\mathrm{tr}_\Gamma) \sim k^{\frac{1}{d}(\frac{n}{2}-s)-\frac{1}{2}}, \quad k \in \mathbb{N},$$

where we make use of $\frac{n}{2} \ge s > \frac{n-d}{2}$. By (5.5) and the multiplication property for approximation numbers, Lemma 4.2(iii), one obtains

$$a_{2k}(T) \le c\, a_k(\mathrm{tr}_\Gamma)a_k(\mathrm{id}_\Gamma) \sim k^{-\frac{1}{d}(2s-n+d)}. \tag{5.9}$$

Since Proposition 4.3(ii) tells us that the approximation numbers of T coincide with its eigenvalues, assertion (5.8) follows from (5.9).

Step 3: To obtain the converse of (5.8) we use the same argument as in Theorem 4.7, Step 2, now with $p = 2$, see [18, 4.8]. Let $J \in \mathbb{N}$ and $c > 0$ be suitably chosen numbers such that there are lattice points

$$\gamma_{j,l} = 2^{(-j-J)a}m \quad \text{with} \ \ m \in \mathbb{Z}^n, \ l = 1, \ldots, M_j, \quad \text{where} \ \ M_j \sim 2^{jd} \tag{5.10}$$

with

$$\mathrm{dist}(\gamma_{j,l}, \Gamma) \le c2^{-j},$$

assuming further that the anisotropic balls $B^a(\gamma_{j,l}, c2^{-j+1})$ are disjoint. Let k be a non-negative C^∞ function in $\mathbb{R}^n$ with

$$\mathrm{supp}\, k \subset \big\{y \in \mathbb{R}^n : |y|_a < 2^J \text{ and } y_j > 0, j = 1, \ldots, n\big\},$$

for some $J \in \mathbb{N}$, and $\sum_{m \in \mathbb{Z}^n} k(x - m) = 1$, $x \in \mathbb{R}^n$. For some $J \in \mathbb{N}$, we put for $j \in \mathbb{N}_0$,

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(x - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \ x \in \mathbb{R}^n. \tag{5.11}$$

Using the same argument as in Step 2 in the proof of [18, Theorem 3.7,p. 319] we obtain that $\left\| f_j^a | L_2(\Gamma) \right\| \geq c \ 2^{-j(s-\frac{n}{2})} \ 2^{-j\frac{d}{2}}$ whenever $\|f_j^a|H_2^{s,a}(\mathbb{R}^n)\| \sim 1$. By (5.6) (with $\widetilde{T}$ replaced by T) this can be rewritten as

$$\|\sqrt{T} f_j^a | H_2^{s,a}(\mathbb{R}^n)\| \geq c 2^{-j(s-\frac{n}{2})} 2^{-j\frac{d}{2}} \quad \text{if } \|f_j^a|H_2^{s,a}(\mathbb{R}^n)\| \sim 1.$$

On the other hand, for an arbitrary operator L with rank $L \leq M_j - 1$ we can always find some f_j^a according to (5.11) such that $\|f_j^a|H_2^{s,a}(\mathbb{R}^n)\| \sim 1$ and $L f_j^a = 0$. This leads to

$$a_{M_j}(\sqrt{T}) \geq c 2^{-j(s-\frac{n}{2})} 2^{-j\frac{d}{2}},$$

see also Theorem 4.7. Since $a_k(\sqrt{T}) = \lambda_k^{\frac{1}{2}}$, we obtain by (5.10) that

$$\lambda_k \geq c k^{-\frac{1}{d}(2s-n+d)}, \quad k \in \mathbb{N}.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.3. (i) Let Γ be the anisotropic d-set considered in [6, Section 3.1]. Farkas proved in [6, Section 4] for the operator (1.4) that $\lambda_k(A^{-1} \circ \text{tr}^\Gamma) \sim c k^{-\frac{1}{d}(d+2t-2)}$. If we take the case $n = 2$ and $t = s$ we have the same result $\lambda_k \sim k^{-\frac{1}{d}(2t-2+d)}$.

(ii) If we restrict ourselves to the case $n = 2$, $s_1 = 1$ and $s_2 = 2$, then

$$Au(x) = -\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^4 u(x)}{\partial x_2^4} + u(x), \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{5.12}$$

Let again Γ be the anisotropic d-set considered in [6, Section 3.1], with respect to the anisotropy $a = (\frac{4}{3}, \frac{2}{3})$. Farkas obtained in [5, Section 4] that $\lambda_k(A^{-1} \circ \text{tr}^\Gamma) \sim c k^{-\frac{1}{d}(d+\frac{2}{3})}$, where tr^Γ is given by (1.5) and A^{-1} is the inverse of (5.12). For this special case our results coincide. Operators of this type have been investigated by Triebel in [20] and by Shevchik in [14].

(iii) In view of the isotropic results [24, Theorem 3, Remark 10] for the operator $B_s = (\text{id} - \Delta)^{-s} \circ \text{tr}^\Gamma$ we have the same results like in the anisotropic case if we restrict the outcome to the classical example of a compact d-set with $0 < d < n$ and $n - d < 2s \leq n$, see (1.3).

References

[1] Dintelmann, P., *On Fourier multipliers between anisotropic weighted function spaces*. PhD thesis. TH Darmstadt, Germany 1995.

[2] Edmunds, D. E. and Triebel, H., Spectral theory for isotropic fractal drums. *C. R. Acad. Sci. Paris Sér. I Math.* 326 (1998), 1269 – 1274.

[3] Edmunds, D. E. and Evans, W. D., *Spectral Theory and Differential Operators.* Oxford: Oxford Univ. Press 1987.

[4] Farkas, W. and Triebel, H., The distribution of eigenfrequencies of anisotropic fractal drums. *J. London Math. Soc. (2)* 60 (1999), 224 – 236.

[5] Farkas, W., The behaviour of the eigenvalues for a class of operators related to some self-affine fractals in $\mathbb{R}^2$. *Z. Anal. Anwendungen* 18 (1999), 875 – 893.

[6] Farkas, W., Eigenvalue distribution of some fractal semi-elliptic differential operators. *Math. Z.* 236 (2001), 291 – 320.

[7] Fujita, T., A Fractional Dimension, Self-similarity and a Generalized diffusion Operator. In: *Probabilistic Methods in Mathematical Physics* (Proceedings of Taniguchi International Symposium Katata and Kyoto 1985; eds.: K. Ito and N. Ikeda). Boston: Academic Press 1987, pp. 83 – 90.

[8] Kigami, J. and Lapidus, M. L., Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals. *Comm. Math. Phys.* 158 (1993), 93 – 125.

[9] Lapidus, M. L., Vibrations of Fractal Drums, the Riemann Hypothesis, Waves in Fractal Media, and the Weyl–Berry Conjecture. In: *Ordinary and partial differential equations* Vol. IV (Dundee 1992; eds.: B. D. Sleeman and R. J. Jarvis). Pitman Res. Notes in Math. 289. London: Longman Scientific and Technical 1993, pp. 126 – 209.

[10] Leopold, H.-G., Boundedness of anisotropic pseudodifferential operators in function spaces of Besov-Hardy-Sobolev type. *Z. Anal. Anwendungen* 5 (1986), 409 – 417.

[11] Naimark, K. and Solomyak, M., On the eigenvalue behaviour for a class of operators related to self-similar measures on $\mathbb{R}^d$. *C. R. Acad. Sci. Paris Sér. I Math.* 319 (1994), 837 – 842.

[12] Naimark, K. and Solomyak, M., The eigenvalue behaviour for the boundary value problems related to self-similar measures on $\mathbb{R}^d$. *Math. Res. Lett.* 2 (1995), 279 – 298.

[13] Schmeisser, H.-J. and Triebel, H., *Topics in Fourier Analysis and Function Spaces.* Chichester: Wiley 1987.

[14] Shevchik, V., Spectral properties of some semi-elliptic operators in L_p spaces. *Math. Nachr.* 202 (1999), 151 – 162.

[15] Solomyak, M. and Verbitsky, I., On a spectral problem related to self-similar measures. *Bull. London Math. Soc.* 27 (1995), 242 – 248.

[16] Stein, E. M. and Wainger, S., Problems in harmonic analysis related to curvature. *Bull. Amer. Math. Soc.* 84 (1978)(6), 1239 – 1295.

[17] Stöckert, B. and Triebel, H., Decomposition methods for function spaces of $B_{p,q}^s$ type and $F_{p,q}^s$ type. *Math. Nachr.* 89 (1979), 247 – 267.

[18] Tamási, E., Anisotropic Besov Spaces and approximation numbers of traces on related fractal sets. *Rev. Mat. Complut.* 19 (2006)(2), 297 – 321.

[19] Triebel, H., *Fourier Analysis and Function Spaces.* Leipzig: Teubner 1977.

[20] Triebel, H., A priori estimates and boundary value problems for semi-elliptic differential equations: a model case. *Comm. Partial Diff. Equ.* 8 (1983)(15), 1621 – 1664.

[21] Triebel, H., *Theory of Function Spaces.* Basel: Birkhäuser 1983.

[22] Triebel, H., *Higher Analysis.* Leipzig: Barth 1992.

[23] Triebel, H., *Fractals and Spectra.* Basel: Birkhäuser 1997.

[24] Triebel, H., Approximation numbers in function spaces and the distribution of eigenvalues of some fractal elliptic operators. *J. Approx. Theory* 129 (2004), 1 – 27.

[25] Triebel, H., *Theory of Function Spaces* III. Basel: Birkhäuser 2006.

[26] Yamazaki, M., A quasihomogeneous version of paradifferential operators I. Boundedness on spaces of Besov type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 33 (1986), 131 – 174.