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Generalized Rademacher-Stepanov Type Theorem and Applications

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Abstract. The main purpose of this article is to generalize a theorem of Stepanov which provides a necessary and sufficient condition for almost everywhere differentiability of functions over Euclidean spaces. We state and prove an L^p-type generalization of the Stepanov theorem and then we extend it to the context of Orlicz spaces. Then, this generalized Rademacher–Stepanov type theorem is applied to the Sobolev and bounded variation maps with values into a metric space. It is shown that several generalized differentiability type theorems are valid for the Sobolev maps from a Lipschitz manifold into a metric space. As a byproduct, it is shown that the Sobolev spaces of Korevaar–Schoen and Reshetnyak are equivalent.

Keywords. Rademacher and Stepanov theorems, Sobolev and bounded variation spaces, generalized differentiability, Lipschitz manifolds, Orlicz spaces

Mathematics Subject Classification (2000). Primary 58C20, 46E30, secondary 46E35

1. Introduction

In this article, we generalize a theorem of Stepanov about differentiability of functions on the finite dimensional Euclidean spaces. Then, this generalized Stepanov type theorem is applied to the Sobolev and bounded variation (BV) maps. The Stepanov theorem provides a necessary and sufficient condition for almost everywhere (a.e.) differentiability of real-valued functions on $\mathbb{R}^k$. It is a generalized version of the Rademacher theorem, namely, every Lipschitz function over a finite dimensional Euclidean space is differentiable a.e. The precise statement of the Stepanov theorem is as the following (see [4, p. 218], [16, p. 250] and [17, p. 97]): *Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a measurable function on $\mathbb{R}^k$ with the standard Euclidean metric and Lebesgue measure. Suppose that*

$$\limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} < \infty,$$

for a.e. $x \in \mathbb{R}^k$. Then, f is differentiable a.e.

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Our generalized Stepanov type theorem is based on the following L^p-type differentiability concept.

Definition 1.1. Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ belong to $L^p_{loc}(\mathbb{R}^k)$, for some $p \geq 1$. We say that f is *L^p-differentiable* at point $x \in \mathbb{R}^k$, if there is a linear functional $Df(x) : \mathbb{R}^k \longrightarrow \mathbb{R}$ such that (see [16, p. 242])

$$\lim_{r\to 0} \fint_{B(x,r)} \frac{|f(y) - f(x) - Df(x)\cdot(y-x)|^p}{r^p}\, dy = 0,$$

where $B(x,r)$ denotes the closed ball of radius $r > 0$ with center at x and

$$\fint_{B(x,r)} g(y)\, dy := \frac{1}{\mathrm{vol}(B(x,r))} \int_{B(x,r)} g(y)\, dy.$$

We call $Df(x)$ the *L^p-differential* of f at x.

Notice that if f is L^p-differentiable at x, then it is *approximately differentiable* at x, i.e., for every $\epsilon > 0$, we have (see [3, p. 233] and [4, p. 212])

$$\lim_{r\to 0} \frac{\mathrm{vol}\big(\{y \in B(x,r) : |f(y) - f(x) - Df(x)\cdot(y-x)| \geq \epsilon\, |y-x|\}\big)}{\mathrm{vol}(B(x,r))} = 0.$$

In the first section, we state and prove the following L^p-type differentiability theorem which can be interpreted as an L^p-type generalization of the Stepanov (and Rademacher) theorem (compare [18, Theorems 2.1.6, 3.5.7]):

Theorem 1.2. *Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ belong to $L^p_{loc}(\mathbb{R}^k)$, for some $p \geq 1$. Suppose that*

$$\limsup_{\epsilon\to 0} \fint_{B(x,\epsilon)} \frac{|f(y) - f(x)|^p}{\epsilon^p}\, dy < \infty,$$

for a.e. $x \in \mathbb{R}^k$. Then, f is L^p-differentiable a.e.

Notice that if a function $f : \mathbb{R}^k \longrightarrow \mathbb{R}$, for some positive numbers M and $p \geq 1$, satisfies the condition

$$\fint_{B(x,r)} \frac{|f(y) - f(x)|^p}{r^p}\, dy \leq M,$$

for a.e. $x \in \mathbb{R}^k$ and all $r > 0$, then by Campanato's theorem (see for example [10, p. 31]), f is Lipschitz and therefore f is differentiable a.e.

In the second section, a Rademacher differentiability type theorem is proved for Lipschitz maps into a Banach space. It is shown that if $f : \mathbb{R}^k \longrightarrow Y$ is a map into a metric space such that

$$\limsup_{\epsilon\to 0} \fint_{B(x,\epsilon)} \frac{d^p(f(y), f(x))}{\epsilon^p}\, dy < \infty,$$

for a.e. $x \in \mathbb{R}^k$, then $\lim_{\epsilon\to 0} \fint_{B(x,\epsilon)} \frac{d^p(f(y),f(x))}{\epsilon^p}\,dy$ exists, for a.e. $x \in \mathbb{R}^k$ (see Theorem 3.5). This can be interpreted as a generalized L^p-type differentiability theorem for maps with values into a metric space.

In the third section, these generalized differentiability concepts are applied to the Sobolev and BV type maps with values in a metric space. As a byproduct, it is shown that the Sobolev type spaces of Korevaar–Schoen and Reshetnyak are equivalent (this has been proved in [7, Theorem 5.1] with an additional assumption that $\partial\Omega$ is smooth, see also [14]). In fact, when the range of maps is into $\ell^\infty(\mathbb{N})$, we compute the energy norms for the Sobolev type maps by the definitions of Korevaar–Schoen and Reshetnyak (see Theorem 4.2, Remark 4.3 and Corollary 4.4). These computations provide the best estimate of the energy norms for the Sobolev type maps in the sense of Korevaar–Schoen and Reshetnyak with general metric space targets (compare with [7, Theorem 5.1]).

In the fourth section, the main result of Gregori [6] for the Sobolev type maps over Lipschitz manifolds (that is an extension of the Korevaar–Schoen result in [9]) is improved, see Corollary 5.2. It is shown that, in the convergence of the family of the approximate energy density functions to the energy density function, in fact, the (strong) limit exists (see [9] or [6] for definitions). Therefore, we are able to obtain another generalized differentiability type theorem.

Finally, in the last section, a generalized differentiability theorem is introduced in the context of Orlicz's spaces.

2. Generalized differentiability for real-valued maps

In this section, we prove Theorem 1.2 and then this theorem is applied to the Sobolev and bounded variation (BV) functions in order to show the generalized differentiability for such functions.

First, we recall the concept of density in measure theory which has a crucial role in the proofs. Let G be a subset of $\mathbb{R}^k$. It is said that $x \in \mathbb{R}^k$ is a *point of density one* for G, if

$$\lim_{r\to 0} \frac{\operatorname{vol}(B(x,r)\cap G)}{\operatorname{vol}(B(x,r))} = 1.$$

By the Lebesgue differetiation theorem, almost every $x \in G$ is a point of density one for G.

The proof of Theorem 1.2 is based on the following lemma which is a combination of the proofs of Campanato and Stepanov theorems; compare with [4, Lemma 3.1.5].

Lemma 2.1. *Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ belong to $L^p_{loc}(\mathbb{R}^k)$, for some $p \geq 1$. For positive numbers α, M, R and T, we define the set E as the following:*

$$E := \left\{ x \in B(0,T) \,:\, \sup_{0<r\leq R} \fint_{B(x,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}}\,dy \,\leq M^p \right\}.$$

Then, we have:

(i) *The restriction of f to E is α-Hölder, except on a set of measure zero.*

(ii) *The restriction of f to E is equal to an α-Hölder function whose domain is entire $\mathbb{R}^k$, except on a set of measure zero.*

(iii) *Let $\alpha = 1$, then f is L^p-differentiable, for a.e. $x \in E$.*

Proof. (i) Let $Q_r(w)$ denote $\fint_{B(w,r)} f(z)\,dz$, where $w \in \mathbb{R}^k$ and $r > 0$. Set $r_i := \frac{R}{2^i}$, for non-negative integer i. Suppose that $x \in E$, then we have

$$
\begin{aligned}
\big|Q_{r_i}(x) - Q_{r_{i+1}}(x)\big| &\leq C \fint_{B(x,r_i)} \fint_{B(x,r_i)} \big|f(z) - f(w)\big|\,dz\,dw \\
&\leq C \fint_{B(x,r_i)} \fint_{B(x,r_i)} \big|(f(z) - f(x)) - (f(w) - f(x))\big|\,dz\,dw \\
&\leq 2C \fint_{B(x,r_i)} \big|f(z) - f(x)\big|\,dz \\
&\leq 2C \left(\fint_{B(x,r_i)} \big|f(z) - f(x)\big|^p\,dz \right)^{\frac{1}{p}} \\
&\leq CMr_i^{\alpha},
\end{aligned}
$$

for every non-negative integer i and some (universal) constant C which depends on k. Then, for all $j > i \geq 0$, we have

$$
\big|Q_{r_j}(x) - Q_{r_i}(x)\big| \leq \sum_{m=i}^{j-1} \big|Q_{r_m}(x) - Q_{r_{m+1}}(x)\big| \leq CM \sum_{m=i}^{j-1} r_m^{\alpha}. \tag{2.1}
$$

This implies that $\big\{Q_{r_i}(x)\big\}_{i\geq 1}$ is a Cauchy sequence. Hence, there exists a real number, say $f_0(x)$, such that $f_0(x) := \lim_{i\to\infty} Q_{r_i}(x)$. Moreover, from (2.1) we get

$$
\big|Q_{r_i}(x) - f_0(x)\big| \leq CM \sum_{m=i}^{\infty} r_m^{\alpha}, \tag{2.2}
$$

for non-negative integer i. Suppose that $r_{i_0+1} \leq |x - y| \leq r_{i_0}$, where $x, y \in E$ and $i_0 \geq 1$ is an integer. Then, we have

$$
\begin{aligned}
\big|Q_{r_{i_0}}(x) - Q_{r_{i_0}}(y)\big| &\leq C \fint_{B(x,2r_{i_0})} \fint_{B(x,2r_{i_0})} \big|f(z) - f(w)\big|\,dz\,dw \\
&\leq 2C \fint_{B(x,2r_{i_0})} \big|f(z) - f(x)\big|\,dz \\
&\leq 2C \left(\fint_{B(x,2r_{i_0})} \big|f(z) - f(x)\big|^p\,dz \right)^{\frac{1}{p}}.
\end{aligned}
$$

Hence

$$\left|Q_{r_{i_0}}(x) - Q_{r_{i_0}}(y)\right| \leq CMr_{i_0}^{\alpha}. \tag{2.3}$$

Therefore, from (2.2) and (2.3), we see that $\left|f_0(x) - f_0(y)\right| \leq CM\sum_{m=i_0}^{\infty} r_m^{\alpha}$. Since $\frac{R}{2^{i_0+1}} = r_{i_0+1} \leq |x-y| \leq r_{i_0} = \frac{R}{2^{i_0}}$, we obtain

$$\left|f_0(x) - f_0(y)\right| \leq CK_{\alpha}M\,|x-y|^{\alpha}, \tag{2.4}$$

where K_{α} is a constant which depends on α. Let x and y be two Lebesgue points of f, then, by the Lebesgue differentiation theorem, we have

$$\left|f(x) - f(y)\right| \leq CK_{\alpha}M\,|x-y|^{\alpha}, \tag{2.5}$$

for a.e. $x, y \in E$ whenever $|x-y| \leq \frac{R}{2}$. This completes the proof of part (i).

(ii) It is a straightforward generalization of Theorem 1 in [3, p. 80]; see also [4, p. 201].

(iii) By part (ii), we know that the restriction of f to E, except for a set of measure zero, is equal to a Lipschitz function g whose domain is entire $\mathbb{R}^k$. Then, by the Rademacher theorem, g is differentiable a.e.

Let $x \in E$ be a point of density one for E, suppose that g is differentiable at x and $f(x) = g(x)$. We show that f is L^p-differentiable at point x. Let $0 < \epsilon < \frac{1}{4}$. Since x is a point of density one for E, there is $0 < \delta < R$, depending on ϵ, such that $\frac{\mathrm{vol}(B(x,r)\cap E^c)}{\mathrm{vol}(B(x,r))} < \epsilon$, for all $0 < r \leq \delta$, where $E^c := \mathbb{R}^k \backslash E$, and also $|g(w) - g(x) - Dg(x)\cdot(w-x)| \leq \epsilon\,|w-x|$, for all $w \in B(x,\delta)$. Set $\delta_1 := \epsilon^{\frac{1}{k}}\delta$, $\delta_2 := \delta - \delta_1$ and $\delta_3 := \delta - 2\delta_1$. Then, for every $y \in B(x,\delta_2)$, we show that

$$B(y,\delta_1) \cap E \neq \emptyset. \tag{2.6}$$

By contradiction, suppose that $B(y,\delta_1) \cap E = \emptyset$. Then, we have

$$\epsilon > \frac{\mathrm{vol}(B(x,\delta)\cap E^c)}{\mathrm{vol}(B(x,\delta))} \geq \frac{\mathrm{vol}(B(y,\delta_1))}{\mathrm{vol}(B(x,\delta))} = \frac{\delta_1^k}{\delta^k} = \epsilon.$$

It is a contradiction. Therefore, for every $y \in B(x,\delta_2)$, there exists some $z = z_y \in B(y,\delta_1) \cap E$. This implies (2.6). On the other hand, we have

$$\begin{aligned}
|f(y) - f(x) - Dg(x)\cdot(y-x)| &\leq |f(y) - g(z)| + |g(z) - g(x) - Dg(x)\cdot(z-x)| \\
&\quad + |Dg(x)\cdot(z-x) - Dg(x)\cdot(y-x)| \\
&\leq |f(y) - g(z)| + \epsilon\,|z-x| + L\,|z-y| \\
&\leq |f(y) - g(z)| + \epsilon\,\delta + L\,\delta_1 \\
&\leq |f(y) - g(z)| + \epsilon\,\delta + L\,\epsilon^{\frac{1}{k}}\delta \\
&\leq |f(y) - g(z_y)| + (1+L)\,\epsilon^{\frac{1}{k}}\,\delta,
\end{aligned}$$

where L is a constant which depends on the Lipschitz constant of g. Suppose that $z' \in B(x,\delta_3) \cap E$, $f(z') = g(z')$ and $y \in B(z',\delta_1)$. Then, we have

$$\begin{aligned}
|f(y)-f(x)-Dg(x)\cdot(y-x)| &\le |f(y)-g(z_y)| + (1+L)\,\epsilon^{\frac{1}{k}}\,\delta \\
&\le |f(y)-g(z')| + |g(z_y)-g(z')| + (1+L)\,\epsilon^{\frac{1}{k}}\delta \\
&\le |f(y)-g(z')| + L\,(|z_y-y|+|y-z'|) + (1+L)\,\epsilon^{\frac{1}{k}}\delta \\
&\le |f(y)-g(z')| + 2\,\delta_1\,L + (1+L)\,\epsilon^{\frac{1}{k}}\,\delta \\
&\le |f(y)-g(z')| + (1+3\,L)\,\epsilon^{\frac{1}{k}}\,\delta.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\fint_{B(z',\delta_1)} |f(y)-f(x)-Dg(x)\cdot(y-x)|^p\,dy \\
&\le 2^p \fint_{B(z',\delta_1)} |f(y)-g(z')|^p\,dy + (1+3\,L)^p\,\epsilon^{\frac{p}{k}}\,\delta^p \\
&= 2^p \fint_{B(z',\delta_1)} |f(y)-f(z')|^p\,dy + (1+3\,L)^p\,\epsilon^{\frac{p}{k}}\,\delta^p \\
&\le 2^p \left[M^p \delta_1^p + (1+3\,L)^p\,\epsilon^{\frac{p}{k}}\,\delta^p\right] \\
&\le 2^p \left[M^p \epsilon^{\frac{p}{k}} + (1+3\,L)^p\,\epsilon^{\frac{p}{k}}\right]\delta^p \\
&= \lambda(\epsilon)\,\delta_3^p,
\end{aligned}$$

where λ is a function such that $\lim_{s\to 0}\lambda(s) = 0$. Since we can cover $B(x,\delta_3)$ by a minimum number of balls whose centers are in E and their radii are equal to δ_1 (f and g are equal at such points), we get

$$\fint_{B(x,\delta_3)} |f(y)-f(x)-Dg(x)\cdot(y-x)|^p\,dy \le C\,\lambda(\epsilon)\,\delta_3^p,$$

where C is a (universal) constant which depends on k. Letting $\epsilon \to 0$, this completes the proof of Lemma 2.1. □

Proof of Theorem 1.2:. Let m and n be two positive integer numbers. We define the set $E_{m,n}$ as the following:

$$E_{m,n} := \left\{ x \in B(0,m) \;:\; \sup_{0<r\le\frac{1}{n}} \fint_{B(x,r)} \frac{|f(y)-f(x)|^p}{r^p}\,dy \;\le m \right\}.$$

It is clear that $\mathbb{R}^k = \bigcup_{m,n} E_{m,n}$, except for a set of measure zero. Now, the assertion follows from Lemma 2.1. □

Remark 2.2. We can show that Theorem 1.2 (and Definition 1.1) is valid (and meaningful) for $p = \infty$, if the *average integral* $⨍$ is replaced by the essential supremum or supremum norm. So, the proof of Theorem 1.2 provides a proof for the Stepanov theorem (using the Rademacher theorem), as well. Moreover, we can obtain [4, Theorem 3.1.8]; see also [16, p. 250].

Corollary 2.3. *Let* $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ *belong to* $L^p_{loc}(\mathbb{R}^k)$*, for some* $p \geq 1$*. Suppose that* $\limsup_{\epsilon \to 0} ⨍_{B(x,\epsilon)} \frac{|f(y)-f(x)|^p}{\epsilon^p}\, dy < \infty$*, for a.e.* $x \in \mathbb{R}^k$*. Then*

$$\lim_{\epsilon \to 0} ⨍_{B(x,\epsilon)} \frac{|f(y) - f(x)|^p}{\epsilon^p}\, dy = ⨍_{B(0,1)} |Df(x) \cdot v|^p\, dv,$$

for a.e. $x \in \mathbb{R}^k$*, where* $Df(x)$ *is the* L^p*-differential of* f *at* x*.*

Proof. Since L^p is a norm space, then by the triangle inequality, we have

$$\left| \left(⨍_{B(x,\epsilon)} |f(y) - f(x)|^p\, dy \right)^{\frac{1}{p}} - \left(⨍_{B(x,\epsilon)} |Df(x) \cdot (y-x)|^p\, dy \right)^{\frac{1}{p}} \right|$$
$$\leq \left(⨍_{B(x,\epsilon)} |f(y) - f(x) - Df(x) \cdot (y-x)|^p\, dy \right)^{\frac{1}{p}},$$

for a.e. $x \in \mathbb{R}^k$ and all $\epsilon > 0$, where $Df(x)$ is the L^p-differential of f at x (using Theorem 1.2). Since $Df(x)$ is linear, by the change of variables formula, we have

$$⨍_{B(x,r)} \frac{|Df(x) \cdot (y-x)|^p}{r^p}\, dy = ⨍_{B(0,s)} \frac{|Df(x) \cdot v|^p}{s^p}\, dv,$$

for all positive numbers r and s. This implies that

$$\lim_{\epsilon \to 0} ⨍_{B(x,\epsilon)} \frac{|f(x) - f(y)|^p}{\epsilon^p}\, dy = ⨍_{B(0,1)} |Df(x) \cdot v|^p\, dv,$$

for a.e. $x \in \mathbb{R}^k$. This completes the proof of Corollary 2.3. □

Remark 2.4. Notice that if f is differentiable at point x, then we immediately obtain the conclusion of Corollary 2.3 at such a point. Also, the function f (under the assumptions of Corollary 2.3) is not necessarily differentiable a.e. (in the usual sense).

Question 2.5. Is there any direct proof for Corollary 2.3, without using Theorem 1.2 or Lemma 2.1? Namely, is there any proof for Corollary 2.3 without appealing to the differentiability of Lipschitz functions?

Next, we provide a few applications for this generalized Stepanov type theorem in the Sobolev and BV spaces. For the basic properties of the Sobolev and BV spaces, see for example [1] and [3].

Proposition 2.6. *Suppose that $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ belongs to $L^p_{loc}(\mathbb{R}^k)$ and ν is a (Radon) measure on $\mathbb{R}^k$ in such a way that*

$$\fint_{B(x,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}}\, dy \le \left(\frac{\nu(B(x,r))}{r^k}\right)^{\beta p} < \infty,$$

for all $0 < r \le R$ and a.e. $x \in \mathbb{R}^k$ (with respect to the Lebesgue measure), where $p \ge 1$, $R > 0$, $\beta \ge 0$ and $\alpha \ge 1$. Then:

(i) *f is L^p-differentiable a.e. In particular, we have*

$$\lim_{\epsilon\to 0} \fint_{B(x,\epsilon)} \frac{|f(y)-f(x)|^p}{\epsilon^p}\, dy = \fint_{B(0,1)} |Df(x)\cdot v|^p\, dv,$$

for a.e. $x \in \mathbb{R}^k$.

(ii) *Furthermore, if $\alpha - k\beta > 0$, then f is L^∞-differentiable a.e. (see Remark 2.2).*

Proof. (i) We have

$$\fint_{B(x,r)} \frac{|f(y)-f(x)|^p}{r^{p}}\, dy \le \fint_{B(x,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}}\, dy \le \left(\frac{\nu(B(x,r))}{r^k}\right)^{\beta p},$$

for all $0 < r \le \min\{1, R\}$ and a.e. $x \in \mathbb{R}^k$. On the other hand, by [3, p. 38, Theorem 1], we know that

$$\limsup_{r\to 0} \frac{\nu(B(x,r))}{r^k} < \infty, \tag{2.7}$$

for a.e. $x \in \mathbb{R}^k$ (with respect to the Lebesgue measure). Therefore, we obtain $\limsup_{\epsilon\to 0} \fint_{B(x,\epsilon)} \frac{|f(y)-f(x)|^p}{\epsilon^p}\, dy < \infty$, for a.e. $x \in \mathbb{R}^k$. From Corollary 2.3, we obtain the assertion of part (i).

(ii) By the assumptions, we have

$$\fint_{B(x,r)} |f(y)-f(x)|^p\, dy \le \nu(B(x,r))^{\beta p}\, r^{(\alpha - k\beta)p},$$

for all $0 < r \le R$ and a.e. $x \in \mathbb{R}^k$. Then, similar to the proof of Lemma 2.1 (by inequality (2.5)), we have (or we can directly apply Campanato's theorem [10, p. 31]) $|f(y) - f(x)| \le \hat{C}\, \nu(B(x,r))^\beta r^{\alpha - k\beta} = \hat{C}\big(\frac{\nu(B(x,r))}{r^k}\big)^\beta r^\alpha$, for a.e. $y \in B(x, \frac{r}{2})$, a.e. $x \in \mathbb{R}^k$ and all $0 < r \le R$, where $\hat{C}$ is a constant which does not depend on f and r. Therefore, by (2.7) and $\alpha \ge 1$, we obtain

$$\limsup_{r\to 0} \left\{ \frac{1}{r} \operatorname*{ess\,sup}_{y\in B(x,r)} |f(y)-f(x)| \right\} < \infty,$$

for a.e. $x \in \mathbb{R}^k$. Therefore, the assertion of part (ii) follows from Theorem 1.2 and Remark 2.2. □

Now, we replace the boundedness condition $\limsup_{\epsilon\to 0} ⨍_{B(x,\epsilon)} \frac{|f(y)-f(x)|^p}{\epsilon^p}\, dy < \infty$ in Theorem 1.2, Corollary 2.3 and Proposition 2.6 by a slightly weaker assumption.

Theorem 2.7. *Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ belong to $L^p_{loc}(\mathbb{R}^k)$, for some $p \geq 1$. Then:*

(i) *If*

$$\limsup_{\epsilon\to 0} ⨍_{B(x,\epsilon)} ⨍_{B(x,\epsilon)} \frac{|f(z)-f(w)|^p}{\epsilon^p}\, dz\, dw < \infty,$$

for a.e. $x \in \mathbb{R}^k$, then f is L^p-differentiable a.e. and

$$\lim_{\epsilon\to 0} ⨍_{B(x,\epsilon)} ⨍_{B(x,\epsilon)} \frac{|f(z)-f(w)|^p}{\epsilon^p}\, dz\, dw = ⨍_{B(0,1)} ⨍_{B(0,1)} |Df(x)\cdot(v-u)|^p dv\, du,$$

for a.e. $x \in \mathbb{R}^k$.

(ii) *If there exists a (Radon) measure ν on $\mathbb{R}^k$ in such a way that*

$$⨍_{B(x,r)} ⨍_{B(x,r)} \frac{|f(z)-f(w)|^p}{r^{\alpha p}}\, dz\, dw \leq \left(\frac{\nu(B(x,r))}{r^k}\right)^{\beta p} < \infty,$$

for all $0 < r \leq R$ and a.e. $x \in \mathbb{R}^k$, where $p \geq 1$, $R > 0$, $\beta \geq 0$ and $\alpha \geq 1$, then f is L^p-differentiable a.e. and

$$\lim_{\epsilon\to 0} ⨍_{B(x,\epsilon)} ⨍_{B(x,\epsilon)} \frac{|f(z)-f(w)|^p}{\epsilon^p}\, dz\, dw = ⨍_{B(0,1)} ⨍_{B(0,1)} |Df(x)\cdot(v-u)|^p dv\, du,$$

for a.e. $x \in \mathbb{R}^k$. Furthermore, if $\alpha - k\beta > 0$, then f is L^∞-differentiable a.e.

Proof. It is similar to the proofs of Theorem 1.2, Corollary 2.3 and Proposition 2.6 (with minor changes). □

Corollary 2.8. *Let $W^{1,p}_{loc}(\mathbb{R}^k)$ denote the Sobolev space, i.e., the set of all functions in $L^p_{loc}(\mathbb{R}^k)$ whose (first order) weak derivatives are in $L^p_{loc}(\mathbb{R}^k)$, for some $p \geq 1$. Suppose that $f \in W^{1,p}_{loc}(\mathbb{R}^k)$. Then:*

(i) *f is L^p-differentiable a.e.*

(ii) *If $p < k$, then f is L^q-differentiable a.e., where $q = \frac{kp}{k-p}$.*

(iii) *If $k < p$, then f is differentiable a.e. (after changing f on a set of measure zero).*

Proof. (i) It follows from Theorem 2.7 and the Poincaré inequality.

(ii) It follows from Theorem 2.7 and the Sobolev inequality.

(iii) It follows from Theorem 2.7 and the Poincaré inequality. □

Corollary 2.9. *Let $BV_{loc}(\mathbb{R}^k)$ denote the set of all locally bounded variation functions on $\mathbb{R}^k$. Suppose that $f \in BV_{loc}(\mathbb{R}^k)$. Then f is $L^{\frac{k}{k-1}}$-differentiable a.e.*

Proof. It follows from Theorem 2.7 and the Sobolev inequality. □

Corollary 2.8 and Corollary 2.9 are well-known, see [16, p. 242] and [3, Chapter 6]; compare with Theorem 6.3.

3. Generalized differentiability for maps with values in a metric space

We extend a consequence of the main result of previous section to maps with values into a metric space. Let $\ell^\infty = \ell^\infty(\mathbb{N})$ denote the Banach space of all bounded sequences of real numbers with sup-norm $\|\cdot\|_\infty$. First, we state and prove a Rademacher differentiability type theorem for Lipschitz maps into the Banach space ℓ^∞. Then, we generalize Corollary 2.3 to maps with range in a metric space.

We start this section with the following lemma which has a crucial role in order to study the generalized differentiability properties for maps with values into a metric space. In fact, this lemma is obvious for Lipschitz maps with values into a finite dimensional norm space. Moreover, Lemma 3.1, without computing its limit, was proved by Kirchheim [8], see also Theorem 3.3.

Lemma 3.1. *Let $f : \mathbb{R} \longrightarrow \ell^\infty$ be a Lipschitz map. Then*

$$\lim_{t\to 0^+} \frac{\|f(x+t) - f(x)\|_\infty}{t} = \sup_{i\in\mathbb{N}} |f_i'(x)|,$$

for a.e. $x \in \mathbb{R}$, where $f = (f_i)_{i\in\mathbb{N}}$.

Proof. Since f is a Lipschitz map, there exists $M > 0$ such that $|f_i(x)-f_i(y)| \leq M|x-y|$, for all $x, y \in \mathbb{R}$ and $i \in \mathbb{N}$ (f is called *M-Lipschitz*). Since every Lipschitz function on $\mathbb{R}$ is a.e. differentiable, there exists a subset $A \subset \mathbb{R}$ of measure zero such that $f_i'(x)$ exists for all $i \in \mathbb{N}$ and $x \in \mathbb{R}\backslash A$. Define the measurable function

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \qquad g(x) := \sup_i |f_i'(x)|.$$

By [3, p. 47], we know that g is *approximately continuous* for a.e. $x \in \mathbb{R}$. So, without loss of generality, we may assume that g is approximately continuous on $\mathbb{R}\backslash A$. Let $x_0 \in \mathbb{R}\backslash A$. We show that

$$\lim_{t\to 0^+} \frac{\sup_i |f_i(x_0+t) - f_i(x_0)|}{t} = g(x_0).$$

Let ϵ and η be arbitrary positive numbers. Since g is approximately continuous at x_0, there exists $\delta > 0$ such that

$$\frac{\operatorname{vol}\big(\{y \in B(x_0, r) : |g(x_0) - g(y)| \geq \epsilon\}\big)}{\operatorname{vol}(B(x_0, r))} \leq \eta, \tag{3.1}$$

for all $0 < r \leq \delta$. Moreover, there exists $i_0 \in \mathbb{N}$ such that

$$g(x_0) - \eta \leq |f'_{i_0}(x_0)| \leq g(x_0). \tag{3.2}$$

Since f_i is Lipschitz, we have $f_i(y) - f_i(x) = \int_x^y f'_i(s)\, ds$, for all $x, y \in \mathbb{R}$ and $i \in \mathbb{N}$. Then

$$|f_i(y) - f_i(x)| \leq \int_x^y g(s)\, ds, \tag{3.3}$$

for all $y \geq x$ and $i \in \mathbb{N}$. Since f_{i_0} is differentiable at x_0, there exists $\delta' > 0$ such that $\big|\frac{f_{i_0}(x) - f_{i_0}(x_0)}{x - x_0} - f'_{i_0}(x_0)\big| \leq \eta$, for all $x \in B(x_0, \delta')$. Hence

$$\Big[|f'_{i_0}(x_0)| - \eta\Big]|x - x_0| \leq |f_{i_0}(x) - f_{i_0}(x_0)| \leq \Big[|f'_{i_0}(x_0)| + \eta\Big]|x - x_0|, \tag{3.4}$$

for all $x \in B(x_0, \delta_0)$, where $\delta_0 := \min\{\delta, \delta'\}$. Then, from (3.2) and (3.4), we have

$$\Big[g(x_0) - 2\eta\Big]|x - x_0| \leq |f_{i_0}(x) - f_{i_0}(x_0)| \leq \sup_i |f_i(x) - f_i(x_0)|, \tag{3.5}$$

for all $x \in B(x_0, \delta_0)$. Also, from (3.1) and (3.3), we get

$$\begin{aligned} |f_i(x) - f_i(x_0)| &\leq (g(x_0) + \epsilon)\,|x - x_0| \\ &\qquad + M \operatorname{vol}\big(\{y \in B(x_0, r) : |g(x_0) - g(y)| \geq \epsilon\}\big) \\ &\leq (g(x_0) + \epsilon)\,|x - x_0| + M\,\eta \operatorname{vol}(B(x_0, r)), \end{aligned} \tag{3.6}$$

for all $x \in B(x_0, \delta_0)$ and $i \in \mathbb{N}$, where $r = |x - x_0|$. Therefore, from (3.5) and (3.6), we obtain

$$\Big[g(x_0) - 2\eta\Big]|x - x_0| \leq \sup_i |f_i(x) - f_i(x_0)| \leq \Big[(g(x_0) + \epsilon) + 2M\eta\Big]|x - x_0|,$$

for all $x \in B(x_0, \delta_0)$. Letting $\epsilon, \eta \to 0$, this implies the assertion. □

Remark 3.2. In Lemma 3.1, the Lipschitz map f is not necessarily differentiable a.e. For example, consider $f_i(x) := \frac{\sin(ix)}{i}$. See also [2, Chapter 5], about the Radon-Nikodým property (RNP) and its relation with differentiability of Lipschitz maps with values into a Banach space.

Next, we extend Lemma 3.1 to the Lipschitz maps from $\mathbf{R}^k$ into ℓ^∞. In fact, the following result can be interpreted as an extension of Rademacher's differentiability theorem for maps with values into a Banach space.

Theorem 3.3. *Let $f : \mathbb{R}^k \longrightarrow \ell^\infty$ be a Lipschitz map. Then*

$$\lim_{h\to 0} \frac{\sup_l |f_l(x+h) - f_l(x)| - \sup_l |Df_l(x) \cdot h|}{|h|} = 0,$$

for a.e. $x \in \mathbb{R}^k$, where $f = (f_l)_{l\in\mathbb{N}}$.

Proof. The restriction of f to any line satisfies the assumptions of Lemma 3.1; then for any $v \in \mathbb{R}^k$, there exists a subset $A_v \subset \mathbb{R}^k$ of measure zero such that $\lim_{t\to 0^+}\{\frac{1}{t}\sup_l |f_l(x+tsv) - f_l(x)|\}$, exists, for all $s > 0$ and $x \in \mathbb{R}^k\backslash A_v$. Now, let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence of elements of $S^{k-1} := \{z \in \mathbb{R}^k : |z| = 1\}$ which is dense in S^{k-1}. Then, there is a subset $A \subset \mathbb{R}^k$ of measure zero such that

$$g_{sv_i}(x) := \lim_{t\to 0^+} \frac{\sup_l |f_l(x+tsv_i) - f_l(x)|}{t},$$

exists, for all $i \in \mathbb{N}$, $s > 0$ and $x \in \mathbb{R}^k\backslash A$. Notice that $g_{sv}(x) = sg_v(x)$. Since f is M-Lipschitz, for some $M > 0$, we have $\big||f_l(x+tsv) - f_l(x)| - |f_l(x+tsw) - f_l(x)|\big| \le M\,t\,s\,|v - w|$, for all $l \in \mathbb{N}$; $s, t \in]0, \infty[$ and $x, v, w \in \mathbb{R}^k$. By choosing $v = v_i$ and $w = v_j$, we have

$$\left|\frac{\sup_l |f_l(x+tsv_i) - f_l(x)|}{t} - \frac{\sup_l |f_l(x+tsv_j) - f_l(x)|}{t}\right| \le M\,s\,|v_i - v_j|, \quad (3.7)$$

for all $i, j \in \mathbb{N}$; $s, t \in]0, \infty[$ and $x \in \mathbb{R}^k\backslash A$. Letting $t \to 0^+$, then

$$|g_{sv_i}(x) - g_{sv_j}(x)| \le M\,s\,|v_i - v_j|, \quad (3.8)$$

for all $i, j \in \mathbb{N}$, $s > 0$ and $x \in \mathbb{R}^k\backslash A$. On the other hand, for every $v \in S^{k-1}$, there exists a subsequence $\{v_{i_k}\}$ which converges to v. From (3.8), for any $s > 0$ and $x \in \mathbb{R}^k\backslash A$, the sequence $\{g_{sv_{i_k}}(x)\}$ converges to a real number, say a. Also, (3.7) implies that $\lim_{t\to 0^+}\{\frac{1}{t}\sup_l |f_l(x+tsv) - f_l(x)|\}$, exists and it is equal to $a = g_{sv}(x)$, whenever $x \in \mathbb{R}^k\backslash A$, $v \in S^{k-1}$ and $s > 0$.

Fix $\epsilon > 0$ and $x \in \mathbb{R}^k\backslash A$. There exists a finite sequence of elements $\{v_i\}_{i\in\mathbb{N}}$ (or S^{k-1}), say $\{w_i\}_{1\le i\le N}$, such that $\{B(w_i, \epsilon)\}_{1\le i\le N}$ covers S^{k-1}. Also, there exists $\delta > 0$ such that

$$\left|\frac{\sup_l |f_l(x+tsw_i) - f_l(x)|}{t} - g_{sw_i}(x)\right| \le \epsilon, \quad (3.9)$$

for all $0 < t < \delta$, $0 \le s \le 1$ and $1 \le i \le N$. Let $h \in \mathbb{R}^k$ and $0 < |h| < \delta$. Set $w := \frac{h}{|h|}$ and $t := |h|$. There is $1 \le i_0 \le N$ such that $w \in B(w_{i_0}, \epsilon)$. Hence, by (3.7), (3.8) and 3.9, we get

$$\begin{aligned}
&\left| \frac{\sup_l |f_l(x+tsw) - f_l(x)|}{t} - g_{sw}(x) \right| \\
&\le \left| \frac{\sup_l |f_l(x+tsw) - f_l(x)|}{t} - \frac{\sup_l |f_l(x+tsw_{i_0}) - f_l(x)|}{t} \right| \\
&\quad + \left| \frac{\sup_l |f_l(x+tsw_{i_0}) - f_l(x)|}{t} - g_{sw_{i_0}}(x) \right| + |g_{sw_{i_0}}(x) - g_{sw}(x)| \\
&\le M\, s\, |w - w_{i_0}| + \epsilon + M\, s\, |w - w_{i_0}| \\
&\le 2\, M\, \epsilon + \epsilon,
\end{aligned}$$

for all $w \in S^{k-1}$, $0 \le s \le 1$, $0 < |h| < \delta$ and $x \in \mathbb{R}^k \backslash A$. By Lemma 3.1, this implies that

$$\lim_{h \to 0} \frac{\sup_l |f_l(x+h) - f_l(x)| - \sup_l |Df_l(x) \cdot h|}{|h|} = 0,$$

for all $x \in \mathbb{R}^k \backslash A$. □

Corollary 3.4. *Let $f : \mathbb{R}^k \longrightarrow \ell^\infty$ be a Lipschitz map and $p \ge 1$. Then*

$$\lim_{\epsilon \to 0} \fint_{B(x,\epsilon)} \frac{\|f(y) - f(x)\|_\infty^p}{\epsilon^p}\, dy = \fint_{B(0,r)} \frac{\sup_l |Df_l(x) \cdot v|^p}{r^p}\, dv,$$

for a.e. $x \in \mathbb{R}^k$ and all $r > 0$, where $f = (f_l)_{l \in \mathbb{N}}$.

Proof. It is similar to the proof of Corollary 2.3. Notice that the map $v \longmapsto \sup_l |Df_l(x) \cdot v|$, is not necessarily a linear map, but it is positively homogeneous of degree one, i.e., $\sup_l |Df_l(x) \cdot (\rho\, v)| = \rho \sup_l |Df_l(x) \cdot v|$, for all $\rho > 0$. □

Let (Y, d) be a metric space and let $p \ge 1$. Let $L^p_{loc}(\mathbb{R}^k, Y)$ denote the set of all Borel measurable maps $f : \mathbb{R}^k \longrightarrow Y$ with separable (or essentially separable) range such that $\int_B d^p(f(y), Q)\, dy < \infty$, for all (or some) $Q \in Y$ and all balls B in $\mathbb{R}^k$ (see [9, p. 571], [2, p. 100] and [7]).

Theorem 3.5. *Let (Y, d) be a metric space and let $f \in L^p_{loc}(\mathbb{R}^k, Y)$, for some $p \ge 1$. Suppose that $\limsup_{\epsilon \to 0} \fint_{B(x,\epsilon)} \frac{d^p(f(y), f(x))}{\epsilon^p}\, dy < \infty$, for a.e. $x \in \mathbb{R}^k$. Then $\lim_{\epsilon \to 0} \fint_{B(x,\epsilon)} \frac{d^p(f(y), f(x))}{\epsilon^p}\, dy$ exists, for a.e. $x \in \mathbb{R}^k$. In particular, when $Y = \ell^\infty$, we have*

$$\lim_{\epsilon \to 0} \fint_{B(x,\epsilon)} \frac{d^p(f(y), f(x))}{\epsilon^p}\, dy = \fint_{B(0,r)} \frac{\sup_l |Df_l(x) \cdot v|^p}{r^p}\, dv,$$

for a.e. $x \in \mathbb{R}^k$ and all $r > 0$, where $f = (f_l)_{l \in \mathbb{N}}$.

Proof. Since the image of f, say $\mathrm{Im}(f)$, is separable (or essentially separable), there is an isometric embedding from $\mathrm{Im}(f)$ into the Banach space $\ell^\infty = \ell^\infty(\mathbb{N})$. So, without loss of generality, we can assume that $Y = \ell^\infty$. On the other hand, we know that every Lipschitz map from a subset of $\mathbb{R}^k$ into ℓ^∞ can be extended to a Lipschitz map whose domain is entire $\mathbb{R}^k$, see [2, Lemma 1.1 (ii)]. Also, we can easily extend Lemma 2.1 and Corollary 2.3 to maps with range in ℓ^∞, see also [11, Chapter 1]. Now, the rest of proof is similar to the proof of Corollary 3.4. □

4. Sobolev type spaces for maps with values in a metric space

In this section, we apply the previous section results to the Sobolev type maps with values into a metric space. We compute the energy norms for the Sobolev type maps in the sense of Korevaar–Schoen and Reshetnyak. Then, it is shown that the Sobolev type spaces with definitions of Korevaar–Schoen and Reshetnyak are equivalent; this fact is announced in [13] and it is also proved in [7, Theorem 5.1] with an additional assumption that $\partial\Omega$ is smooth, see also [14]. Indeed, we find the best constants in order to compare the energy norms of the Sobolev type maps by the definitions of Korevaar–Schoen and Reshetnyak. Also, these computations, let us to modify Reshetnyak's definition of the energy norm in such a way that modified energy norm becomes equal (up to a multiple universal constant) to Korevaar–Schoen's definition (see Remark 4.5).

Now, we recall the definitions of Korevaar–Schoen and Reshetnyak for the Sobolev classes maps with values into a metric space. Let (Y, d) be a metric space and let Ω be an open domain (connected and bounded) in the n-dimensional Euclidean space (or in a Riemannian manifold, see Remark 4.5). Let $u : \Omega \longrightarrow Y$ belong to $L^p(\Omega, Y)$, for some $p \geq 1$; i.e., u is a Borel measurable map with separable (or essentially separable) range for which $\int_\Omega d^p(u(y), Q)dy < \infty$, for all $Q \in Y$ (see [9] and [7] for the basic concepts). Denote the set of all compactly supported continuous functions $\phi : \Omega \longrightarrow [0, 1]$ by $C_c(\Omega, [0, 1])$. For $\epsilon > 0$, define

$$\Omega_\epsilon := \big\{x \in \Omega \,:\, d(x, \partial\Omega) > \epsilon\big\}$$
$$E_\epsilon(\phi) := \int_{\Omega_\epsilon} \phi(x)\left(\fint_{B(x,\epsilon)} \frac{d^p(u(x), u(y))}{\epsilon^p}\, dy\right) dx,$$

where $\phi \in C_c(\Omega, [0, 1])$. We say that u belongs to the Sobolev type (BV when $p = 1$) space of Korevaar–Schoen [9], if $u \in L^p(\Omega, Y)$ and also

$$\sup_{\phi \in C_c(\Omega,[0,1])} \left(\limsup_{\epsilon \longrightarrow 0} E_\epsilon(\phi)\right) < \infty.$$

When $p > 1$, it is shown in [9] that there exists a non-negative function $g \in L^p(\Omega)$ such that

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \phi(x) \left(\fint_{B(x,\epsilon)} \frac{d^p(u(x), u(y))}{\epsilon^p} \, dy \right) dx = \int_\Omega \phi(x)\, g^p(x)\, dx,$$

for all $\phi \in C_c(\Omega, [0,1])$ and furthermore, the family of measures

$$\left(\fint_{B(x,\epsilon)} \frac{d^p(u(x), u(y))}{\epsilon^p} \, dy \right) dx,$$

converges weakly to the measure $g^p(x)\, dx$, as $\epsilon \to 0$. We call $\int_\Omega g^p(x)\, dx$, the *p-energy norm (energy norm)* of u (over Ω).

The map $u \in L^p(\Omega, Y)$, for some $p \geq 1$, belongs to the Sobolev type space in the sense of Reshetnyak, if there exists a non-negative function $w \in L^p(\Omega)$ such that for any $Q \in Y$, the real-valued function

$$x \longmapsto u_Q(x) := d(u(x), Q),$$

belongs to the (classical) Sobolev space $W^{1,p}(\Omega)$, furthermore,

$$|Du_Q(x)| \leq w(x), \tag{4.1}$$

for all $Q \in Y$ and a.e. $x \in \Omega$, where D denotes the differential operator. Define the *p-energy norm (energy norm)* of u (over Ω) by $\inf_w \int_\Omega w^p(x)\, dx$, where the infimum is taken over all functions w as above. One can show that the above infimum attains by a unique function (up to a set of measure zero). Let $\mathcal{E}_p^{KS}[u,\Omega]$ and $\mathcal{E}_p^R[u,\Omega]$ denote the p-energy norms for the Sobolev map $u : \Omega \longrightarrow Y$ in the sense of Korevaar–Schoen and Reshetnyak, respectively.

Theorem 4.1. *Let $u : \mathbb{R}^n \longrightarrow \ell^\infty$ be a Lipschitz map and let $p \geq 1$. Then*

$$\mathcal{E}_p^{KS}[u,\Omega] = \int_\Omega \left(\fint_{B(0,1)} \sup_i |Du_i(x) \cdot v|^p \, dv \right) dx, \quad \mathcal{E}_p^R[u,\Omega] = \int_\Omega \sup_i |Du_i(x)|^p \, dx,$$

where $u = \{u_i\}_{i \in \mathbb{N}}$ and Ω is an open domain in $\mathbb{R}^n$. Moreover, there exists a positive constant $c_{n,p}$, depending on n and p, such that

$$c_{n,p}\, \mathcal{E}_p^R[u,\Omega] \leq \mathcal{E}_p^{KS}[u,\Omega] \leq \mathcal{E}_p^R[u,\Omega].$$

Proof. Suppose that u is M-Lipschitz, for some $M > 0$. By Corollary 3.4, we have

$$\lim_{\epsilon \to 0} \fint_{B(x,\epsilon)} \frac{\sup_i |u_i(y) - u_i(x)|^p}{\epsilon^p} \, dy = \fint_{B(0,1)} \sup_i |Du_i(x) \cdot v|^p \, dv,$$

for a.e. $x \in \mathbb{R}^n$. Therefore, by Lebesgue's dominated convergence theorem, we obtain $\mathcal{E}_p^{KS}[u,\Omega] = \int_\Omega \left(\fint_{B(0,1)} \sup_i |Du_i(x) \cdot v|^p \, dv \right) dx$.

By [13, p. 579, Theorem 5.1], for any 1-Lipschitz function $\zeta : \ell^\infty \longrightarrow \mathbb{R}$, we have $|D(\zeta \circ u)(z)| \leq w(z)$, for a.e. $z \in \mathbb{R}^n$, whenever w satisfies (4.1). Notice that u has a separable range but ℓ^∞ is not a separable space. Suppose that a sequence of real numbers $a = \{a_i\}$ belongs to ℓ^1 with norm less than or equal to 1, i.e., $\|a\|_1 := \sum_{i\in\mathbb{N}} |a_i| \leq 1$. Define the 1-Lipschitz function

$$\zeta_a : \ell^\infty \longrightarrow \mathbb{R}, \qquad \zeta_a(\{x_i\}) := \sum_{i\in\mathbb{N}} a_i x_i.$$

Notice that ζ_a is a linear functional over ℓ^∞ with norm $\|a\|_1 \leq 1$ and $|D(\zeta_a \circ u)(z)| \leq w(z)$, for a.e. $z \in \mathbb{R}^n$. Hence, since $\lim_{j\to\infty} \sum_{i=j}^\infty |a_i| = 0$, and $\sup_{i\in\mathbb{N}} |Du_i(z)| \leq M$, for a.e. $z \in \mathbb{R}^n$, we obtain

$$\left| \sum_{i\in\mathbb{N}} a_i \, Du_i(z) \right| \leq w(z), \tag{4.2}$$

for a.e. $z \in \mathbb{R}^n$ and all $a \in \ell^1$ with norm less than or equal to 1. From (4.2), we get

$$\sup_{i\in\mathbb{N}} |Du_i(z)| \leq w(z), \tag{4.3}$$

for a.e. $z \in \mathbb{R}^n$.

On the other hand, consider an arbitrary 1-Lipschitz function $\eta : \ell^\infty \longrightarrow \mathbb{R}$. We show that

$$|D(\eta \circ u)(z)| \leq \sup_{i\in\mathbf{N}} |Du_i(z)|, \tag{4.4}$$

for a.e. $z \in \mathbb{R}^n$. We know that

$$\frac{|(\eta \circ u)(y) - (\eta \circ u)(x)|}{|y-x|} \leq \frac{\|u(y) - u(x)\|_\infty}{|y-x|} = \frac{\sup_i |u_i(y) - u_i(x)|}{|y-x|}, \tag{4.5}$$

for all $x, y \in \mathbb{R}^n$ and $x \neq y$. Suppose that the real-valued functions $\eta \circ u$; u_1, u_2, ... are differentiable at point $z_0 \in \mathbb{R}^n$. There is a unit vector $v_0 \in \mathbb{R}^n$ such that $|D(\eta \circ u)(z_0)| = D(\eta \circ u)(z_0) \cdot v_0$, and then, by (4.5) and Theorem 3.3 (or Lemma 3.1), we have

$$|D(\eta \circ u)(z_0)| = \lim_{t\to 0^+} \frac{\eta \circ u(z_0 + tv_0) - \eta \circ u(z_0)}{t} \leq \sup_{i\in\mathbb{N}} |Du_i(z_0) \cdot v_0|.$$

By the Cauchy-Schwarz inequality, we get $|D(\eta \circ u)(z_0)| \leq \sup_{i\in\mathbb{N}} |Du_i(z_0)| \, |v_0|$ $= \sup_{i\in\mathbb{N}} |Du_i(z_0)|$. This proves (4.4). Therefore, by (4.3) and (4.4), we obtain $\mathcal{E}_p^R[u,\Omega] = \int_\Omega \sup_i |Du_i(x)|^p \, dx$.

Finally, we show that $c_{n,p}\,\mathcal{E}_p^R[u,\Omega] \le \mathcal{E}_p^{KS}[u,\Omega] \le \mathcal{E}_p^R[u,\Omega]$, for some positive constant $c_{n,p}$ which depends on n and p. To do this, for $A \in \mathbb{R}^n \backslash \{0\}$, we define

$$c_{n,p} := \fint_{B(0,1)} \frac{|A \cdot v|^p}{|A|^p}\, dv, \tag{4.6}$$

where $c_{n,p}$ is a positive constant which depends on n and p and it does not depend on A. Then, by the Cauchy-Schwarz inequality, we have

$$c_{n,p}\,|A|^p = \fint_{B(0,1)} |A \cdot v|^p\, dv \le \fint_{B(0,1)} (|A|\ |v|)^p\, dv \le |A|^p,$$

for all $A \in \mathbb{R}^n$. Therefore, we obtain

$$c_{n,p}\, \sup_i |Du_i(x)|^p \le \fint_{B(0,1)} \sup_i |Du_i(x) \cdot v|^p\, dv \le \sup_i |Du_i(x)|^p,$$

for a.e. $x \in \mathbb{R}^n$. By taking integral with respect to x, we have

$$c_{n,p}\int_\Omega \sup_i |Du_i(x)|^p dx \le \int_\Omega \left(\fint_{B(0,1)} \sup_i |Du_i(x) \cdot v|^p dv \right) dx \le \int_\Omega \sup_i |Du_i(x)|^p dx.$$

This completes the proof of Theorem 4.1. □

Next, we extend Theorem 4.1 to the Sobolev type maps instead of Lipschitz maps; see also Remark 4.3.

Theorem 4.2. *Suppose that $u \in L^p(\Omega, \ell^\infty)$, for some $p > 1$. Then, u belongs to the Sobolev type space in the sense of Korevaar–Schoen iff u belongs to the Sobolev type space in the sense of Reshetnyak. Moreover, there exists a positive constant $c_{n,p}$, depending on n and p, such that*

$$c_{n,p}\,\mathcal{E}_p^R[u,\Omega] \le \mathcal{E}_p^{KS}[u,\Omega] \le \mathcal{E}_p^R[u,\Omega]. \tag{4.7}$$

Proof. It is clear that if the image of u is included in a line, then the conclusion of theorem is valid for such a function (in fact, u belongs to the classical Sobolev space, see [9, Theorem 1.6.2]).

Suppose that $u = \{u_i\}_{i\in\mathbb{N}}$ belongs to the Sobolev type space in the sense of Reshetnyak. Then, there is a non-negative function $w \in L^p(\Omega)$ such that w satisfies (4.1). By [13, p. 579, Theorem 5.1], for any 1-Lipschitz function $\zeta : \ell^\infty \longrightarrow \mathbb{R}$, we have $|D(\zeta \circ u)(x)| \le w(x)$, for a.e. $x \in \Omega$. Therefore, we obtain $\sup_{i\in\mathbb{N}} |Du_i(x)| \le w(x)$, for a.e. $x \in \Omega$. For $m \in \mathbb{N}$, define

$$\pi_m : \ell^\infty \longrightarrow \ell^\infty, \qquad \pi_m(\{x_i\}_{i\in\mathbb{N}}) := (x_1, x_2, \ldots, x_m, 0, 0, 0, \ldots).$$

Since $\|\pi_m \circ u(y) - \pi_m \circ u(x)\|_\infty^p \le \sum_{i=1}^m |u_i(y) - u_i(x)|^p$, and each u_i belongs to the Sobolev type space in the sense of Korevaar–Schoen (or equivalently the classical Sobolev space), we see that $\pi_m \circ u$ belongs to the Sobolev type space in the sense of Korevaar–Schoen. Suppose that U is an open subset (ball) in Ω, by [9], there exist $\epsilon_0 > 0$ and non-negative function $g_m \in L^p(\Omega)$ such that

$$\int_{U_\epsilon} \left(\fint_{B(x,\epsilon)} \frac{\|\pi_m \circ u(y) - \pi_m \circ u(x)\|_\infty^p}{\epsilon^p}\, dy \right) dx \le (1 + C\epsilon) \int_U g_m^p(x)\, dx, \quad (4.8)$$

for all $0 < \epsilon < \epsilon_0$ and $m \in \mathbb{N}$, where C is a constant which depends on the dimension of domain, furthermore

$$\lim_{\epsilon \to 0} \int_{U_\epsilon} \left(\fint_{B(x,\epsilon)} \frac{\|\pi_m \circ u(y) - \pi_m \circ u(x)\|_\infty^p}{\epsilon^p}\, dy \right) dx = \int_U g_m^p(x)\, dx.$$

Hence, by Corollary 2.8 (i) and Theorem 3.5, we obtain

$$\lim_{\epsilon \to 0} \fint_{B(x,\epsilon)} \frac{\|\pi_m \circ u(y) - \pi_m \circ u(x)\|_\infty^p}{\epsilon^p}\, dy \le \sup_{1 \le i \le m} |Du_i(x)|^p \le w^p(x), \quad (4.9)$$

for a.e. $x \in \Omega$. From (4.9) and this fact that the measures

$$\left(\fint_{B(x,\epsilon)} \frac{\|\pi_m \circ u(y) - \pi_m \circ u(x)\|_\infty^p}{\epsilon^p}\, dy \right) dx,$$

converge weakly to the measure $g_m^p(x)\, dx$ (as $\epsilon \to 0$), we get

$$\int_U g_m^p(x)\, dx \le \int_U \sup_{1 \le i \le m} |Du_i(x)|^p\, dx \le \int_U w^p(x)\, dx.$$

Since $\|\pi_m \circ u(y) - \pi_m \circ u(x)\|_\infty^p \le \|\pi_{m+1} \circ u(y) - \pi_{m+1} \circ u(x)\|_\infty^p$, for all $m \in \mathbb{N}$, by (4.8), (4.9) and Fatou's lemma, we have

$$\int_{\Omega_\epsilon} \left(\fint_{B(x,\epsilon)} \lim_{m \to \infty} \frac{\|\pi_m \circ u(y) - \pi_m \circ u(x)\|_\infty^p}{\epsilon^p}\, dy \right) dx \le (1 + C\epsilon) \int_\Omega w^p(x)\, dx,$$

or

$$\int_{\Omega_\epsilon} \left(\fint_{B(x,\epsilon)} \frac{\|u(y) - u(x)\|_\infty^p}{\epsilon^p}\, dy \right) dx \le (1 + C\epsilon) \int_\Omega w^p(x)\, dx,$$

for all $0 < \epsilon < \epsilon_0$. Then

$$\limsup_{\epsilon \to 0} \int_{\Omega_\epsilon} \left(\fint_{B(x,\epsilon)} \frac{\|u(y) - u(x)\|_\infty^p}{\epsilon^p}\, dy \right) dx \le \int_\Omega w^p(x)\, dx < \infty.$$

This implies that u belongs to the Sobolev type space in the sense of Korevaar–Schoen and $\mathcal{E}_p^{KS}[u, \Omega] \le \mathcal{E}_p^R[u, \Omega]$.

On the other hand, suppose that u belongs to the Sobolev type space in the sense of Korevaar–Schoen, we show that u belongs to the Sobolev type space in the sense of Reshetnyak. Consider an arbitrary 1-Lipschitz function $\zeta : \ell^\infty \longrightarrow \mathbb{R}$, then

$$\int_{U_\epsilon}\left(⨍_{B(x,\epsilon)} \frac{|\zeta\circ u(y) - \zeta\circ u(x)|^p}{\epsilon^p}\,dy\right)dx \le \int_{U_\epsilon}\left(⨍_{B(x,\epsilon)} \frac{\|u(y) - u(x)\|_\infty^p}{\epsilon^p}\,dy\right)dx,$$

for all open subsets (balls) $U \subset \Omega$ and $\epsilon > 0$ small enough. Therefore, $\zeta \circ u$ belongs to the Sobolev type space in the sense of Korevaar–Schoen (or equivalently the classical Sobolev space). Letting $\epsilon \to 0$, by Corollary 2.8 (i) and Fatou's lemma (see also Proposition 2.6 (i)), we obtain

$$\int_U\left(⨍_{B(0,1)} |D(\zeta\circ u)(x)\cdot v|^p\,dv\right)dx \le \int_U g^p(x)\,dx,$$

and then $⨍_{B(0,1)}|D(\zeta\circ u)(x)\cdot v|^p\,dv \le g^p(x)$, for a.e. $x \in \Omega$, where g is defined by

$$\lim_{\epsilon\to 0}\int_{U_\epsilon}\left(⨍_{B(x,\epsilon)} \frac{\|u(y) - u(x)\|_\infty^p}{\epsilon^p}\,dy\right)dx = \int_U g^p(x)\,dx.$$

This implies that $c_{n,p}\,|D(\zeta\circ u)(x)|^p \le g^p(x)$, for a.e. $x \in \Omega$, where the constant $c_{n,p}$ is defined as in (4.6). So, u belongs to the Sobolev type space in the sense of Reshetnyak and $c_{n,p}\,\mathcal{E}_p^R[u,\Omega] \le \mathcal{E}_p^{KS}[u,\Omega]$. This completes the proof of Theorem 4.2. □

Remark 4.3. Suppose that $u \in L^p(\Omega, \ell^\infty)$, for some $p > 1$. If u belongs to the Sobolev type space (in the sense of Korevaar–Schoen or Reshetnyak), then we can show that

$$\mathcal{E}_p^{KS}[u,\Omega] = \int_\Omega\left(⨍_{B(0,1)} \sup_i |Du_i(x)\cdot v|^p\,dv\right)dx, \quad \mathcal{E}_p^R[u,\Omega] = \int_\Omega \sup_i |Du_i(x)|^p\,dx.$$

Compare with Theorem 4.1 and Theorem 5.2. Moreover, we can easily show (by considering linear maps) that the energy norms $\mathcal{E}_p^{KS}[u,\Omega]$ and $\mathcal{E}_p^R[u,\Omega]$ are not equal (up to a multiple universal constant) and the inequalities in (4.7) are sharp.

Corollary 4.4. *Suppose that (Y,d) is a metric space and Ω is an open domain in $\mathbb{R}^n$. Let $u : \Omega \longrightarrow Y$ belong to $L^p(\Omega, Y)$, for some $p > 1$. Then, u belongs to the Sobolev type space in the sense of Korevaar–Schoen iff u belongs to the Sobolev type space in the sense of Reshetnyak. Also, if u is a Sobolev type map, then*

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left(⨍_{B(x,\epsilon)} \frac{d^p(u(y),u(x))}{\epsilon^p}\,dy\right)dx,$$

exists. Moreover, there exists a positive constant $c_{n,p}$, depending on n and p, such that $c_{n,p}\,\mathcal{E}_p^R[u,\Omega] \le \mathcal{E}_p^{KS}[u,\Omega] \le \mathcal{E}_p^R[u,\Omega]$.

Proof. Since the image of u, say $\mathrm{Im}(u)$, is separable (or essentially separable), there is an isometric embedding from $\mathrm{Im}(u)$ into ℓ^∞. Therefore, without loss of generality, we may assume that $Y = \ell^\infty$. The existence of limit follows from Theorem 4.2 and [9, Theorem 1.5.1]. Moreover, the rest of assertion follows from Theorem 4.2. □

Remark 4.5. All results of this section are valid, if the domain of maps, instead of Euclidean space, replaces by a Riemannian domain, i.e., a connected open subset of an n-dimensional Riemannian manifold with compact closure. Also, in Reshetnyak's definition, suppose that the condition $|Du_Q(x)| \leq w(x)$, for all $Q \in Y$ and a.e. $x \in \Omega$, replaces by the condition

$$\fint_{B(0,1)} \sup_{\zeta} |D(\zeta \circ u)(x) \cdot v|^p \, dv \leq w(x),$$

for a.e. $x \in \Omega$, where supremum is taken over all 1-Lipschitz functions $\zeta : \ell^\infty \longrightarrow \mathbb{R}$. Then, we can show that the energy norm, by this definition, is equal (up to a multiple universal constant) to the energy norm by the definition of Korevaar–Schoen.

5. Sobolev type spaces for maps whose domain is a Lipschitz manifold

In this section, we strengthen the main result of Gregori [6] about the Sobolev type maps. Gregori generalized Korevaar–Schoen's work [9] to maps whose domain is a Lipschitz manifold. In [9] and [6], it is shown that the approximate energy functions, for the Sobolev maps, converges weakly to the energy density. We improve this result by showing that the (strong) convergence holds (see Theorem 5.2). In [9], the proof of the weak convergence of the approximate energy functions is based on the sub-partition estimate (see [9, Lemma 1.3.1]). Also, Gregori [6] generalized the sub-partition estimate for maps whose domain is a Lipschitz manifold. Here, in the proof of the (strong) convergence of the approximate energy functions, we did not use the (generalized) sub-partition estimate for maps whose domain is a Lipschitz manifold. Indeed, the proof of the (strong) convergence of the approximate energy functions is based on an extension of Theorem 3.5 for maps whose domain is a Lipschitz manifold (and [6, Lemma 2], see also the proof of Theorem 5.2). Moreover, we may interpret this result as another generalized differentiability property for the Sobolev type maps.

It is said that a subset X of Euclidean space $\mathbb{R}^{n+k}$ is an n-dimensional *Lipschitz Manifold*, if for any $x \in X$, there exists an open neighborhood $U_x \subset X$ of x such that U_x is bi-Lipschitz equivalent to an open ball in $\mathbb{R}^n$, i.e., there exist an open ball V in $\mathbb{R}^n$, positive number M and bijection $\psi : U_x \longrightarrow V$ such that

$$M^{-1}\,|z-w| \leq |\psi(z)-\psi(w)| \leq M\,|z-w|,$$

for all $z, w \in X$ (ψ is called an *M-bi-Lipschitz* map). We consider on X, the induced Euclidean metric (from $\mathbb{R}^{n+k}$) and n-dimensional Hausdorff measure $\mathcal{H} = \mathcal{H}^n$. We denote the closed ball (in X) of radius $r > 0$ with center at $x \in X$ by $B_X(x, r)$. Similar to the previous section, we can define the average integral, energy norm and other concepts for maps from a Lipschitz manifold into a metric space, see also [11].

Lemma 5.1. *Let Ω be an open domain in a Lipschitz manifold X and let (Y, d) be a metric space. Suppose that $u \in L^p(\Omega, Y)$ belongs to the Sobolev type space in the sense of Korevaar–Schoen, for some $p \geq 1$. Then*

$$\lim_{\epsilon\to 0} \fint_{B_X(x,\epsilon)} \frac{d^p(u(y), u(x))}{\epsilon^p}\, d\mathcal{H}(y),$$

exists, for a.e. $x \in \Omega$.

Proof. First, we assume that $Y = \mathbb{R}$ and also there is an M-bi-Lipschitz map ψ from an open ball $U \subset \mathbb{R}^n$ onto Ω (i.e., $M^{-1}\,|z-w| \leq |\psi(z)-\psi(w)| \leq M\,|z-w|$) and u is Lipschitz. Then, $v := u \circ \psi$ is a Lipschitz function on an open ball in Euclidean space. Therefore, by the Rademacher theorem, ψ and v are differentiable a.e. (with respect to the n-dimensional Lebesgue measure).

Suppose that the derivative of ψ is approximately continuous at $z_0 \in U$ (see [3, p. 47]) and also v is differentiable at z_0. We show that

$$\lim_{\epsilon\to 0} \fint_{B_X(x_0,\epsilon)} \frac{|u(y)-u(x_0)|^p}{\epsilon^p}\, d\mathcal{H}(y),$$

exists, where $x_0 := \psi(z_0) \in \Omega$. Without loss of generality, we may assume that $z_0 = 0 \in \mathbb{R}^n$. There exist a linear isometry $B : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k}$ and an invertible linear (affine) map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $B \circ D\psi(z_0) \circ A(v) = (v, 0)$, for all $v \in \mathbb{R}^n$. Then, we have $\lim_{z\to z_0} \frac{|z-z_0|}{|\psi\circ A(z)-\psi\circ A(z_0)|} = 1$, and furthermore, the (absolute value of) Jacobian of $\psi \circ A$ is approximately continuous at z_0, with value 1. Hence $\lim_{r\to 0} \frac{\mathcal{H}^n(B_X(x_0,r))}{\mathrm{vol}(B(z_0,r))} = 1$. Therefore, by the change of variables formula, $z \longmapsto \psi \circ A(z)$, we have

$$\lim_{\epsilon\to 0} \fint_{B_X(x_0,\epsilon)} \frac{|u(y)-u(x_0)|^p}{\epsilon^p}\, d\mathcal{H}(y) = \lim_{\epsilon\to 0} \fint_{B(z_0,\epsilon)} \frac{|v\circ A(z)-v\circ A(z_0)|^p}{\epsilon^p}\, dz.$$

Notice that, since $v \circ A$ is differentiable at z_0, by Corollary 2.3 and Remark 2.4, we know that $\lim_{\epsilon\to 0} \fint_{B(z_0,\epsilon)} \frac{|v\circ A(z)-v\circ A(z_0)|^p}{\epsilon^p}\, dz$ exists. This implies that

$\lim_{\epsilon\to 0} \fint_{B_X(x,\epsilon)} \frac{d^p(u(y),u(x))}{\epsilon^p}\, d\mathcal{H}(y)$ exists, for a.e. $x \in \Omega$, whenever u is a Lipschitz and real-valued function. For the map u, under the assumptions of Lemma 5.1, the proof is similar to the proofs of Theorem 3.3, Corollary 3.4 and Theorem 3.5. □

Theorem 5.2. *Let Ω be an open domain in a Lipschitz manifold X and let (Y, d) be a metric space. Suppose that $u \in L^p(\Omega, Y)$ belongs to the Sobolev type space in the sense of Korevaar–Schoen, for some $p > 1$. Then*

$$g(x) := \lim_{\epsilon\to 0} \fint_{B_X(x,\epsilon)} \frac{d^p(u(y), u(x))}{\epsilon^p}\, d\mathcal{H}(y)$$

exists, for a.e. $x \in \Omega$, and

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \left| \left(\fint_{B_X(x,\epsilon)} \frac{d^p(u(y), u(x))}{\epsilon^p}\, d\mathcal{H}(y) \right) - g(x) \right| d\mathcal{H}(x) = 0.$$

Proof. By [6, Lemma 2] (it can be proved by Theorem 1.5.1 and Theorem 1.10 in [9]), for any $\eta > 0$ there is $\delta > 0$ such that

$$\limsup_{\epsilon\to 0} \int_A \left(\fint_{B_X(x,\epsilon)} \frac{d^p(u(y), u(x))}{\epsilon^p}\, d\mathcal{H}(y) \right) d\mathcal{H}(x) \le \eta,$$

whenever A is a compact (measurable) subset in Ω so that $\mathcal{H}(A) \le \delta$. On the other hand, Lemma 5.1 implies that

$$g(x) := \lim_{\epsilon\to 0} \fint_{B_X(x,\epsilon)} \frac{d^p(u(y), u(x))}{\epsilon^p}\, d\mathcal{H}(y)$$

exists, for a.e. $x \in \Omega$. Then, by the Egorov theorem, for any $\delta' > 0$, there exists measurable set $B \subset \Omega$ such that $\mathcal{H}(\Omega \backslash B) \le \delta'$ and

$$\fint_{B_X(b,\epsilon)} \frac{d^p(u(y), u(b))}{\epsilon^p}\, d\mathcal{H}(y) \longrightarrow g(b),$$

uniformly on $b \in B$, as $\epsilon \to 0$. Therefore, for any $\eta > 0$ and $\delta_0 > 0$, there exist $\epsilon_0 > 0$ and a compact (measurable) subset G in Ω such that $\mathcal{H}(\Omega \backslash G) < \delta_0$ and

$$\left| \fint_{B_X(x,\epsilon)} \frac{d^p(u(y), u(x))}{\epsilon^p}\, d\mathcal{H}(y) - g(x) \right| \le \eta,$$

for all $x \in G$ and $0 < \epsilon < \epsilon_0$. Furthermore, for any compact (measurable) subset $F \subset (\Omega \backslash G)$, we have

$$\limsup_{\epsilon\to 0} \int_F \left(\fint_{B_X(x,\epsilon)} \frac{d^p(u(y), u(x))}{\epsilon^p}\, d\mathcal{H}(y) \right) d\mathcal{H}(x) \le \eta.$$

For ϵ small enough, we know that $G \subset \Omega_\epsilon$. Then

$$\begin{aligned}&\int_{\Omega_\epsilon} \fint_{B_X(x,\epsilon)} \frac{d^p(u(y),u(x))}{\epsilon^p}\, d\mathcal{H}(y)\, d\mathcal{H}(x)\\ &= \int_G \fint_{B_X(x,\epsilon)} \frac{d^p(u(y),u(x))}{\epsilon^p}\, d\mathcal{H}(y)\, d\mathcal{H}(x)\\ &\quad + \int_{\Omega_\epsilon \setminus G} \fint_{B_X(x,\epsilon)} \frac{d^p(u(y),u(x))}{\epsilon^p}\, d\mathcal{H}(y)\, d\mathcal{H}(x),\end{aligned}$$

for $\epsilon > 0$ small enough. Also, by choosing δ_0 small enough, we have that $\int_{\Omega \setminus G} g(x)\, d\mathcal{H}(x) \le \eta$. Finally, we obtain

$$\int_{\Omega_\epsilon} \left| \left(\fint_{B_X(x,\epsilon)} \frac{d^p(u(y),u(x))}{\epsilon^p}\, d\mathcal{H}(y) \right) - g(x) \right| d\mathcal{H}(x) \le \eta\, \mathcal{H}(\Omega) + 2\eta,$$

for $\epsilon > 0$ small enough. Letting $\eta \to 0$, then the assertion follows (as $\epsilon \to 0$). □

Question 5.3. Under the assumptions of Theorem 5.2, for $p = 1$, is it possible to prove the existence of the following limit:

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \left(\fint_{B_X(x,\epsilon)} \frac{d^p(u(y),u(x))}{\epsilon^p}\, d\mathcal{H}(y) \right) d\mathcal{H}(x)?$$

Remark 5.4. We can extend Theorem 4.1, Theorem 4.2 and Corollary 4.4 to maps whose domain is a Lipschitz manifold.

6. Orlicz spaces and L^Φ-differentiability

In this section, we extend the concept of L^p-differentiability to Orlicz's spaces context. For the basic concepts of Orlicz's spaces, see for example [1, Chapter VIII] or [12].

Definition 6.1. Let $\Phi : [0,\infty[\longrightarrow [0,\infty[$ and $f : \mathbb{R}^k \longrightarrow \mathbf{R}$ be (Borel) measurable functions. We say that f is L^Φ-*differentiable* at point $x \in \mathbb{R}^k$, if there exist a linear functional $Df(x) : \mathbb{R}^k \longrightarrow \mathbb{R}$ and a positive number a such that

$$\lim_{\epsilon \to 0} \fint_{B(x,\epsilon)} \Phi\left(\frac{|f(y) - f(x) - Df(x)\cdot(y-x)|}{a\,\epsilon} \right) dy = 0.$$

Theorem 6.2. *Let $\Phi : [0,\infty[\longrightarrow [0,\infty[$ be a convex function with the following properties:*

– *Φ is an increasing and invertible function,*

– $\lim_{s\to 0} \Phi(s) = \Phi(0) = 0$.

Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ *be a measurable function on* $\mathbb{R}^k$*, then* f *is* L^Φ*-differentiable a.e. iff*

$$\limsup_{\epsilon\to 0} ⨍_{B(x,\epsilon)} \Phi\left(\frac{|f(y)-f(x)|}{a_x\,\epsilon}\right)\,dy < \infty, \tag{6.1}$$

for a.e. $x \in \mathbb{R}^k$*, where* a_x *is a positive number. Moreover, similar to Theorem* 2.7*, we can replace the condition* (6.1) *with the following condition:*

$$\limsup_{\epsilon\to 0} ⨍_{B(x,\epsilon)} ⨍_{B(x,\epsilon)} \Phi\left(\frac{|f(z)-f(w)|}{a_x\,\epsilon}\right)\,dz\,dw < \infty. \tag{6.2}$$

Proof. By the convexity of Φ, it is clear that the L^Φ-differentiability (a.e.) condition implies (6.1). On the other hand, since Φ is a convex function by the Jensen inequality, we have

$$\Phi\left(⨍_{B(x,\epsilon)} \frac{|f(z)-f(x)|}{a_x\,\epsilon}\,dz\right) \le ⨍_{B(x,\epsilon)} \Phi\left(\frac{|f(z)-f(x)|}{a_x\,\epsilon}\right)\,dz,$$

for a.e. $x \in \mathbb{R}^k$, where a_x is a positive number which satisfies (6.1). Then, we get $\limsup_{\epsilon\to 0} ⨍_{B(x,\epsilon)} \frac{|f(z)-f(x)|}{\epsilon}\,dz < \infty$, for a.e. $x \in \mathbb{R}^k$. Therefore, by Theorem 1.2, we know that f is L^1-differentiable a.e.

Let the notations be as in the proof of Lemma 2.1, except the set E is defined as the following:

$$E := \left\{ x \in B(0,T) \,:\, \sup_{0<r\le R} ⨍_{B(x,r)} \Phi\left(\frac{|f(y)-f(x)|}{A\,r}\right)\,dy \;\le \Phi(M) \right\},$$

where T, R, M and A are positive numbers. Then, similar to the proof of Lemma 2.1 (iii), we obtain $|f(y)-f(x)-Dg(x)\cdot(y-x)| \le |f(y)-g(z')| + (1+3\,L)\,\epsilon^{\frac{1}{k}}\delta$. Since Φ is a convex and increasing function and $\Phi(0)=0$, we have

$$\begin{aligned}\Phi\left(\frac{|f(y)-f(x)-Dg(x)\cdot(y-x)|}{2A\,\delta}\right) &\le \frac{1}{2}\Phi\left(\frac{|f(y)-g(z')|}{A\,\delta}\right) + \frac{1}{2}\Phi\left(\frac{1+3L}{A}\,\epsilon^{\frac{1}{k}}\right)\\ &\le \frac{1}{2}\epsilon^{\frac{1}{k}}\,\Phi\left(\frac{|f(y)-g(z')|}{A\,\delta_1}\right) + \frac{1}{2}\epsilon^{\frac{1}{k}}\,\Phi\left(\frac{1+3L}{A}\right).\end{aligned}$$

Notice that $\Phi(\lambda t) \le \lambda\Phi(t)$ for all $0 \le \lambda \le 1$ and $t \ge 0$. By integrating with respect to y , we obtain

$$
\begin{aligned}
&\fint_{B(z',\delta_1)} \Phi\left(\frac{|f(y)-f(x)-Dg(x)\cdot(y-x)|}{2A\,\delta}\right) dy \\
&\le \frac{\epsilon^{\frac{1}{k}}}{2}\left[\fint_{B(z',\delta_1)} \Phi\left(\frac{|f(y)-g(z')|}{A\,\delta_1}\right) dy + \Phi\left(\frac{1+3\,L}{A}\right)\right] \\
&\le \frac{\epsilon^{\frac{1}{k}}}{2}\left[\Phi(M) + \Phi\left(\frac{1+3\,L}{A}\right)\right] \\
&= \lambda(\epsilon),
\end{aligned}
$$

where λ is a function such that $\lim_{s\to 0}\lambda(s)=0$. Since we can cover $B(x,\delta_3)$ by a minimum number of balls whose centers are in E and their radii are equal to δ_1 (f and g are equal at such points), we get

$$
\fint_{B(x,\delta_3)} \Phi\left(\frac{|f(y)-f(x)-Dg(x)\cdot(y-x)|}{2A\,\delta}\right) dy \le C\,\lambda(\epsilon),
$$

where C is a constant which depends on k. Then, we have

$$
\fint_{B(x,\delta_3)} \Phi\left(\frac{|f(y)-f(x)-Dg(x)\cdot(y-x)|}{2(1+A)\,\delta_3}\right) dy \le C\,\lambda(\epsilon),
$$

for ϵ small enough (depending on k). Letting $\epsilon \to 0$, this implies that f is L^{Φ}-differentiable at x. Now, the rest of proof is similar to the proof of Theorem 1.2. Also, similar to Theorem 2.7, we can replace the condition (6.1) with (6.2). □

Next, we apply the previous theorem to the Sobolev functions.

Theorem 6.3. *Suppose that $u \in W^{1,n}_{loc}(\mathbb{R}^n)$, for an integer number $n > 1$. Then, u is L^{Φ}-differentiable a.e., where $\Phi(t) := \exp\left(t^{\frac{n}{n-1}}\right) - 1$. In particular, u is L^p-differentiable a.e., for all $p \ge 1$.*

Proof. By the Trudinger inequality (see for example [5, Theorem 7.15]), there exist positive constants c_1 and c_2, depending only on n, such that

$$
\fint_B \Phi\left(\frac{|u(y)-u_B|}{c_1\,\|Du\|_{L^n(B)}}\right) dy \le c_2,
$$

for all balls $B \subset \mathbb{R}^n$, where $u_B := \fint_B u$. Suppose that a point $x \in \mathbb{R}^n$ satisfies

$$
\lim_{\epsilon\to 0}\left(\fint_{B(x,\epsilon)} |Du(y)|^n\,dy\right)^{\frac{1}{n}} = |Du(x)| < \infty. \tag{6.3}
$$

Then, we have $\lim_{\epsilon\to 0}\frac{\|Du\|_{L^n(B(x,\epsilon))}}{\epsilon} = \omega_n\,|Du(x)|$, where $\omega_n := [\mathrm{vol}(B(0,1))]^{\frac{1}{n}}$.

From the convexity of Φ, we know that

$$\Phi\left(\frac{|u(z)-u(w)|}{2c_1\,\|Du\|_{L^n(B)}}\right)\le\frac{1}{2}\left[\Phi\left(\frac{|u(z)-u_B|}{c_1\,\|Du\|_{L^n(B)}}\right)+\Phi\left(\frac{|u(w)-u_B|}{c_1\,\|Du\|_{L^n(B)}}\right)\right],$$

for all $z,w\in\mathbb{R}^n$ and ball $B\subset\mathbb{R}^n$. Then, by choosing $B=B(x,\epsilon)$ and taking integral over $B(x,\epsilon)$, we obtain

$$\begin{aligned}
&⨍_{B(x,\epsilon)}⨍_{B(x,\epsilon)}\Phi\left(\frac{|u(z)-u(w)|}{2c_1\,\|Du\|_{L^n(B(x,\epsilon))}}\right)dz\,dw\\
&\le\frac{1}{2}\left[⨍_{B(x,\epsilon)}⨍_{B(x,\epsilon)}\Phi\left(\frac{|u(z)-u_{B(x,\epsilon)}|}{c_1\,\|Du\|_{L^n(B(x,\epsilon))}}\right)dz\,dw\right.\\
&\quad\left.+⨍_{B(x,\epsilon)}⨍_{B(x,\epsilon)}\Phi\left(\frac{|u(w)-u_{B(x,\epsilon)}|}{c_1\,\|Du\|_{L^n(B(x,\epsilon))}}\right)dz\,dw\right]\\
&\le⨍_{B(x,\epsilon)}\Phi\left(\frac{|u(y)-u_{B(x,\epsilon)}|}{c_1\,\|Du\|_{L^n(B(x,\epsilon))}}\right)dy\\
&\le c_2.
\end{aligned}$$

Therefore, if $x\in\mathbb{R}^n$ satisfies (6.3), we have

$$\limsup_{\epsilon\to0}⨍_{B(x,\epsilon)}⨍_{B(x,\epsilon)}\Phi\left(\frac{|u(z)-u(w)|}{3\,c_1\,\omega_n\,\max\{|Du(x)|\,,\,1\}\,\epsilon}\right)dz\,dw\le c_2.$$

By Lebesgue's differentiation theorem, we know that the condition (6.3) holds for a.e. $x\in\mathbb{R}^n$, then the assertion follows from Theorem 6.2. □

Remark 6.4. When $u\in W^{1,n}_{loc}(\mathbb{R}^n)$ (for an integer $n\ge1$), by the John-Nirenberg inequality [5, Theorem 7.15], we can show that u is L^Ψ-differentiable a.e., where $\Psi(t):=e^t-1$. Also, we can extend Theorem 6.2 and Theorem 6.3 to maps with values in a metric space (as before).

Question 6.5. Is it possible to extend the conclusions of Corollary 2.8 (ii) and Theorem 6.3 to the L^p-differentiable functions instead of the Sobolev functions? With some extra assumptions?

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