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On Nontrivial Solutions of Variational-Hemivariational Inequalities with Slowly Growing Principal Parts

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Abstract. This paper is concerned with the inclusion

$$-\operatorname{div}(a(|\nabla u|)\nabla u) + \partial_u G(x,u) \ni 0 \quad \text{in } \Omega,$$

with Dirichlet boundary condition $u = 0$ on $\partial\Omega$, in the case where the higher order part has slow growth and the lower order part is locally Lipschitz. By using a Mountain Pass theorem for variational-hemivariational inequalities without the Palais–Smale condition in Orlicz–Sobolev spaces, we show the existence of nontrivial solutions of the above inclusion.
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1. Introduction

This paper is about a variational-hemivariational inequality arising from the following inclusion:

$$-\operatorname{div}(a(|\nabla u|)\nabla u) + \partial_u G(x,u) \ni 0 \quad \text{in } \Omega, \tag{1}$$

with boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. We are interested here in the case where the function $a(t)t$ has very slow growth and $G(x,u)$ is a Carathéodory function that is locally Lipschitz in u and the lower order term

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$\partial_u G(x,u)$ in (1) is the generalized gradient of G with respect to u (cf. [3]). The principal (higher order) part of the equation is represented by the function $\phi : t \mapsto a(t)t,\ t \in \mathbb{R}$, which is supposed to be increasing, continuous, odd, and vanishing at 0.

Let Φ denote the antiderivative of ϕ, $\Phi(t) = \int_0^t \phi(s)ds$ ($t \in \mathbb{R}$). The classical case $\Phi(t) = t^2$ corresponds to the semilinear Laplace inclusion. We are concerned here with the situation where Φ is growing very slowly, that is, $\Phi(t) = o(t^p)$ as $t \to \infty$ for all $p > 1$. In this case, Orlicz–Sobolev spaces rather than regular Sobolev spaces are more suitable as function spaces for the study of (1)–(2). Since the Hölder conjugate $\overline{\Phi}$ of Φ does not satisfy a Δ_2 condition (see section 2 for more details on $\overline{\Phi}$ and Δ_2 condition), the functional $u \mapsto \int_\Omega \Phi(|\nabla u|)\, dx$, does not belong to class C^1. Moreover, the integral given by the lower order term is not differentiable in general. Therefore, problem (1)–(2) is formulated, in the weak form, not as a variational equation but naturally as a variational-hemivariational inequality in an appropriate Orlicz–Sobolev space. In a previous paper (cf. [9]), problem (1)–(2) was studied in the particular case where $G(x,u)$ is of class C^1 in u. In that case, the functional defined by the integral is also of class C^1 and the problem is therefore formulated as a variational inequality.

To study the existence of nontrivial solutions, we shall use a version of the Mountain Pass theorem for variational-hemivariational inequalities. Note that in the case both Φ and $\overline{\Phi}$ satisfy Δ_2 conditions, we could prove a compactness condition for equations in $W_0^1 L_\Phi$, which implies the Palais–Smale (PS) condition (cf. [8]). However, there has not been proved a similar result in Orlicz–Sobolev spaces when either Φ or $\overline{\Phi}$ fails to satisfy this condition. We could in fact prove the existence and boundedness of Palais–Smale sequences $\{u_n\}$ of the variational-hemivariational inequality associated with problem (1)–(2). However, the convergence of the integrals $\{\int_\Omega \Phi(|\nabla u_n|)dx\}$ is, in our case, not strong enough to allow us to conclude the strong convergence of a subsequence of $\{u_n\}$ in Orlicz–Sobolev spaces. Therefore, we need here a version of the Mountain Pass theorem for variational-hemivariational inequalities without the (PS) condition.

The paper is organized as follows. In Section 2, basic concepts and results related to Orlicz–Sobolev spaces are presented. Next, in Section 3, we state and prove a general linking theorem for variational-hemivariational inequalities without the (PS) condition whose corollary, a Mountain Pass theorem for variational-hemivariational inequalities, will be needed for our investigation of problem (1)–(2). Although being abstract preparatory results for our existence theorem later, these versions of linking and Mountain Pass theorems have their own interests and would be useful in other situations as well. In Section 4, we apply the abstract version of Mountain Pass theorem established in Section 3 to prove the existence of nontrivial solutions of the variational-hemivariational inequality that formulates (1)–(2). Note that in the particular case where $G(x,u)$ is of class C^1 in u, then our theorem here reduces to that in [9]. Hence, the results here generalize those in that paper to the case of locally Lipschitz lower order terms. We also observe that the arguments in our case also apply to inequalities in which the principal operators have not very fast growth. Therefore, when both Φ and $\overline{\Phi}$ satisfy Δ_2 conditions, our results here give an alternate and generalization of some existence results in [4] in cases where the equations contain locally Lipschitz lower order terms.

2. Problem setting – preliminaries on Orlicz-Sobolev spaces

The inclusion (1)–(2) can be formulated (in the weak form) as the inequality

$$\langle J'(u), v\rangle + \int_\Omega G^o(x,u;v)\,dx \geq 0,$$

for all test functions v, where J is the potential functional associated with the principal part:

$$J(u) = \int_\Omega \Phi(|\nabla u|)\,dx, \tag{3}$$

J' is the Gâteaux derivative of J, $\langle\cdot,\cdot\rangle$ is the pairing between the space of test functions and its dual, and $G^o(x,u;v)$ stands for the generalized directional derivative of G (with respect to u) in the direction of v. Since the growth of the principal term is represented by Φ, we choose the function space for the solutions and test functions as the first-order Orlicz–Sobolev space $W_0^1 L_\Phi$. In this space, we write the above inequality as the following variational-hemivariational inequality:

$$\begin{cases} J(v) - J(u) + \displaystyle\int_\Omega G^o(x,u;v-u)\,dx \geq 0, \quad \forall v \in W_0^1 L_\Phi \\ u \in W_0^1 L_\Phi. \end{cases} \tag{4}$$

We recall that $W^1 L_\Phi$ is the Orlicz–Sobolev space of functions $u \in L_\Phi$ such that $\nabla u \in (L_\Phi)^N$. L_Φ is the usual Orlicz space associated with the Young function Φ with the (Luxemburg) norm $\|\cdot\|_\Phi$ defined by:

$$\|u\|_\Phi = \|u\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

A Young function Φ is said to satisfy a Δ_2 condition (on $\mathbb{R}$) if there exists $k > 0$ such that $\Phi(2t) \leq k\Phi(t)$ for all $t > 0$. Since Φ is assumed here to satisfy a

Δ_2 condition, $L_\Phi = E_\Phi = \tilde{L}_\Phi$, where E_Φ is the closure of $L^\infty(\Omega)$ in L_Φ (with respect to the norm-topology) and

$$\tilde{L}_\Phi := \left\{ v : \Omega \to \mathbb{R} : v \text{ is measurable on } \Omega \text{ and } \int_\Omega \Phi(v)dx < \infty \right\}.$$

The corresponding norm on W^1L_Φ is given by

$$\|u\|_{1,\Phi} = \|u\|_{W^1L_\Phi} = \|u\|_\Phi + \sum_{j=1}^N \|\partial_j u\|_\Phi.$$

Properties of the Orlicz space L_Φ and of the Orlicz–Sobolev spaces W^1L_Φ and $W_0^1L_\Phi$ when Φ and/or $\overline{\Phi}$ satisfies a Δ_2 condition are presented in detail in [1, 5–7]. It is known (cf. [6, 7]) that L_Φ is the dual space of $E_{\overline{\Phi}}$, i.e., $L_\Phi = (E_{\overline{\Phi}})^*$, and $L_{\overline{\Phi}} = (E_\Phi)^*$, where $\overline{\Phi}$ is the Hölder conjugate function of Φ, defined by $\overline{\Phi}(t) = \sup\{ts - \Phi(s) : s \in \mathbb{R}\}$. The space W^1L_Φ and W^1E_Φ can be identified with closed subspaces of the products $\prod_{i=0}^N L_\Phi$ and $\prod_{i=0}^N E_\Phi$, respectively. We have $\prod_{i=0}^N L_\Phi = \left(\prod_{i=0}^N E_{\overline{\Phi}}\right)^*$ and if we denote by $\tau = \sigma\left(\prod L_\Phi, \prod E_{\overline{\Phi}}\right)$ the weak* topology in $\prod L_\Phi$ and also the restriction of τ to the closed subspace W^1L_Φ, then W^1L_Φ is closed under weak* convergence of $\prod L_\Phi$. Since $\prod E_{\overline{\Phi}}$ is separable, we have the following properties of W^1L_Φ (cf. [5]):

If $\{u_n\}$ is a bounded sequence in W^1L_Φ (with respect to $\|\cdot\|_{1,\Phi}$), then $\{u_n\}$ has a subsequence which converges with respect to the topology τ to some $u \in W^1L_\Phi$, i.e., a bounded set in W^1L_Φ is relatively sequentially compact with respect to the weak* topology τ.

We denote by $W_0^1L_\Phi$ the closure of $C_0^\infty(\Omega)$ with respect to the weak* topology τ. By a Poincaré inequality for Orlicz-Sobolev spaces (see [5]), we know that on $W_0^1L_\Phi$ the norm $\|\cdot\|_{W^1L_\Phi}$ is equivalent to the norm $\|\cdot\|_{W_0^1L_\Phi}$ given by

$$\|u\|_{W_0^1L_\Phi} = \|\,|\nabla u|\,\|_{L_\Phi}.$$

A Young function Φ_1 is said to grow (essentially) more slowly than another Young function Φ_2 (at infinity) (cf. [1, 6, 7]), abbreviated by $\Phi_1 \ll \Phi_2$, if $\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(kt)} = 0$, for all $k > 0$. Let us denote by Φ^* the Sobolev conjugate of Φ (in $\mathbb{R}^N$), with $(\Phi^*)^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds$, provided that

$$\int_1^\infty \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds = \infty. \tag{5}$$

We have the following embeddings, similar to those among Sobolev spaces:

- The embedding $W_0^1L_\Phi \hookrightarrow L_{\Phi^*}$ is continuous.
- If $\Psi \ll \Phi^*$, then the embedding $W^1L_\Phi \hookrightarrow L_\Psi$ is compact. In particular, since $\Phi \ll \Phi^*$ (cf. [5, Lemma 4.14]), the embedding $W^1L_\Phi \hookrightarrow L_\Phi$ is compact.

Moreover, it is shown that W^1L_Φ is continuously embedded in $L^\infty(\Omega)$ (cf. [1,5]) if $\int_1^\infty \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds < \infty$ in (5).

We are interested here in problems with principal terms given by a Young function Φ growing more slowly than any power $t^{1+\epsilon}$ ($\epsilon > 0$), that is $\Phi(t) \ll t^{1+\epsilon}$, for all $\epsilon > 0$. Typical examples of such Young functions are

$$\Phi(t) = \int_0^{|t|} [\ln(1+s)]^\beta \, ds \quad (t \in \mathbb{R}) \tag{6}$$

($\Phi(t) = (|t|+1)\ln(|t|+1) - |t|$, when $\beta = 1$), or

$$\Phi(t) = |t|[\ln(|t|+1)]^\beta \quad (t \in \mathbb{R}), \tag{7}$$

(with β being a fixed positive constant). It is easy to check that in both cases

$$\lim_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)} = 1. \tag{8}$$

For such functions, their conjugates $\overline{\Phi}$ do not satisfy a Δ_2 condition. We refer to [6] (or [1,7]) for basic properties of Δ_2 condition. In what follows, we assume that Φ satisfies the growth condition determined by (8).

3. Linking and Mountain Pass theorems

We shall need here a version of the Mountain Pass theorem for variational-hemivariational inequalities without the Palais–Smale condition. Since this compactness condition is not imposed, we obtain, instead of the existence of critical points of the associated functionals, only that of (PS) sequences in the sense of (12) and (13) below. Note that in the problem we are interested in here, the functional J is convex and finite everywhere, hence locally Lipschitz on X. We state the theorem for the general case of sums of convex, lower semicontinuous and locally Lipschitz functionals, due to its own interest and applicability in other situations as well.

Furthermore, we shall first establish a more general linking theorem which contains the needed Mountain Pass theorem as a particular case. Let us start with the definition of linking that we are interested in.

Definition 1 ([12, Definition 3.3]). Let S be a nonempty closed subset of a Banach space X and let Q be a compact topological submanifold of X with nonempty boundary ∂Q (in the sense of manifolds with boundary). We say that *S and Q link* if the next properties hold:

$$S \cap \partial Q = \emptyset, \quad f(Q) \cap S \neq \emptyset$$

whenever $f \in \Gamma$, where

$$\Gamma = \{f \in C(Q, X) : f|_{\partial Q} = id_{\partial Q}\}. \tag{9}$$

We are now ready to state and prove the following general minimax theorem with the above type of linking for functionals which are sums of convex and locally Lipschitz ones.

Theorem 1. *Let the functional* $I : X \to \mathbb{R} \cup \{+\infty\}$ *on the Banach space* $(X, \|\cdot\|)$ *satisfy the following assumption:*

(H) $I = P + \psi$, *where* $P : X \to \mathbb{R}$ *is a locally Lipschitz functional and* $\psi : X \to \mathbb{R} \cup \{+\infty\}$ *is convex, proper, and lower semicontinuous.*

Let the closed sets S *and* Q *link in* X *in the sense of Definition* 1. *Assume*

$$\sup_{x\in Q} I(x) \in \mathbb{R},\ a := \sup_{x\in\partial Q} I(x) < b := \inf_{x\in S} I(x). \tag{10}$$

Then the number

$$c = \inf_{f\in\Gamma} \sup_{x\in Q} I(f(x)), \tag{11}$$

where Γ *is given by* (9), *satisfies the following property: There exist sequences* $\{u_n\}$ *in* X *and* $\{\epsilon_n\}$ *in* $(0, +\infty)$ *such that*

$$\epsilon_n \downarrow 0,\ I(u_n) \to c \quad \text{as } n \to \infty, \tag{12}$$

and

$$P^o(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\epsilon_n \|v - u_n\|, \quad \forall v \in X, \forall n \in \mathbb{N}. \tag{13}$$

Proof. First, we see that thanks to (10) and the linking hypothesis we have that $\bar{\epsilon} := c - a > 0$. Arguing by contradiction, we assume that there exists $\epsilon \in (0, \bar{\epsilon})$ such that whenever $u \in X$, we have either

$$I(u) \notin [c - \epsilon, c + \epsilon], \tag{14}$$

or one can find $v = v(u) \in X$ satisfying

$$P^o(u; v - u) + \psi(v) - \psi(u) < -3\epsilon\|v - u\|. \tag{15}$$

In particular, $K_c(I) = \emptyset$, where $K_c(I)$ is the set of all critical points of I at the level c: $K_c(I) = K(I) \cap I^{-1}(c)$, where $K(I) = \{u \in X : P^o(u; v - u) + \psi(v) - \psi(u) \geq 0, \text{ for all } v \in X\}$. For $d \in \mathbb{R}$, we also use the notation $I_d = \{u \in X : I(u) \leq d\}$.

We claim that for each $u_0 \in I_{c+\epsilon}$, there exist $v_0 = v_0(u_0) \in X$ and a neighborhood U_0 of u_0 in X such that

$$P^o(u; v_0 - u) + \psi(v_0) - \psi(w) \leq K(\|u - v_0\| + \|w - v_0\|), \quad \forall u, w \in U_0, \tag{16}$$

with some constant $K > 0$, and

$$P^o(u; v_0 - u) + \psi(v_0) - \psi(w) \leq -3\epsilon\|w - v_0\|, \tag{17}$$

for all $u, w \in U_0$ with $I(w) \geq c - \epsilon$. Moreover, we claim that if $u_0 \in K(I)$, then one can take $v_0 = u_0$, and if $u_0 \notin K(I)$, then one can choose v_0 and U_0 so that $v_0 \notin \overline{U_0}$ and, for some $\delta_0 > 0$,

$$P^o(u; v_0 - u) + \psi(v_0) - \psi(w) \leq -\delta_0 \|w - v_0\|, \quad \forall u, w \in U_0. \tag{18}$$

To justify the claim, we notice that if $u_0 \in K(I)$, then (15) does not hold for $u = u_0$ and it follows from (14) and $u_0 \in I_{c+\epsilon}$ that $I(u_0) < c - \epsilon$. At this point, we can further proceed as in page 67 of [12] to check that $v_0 = u_0$ fulfills (16) and (17). In the case where $u_0 \notin K(I)$, by treating separately the situations where $I(u_0) < c - \epsilon$ and $I(u_0) \geq c - \epsilon$, we may prove the claim using the same arguments as in pages 67–69 of [12], taking the neighborhood U of $K_c(I)$ therein to be just $U = \emptyset$, which is possible because $K_c(I) = \emptyset$.

The next step in the proof is to show that for every compact subset A of X which satisfies

$$c \leq \sup_{x \in A} I(x) \leq c + \epsilon, \tag{19}$$

there exists $\alpha \in C(W \times [0, \bar{s}], X)$, with $\bar{s} > 0$ and W being a closed neighborhood of A in X, such that $\alpha(\cdot, 0) = id_W$,

$$\|u - \alpha(u, s)\| \leq s, \quad \forall s \in [0, \bar{s}],\ u \in W, \tag{20}$$

and

$$\sup_{u \in A} I(\alpha(u, s)) - \sup_{u \in A} I(u) \leq -2\epsilon s, \quad \forall s \in [0, \bar{s}]. \tag{21}$$

In order to establish the above assertions, we first observe that, due to (19), we may apply the properties (16)–(18), referring to any $u_0 \in A$. This fact, combined with the compactness of A, enables us to construct as in [12, pages 69–70], a radial deformation of type

$$\alpha(u, s) = u + s\bar{w}, \tag{22}$$

with $\bar{w} = \bar{w}(u)$, around $A \times \{0\}$ for which the relations (20) and (21) hold. The construction of the mapping $\bar{w} = \bar{w}(u)$ relies on the mapping $v_0 = v_0(u_0)$ for which the relations (16) and (17) are true. A main point in the argument to obtain (20)–(21) from (22) is to remark that if $u \in K(I) \cap A$ then necessarily $I(u) < c - \epsilon$. Furthermore, the proof of (20) and (21) does not make use of the Palais–Smale condition at the level c.

Finally, we conclude the proof of the theorem by enlarging the class Γ in (9) to the larger class Γ_1 defined as the set of all mappings $f \in C(Q, X)$ such that $f|_{\partial Q}$ and $id_{\partial Q}$ are homotopic as maps from ∂Q to $I_{c-\epsilon/4}$ and $f(\partial Q) \subset I_{c-\epsilon/2}$. The reason of this extension from Γ to Γ_1 is that the composition $\alpha(f(\cdot), s)$ belongs to Γ_1 whenever $f \in \Gamma_1$, while Γ does not generally have this property. Moreover, the set Γ_1 is closed in $C(Q, X)$ with respect to the uniform convergence topology on that space (see [12, pages 75–76]).

We define the mapping $\Pi : \Gamma_1 \to \mathbb{R} \cup \{+\infty\}$ by

$$\Pi(f) = \sup_{x \in Q} I(f(x)), \quad \forall f \in \Gamma_1,$$

which is lower semicontinuous. Moreover, if c_1 is defined by

$$c_1 = \inf_{f \in \Gamma_1} \sup_{x \in Q} I(f(x)) = \inf_{f \in \Gamma_1} \Pi(f),$$

then by means of the Homotopy Extension Theorem and the formula of c in (11), we can prove the equality $c = c_1$ (cf. [12, page 75]). This ensures, in particular, that the functional Π is bounded below on Γ_1, and Ekeland's variational principle can thus be applied to Π. Hence, we can produce some $f \in \Gamma_1$ satisfying (19) with $A = f(Q)$ and

$$\Pi(g) \geq \Pi(f) - \epsilon \|g - f\|, \quad \text{for all } g \in \Gamma_1. \tag{23}$$

Since (19) holds, we are allowed to consider the deformation $\alpha \in C(W \times [0, \bar{s}], X)$ corresponding to the compact subset $A = f(Q)$ of X. For a possibly smaller $\bar{s} > 0$ we have that

$$\alpha(f(\cdot), s) \in \Gamma_1, \quad \forall s \in [0, \bar{s}], \tag{24}$$

(see [12, page 77]). In view of (24), we may set $g = \alpha(f(\cdot), s)$ for any $s \in [0, \bar{s}]$ in (23) which, together with (20), leads to $\Pi(\alpha(f(\cdot), s)) - \Pi(f) \geq -\epsilon \|\alpha(f(\cdot), s) - f\| \geq -\epsilon s$. On the other hand, relations (21) and (24) imply that $-2\epsilon s \geq \Pi(\alpha(f(\cdot), s)) - \Pi(f)$. Consequently, a contradiction occurs for any $s > 0$ and completes our proof. □

We illustrate the general minimax principle stated in Theorem 1 with the important, particular case of the Mountain Pass theorem without assuming the (PS) condition, which is formulated in the setting of hypothesis (H) above. This abstract result will be used in our variational approach for studying problem (1)–(2).

Corollary 1. *Let* $(X, \|\cdot\|)$ *be a Banach space and* $I : X \to \mathbb{R} \cup \{+\infty\}$ *satisfy assumption* (H). *Suppose that*

(i) $I(0) = 0$ *and there exist* $\beta, \rho > 0$ *such that*

$$I(u) \geq \beta, \quad \forall u \in X, \|u\| = \rho, \tag{25}$$

(ii) *There exists* $e \in X$ *such that* $\|e\| > \rho$ *and* $I(e) \leq 0$.

Let

$$c = \inf_{f\in\Gamma} \sup_{t\in[0,1]} I(f(t)) \ (\geq \beta), \tag{26}$$

where $\Gamma = \{f \in C([0,1], X) : f(0) = 0, f(1) = e\}$. *Then, there exist sequences* $\{u_n\} \subset X$ *and* $\{\epsilon_n\} \subset (0, +\infty)$ *that satisfy* (12) *and* (13).

Remark 1. Livrea and Molica Bisci ([11]) obtained the results of this section in the following particular case of assumption (H) (with a slight change of notation to relate with our notation here):

(H'_f) ([11, page 250]) $I(x) := P(x) + \psi(x)$ for all $x \in X$, where $P : X \to \mathbb{R}$ is locally Lipschitz continuous while $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper and lower semicontinuous. Moreover, ψ is continuous on any nonempty compact set $A \subset X$ such that $\sup_{x\in A} \psi(x) < +\infty$.

The last condition in (H'_f) is not generally satisfied by convex, proper, and lower semicontinuous functionals (cf. [13] for an explicit example even in a finite dimensional space).

4. Existence of nontrivial solutions via Mountain Pass theorem

In this section, we apply the Mountain Pass theorem in the previous section to prove existence of nontrivial solutions for the inclusion (1)–(2) in the case where $\Phi(t)$ is growing more slowly than any power t^p ($p > 1$).

Let us consider now the necessary assumptions on G and Φ. First, assume that $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $G(x, 0) = 0$ and $G(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz for almost all $x \in \Omega$. Furthermore, the generalized gradient of $G(x, \cdot)$ has the following growth condition:

$$|\xi| \leq a_1 + a_2|s|^{\alpha-1}, \tag{27}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, all $\xi \in \partial_s G(x, s)$, where $1 < \alpha \leq \frac{N}{N-1}$ (subcritical condition). From the embeddings between Orlicz and Orlicz–Sobolev spaces in Section 2 with $\Psi = \Phi_\alpha$ ($\Phi_\alpha(t) = t^\alpha$, $\forall t \geq 0$), we have the compact (hence continuous) embedding

$$W^1 L_\Phi \hookrightarrow L^\alpha(\Omega). \tag{28}$$

On the other hand, we refer to [3] for the basic concepts and results about the nonsmooth analysis of locally Lipschitz functionals.

Next, let us assume the following behavior of $G(x, t)$ when t is small:

$$\liminf_{t\to 0} \frac{G(x,t)}{\Phi(t)} > -\Lambda \quad \text{uniformly for almost all } x \in \Omega, \tag{29}$$

where $\Lambda = \inf_{u\in W_0^1 L_\Phi\setminus\{0\}} \frac{\int_\Omega \Phi(|\nabla u|)\,dx}{\int_\Omega \Phi(|u|)\,dx}$. Note that $\Lambda > 0$ (see e.g. [5]) and from its definition, we have

$$\int_\Omega \Phi(|\nabla u|)\,dx \geq \Lambda \int_\Omega \Phi(|u|)\,dx, \quad \forall u \in W_0^1 L_\Phi.$$

We also suppose that there exist $t_1 > 0$ and $\gamma > 1$ such that

$$\sup_{x\in\Omega} G(x,t_1) < 0 \tag{30}$$

and

$$G^o(x,t;t) \leq \gamma G(x,t) \quad \text{for a.e. } x \in \Omega, \text{ all } t \text{ with } |t| \geq t_1. \tag{31}$$

It follows from this assumption that there exist $a_3, a_4 > 0$ such that

$$G(x,t) \leq -a_3 t^\gamma + a_4 t \quad \text{for a.e. } x \in \Omega,\ \text{all } t \geq 0. \tag{32}$$

In fact, for $x \in \Omega, s \in \mathbb{R}$, and $t > 0$, we have the following formula:

$$\partial_t[G(x,t)t^{-\gamma}] = \gamma t^{-1-\gamma}[\gamma^{-1} t \partial_t G(x,t) - G(x,t)].$$

Thus, for $t > t_1$, from Lebourg's theorem (cf. [10]), there exists $\tilde{t} \in (t_1, t)$ such that

$$\begin{aligned}\left(\frac{t}{t_1}\right)^{-\gamma} G(x,t) - G(x,t_1) &\in \partial_\tau \left[\left(\frac{\tau}{t_1}\right)^{-\gamma} G(x,\tau)\right]_{\tau=\tilde{t}} (t-t_1)\\ &= t_1^\gamma \partial_\tau \left[\tau^{-\gamma} G(x,\tau)\right]_{\tau=\tilde{t}} (t-t_1)\\ &= \gamma t_1^\gamma \tilde{t}^{-1-\gamma} \left[\gamma^{-1}\tilde{t}\partial_t G(x,\tilde{t}) - G(x,\tilde{t})\right](t-t_1).\end{aligned}$$

Hence, from assumption (31),

$$\left(\frac{t}{t_1}\right)^{-\gamma} G(x,t) - G(x,t_1) \leq \gamma t_1^\gamma \tilde{t}^{-1-\gamma}[\gamma^{-1} G^o(x,\tilde{t};\tilde{t}) - G(x,\tilde{t})](t-t_1) \leq 0.$$

Therefore,

$$G(x,t) \leq \frac{G(x,t_1)}{t_1^\gamma} t^\gamma \leq -a_3 t^\gamma \quad \text{for a.e. } x \in \Omega,\ \text{all } t \geq t_1, \tag{33}$$

where, from condition (30), $a_3 = -t_1^{-\gamma} \sup_{x\in\Omega} G(x,t_1)$ is a positive number.

On the other hand, it follows from (27) and Lebourg's theorem that for a.e. $x \in \Omega$, all $t \geq 0$, $|G(x,t)| = |G(x,t) - G(x,0)| = |\xi t|$ for some $\xi \in \partial_s G(x,s)$, $0 \leq s \leq t$, and hence $|G(x,t)| \leq t(a_1 + a_2 s^{\alpha-1}) \leq (a_1 + a_2 t^{\alpha-1})t$. Thus,

$$|G(x,t)| \leq (a_1 + a_2 t_1^{\alpha-1})t, \tag{34}$$

for a.e. $x \in \Omega$, all $t \in [0, t_1]$. Choosing $a_4 = a_1 + a_2 t_1^{\alpha-1} + a_3 t_1^{\gamma-1} (> 0)$, we see that $|G(x,t)| \le a_4 t - a_3 t^\gamma$ for a.e. $x \in \Omega$, all $t \in [0, t_1]$. Combining this estimate with (33), we obtain (32).

Concerning Φ, we assume that

$$\Phi \text{ satisfies a } \Delta_2 \text{ condition (on } \mathbb{R}) \tag{35}$$

and

$$k \ge \sup_{t>0} \frac{\Phi(2t)}{\Phi(t)} (< \infty). \tag{36}$$

Moreover, suppose that

$$k_1 = \frac{\ln k}{\ln 2} < \alpha \tag{37}$$

(or equivalently, $k < 2^\alpha$).

The following theorem is our main existence result for nontrivial solutions of the inequality (4) (or the inclusion (1)–(2)).

Theorem 2. *Suppose Φ is a Young function satisfying a Δ_2 condition and the growth condition* (8)*. Assume G satisfies* (27), (30), *and* (31)*. Let k be given by* (36) *and assume that* (29) *and* (37) *hold. Under these assumptions, the variational-hemivariational inequality* (4) (*with J given by* (3)) *has a nontrivial solution.*

The proof of this result is an application of the Mountain Pass theorem (Corollary 1) stated in Section 2. Although following steps similar to those in the proof of the main theorem in [9], some different calculations and arguments are needed here due to the presence of the nonsmooth term G, and a complete proof of Theorem 2 is given below. We first recall the following lemma concerning an estimate of $\|u\|_\Phi$.

Lemma 1 ([9, Lemma 3])**.** *Assume Φ is a Young function that satisfies a Δ_2 condition on $\mathbb{R}$, that is $\Phi(2t) \le k\Phi(t)$, for all $t > 0$, for some $k > 1$. Then, for each $R > 0$, there is $c > 0$ such that for all u with $\|u\|_\Phi \le R$ or $\int_\Omega \Phi(u)\,dx \le R$, we have*

$$\int_\Omega \Phi(u)\,dx \ge c\|u\|_\Phi^{\frac{\ln k}{\ln 2}}. \tag{38}$$

Proof of Theorem 2. We consider the Banach space $X = W_0^1 L_\Phi$ with norm $\|\cdot\| = \|\cdot\|_{W_0^1 L_\Phi}$. Let us define the functional $I : W_0^1 L_\Phi \to \mathbb{R}$ by $I = J + P$, where J is given by (3) and

$$P(u) = \int_\Omega G(x,u)\,dx, \quad u \in W_0^1 L_\Phi.$$

It follows from (27) and Lebourg's theorem (see also (34)) that G has the growth

$$|G(x,s)| \le a_5 + a_6|s|^\alpha \quad \text{for a.e.} x \in \Omega, \text{ all } s \in \mathbb{R}. \tag{39}$$

Therefore, from the locally Lipschitz property of $G(x,\cdot)$, we see that P is locally Lipschitz on $L^\alpha(\Omega)$ and thus on $W_0^1 L_\Phi$, because of the embedding (28). We show that under the assumptions of Theorem 2, all the assumption of the Mountain Pass theorem in Section 3 (Corollary 1) are fulfilled.

Let us check the first condition (25) in Corollary 1. From (29), there are $s_1 \in (0,\infty)$ and $\epsilon_1 \in (0,\Lambda)$ such that

$$G(x,s) \ge (-\Lambda + \epsilon_1)\Phi(s) \tag{40}$$

for a.e. $x \in \Omega$, all s with $|s| < s_1$. From (39), there exists $a_7 > 0$ such that

$$|G(x,s)| \le a_7 |s|^\alpha \tag{41}$$

for a.e. $x \in \Omega$, all s with $|s| \ge s_1$. Combining (40) and (41), we have

$$G(x,s) \ge (-\Lambda + \epsilon_1)\Phi(s) - a_7|s|^\alpha,$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$. Therefore,

$$\begin{aligned} \int_\Omega G(x,u)\,dx &\ge (-\Lambda + \epsilon_1)\int_\Omega \Phi(u)\,dx - a_7\int_\Omega |u|^\alpha\,dx \\ &\ge \frac{-\Lambda + \epsilon_1}{\Lambda}\int_\Omega \Phi(|\nabla u|)\,dx - a_7 \int_\Omega |u|^\alpha\,dx, \end{aligned} \tag{42}$$

for all $u \in W_0^1 L_\Phi$. For simplicity, if there is no confusion, we shall in the sequel use C to denote a generic positive constant. By means of the continuous embedding from $W_0^1 L_\Phi$ into $L^\alpha(\Omega)$ and Lemma 1 with $R = 1$, we see that there is a constant $C > 0$ such that

$$\int_\Omega |u|^\alpha\,dx \le C\|u\|^\alpha_{W_0^1 L_\Phi} \le C\left[\int_\Omega \Phi(|\nabla u|)\,dx\right]^{\frac{\ln 2}{\ln k}\alpha}, \tag{43}$$

for all $u \in W_0^1 L_\Phi$ with $\|u\|_{W_0^1 L_\Phi} = r < 1$. Combining (42) with (43) yields the following estimate for $I(u)$:

$$\begin{aligned} I(u) &\ge \int_\Omega \Phi(|\nabla u|)\,dx - \frac{\Lambda - \epsilon_1}{\Lambda}\int_\Omega \Phi(|\nabla u|)\,dx - a_7\int_\Omega |u|^\alpha\,dx \\ &\ge \frac{\epsilon_1}{\Lambda}\int_\Omega \Phi(|\nabla u|)\,dx - a_7 C\left[\int_\Omega \Phi(|\nabla u|)\,dx\right]^{\frac{\ln 2}{\ln k}\alpha} \\ &= \left\{\frac{\epsilon_1}{\Lambda} - a_7 C\left[\int_\Omega \Phi(|\nabla u|)\,dx\right]^{\frac{\ln 2}{\ln k}\alpha - 1}\right\}\int_\Omega \Phi(|\nabla u|)\,dx. \end{aligned} \tag{44}$$

Since $\lim_{\|u\|\to 0}\big[\int_\Omega \Phi(|\nabla u|)\,dx\big]^{\frac{\ln 2}{\ln k}\alpha-1}=0$ (note that $\frac{\ln 2}{\ln k}\alpha-1>0$), we have, by choosing $r>0$ sufficiently small, the following estimate:

$$a_7 C\left[\int_\Omega \Phi(|\nabla u|)\,dx\right]^{\frac{\ln 2}{\ln k}\alpha-1}\le \frac{\epsilon_1}{2\Lambda},$$

for all $u\in W_0^1L_\Phi$ with $\|u\|=r$. From (44) and (38) (with u replaced by $|\nabla u|$) ,

$$I(u)\ge \frac{\epsilon_1}{2\Lambda}\int_\Omega \Phi(|\nabla u|)\,dx\ge \frac{\epsilon_1}{2\Lambda}C\|u\|^{\frac{\ln k}{\ln 2}}_{W_0^1L_\Phi}=\frac{C\epsilon_1}{2\Lambda}r^{\frac{\ln k}{\ln 2}}>0,$$

for all $u\in W_0^1L_\Phi$ with $\|u\|=r$. We have checked (25), so (i) of Corollary 1 holds here.

Now, let us check condition (ii) in Corollary 1. Let us fix a number $\gamma_0\in(1,\gamma)$ with $\gamma>1$ given in (31). From (8), there exists $T_0>0$ such that $\frac{t\Phi'(t)}{\Phi(t)}\le\gamma_0$, for all $t\ge T_0$. Hence,

$$\ln\left(\frac{\Phi(t)}{\Phi(T_0)}\right)=\int_{T_0}^t\frac{\Phi'(s)}{\Phi(s)}ds\le\int_{T_0}^t\frac{\gamma_0}{s}ds=\ln\left(\frac{t^{\gamma_0}}{T_0^{\gamma_0}}\right),\quad \forall t\ge T_0, \tag{45}$$

implying that $\Phi(t)\le\frac{\Phi(T_0)}{T_0^{\gamma_0}}t^{\gamma_0}$, for all $t\ge T_0$. Therefore, for some constants $C_1,C_2>0$ we have

$$\Phi(t)\le C_1t^{\gamma_0}+C_2,\quad \forall t\ge 0. \tag{46}$$

Let us fix $\phi_0\in C_0^1(\Omega)$ such that $\phi_0\ge 0$ on Ω and $\phi_0\ne 0$. For $\lambda>0$, let $u=u_\lambda=\lambda\phi_0(\ge 0)$. It follows from (46) and (32) the following estimates:

$$\begin{aligned}&I(u)\\ &\le\int_\Omega(C_1|\nabla u|^{\gamma_0}+C_2)\,dx+\int_\Omega(-a_3u^\gamma+a_4u)\,dx\\ &=-\lambda^\gamma\Big(a_3\int_\Omega\phi_0^\gamma\,dx-C_1\lambda^{\gamma_0-\gamma}\int_\Omega|\nabla\phi_0|^{\gamma_0}\,dx-C_2|\Omega|\lambda^{-\gamma}-a_4\lambda^{-\gamma}\int_\Omega\phi_0\,dx\Big).\end{aligned}\tag{47}$$

As $\lambda\to\infty$, $\lambda^{\gamma_0-\gamma},\lambda^{-\gamma}\to 0$ and the number in the parentheses tends to $a_3\int_\Omega\phi_0^\gamma\,dx$ (note that this number is strictly positive since $a_3>0$ and $\phi_0\ge 0$, $\phi\ne 0$ on Ω). Hence, the right hand side of (47) tends to $-\infty$ as $\lambda\to\infty$. For $\lambda>0$ sufficiently large, $I(u_\lambda)<0$ and u_λ is outside the ball centered at 0 with radius r.

We have checked both conditions in Corollary 1. By that result, there exist a sequence $\{u_n\}$ in $W_0^1L_\Phi$ and a sequence $\{\epsilon_n\}$ in $(0,\infty)$ such that $\epsilon_n\downarrow 0$, $I(u_n)\to c$ (c is given in (26)) and for every $n\in\mathbb{N}$,

$$\int_\Omega\Phi(|\nabla v|)\,dx-\int_\Omega\Phi(|\nabla u_n|)\,dx+P^o(u_n;v-u_n)\ge-\epsilon_n\|v-u_n\|,$$

for all $v \in W_0^1 L_\Phi$. From Aubin–Clarke's theorem (cf. [3]), we have that

$$P^o(u;v) \le \int_\Omega G^o(x,u(x);v(x))dx, \quad \forall u,v \in W_0^1 L_\Phi.$$

Hence,

$$\begin{aligned}&\int_\Omega \Phi(|\nabla v|)\,dx - \int_\Omega \Phi(|\nabla u_n|)\,dx + \int_\Omega G^o(x,u_n(x);v(x)-u_n(x))\,dx \\ &\ge -\epsilon_n\|v-u_n\|, \quad \forall v \in W_0^1 L_\Phi.\end{aligned} \tag{48}$$

We show that the sequence $\{u_n\}$ is bounded in $W_0^1 L_\Phi$. In fact, because $\gamma > 1$, we can choose $\gamma_0 \in (1,\gamma)$ sufficiently close to 1 such that $2^{\gamma_0} < \gamma + 1$. By using calculations as in (45), we have for all $t \ge T_0$,

$$\ln\left(\frac{\Phi(2t)}{\Phi(t)}\right) = \int_t^{2t} \frac{\Phi'(s)}{\Phi(s)}ds \le \int_t^{2t} \frac{\gamma_0}{s}ds = \gamma_0 \ln 2.$$

Hence, $\Phi(2t) \le 2^{\gamma_0}\Phi(t)$, for all $t \ge T_0$, and thus $\Phi(2t) \le 2^{\gamma_0}\Phi(t) + C_3$, for all $t \in \mathbb{R}$, with $C_3 = \sup\{\Phi(2t) : 0 \le t \le T_0\} \in (0,\infty)$. Hence,

$$\int_\Omega \Phi(2|\nabla u|)\,dx \le 2^{\gamma_0}\int_\Omega \Phi(|\nabla u|)\,dx + C_3|\Omega|, \quad \forall u \in W_0^1 L_\Phi. \tag{49}$$

Letting $v = 2u_n$ in (48) and using (49), one gets

$$\begin{aligned}(2^{\gamma_0}-1)\int_\Omega \Phi(|\nabla u_n|)\,dx + C_3|\Omega| &\ge \int_\Omega \Phi(2|\nabla u_n|)\,dx - \int_\Omega \Phi(|\nabla u_n|)\,dx \\ &\ge -\int_\Omega G^o(x,u_n(x);u_n(x))\,dx - \epsilon_n\|u_n\|.\end{aligned} \tag{50}$$

Without loss of generality, we can only consider n such that $\|u_n\| > 1$. It follows from [6, Theorem 9.5, Chapter 2], that

$$\int_\Omega \Phi(|\nabla u_n|)\,dx \ge \||\nabla u_n|\|_\Phi = \|u_n\|. \tag{51}$$

From (31) and (27), there exists a constant $a_8 > 0$ such that

$$G^o(x,t;t) \le \gamma G(x,t) + a_8, \quad \forall t \in \mathbb{R}. \tag{52}$$

From (50)–(52),

$$\begin{aligned}&(2^{\gamma_0}-1)\int_\Omega \Phi(|\nabla u_n|)\,dx + c_3|\Omega| \\ &\ge -\gamma\int_\Omega G(x,u_n)\,dx - a_8|\Omega| - \epsilon_n\int_\Omega \Phi(|\nabla u_n|)\,dx.\end{aligned} \tag{53}$$

On the other hand, since $I(u_n) \to c$,

$$\int_\Omega \Phi(|\nabla u_n|)\,dx + \int_\Omega G(x,u_n)\,dx = c + \delta_n, \tag{54}$$

with $\delta_n \to 0$ as $n \to \infty$. It follows from (53) and (54) that

$$(2^{\gamma_0} - 1 + \epsilon_n)\int_\Omega \Phi(|\nabla u_n|)\,dx \geq \gamma \int_\Omega \Phi(|\nabla u_n|)\,dx - \gamma(c+\delta_n) - (a_8 + c_3)|\Omega|.$$

It follows that

$$(\gamma + 1 - 2^{\gamma_0} - \epsilon_n)\int_\Omega \Phi(|\nabla u_n|)\,dx \leq \gamma(c + \sup \delta_n) + (a_8 + c_3)|\Omega| < \infty, \tag{55}$$

for all $n \in \mathbb{N}$. On the other hand, by the choice of γ_0, because $\epsilon_n \to 0$ as $n \to \infty$, we have $\gamma + 1 - 2^{\gamma_0} - \epsilon_n \geq \epsilon_0 > 0$, for all n large. Estimate (55) shows that the sequence $\left\{\int_\Omega \Phi(|\nabla u_n|)\,dx\right\}$ is bounded, implying the boundedness of $\{u_n\}$ in $W_0^1 L_\Phi$ (see (51)).

Since $\{u_n\}$ is bounded in $W_0^1 L_\Phi$, by passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup^* u$ in $W_0^1 L_\Phi$. This implies that $\int_\Omega u_n \phi\,dx \to \int_\Omega u\phi\,dx$ and $\int_\Omega \partial_i u_n \phi\,dx \to \int_\Omega \partial_i u \phi\,dx$, for all $\phi \in E_{\overline{\Phi}}$, and thus for all $\phi \in L^\infty(\Omega)$, i.e., $\nabla u_n \rightharpoonup \nabla u$ in $[L^1(\Omega)]^N$(-weak). Since the function $\xi \mapsto \Phi(|\xi|)$ is convex, continuous on $\mathbb{R}^N$ and $\Phi(|\xi|) \geq 0$, for all $\xi \in \mathbb{R}^N$, we have

$$\int_\Omega \Phi(|\nabla u|)\,dx \leq \liminf \int_\Omega \Phi(|\nabla u_n|)\,dx, \tag{56}$$

(cf. e.g. [2]). Put $\Phi_1^*(t) = |t|^{\frac{N}{N-1}}$, $t \in \mathbb{R}$. (We assume here that $N > 1$, thus $p^* = \frac{N}{N-1}$ is the Sobolev conjugate exponent of $p = 1$ and Φ_1^* is a Young function; trivial modifications are needed for the case $N = 1$.) Straightforward calculations show that $\Phi_1^* \ll \Phi^*$ (Φ^* is the Sobolev conjugate of Φ). Consequently, the embedding $W_0^1 L_\Phi \hookrightarrow L_{\Phi_1^*}(= L^{N/(N-1)}(\Omega))$ is compact. It follows that $u_n \to u$ in $L^{N/(N-1)}(\Omega)$ and thus in $L^\alpha(\Omega)$. From the growth condition (27) and Fatou's lemma, we see that

$$\limsup \int_\Omega G^o(x, u_n(x); v(x) - u_n(x))\,dx \leq \int_\Omega G^o(x, u(x); v(x) - u(x))\,dx. \tag{57}$$

Also, since the sequence $\{u_n\}$ is bounded in $W_0^1 L_\Phi$, we have

$$\epsilon_n \|v - u_n\| \to 0. \tag{58}$$

Letting $n \to \infty$ in (48) and noting (56), (57), and (58), we have that u is a solution of (4). Note that $u \neq 0$. In fact, suppose by contradiction that $u = 0$. Letting $v = 0$ in (48), one gets

$$\int_\Omega \Phi(|\nabla u_n|)\,dx \leq \int_\Omega G^o(x, u_n; -u_n)\,dx + \epsilon_n \|u_n\|, \quad \forall n \in \mathbb{N}.$$

From (57), we have $\lim_{n\to\infty} \int_\Omega \Phi(|\nabla u_n|)\,dx = 0$. Also, $G(\cdot, u_n) \to G(\cdot, 0) = 0$ in $L^1(\Omega)$ and thus $\int_\Omega G(x, u_n)\,dx \to 0$. Hence,

$$I(u_n) = \int_\Omega \Phi(|\nabla u_n|)\,dx + \int_\Omega G(x, u_n)\,dx \to 0$$

as $n \to \infty$. This contradicts (12) and (26) and completes our proof. □

Let us conclude our paper with some further remarks.

Remark 2. Assumption (31) is an adaptation of the classical "super-quadratic" condition in applications of the Mountain Pass theorem to our nonsmooth problem in Orlicz–Sobolev space, which is in fact a "super-linear" condition (because γ is only assumed to be greater than 1) for generalized directional derivative here.

A point worth mentioning is that in applications of the Ambrosetti–Rabinowitz theorem to (even smooth) boundary value problems, sign conditions similar to (30) are considered (see e.g. condition (p4) in [14, Section 2]). However, those sign conditions are usually imposed for all large values of t . Here in (30) this sign condition is assumed at only one value t_1 of t. Condition (30) is essential for the constant a_3 in estimates (33) and thus (32) to be strictly positive. This sign property plays a crucial role in estimate (47) to conclude that the limit value of $I(u_\lambda)$ is $-\infty$ as $\lambda \to +\infty$. The negative value of $I(u_\lambda)$, in its turn, contributes in an essential way to one of the two geometric conditions of the Mountain Pass theorem.

Remark 3. Related to the assumptions on Φ, we note that with Φ given by (6) or (7), since the function $t\frac{\Phi'(t)}{\Phi(t)}$ is decreasing on $(0, \infty)$, we have

$$\sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)} = \lim_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)} = 1 + \beta = k_1.$$

Hence, one has (35) and (36) with $k = 2^{1+\beta}$. If Φ is given by (6) or (7), then (37) holds if $1 + \beta < \alpha$.

Remark 4. Using similar arguments as in [9], we can extend the above results and assumptions to variational-hemivariational inequalities that contains locally Lipschitz lower order terms and principal terms defined by Young functions Φ with "not very fast" growth (i.e., when Φ satisfies a Δ_2 condition, but $\overline{\Phi}$ may or may not satisfy this condition). Examples of such functions are

$$\Phi(t) = \int_0^{|t|} s^{p-1}[\ln(1+s)]^\beta\,ds, \quad \text{or} \quad \Phi(t) = |t|^p[\ln(1+|t|)]^\beta, \quad t \in \mathbb{R},$$

with $\beta > 0, p \geq 1$. Since these functions Φ are not equivalent to any power functions t^p ($p \in [1, \infty]$), the regular setting in ordinary Sobolev spaces seem not suitable for such inequalities.

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