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On the Mathematical Analysis and Numerical Approximation of a System of Nonlinear Parabolic PDEs

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Abstract. In this paper we consider a boundary value problem for a system of 2 nonlinear parabolic PDEs e.g. arising in the context of flow and transport in porous media. The flow model is based on tho nonlinear Richard's equation problem and is combined with the transport equation through saturation and Darcy's velocity (discharge) terms. The convective terms are approximated by means of the method of characteristics initiated by P. Pironneau [Num. Math. 38 (1982), 871–885] and R. Douglas and T. F. Russel [SIAM J. Num. Anal. 19 (1982), 309–332]. The nonlinear terms in Richard's equation are approximated by means of a relaxation scheme applied by W. Jäger and J. Kačur [RAIRO Model. Math. Anal. Num. 29 (1995), 605–627] and J. Kačur [IMA J. Num. Anal. 19 (1999), 119–154; SIAM J. Num. Anal. 39 (1999), 290–316]. The convergence of the approximation method is proved.

Keywords. Relaxation method, method of characteristics, contaminant transport, convection-diffusion with adsorption

Mathematics Subject Classification (2000). 65M25,65M12

1. Introduction

1.1. The mathematical model. In this paper we deal with the mathematical analysis and with a new numerical approximation scheme of a coupled system of nonlinear parabolic PDEs for a couple of space and time dependent functions $[u, w]$, viz

$$\partial_t b(u) - \operatorname{div}(\bar{F}(x,u) + \bar{A}(x)\nabla u) = f(t,x) \tag{1.1}$$

$$b(u)\partial_t w + \bar{v}(u,\nabla u).\nabla w - \operatorname{div}(D(u,\nabla u)\nabla w) = G(t,x,w), \tag{1.2}$$

$$\text{in } \ (t,x) \in I \times \Omega \equiv Q_T, \quad I = (0,T),\, T < \infty,$$

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along with the boundary conditions

$$u = u^D \quad \text{on } I \times \partial\Omega \tag{1.3}$$

$$w = w^D \quad \text{on } I \times \Gamma_1, \qquad -D(u, \nabla u)\nabla w \cdot \bar{\nu} = 0 \quad \text{on } I \times \Gamma_2, \tag{1.4}$$

($\bar{\nu}$ is the unit outward normal vector to $\partial\Omega$) and along with the initial conditions

$$u = u_0 \quad \text{on } \{0\} \times \Omega \ , \qquad w = w_0 \quad \text{on } \{0\} \times \Omega. \tag{1.5}$$

Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$, Γ_1 and Γ_2 are open parts of $\partial\Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\operatorname{meas}\Gamma_1 + \operatorname{meas}\Gamma_2 = \operatorname{meas}\partial\Omega$. We assume that the scalar function b is positive and satisfies $0 < \delta \le b'(s) \le M < \infty$ for some $M > \delta > 0$, for all $s \in \mathbb{R}$. Furthermore, we assume that $D \equiv D(u, \nabla u)$ is a symmetric positive definite matrix with $\|D(\eta, \xi)\| \le C(1 + |\eta| + |\xi|)$, for all $\eta \in \mathbb{R}$, for all $\xi \in \mathbb{R}^3$. Moreover, we assume that the vector functions $\bar{v}$ and $\bar{F}$ in $\mathbb{R}^3$ obey $|\bar{v}(\eta, \xi)| \le C(1 + |\eta| + |\xi|)$, for all $\eta \in \mathbb{R}$, for all $\xi \in \mathbb{R}^3$ and $|\partial_s \bar{F}(x, s)| \le C$, for all $x \in \Omega$ and for all $s \in \mathbb{R}$. Furthermore, we let $|G(t, x, s)| \le C(1 + |s|)$ for all $t \in I$, for all $x \in \Omega$ and for all $s \in \mathbb{R}$. Here C is a generic constant. Next, the matrix $\overline{A}(x) \in \mathbb{R}^{3\times3}$ is only space dependent. By means of u^D and w^D we prescribe the Dirichlet boundary conditions and we assume that these functions are defined on Q_T.

1.2. A motivating physical example. The mathematical model (1.1)–(1.5) arises e.g. in the modeling of contaminant transport in porous media with unsaturated-saturated flow. In the terminology used further we will stick to this physical context throughout the paper. The flow in an unsaturated porous media Ω is governed by Richard's equation – see [6, 12, 20] etc.:

$$\partial_t \theta = \operatorname{div}(k(h)\bar{A}(x)\nabla(h + Z)), \quad x = (X, Y, Z), \tag{1.6}$$

where θ is the volumetric water content, h is the pressure head, $k(h)$ is the hydraulic permeability, $\bar{A}(x)$ is the permeability in saturated porous media and the gravitation is in the direction of the Z-axis. We can use the Van Genuchten–Mualem model for the retention and permeability curves:

$$\theta = \theta(h) = \theta_r + \frac{\theta_s - \theta_r}{(1 + (\alpha h)^n)^m}$$

$$\widetilde{k}(S) = S^{\frac{1}{2}}\Big(1 - \big(1 - S^{\frac{1}{m}}\big)^m\Big)^2, \qquad S = \frac{\theta - \theta_r}{\theta_s - \theta_r},$$

where S is the effective saturation and $k(h) = \widetilde{k}(\theta(h))$ for $h < 0, k(h) = 1$ for $h \ge 0$. Here, $0 < \theta_r, \theta_s < 1, 1 < n,\ m = 1 - \frac{1}{n},\ \alpha < 0$ are so called soil parameters. The flow in the saturated region $h \ge 0$ is governed by Darcy's law – see [6]:

$$S_e \partial_t h - \operatorname{div}(\bar{A}(x)\nabla(h + Z)) = 0,$$

where S_e is the specific (elastic) storativity coefficient and $\theta = S_e h$. Thus, we can extend (1.6) from the unsaturated to the saturated zone. The Van Genuchten model reflects two fronts (free boundaries) of degeneracy. The first front occurs at the interface between dry ($S = 0$) and wet ($S > 0$) zones. The second front occurs at the boundary between saturated ($h \geq 0$) and unsaturated ($-\infty < h < 0$) zones.

Using Kirchoff's transformation we can transfer all nonlinearities to the parabolic term θ in (1.6). We introduce the new unknown u and the function $b(u)$ by $u := \beta(h) = \int_0^h k(s)ds$, $b(u) := \theta(\beta^{-1}(u))$. Then, we can rewrite (1.6) into the form

$$\partial_t b(u) - \mathrm{div}(\bar{A}(x)\nabla u) - \mathrm{div}\bar{K}(x, b(u)) = 0, \tag{1.7}$$

where $\bar{K}(x, b(u)) = \bar{A}(x)\widetilde{k}(b(u)).\bar{e}_Z$, $\bar{e}_Z$ being the unit vector in direction Z. The unknown u varies in $(u^\star, 0)$ with $u^\star > -\infty$. The same transformation can be used for $h > 0$. We obtain $u = h, b(u) = S_e u$. Then, (1.7) is of the form (1.1). We can verify that $|\partial_s \bar{K}(x, b(s))| \leq C$ for $n \geq 2$ (in Van Genuchten's model) and $b'(u^\star) = \infty, b'(0) = 0$ so that b is not globally Lipschitz continuous. If the data's (initial wetness, boundary conditions) guarantee that $h \geq h_0 > -\infty$, i.e., $u \geq u^\star + \varepsilon$ for some $\varepsilon > 0$, then we can assume that b is Lipschitz continuous, since $b'(s)$ is decreasing for $0 \geq u \geq u^\star$. This assumption is based on the comparison principle for the problem (1.1),(1.3),$(1.5)_1$, proved in [1, Theorem 2.2]. To guarantee that $0 < \delta \leq b'(s)$ we have to regularize $\theta(h)$ and $\widetilde{k}(S)$ in a small neighbourhood of $h = 0$ and $S = 1$, respectively. In that case we can take $\theta_\eta(h) = \theta(h)$ and $\widetilde{k}_\eta(S) = \widetilde{k}(S)$ for $h \in (-\infty, -\eta)$ and $S \in (0, 1-\eta)$. We extend the graphs of θ_η and $\widetilde{k}_\eta$ by lines connecting the points $(-\eta, \theta(-\eta))$ and $(0, \theta_s)$ and $(-\eta, \widetilde{k}(-\eta))$ and $(0, 1)$, respectively. This regularization (for any small η) will guarantee that $b'(s) \geq \delta > 0$ and $|\partial_s \bar{K}(x, b(s))| \leq C < \infty$. We can expect the solution u_η to this regularized problem to converge to u as $\eta \to 0$. This has been discussed (with a similar regularization) in [2]. The transport equation for the contaminant is considered in the form – see e.g. [6, 20, 24]:

$$\partial_t(\theta w) + \mathrm{div}(\bar{v}w - D\nabla w) = G(t, x, w),$$

where $\bar{v} = -(\bar{A}\nabla u + \bar{K}(x, b(u))$ is Darcy's velocity . The matrix D is of the form – see [6]:

$$D_{ij} = \{D_{\mathrm{mol}} + \alpha_T|\bar{v}|\}\delta_{ij} + (\alpha_L - \alpha_T)\frac{v_i v_j}{|\bar{v}|}\}, \quad i, j = 1, \ldots, 3,$$

where D_{mol} is the molecular diffusivity of the contaminant in the fluid and α_T and α_L are the transversal and longitudinal length's, respectively. On account of Richard's equation , we can rewrite the transport equation in the form

$$\theta\partial_t w + \bar{v}\nabla w - \mathrm{div}(D\nabla w) = G(t, x, w),$$

which is of the type (1.2).

1.3. Basic ideas of a new method of discretization in time. Evidently, the problem (1.1)–(1.5) cannot be solved exactly. So far, the convergence of a semi-discrete numerical method, viz a method of discretization in time, has been studied separately for the flow and for the transport problems. The convergence of the numerical solution for the flow problem has been discussed in [12], where Lipschitz continuity of $b(u)$ has been assumed. The convergence of the numerical solution of the contaminant transport (even with adsorption) has been discussed in [4, 5, 8, 9, 19, 24], among others. In the papers mentioned it was substantially assumed that the flow velocity $\bar{v}$ and the saturation $\theta \geq \theta_0 > 0$ are Lipschitz continuous. However, this assumption cannot be guaranteed for θ and $\bar{v}$ generated by the solution u of Richard's equation. Moreover, in order to establish a practical numerical approximation, the coupled flow-transport problem is not splitted into the flow and next into the transport problem, but we solve them simultaneously using a time stepping procedure. The exact mathematical proof of the convergence of such numerical approximation is discussed in the present paper. This proof is not a straightforward combination of the technical tools which we have developed in [18, 19]. Indeed, we do not a priori assume regularity of the velocity field governed by the flow problem. In our concept of numerical approximation of (1.1)–(1.5) we control the convective terms by the method of characteristics, initiated in [23] and [10] and dynamically developed in the last decade in [4, 7, 9, 11, 18], among others. We follow the idea of [18] using a regularization of the approximated characteristics. The nonlinear parabolic term in (1.1) is controlled by the relaxation method developed in [14, 16, 17]. Let $u_i \approx u(x,t_i), w_i \approx w(x,t_i)$ be approximations on the time level $t = t_i = i\tau$, where $\tau = \frac{T}{n}$ is the time step and $i = 1, \ldots, n$ ($n \in \mathbb{N}$). If $\bar{V}(x,t)$ is a velocity field, then the characteristics (in the time interval (t_{i-1}, t_i)) are the curves governed by the ODE

$$\frac{d\mathcal{H}(s;t_i,x)}{ds} = \bar{V}(\mathcal{H}(s;t_i,x), s),$$

with the initial condition $\mathcal{H}(t_i;t_i,x) = x$. Then, we denote $\widetilde{\varphi}^i(x) \equiv \mathcal{H}(t_{i-1};t_i,x)$. Let $\bar{\varphi}^i$ be the (backward Euler) approximation of $\widetilde{\varphi}^i(x)$ expressed in the form $\bar{\varphi}^i(x) := x - \tau\bar{V}(x,t_i)$. The convection-diffusion process described by (1.1) and (1.2) is approximated in the time interval (t_{i-1}, t_i) by superposition of the transport (convection) and the diffusion (without convection). The transport part can be realized by shifting of u_{i-1} and w_{i-1} along the corresponding characteristic, respectively. For this purpose it is crucial that the characteristics (which are also approximated) are not intersecting along $t \in (t_{i-1}, t_i)$ for all $i = 1, \ldots, n$. This in turn requires that the maps $\bar{\varphi}^i$ and their inverse functions are Lipschitz continuous. To guarantee this, we introduce the regularized velocity field

$$\bar{V}_{h,i} := \omega_h * \bar{V}_i \quad (t = t_i)$$

and put $\varphi^i(x) := x - \tau\bar{V}_{h,i}(x)$, where ω_h is the mollifier. We can take $\omega_h(x) = \frac{1}{h^3}\omega_1\left(\frac{x}{h}\right)$, with

$$\omega_1(x) = \frac{1}{\kappa}\begin{cases}\exp\left(\frac{|x|^2}{|x|^2-1}\right) & |x| \le 1\\ 0, & |x| \ge 1,\end{cases}$$

where κ is a scaling parameter such that $\int_{\mathbb{R}^3}\omega_1(x)\,dx = 1$. Recall that the convolution is defined by $g * z(x) = \int_{\mathbb{R}^3} g(x-\xi)z(\xi)\,d\xi$. We will take $h = \tau^\rho$ with a fixed parameter $\rho \in (0, \frac{2}{3})$. Then, if the $L_2(\Omega)$-norm of $\bar{V}$ is bounded (uniformly for $t \in I$) we will prove that the maps φ^i and their inverse are one to one and Lipschitz continuous maps for all $i = 1, \ldots, n$, provided $\tau \le \tau_0$. The transport parts (without diffusion) in (1.1)–(1.2) can be approximated by $u_{i-1}\circ\varphi_1^i \equiv u_{i-1}(\varphi_1^i(x))$ and $w_{i-1}\circ\varphi_2^i$, respectively, where φ_1^i and φ_2^i correspond to

$$\bar{V}_{1,i} \equiv \left.\frac{\partial_s\bar{F}(x,s)}{b'(s)}\right|_{s=u_{i-1}} \qquad\text{and}\qquad \bar{V}_{2,i} \equiv \frac{\bar{v}(u_i, \nabla u_i)}{b(u_i)}.$$

Then we introduce new approximate solutions u_i and w_i on the time level $t = t_i$ by means of the elliptic equations

$$\lambda_i\frac{u_i - u_{i-1}\circ\varphi_1^i}{\tau} - \operatorname{div}(\bar{A}\nabla u_i) = f_i - \operatorname{div}_x\bar{F}(x, u_{i-1}) \tag{1.8}$$

and

$$b(u_i)\,\frac{w_i - w_{i-1}\circ\varphi_2^i}{\tau} - \operatorname{div}(D_i^L\nabla w_i) = G(t_i, x, w_{i-1}). \tag{1.9}$$

Here, $0 \le \lambda_i \in L_\infty(\Omega)$ is a relaxation function which has to satisfy the "convergence condition"

$$\left|\lambda_i - \frac{b(u_i) - b(u_{i-1}\circ\varphi_1^i)}{u_i - u_{i-1}\circ\varphi_1^i}\right| \le \tau. \tag{1.10}$$

Moreover, let $L > 0$ be a truncation parameter. We define a truncation function σ_L

$$D_i^L := D(\sigma_L(u_i), \sigma_L(\nabla u_i)), \quad \sigma_L(s) := \min\left\{1, \tfrac{L}{|s|}\right\}s \tag{1.11}$$

Along with (1.8)–(1.9) we consider the boundary conditions

$$u_i = u_i^D \quad \text{on } \partial\Omega \tag{1.12}$$

$$w_i = w_i^D \quad \text{on } \Gamma_1, \qquad D_i^L\nabla w_i\cdot\nu = 0 \quad \text{on } \Gamma_2. \tag{1.13}$$

The approximation scheme (1.8)–(1.13) is implicit, since λ_i is related to the unknown function u_i in (1.10). We can determine u_i and w_i from (1.8)–(1.13) by using relaxation iterations in (1.8)–(1.10) as follows. We define $u_{i,k}$, $(k = 1, \ldots)$, by means of the elliptic equation

$$\lambda_{i,k-1}\frac{u_{i,k} - u_{i-1}\circ\varphi_1^i}{\tau} - \operatorname{div}(\bar{A}\nabla u_{i,k}) = f_i - \operatorname{div}_x\bar{F}(x, u_{i-1}), \tag{1.14}$$

along with the boundary condition (1.12) where we put $u_{i,k}$ instead of u_i. Next, we take

$$\lambda_{i,k} := \frac{b(u_{i,k}) - b(u_{i-1} \circ \varphi_1^i)}{u_{i,k} - u_{i-1} \circ \varphi_1^i} .$$

If $|\lambda_{i,k_0} - \lambda_{i,k_0-1}| < \tau$, we put $\lambda_i := \lambda_{i,k_0-1}$ and $u_i := u_{i,k_0}$. The efficiency of the relaxation method has been discussed in [3, 13, 14, 17] among others.

1.4. Outline of the paper. In Section 2 we first state the precise assumptions on the data. Next, we introduce a truncated problem P^L related to the original problem (1.1)–(1.5), denoted by problem P. We then show the existence and uniqueness of a suitably introduced variational solution. In Section 3 some auxiliary results are proved, such as the bijective property of the characteristic map φ^i introduced in Section 1.3. We also prove the L_∞-boundedness of the sequences $\{u_i\}$ and $\{w_i\}$ obtained from (1.8)–(1.13) by means of the relaxation iterations based on (1.14). In Section 4 we prove the convergence of the semi-discrete method, described in Section 1.3. In Section 5 we briefly discuss a fully discrete approximation method, which is obtained when the elliptic boundary value problems, that arise at each time point $t = t_i$ from the discretization in time, are approximated by passing to suitable finite dimensional spaces, such as finite element spaces, in the variational formulation.

2. Variational formulation, existence and uniqueness

2.1. Assumption, notations and definitions. Let C denotes a generic positive constant. We shall assume:

- H$_1$) $b(s) \geq \delta > 0$ is Lipschitz continuous, satisfying $0 < \delta < b'(s) \leq M < \infty$ for small δ;
- H$_2$) $\bar{F}(x,s)$ is continuous and satisfies $|\partial_s \bar{F}(x,s)| \leq C$ and $|\mathrm{div}_x F(x,s)| \leq C(1+|s|)$ in $\Omega \times \mathbb{R}$;
- H$_3$) $\bar{v}(\eta,\xi)$ is continuous and satisfies $|\bar{v}(\eta,\xi)| \leq C(1+|\eta|+|\xi|)$ on $\mathbb{R} \times \mathbb{R}^3$; $D(x,\zeta,\xi)$ is continuous, it is a symmetric 3×3 matrix and satisfies $\|D(x,\eta,\xi)\| \leq C(1+|\eta|+|\xi|)$ in $\Omega \times \mathbb{R} \times \mathbb{R}^3$;
- H$_4$) $(\bar{A}(x)\xi,\xi) \geq C_A|\xi|^2$; $(D(x,\zeta,\xi)\eta,\eta) \geq C_D|\eta|^2$ for all $\zeta \in \mathbb{R}$, for all η, $\xi \in \mathbb{R}^3$ and for all $x \in \Omega$;
- H$_5$) G and f are continuous functions in their variables and $|\partial_s G(t,x,s)| \leq C$; f, $\partial_t f \in L_2(I,L_2)$;
- H$_6$) u_0, $w_0 \in L_\infty(\Omega) \cap W_2^1(\Omega)$; u^D, $w^D \in L_\infty(I,W_\infty^1)$; $\partial_t u^D$, $\partial_t w^D \in L_2(I,W_2^1)$.

We introduce the following subspaces of the Sobolev space W_2^1: $V_1 = \{v \in W_2^1 : v = 0 \text{ on } \partial\Omega\}$, $V_2 = \{v \in W_2^1 : v = 0 \text{ on } \Gamma_1\}$. By $L_\infty, L_2, W_2^1, L_2(I,L_2) \equiv$

$L_2(I \times \Omega)$ and $L_2(I, V_l)$, $(l = 1, 2)$, we denote standard functional spaces – see [21]. We denote $(u, v) = \int_\Omega uv\,dx$, $(u, v)_\Gamma = \int_\Gamma uv\,dx$ $(\Gamma \subset \partial\Omega)$. By $\|\cdot\|_\infty, \|\cdot\|_0, \|\cdot\|, \|\cdot\|_\Gamma$ and $\|\cdot\|_*$ we denote the norms in $L_\infty(\Omega)$, $L_2(\Omega)$,$W_2^1(\Omega)$, $L_2(\Gamma)$ and V_l^*, respectively. Here V_l^* is the dual space to $V_l, (l = 1, 2)$, and $\langle w, v\rangle$ is the duality pairing between V_2^* and V_2. The model problem (1.1)–(1.5) is called Problem P, while Problem P^L will denote the truncated problem, where the matrix D is replaced by the truncated matrix D^L, as defined in Section 1.3.

Definition 2.1. $\{u, w^L\}$ is a variational solution to Problem P^L, iff

i) $u - u^D \in L_\infty(I, V_1)$, $u \in L_\infty(Q_T)$, $\quad \partial_t b(u) \in L_2(I, L_2)$,
$w - w^D \in L_2(I, V_2)$, $w \in L_\infty(Q_T)$, $\partial_t(b(u)w) \in L_2(I, V_2^*)$;

ii) the following identities hold:

$$\begin{aligned} &\int_I \big(\partial_t b(u), \xi\big)dt + \int_I \big(\bar{A}\nabla u + \bar{F}(t, u), \nabla\xi\big)dt \\ &= \int_I (f, \xi)dt, \quad \forall \xi \in L_2(I, V_1) \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\int_I \langle \partial_t(b(u)w), \eta\rangle dt - \int_I \big(\partial_t b(u)w, \eta\big)dt \\ &+ \int_I \big(\bar{v}(u, \nabla u)\nabla w, \eta\big)dt + \int_I \big(D^L(u, \nabla u)\nabla w, \nabla\eta\big)dt \\ &= \int_I \big(G(t, x, w), \eta\big)dt, \quad \forall \eta \in L_2(I, V_2) \cap L_\infty(Q_T) \end{aligned} \tag{2.2}$$

iii) $u(0) = u_0$ in L_2 and

$$\begin{aligned} &\int_I \langle \partial_t(b(u)w), \zeta\rangle dt = -\int_I \int_\Omega (b(u)w - b(u_0)w_0)\partial_t\zeta dx\,dt \\ &\qquad \forall \xi \in L_2(I, V_2) \;\text{ with }\; \partial_t\zeta \in L_\infty(Q_T),\ \zeta(x, T) = 0\,. \end{aligned} \tag{2.3}$$

Definition 2.2. $\{u, v\}$ is a variational solution to Problem P, iff

i) condition i) and identity (2.1) from Definition 2.1 are satisfied;

ii) the following identity holds:

$$\begin{aligned} &-\big(b(u_0)w_0, \eta(0)\big) - \int_I \big(\partial_t b(u)w, \eta\big) - \int_I \big(b(u)w, \partial_t\eta\big)dt \\ &+ \int_I \big(\bar{v}(u, \nabla u)\nabla w, \eta\big)dt + \int_I \big(D(u, \nabla u)\nabla w, \nabla\eta\big)dt \\ &= \int_I \big(G(t, x, w), \eta\big)dt \\ &\forall \eta \in L_\infty(I, V_2) \cap L_\infty(Q_T) \quad \text{with } \;\partial_t\eta \in L_2(I, L_2), \eta(x, T) = 0. \end{aligned} \tag{2.4}$$

Remark 2.3. Existence and uniqueness of the solution to the flow problem (1.1), (1.3) and (1.5) is guaranteed by [1]. The uniqueness of the solution to the transport problem has been analysed in [20] under the assumption that D is a time independent matrix. On the other hand a more general structure of the adsorption term has been considered there.

We shall prove the uniqueness of the variational solution (in the sense of Definition 2.1) to the Problem P^L. We need an integration by parts formula.

Lemma 2.4. *If* $\{u, w\}$ *are as in Definition* 2.1, *then*

$$\begin{aligned}&\int_0^t \langle \partial_t(b(u)w), w\rangle dt \\ &= \frac{1}{2}\int_\Omega b(u(t))w^2(t)\,dx - \frac{1}{2}\int_\Omega b(u_0)w_0^2\,dx + \frac{1}{2}\int_0^t\int_\Omega \partial_t b(u)w^2\,dx\,dt.\end{aligned} \tag{2.5}$$

Proof. Since $\partial_t(b(u)w) \in L_2(I, V_2^*)$ we have (see, e.g., [15])

$$\frac{b(u(\cdot+h))w(\cdot+h) - b(u(\cdot))w(\cdot)}{h} \to \partial_t(b(u)w) \quad \text{in } L_2(I, V_2^*) \text{ for } h \to 0.$$

Moreover, from $\partial_t b(u) \in L_2(I, L_2)$ and $w \in L_\infty(Q_T)$ we obtain also

$$\frac{b(u(\cdot+h))w(\cdot) - b(u(\cdot))w(\cdot-h)}{h} \to \partial_t(b(u)w) \quad \text{in } L_2(I, V_2^*), \tag{2.6}$$

since $\frac{b(u(\cdot+h))-b(u(\cdot))}{h}w(\cdot) - \frac{b(u(\cdot))-b(u(\cdot-h))}{h}w(\cdot-h) \to 0$ for $h \to 0$ in $L_2(I, L_2)$ and hence also in $L_2(I, V_2^*)$. Introduce

$$J_h = \int_0^t\int_\Omega \frac{b(u(t+h))w(t+h) - b(u(t))w(t)}{h} w(t)dx\,dt.$$

We have

$$\begin{aligned}J_h :=& -\int_0^t \left(b(u(t))w(t), \frac{w(t)-w(t-h)}{h}\right)dt \\ &- \frac{1}{h}\int_0^h\int_\Omega b(u(t))w(t)w(t-h)\Big)\,dx\,dt \\ &+ \frac{1}{h}\int_t^{t+h}\int_\Omega b(u(t))w(t)w(t-h)\,dx\,dt\,,\end{aligned} \tag{2.7}$$

where we used the extensions $u(s) = u_0$ and $w(s) = w_0$ for $s \in (-h, 0)$. For the first term we have

$$\begin{aligned}\tilde{J}_h :=& -\int_0^t\int_\Omega \frac{w^2(t) - w(t)w(t-h)}{h} b(u(t))\,dx\,dt \\ =& -\int_0^t\int_\Omega \frac{w^2(t) - w^2(t-h)}{h} b(u(t))\,dx\,dt \\ &+ \int_0^t\int_\Omega \frac{w(t)-w(t-h)}{h} b(u(t))w(t-h)\,dx\,dt \\ \equiv&\ J_{1,h} + J_{2,h}\,.\end{aligned} \tag{2.8}$$

Similarly as J_h we may rearrange $J_{1,h}$ and $J_{2,h}$ in the forms

$$\begin{aligned} J_{1,h} = &\int_0^t \int_\Omega \frac{b(u(t+h)) - b(u(t))}{h} w^2(t)\, dx\, dt \\ &- \frac{1}{h}\int_{t-h}^t \int_\Omega b(u(t+h))w^2(t)\, dx\, dt + \frac{1}{h}\int_{-h}^0 \int_\Omega b(u(t+h))w^2(t)\, dx\, dt, \end{aligned}$$

and

$$\begin{aligned} J_{2,h} = &- \int_0^t \int_\Omega \frac{b(u(t+h))w(t) - b(u(t))w(t-h)}{h} w(t)\, dx\, dt \\ &+ \frac{1}{h}\int_{t-h}^t \int_\Omega b(u(t+h))w^2(t)\, dx\, dt + \frac{1}{h}\int_0^{-h} \int_\Omega b(u(t+h))w^2(t)\, dx\, dt. \end{aligned}$$

We substitute $J_{1,h}$ and $J_{2,h}$ into (2.7) and make use of (2.6). Then, taking the limit for $h \to 0$, we arrive at (2.5). □

Theorem 2.5. *If* $\Gamma_1 = \partial\Omega$ *and* $f \in L_\infty$*, then there exists a unique variational solution (in the sense of Definition* 2.1*) to Problem* P^L*.*

Proof. The uniqueness of the variational solution u is guaranteed by [1]. The uniqueness of the variational solution $w = w^L$ is obtained in the following way. Let w_1 and w_2 be two variational solutions. Then, for $w = w_1 - w_2$, we obtain from (2.2)

$$\begin{aligned} &\int_0^t \langle \partial_t(b(u)w), w\rangle\, dt - \int_0^t (\partial_t b(u)w, w)\, dt \\ &+ \int_0^t \big(\bar{v}(u, \nabla u)\nabla w, w\big) dt + \int_0^t \big(D^L(u, \nabla u)\nabla w, \nabla w\big) dt \\ &= \int_0^t \big(G(t, x, w_1) - G(t, x, w_2), w\big) dt\,. \end{aligned} \tag{2.9}$$

We use the integration by parts formula (2.5). Notice that $w_0 = w(0) = 0$. Invoking hypotheses H_3–H_5 we obtain from (2.9) and (2.1) (with $\xi = w^2$) that

$$\begin{aligned} \frac{1}{2}\int_\Omega b(u(t))w^2(t)\, dx \le &\int_0^t \int_\Omega |G'(w_1 + r(x)(w_2 - w_1)| w^2 dx\, dt \\ &+ \int_0^t \int_\Omega |f| w^2 dx\, dt \\ \le &\; C \int_0^t \int_\Omega \frac{1}{b(u)} b(u) w^2 dx\, dt \\ \le &\; C \int_0^t \int_\Omega b(u) w^2 dx\, dt, \end{aligned}$$

where we have taken into account that

$$\int_0^t \int_\Omega (-\partial_t b(u) w^2 + \bar{v} \nabla w^2)\, dx\, dt = \int_0^t \int_\Omega f w^2\, dx\, dt.$$

Then, Gronwall's argument implies that $\int_\Omega b(u(t)) w^2(t) dx = 0$ for a.e. $t \in I$. Since $b(u) \geq \delta > 0$, we obtain that $w = 0$ a.e. in I and the proof is complete. □

3. Method of discretization in time: auxiliary results

The implementation of the method of characteristics mentioned in Section 1.3 is based on the following lemma.

Lemma 3.1. *If $\bar{V} \in L_\infty(I, L_2(\Omega))$ and $\rho \in (0, \frac{2}{3})$, then it holds*

$$\frac{1}{2}|x - y| \leq |\varphi^i(x) - \varphi^i(y)| \leq 2|x - y|, \quad \forall x, y \in \Omega,\ \forall i = 1, \ldots, n,$$

where $\varphi^i(x) = x - \tau\ \omega_h * \bar{V}_i$, $\bar{V}_i = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \bar{V}(t, x) dt$, $h = \tau^\rho$ and $\tau \leq \tau_0$, τ_0 being sufficiently small.

Proof. We start from

$$\varphi^i(x) - \varphi^i(y) = x - y - \tau(x - y) \int_0^1 \int_{R^3} \omega_h'(y + s(x - y) - \xi) \bar{V}_i(\xi)\, ds\, d\xi.$$

Then, from the estimate

$$\begin{aligned}
&\left| \int_0^1 \int_{R^3} \omega_h'(y + s(x - y) - \xi) \bar{V}_i(\xi)\, ds\, d\xi \right| \\
&\leq \int_0^1 \left(\int_{R^3} \omega_h'(y + s(x - y) - \xi)^2 d\xi \right)^{\frac{1}{2}} \left(\int_\Omega |\bar{V}_i(\xi)|^2 d\xi \right)^{\frac{1}{2}} ds \\
&\leq C \int_0^1 \left(\int_{R^3} \omega_h'(y + s(x - y) - \xi)^2 d\xi \right)^{\frac{1}{2}} ds \\
&\leq C h^{-\frac{3}{2}} \int_0^1 \left(\int_{R^3} \omega_1'(z)^2 dz \right)^{\frac{1}{2}} ds \\
&\leq C h^{-\frac{3}{2}},
\end{aligned}$$

we obtain the desired result, since $\tau h^{-\frac{3}{2}} = \tau^{1 - \frac{3}{2}\ \rho}$. □

The solution to the problems (1.8), (1.12) and (1.9), (1.13) are understood in variational sense, where

$$\left(\lambda_i \frac{u_i - u_{i-1} \circ \varphi_1^i}{\tau}, v \right) + (\bar{A} \nabla u_i, \nabla v) = \big(f_i - \mathrm{div}_x \bar{F}(x, u_{i-1}), v \big), \quad \forall v \in V_1 \quad (3.1)$$

and

$$\left(b(u_i)\frac{w_i - w_{i-1} \circ \varphi_2^i}{\tau}, \psi\right) + (D_i^L \nabla w_i, \nabla \psi) = (G(t_i, x, w_{i-1}), \psi), \quad \forall \psi \in V_2 \quad (3.2)$$

and where $\lambda_i \in L_\infty(\Omega)$ satisfies (1.10). The characteristic map φ^i maps Ω into $\Omega_i \subset \Omega^*$, where $\Omega^* \supset \bar{\Omega}$ is a small neighbourhood of $\bar{\Omega}$ provided that $\tau \le \tau_0$. If $\varphi^i(x) \notin \Omega$, we prolongate $u_{i-1} \in W_2^1$ into $\tilde{u}_{i-1} \in W_2^1(\Omega^*)$ so that (see [22] – prolongation of Nikolskij)

$$\|\tilde{u}_{i-1}\|_{W_2^1(\Omega^*)} \le C\|u_{i-1}\|_{W_2^1(\Omega)}.$$

Similarly we prolongate w_{i-1} to $\tilde{w}_{i-1}$ so that $\tilde{w}_{i-1} \circ \varphi_2^i(x)$ is defined for all $x \in \Omega$ also in the case when $\varphi_2^i(x) \in \Omega^* \setminus \bar{\Omega}$. Then Lemma 3.1 guarantees that

$$\|\tilde{z}_{i-1} \circ \varphi^i\|_0 \le C\|z_{i-1}\|_0 \quad \text{and} \quad \|\nabla \tilde{z}_{i-1} \circ \varphi^i\|_0 \le C\|\nabla z_{i-1}\|_0 \quad (3.3)$$

for $z_{i-1} \in W_2^1(\Omega)$. The existence of $u_i \in V_1$ and $w_i \in V_2$, $(i = 1, \dots n)$, in (3.1) and (3.2) is guaranteed by the Lax–Milgram argument, provided that $\lambda_i \in L_\infty(\Omega)$ is given, on account of (3.3) where we replace z by u and w, respectively. In fact, the convective parts in (1.1) and (1.2) can be approximated more precisely, if we approximate the characteristic curves $H(s; t, x)$ by an explicit Euler method but with smaller time steps $\tau_l = t_{i-1}^{(l)} - t_{i-1}^{(l-1)}$, $l = 1, \dots, m$, $(t_{i-1}^{(m)} = t_i,\ t_{i-1}^{(0)} = t_{i-1})$, throughout the time interval (t_{i-1}, t_i). We denote

$$z(x) := \bar{V}(t_i, x), \quad z^h := \omega_h * z \equiv z_0^h, \quad z_1^h = x - (t_{i-1}^{(1)} - t_{i-1}) z_0^h$$

and

$$z_l^h = z_{l-1}^h(x) - (t_{i-1}^{(l)} - t_{i-1}^{(l-1)}) z^h(z_{l-1}^h(x)).$$

Then we put

$$\hat{\varphi}^i := z_{m-1}^h(x) - (t_i - t_{i-1}^{(m-1)}) z^h(z_{m-1}^h(x)). \quad (3.4)$$

Here, $\hat{\varphi}^i(x)$ represents the position of the initial point x after m smaller time steps $\sum_{l=1}^m \tau_l = \tau$. By the same arguments as in Lemma 3.1 we can prove that $\hat{\varphi}^i(x)$ and its inverse are Lipschitz continuous uniformly for $i = 1, \dots, n$.

Lemma 3.2. *If $\rho \in (0, \frac{2}{3})$, $h = \tau^\rho$ then there exists τ_1 such that*

$$\frac{1}{2}|x - y| \le |\hat{\varphi}^i(x) - \hat{\varphi}^i(y)| \le 2|x - y|, \quad \forall x, y \in \Omega,\ \forall i = 1, \dots, n$$

provided that $\tau \le \tau_1$, where $\hat{\varphi}^i$ is from (3.4).

Proof. Start from $\nabla z_l^h = \nabla z_{l-1}^h(1-\tau_l\nabla_y(z^h))$, $(\tau_l \equiv |t_{i-1}^l - t_{i-1}^{l-1}|, y = \nabla z_{l-1}^h(x))$. We use the estimate $|\nabla\omega_h * g| \le \frac{C}{h}\|g\|_\infty \le \frac{C}{h}$. Then, we obtain

$$\begin{aligned}
|\partial_{x_j}\{\widetilde{\varphi}^i(x)\}_j - 1| &\le \frac{\tau}{h}C\left(1+\frac{\tau}{h}C+\left(\frac{\tau}{h}\right)^2C^2+\cdots+\left(\frac{\tau}{h}\right)^{m-1}C^{m-1}\right)\\
&\le C\,\frac{\tau}{h}\,\frac{1}{1-\frac{\tau}{h}C}\\
|\partial_{x_j}\{\widetilde{\varphi}^i(x)\}_k| &\le C\,\frac{\tau}{h}\,\frac{1}{1-\frac{\tau}{h}C}
\end{aligned}$$

for any j and $k = 1,\ldots,N,\ j\neq k$, where we used the notation $\{\bar{v}\}_j$ for the j-th component of $\bar{v}$. Hence, the required result follows. □

In the next lemma we prove the uniform boundedness of $\{u_i\}$ and $\{w_i\}$.

Lemma 3.3. *The solutions u_i and w_i, $(i = 1,\ldots,n)$, to* (3.1) *and* (3.2), *respectively, are uniformly bounded in $L_\infty(\Omega)$, i.e.,*

$$\|u_i\|_\infty \le C \quad \text{and} \quad \|w_i\|_\infty \le C, \quad \forall i = 1,\ldots,n,\ \forall n\in N.$$

Proof. For fixed i, u_i is a minimizer of the functional

$$\Phi_1(u) = \frac{1}{\tau}\int_\Omega \lambda_i(u-u_{i-1}\circ\varphi_1^i)^2dx + (\bar{A}\nabla u,\nabla u) - 2\int_\Omega(f_i - \mathrm{div}_x\bar{F}(x,u_{i-1})u\,dx$$

on the set $u_i^D + V_1$. Denote by $u_i = z_i + u_i^D$ the unique minimizer and consider the truncation $u_i^S := \sigma_S(u_i)$, where $S \ge \|u_i^D\|_{L_\infty(I\times\partial\Omega)}$. Then, $u_i^S \in u_i^D + V_1$. In what follows we shall construct S such that $\Phi_1(u_i^S) \le \Phi_1(u_i)$, from which we deduce (by the uniqueness argument) that $\|u_i\|_{L_\infty} \le S$. Due to the symmetry and the positive definiteness of $\bar{A}(x)$ we can estimate

$$\begin{aligned}
&\Phi_1(u_i) - \Phi_1(u_i^S)\\
&\ge \int_{E_S}(u_i-u_i^S)\left\{\left(\frac{\lambda_i}{\tau}\left(u_i+u_i^S-2u_{i-1}\circ\varphi_1^i\right)\right) - 2f_i + 2\,\mathrm{div}_x\bar{F}(x,u_{i-1})\right\}dx,
\end{aligned}$$

where $E_S = \{x\in\Omega : |u_i(x)| > S\}$, since

$$\left(\bar{A}\nabla u_i,\nabla u_i\right) - \left(\bar{A}\nabla u_i^S,\nabla u_i^S\right) = \left(\bar{A}\nabla(u_i-u_i^S),\nabla(u_i-u_i^S)\right) \ge 0.$$

We have $\|u_{i-1}\circ\varphi_1^i\|_\infty \le C\|u_{i-1}\|_\infty$ and $|f_i| + |\mathrm{div}_x\bar{F}(t_i,u_{i-1})| \le C_1 + C_2|u_{i-1}|$. Then, choosing $S = \max\{\|u^D\|_\infty, \|u_{i-1}\|_\infty + \tau C(1+\|u_{i-1}\|_\infty)\}$, and noticing that $\lambda_i \ge \frac{\delta}{2}$, we obtain that $\Phi_1(u_i^S) \le \Phi_1(u_i)$. This implies that

$$\|u_i\|_\infty \le \max\left\{\|u^D\|_\infty, (1+\tau C)\|u_{i-1}\|_\infty + C\tau\right\}.$$

From this recurrent inequality we obtain the required result. We proceed analogously for the proof of the boundedness of $\{w_i\}$ in L_∞. □

4. Convergence of the semi-discrete method

4.1. Rothe functions and associated step functions on the time interval I. By means of $\{u_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ we construct approximate solutions $\{\bar{u}^n, \bar{w}^n\}$ and $\{u^n, w^n\}$ to the problem (1.1)–(1.5) as follows.

Definition 4.1. The Rothe function u^n is the piecewise linear function on the time interval I (with values taken on in appropriate function spaces on Ω) defined by

$$u^n(t) := u_{i-1} + (t - t_{i-1})\delta u_i \quad \text{for } t \in (t_{i-1}, t_i),\ i = 1, \dots, n, \tag{4.1}$$

where $\delta u_i := \frac{u_i - u_{i-1}}{\tau}$. We introduce the associated step function on I by

$$\bar{u}^n(t) := u_i \quad \text{for } t \in (t_{i-1}, t_i),\ i = 1, \dots, n.$$

We similarly define $\bar{w}^n(t)$ and $w^n(t)$ by means of $\{w_i\}_{i=1}^n$.

To prove the convergence of $\{\bar{u}^n\}$ and $\{\bar{w}^n\}$ in the corresponding functional spaces for $n \to \infty$, i.e. for $\tau \to 0$, we need a priori estimates for $\{\bar{u}^n(t)\}_{n=1}^\infty$ and $\{\bar{w}^n(t)\}_{n=1}^\infty$.

4.2. A priori estimates.

Lemma 4.2. *The a priori estimates*

$$\sum_{i=1}^n \|\delta u_i\|_0^2 \tau \le C, \quad \|u_i\| \le C\ (i = 1, \dots, n), \quad \sum_{i=1}^n \|\nabla(u_i - u_{i-1})\|_0^2 \le C$$

hold uniformly for n.

Proof. Due to (1.10) we split the first term in (3.1)

$$\begin{aligned} \lambda_i(u_i - u_{i-1} \circ \varphi_1^i) = b(u_i) - b(u_{i-1}) + \tau\chi_i(u_i - u_{i-1} \circ \varphi_1^i) \\ + b(u_{i-1}) - b(u_{i-1} \circ \varphi_1^i), \end{aligned} \tag{4.2}$$

where $\|\chi_i\|_\infty \le 1$. We put $v = u_i - u_{i-1} - (u_i^D - u_{i-1}^D)$ into (3.1) and sum up for $i = 1, \dots, j$. The resulting equation is denoted by $J_1 + J_2 = J_3$. We again split $J_l = J_l^0 + J_l^D$ for $l = 1, 2, 3$ (in J_l^0 we use $v = u_i - u_{i-1}$ and in J_l^D we use $v = -(u_i^D - u_{i-1}^D)$) . Let us denote

$$J_{1,\varphi}^0 = \sum_{i=1}^j \left(\chi_i \frac{u_{i-1} - u_{i-1} \circ \varphi_1^i}{\tau}, \frac{u_i - u_{i-1}}{\tau}\right)\tau$$

$$J_{2,\varphi}^0 = \sum_{i=1}^j \left(\frac{b(u_{i-1}) - b(u_{i-1} \circ \varphi_1^i)}{\tau}, \frac{u_i - u_{i-1}}{\tau}\right)\tau.$$

Then, using H_1 we get

$$J_1^0 = J_{1,\varphi}^0 + J_{2,\varphi}^0 + \sum_{i=1}^{j} \big(\delta b(u_i), \delta u_i\big)\tau \geq J_{1,\varphi}^0 + J_{2,\varphi}^0 + C \sum_{i=1}^{j} \|\delta u_i\|_0^2 \tau\,.$$

To estimate $J_{1,\varphi}^0$ we split $u_i - u_{i-1} \circ \varphi_1^i = u_i - u_{i-1} + u_{i-1} - u_{i-1} \circ \varphi_1^i$, and use the formula

$$\frac{u_{i-1} - u_{i-1} \circ \varphi_1^i}{\tau} = \int_0^1 \nabla \tilde{u}_{i-1}\big(x + s(\varphi_1^i(x) - x)\big) ds \cdot \omega_h * \frac{\bar{F}_u'(x, u_{i-1})}{b'(u_{i-1})}\,. \qquad (4.3)$$

From the estimate (3.3) (replacing z_{i-1} by u_{i-1}), the inequality $\lambda_i \leq M$ and the fact that $\big\|\omega_h * \frac{\bar{F}_u'(x,u_{i-1})}{b'(u_{i-1})}\big\|_\infty \leq C$, we obtain

$$|J_{1,\varphi}^0| \leq (\varepsilon + C\tau) \sum_{i=1}^{j} \|\delta u_i\|_0^2 \tau + C_\varepsilon \sum_{i=1}^{j} \|\nabla u_i\|_0^2 \tau$$

and, similarly,

$$|J_{1,\varphi}^D| \leq (\varepsilon + C\tau) \sum_{i=1}^{j} \|\delta u_i^D\|_0^2 \tau + C_1 \sum_{i=1}^{j} \|\nabla u_i\|_0^2 \tau + C_2.$$

In a similar way we can estimate $J_{2,\varphi}^0$ and $J_{2,\varphi}^D$ since b is Lipschitz continuous. Due to H_6 and Lemma 3.3 we estimate

$$\begin{aligned} |J_1^D| &= \left| \sum_{i=1}^{j} \left(\lambda_i \frac{u_i - u_{i-1} \circ \varphi_1^i}{\tau}, \frac{u_i^D - u_{i-1}^D}{\tau} \right) \right| \tau \\ &\leq C_1 \sum_{i=1}^{j} \|\delta u_i^D\|_0^2 \tau + C_2 \sum_{i=1}^{j} \|\nabla u_i\|_0^2 \tau + C_3 \\ &\leq C_1 + C_2 \sum_{i=1}^{j} \|\nabla u_i\|_0^2 \tau. \end{aligned}$$

Combining this estimate with the one for J_1^0, obtained above, we conclude

$$J_1 \geq C_1 \sum_{i=1}^{j} \|\delta u_j\|_0^2 \tau - C_2 \sum_{i=1}^{j} \|u_i\|^2 \tau - C_3. \qquad (4.4)$$

Due to the symmetry and the positive definiteness of $\bar{A}$ we have

$$\begin{aligned} J_2 &\geq \frac{1}{2} C_A \|\nabla u_j\|_0^2 + \frac{1}{2} C_A \sum_{i=1}^{j} \|\nabla(u_i - u_{i-1})\|_0^2 \\ &\quad - C \sum_{i=1}^{j} \|\nabla u_i\|_0 \|\nabla \delta u_i^D\|_0 \; \tau - C \qquad (4.5) \\ &\geq \frac{1}{2} C_A \|\nabla u_j\|_0^2 - C_1 \sum_{i=1}^{j} \|\nabla u_i\|_0^2 \tau - C_2. \end{aligned}$$

Due to assumption H_5 and Lemma 3.3 we obtain

$$|J_3| \le \varepsilon \sum_{i=1}^{j} \|\delta u_i\|_0^2 \tau + C_\varepsilon. \tag{4.6}$$

Inserting (4.4)–(4.6) in the equality $J_1 + J_2 = J_3$ and invoking Gronwall's argument the required estimates follow. □

Using the estimates of Lemma 4.2 we can derive a priori estimates for $\{w_i\}$.

Lemma 4.3. *The estimates* $\sum_{i=1}^{n} \|\nabla w_i\|^2 \tau \le C$, *uniformly for* L, *and*

$$\sum_{j=1}^{n-k} \|b(u_{j+k})w_{j+k} - b(u_j)w_j\|_0^2 \tau \le Ck\tau, \quad \textit{where } C = C(L),$$

take place uniformly for n. *Here* L *is the truncation parameter introduced in* (1.11).

Proof. We put $\psi = (w_i - w_i^D)\tau$ into (3.2) and sum up for $i = 1, \dots, j$. The resulting equation is denoted by $J_1 + J_2 = J_3$. We split $J_l = J_l^0 + J_l^D, l = 1, 2$ and 3, as in the proof of Lemma 4.2. Moreover, we split $w_i - w_{i-1} \circ \varphi_2^i = w_i - w_{i-1} + w_{i-1} - w_{i-1} \circ \varphi_2^i$ and correspondingly we write $J_1 = J_{1,1} + J_{1,2}$. First we estimate

$$\begin{aligned}
J_{1,1}^0 &= \sum_{i=1}^{j} (b(u_i)(w_i - w_{i-1}), w_i) \\
&= \frac{1}{2}\int_\Omega b(u_j) w_j^2 dx - \frac{1}{2}\int_\Omega b(u_0) w_0^2 dx + \frac{1}{2}\sum_{i=1}^{j}\int_\Omega b(u_i)(w_i - w_{i-1})^2 dx \\
&\quad - \frac{1}{2}\sum_{i=1}^{j} \tau \int_\Omega \frac{b(u_i) - b(u_{i-1})}{\tau}\, w_{i-1}^2 dx \\
&\ge \frac{1}{2}\int_\Omega b(u_j) w_j^2 dx - \frac{1}{2}\int_\Omega b(u_0) w_0^2 dx - C\sum_{i=1}^{j} \|\delta b(u_i)\|_0^2 \tau - C.
\end{aligned}$$

Similarly, using Lemma 3.3 and H_6, we obtain

$$\begin{aligned}
|J_{1,1}^D| &\le \left| \sum_{i=1}^{j} \big(b(u_i)w_i - b(u_{i-1})w_{i-1}, w_i^D\big) \right| + \sum_{i=1}^{j} \left| \big((b(u_i) - b(u_{i-1}))w_{i-1}, w_i^D\big) \right| \\
&\le \left| \big(b(u_j)w_j, w_j^D\big) \right| + \left| \big(b(u_0)w_0, w_0^D\big) \right| + \left| \sum_{i=1}^{j} \left(b(u_{i-1})w_{i-1}, \frac{w_i^D - w_{i-1}^D}{\tau}\tau \right) \right| + C \\
&\le C.
\end{aligned}$$

Thus we get

$$J_{1,1} \geq \frac{1}{2}\int_\Omega b(u_j)w_j^2 dx - C. \tag{4.7}$$

To estimate $J_{1,2}$ we use formula (4.3) (where we replace u_{i-1} by w_{i-1} and $\bar{V}^i_{1,h}$ by $\bar{V}^i_{2,h}$), together with (3.3) and Lemma 3.1. Then we have

$$\begin{aligned}|J^0_{1,2}| &\leq C\sum_{i=1}^{j}\tau\int_\Omega\left|b(u_i)\int_0^1\nabla\tilde{w}_{i-1}\big(x+s(\varphi_2^i(x)-x)\big)ds\cdot\omega_h * \frac{\bar{v}(u_i,\nabla u_i)}{b(u_i)}\right|dx\\ &\leq \varepsilon\sum_{i=1}^{j}\|\nabla w_i\|_0^2\tau + C_\varepsilon\sum_{i=1}^{j}\|b(u_i)\|_0^2\tau\\ &\leq \varepsilon\sum_{i=1}^{j}\|\nabla w_i\|_0^2\tau + C_\varepsilon,\end{aligned}$$

since $\left\|\omega_h * \frac{\bar{v}(u_i,\nabla u_i)}{b(u_i)}\right\|_0 \leq C(1+\|u_i\|) \leq C$. Similarly we estimate $|J^D_{1,2}|$ from above. Combining the resulting estimate for $J_{1,2}$ with (4.7) we have

$$J_1 \geq \frac{1}{2}\int_\Omega b(u_j)w_j^2 dx - \varepsilon\sum_{i=1}^{j}\|\nabla w_i\|_0^2\tau - C_\varepsilon. \tag{4.8}$$

Using the positive definiteness of D_i^n we find

$$J_2^0 \geq C_A\sum_{i=1}^{j}\|\nabla w_i\|_0^2\tau.$$

To estimate J_2^D we use H_6 and the growth conditions on D in H_3. We deduce that

$$\begin{aligned}|J_2^D| &\leq C\sum_{i=1}^{j}\int_\Omega\left|D_i^n\nabla w_i\right|dx\\ &\leq \varepsilon\sum_{i=1}^{j}\|\nabla w_i\|_0^2\tau + C_\varepsilon\left(1+\sum_{i=1}^{j}\|u_i\|_0^2\tau + \sum_{i=1}^{j}\|\nabla u_i\|_0^2\tau\right).\end{aligned}$$

Combining this inequality with the one above for J_2^0 and using Lemma 4.2 we arrive at

$$J_2 \geq C_1\sum_{i=1}^{j}\|\nabla w_i\|_0^2\tau - C_2. \tag{4.9}$$

From Lemma 3.3 and using H_6 we obtain

$$|J_3| \leq C. \tag{4.10}$$

Inserting the estimates (4.8)–(4.10) in the equality $J_1 + J_2 = J_3$ the first a priori estimate of this Lemma follows.

To obtain the second a priori estimate we rewrite the first term in (3.2) in the form

$$b(u_i)\frac{w_i - w_{i-1}\circ\varphi_2^i}{\tau} = \frac{b(u_i)w_i - b(u_{i-1})w_{i-1}}{\tau} - \frac{b(u_i)-b(u_{i-1})}{\tau}w_{i-1} + b(u_i)\frac{w_{i-1} - w_{i-1}\circ\varphi_2^i}{\tau}. \tag{4.11}$$

We multiply (3.2) by τ and sum up for $i = j+1,\dots,j+k$. Then we put $\psi = [b(u_{j+k})w_{j+k} - b(u_j)w_j - (b(u_{j+k})w^D_{j+k} - b(u_j)w^D_j)]\tau$ and sum up for $j = 1,\dots,n-k$. We denote the resulting equation by $J_1 + J_2 = J_3$ and again split $J_l = J^0_l + J^D_l$, ($l = 1, 2$ and 3), as in the proof of the previous lemma. We have $J_1 = J_{1,1} + J_{1,2} + J_{1,3}$ due to the splitting (4.11). We first get

$$J^0_{1,1} \ge \sum_{j=1}^{n-k} \|b(u_{j+k})w_{j+k} - b(u_j)w_j\|_0^2\tau.$$

Due to H_6 and the estimates of Lemma 3.3 and Lemma 4.2 we find

$$\begin{aligned}|J^D_{1,1}| &\le \varepsilon\sum_{j=1}^{n-k}\|b(u_{j+k})w_{j+k} - b(u_j)w_j\|_0^2\tau - C_\varepsilon\sum_{j=1}^{n-k}\sum_{i=j}^{j+k}\|\delta b(u_i)w^D_i\|_0\|b(u_{i-1})\delta w^D_i\|_0\tau^2\\ &\le \varepsilon\sum_{j=1}^{n-k}\|b(u_{j+k})w_{j+k} - b(u_j)w_j\|_0^2\tau - C_\varepsilon k\tau\,.\end{aligned}$$

Due to the estimates of Lemma 4.2 and Lemma 3.3 we obtain $|J_{1,2}| \le Ck\tau$. Using H_6, formula (4.3), the estimate (3.3) and the estimates of Lemmas 3.1, 3.3 and 4.2, we get $|J_{1,3}| \le Ck\tau$. Combining this estimate with the ones for $J_{1,1}$ and $J_{1,2}$ we arrive at

$$J_1 \ge \sum_{i=1}^{j} \|b(u_{j+k})w_{j+k} - b(u_j)w_j\|_0^2\tau - Ck\tau. \tag{4.12}$$

To estimate the term J_2 we use the fact that $\|\nabla(b(u_i)w_i)\|_0 \le C_1\|\nabla w_i\|_0 + \|C_2\|u_i\|$ on account of the estimates of Lemma 3.3 and Lemma 4.2 and on account of the Lipschitz continuity of b. Noticing that $|(D^L_i\nabla w_i, \nabla w_j)| \le C(L)\|\nabla w_i\|_0\|\nabla w_j\|_0$ and invoking the first estimate of this Lemma, we get

$$|J_2| \le Ck\tau, \quad |J_3| \le Ck\tau. \tag{4.13}$$

Finally, from (4.12)–(4.13) the second estimate of Lemma 4.3 follows. □

4.3. Compactness results. Next, we prove the compactness of $\{\bar{u}^n\}$ and $\{\bar{w}^n\}$. By $\{\bar{n}\}$ we denote a subsequence of $\{n\}$.

Lemma 4.4. *There exist $u \in L_\infty(Q_T) \cap L_2(I, W_2^1)$ and $w^L \in L_\infty(Q_T) \cap L_2(I, W_2^1)$ such that $\bar{u}^{\bar{n}} \to u$ in $L_r(Q_T)$, for all r, $1 < r < \infty$, $\bar{u}^{\bar{n}} \rightharpoonup u$ in $L_2(I, W_2^1)$, $\delta b(\bar{u}^{\bar{n}}) \rightharpoonup \partial_t b(u)$, $\delta \bar{u}^{\bar{n}} \rightharpoonup \partial_t u$ in $L_2(I, L_2)$ and $\bar{w}^{\bar{n}} \to w^L$ in $L_r(Q_T)$, for all r, $1 < r < \infty$, $\bar{w}^{\bar{n}} \rightharpoonup w^L$ in $L_2(I, W_2^1)$, $\delta(b(\bar{u}^{\bar{n}})\bar{w}^{\bar{n}}) \rightharpoonup \partial_t(b(u)w)$ in $L_2(I, V_2^*)$ when $n \to \infty$.*

Proof. From Lemma 4.2 it follows that $\{\bar{u}^n\}$ is compact in $L_2(I, L_2)$ (see, e.g., [15]). The second estimate in Lemma 4.3 can be rewritten in the form

$$\int_0^{T-z} \|b(\bar{u}^{\bar{n}}(t+z))w^{\bar{n}}(t+z) - b(\bar{u}^{\bar{n}}(t))\bar{w}^{\bar{n}}(t)\|_0^2 \, dt \le Cz, \tag{4.14}$$

where $k\tau \le z \le (k+1)\tau$. Due to Lemma 3.3 and Lemma 4.2 we have $b(\bar{u}^{\bar{n}}(t))\bar{w}^{\bar{n}}(t) \in W_2^1(\Omega)$, and

$$\|b(\bar{u}^{\bar{n}})\bar{w}^{\bar{n}}\|_{L_2(I,W_2^1)} \le C_1\|\bar{u}^{\bar{n}}\|_{L_2(I,W_2^1)} + C_2\|\bar{w}^{\bar{n}}\|_{L_2(I,W_2^1)}.$$

Following [22] we readily find

$$\int_I\int_\Omega \big(b(\bar{u}^{\bar{n}}(t,x+y))\bar{w}^{\bar{n}}(t,x+y) - b(\bar{u}^{\bar{n}}(t,x))\bar{w}^{\bar{n}}(t,x)\big)^2 dx\, dt$$
$$\le |y|\big(C_1\|\nabla\bar{w}^{\bar{n}}\|_{L_2(I,L_2)} + C_2\|\nabla\bar{u}^{\bar{n}}\|_{L_2(I,L_2)}\big).$$

This estimate together with (4.14) guarantees the compactness of $\{b(\bar{u}^{\bar{n}})\bar{w}^{\bar{n}}\}$ in $L_2(I, L_2)$, because of Kolmogorov's compactness argument – see [22]. Since $\{b(\bar{u}^n)\}$ is compact in $L_2(I, L_2)$ and since $b(s) \ge \delta > 0$ (see H_1) we conclude that $\bar{w}^{\bar{n}}(t,x) \to w^L(t,x)$ for a.e. $(t,x) \in Q_T$. This convergence and Lemma 3.3 imply $L_r(Q_T)$-convergence $\bar{w}^{\bar{n}} \to w$ (for all $r > 1$), where $w \in L_\infty(Q_T)$.

To prove that $\delta(b(\bar{u}^{\bar{n}})\bar{w}^{\bar{n}}) \rightharpoonup \partial_t(b(u)w)$ in $L_2(I, V_2^*)$, we apply the duality argument in (3.2). Using (4.11) we rewrite (3.2) in the form

$$\begin{aligned}\left(\frac{b(u_i)w_i - b(u_{i-1})w_{i-1}}{\tau}, v\right) &= \left(\frac{b(u_i) - b(u_{i-1})}{\tau} w_{i-1}, v\right) \\ &\quad - (D^L_{n,i}\nabla w_i, \nabla v) + (G(t_i, x, w_{i-1}), v) \\ &\quad - \left(b(u_i)\frac{w_{i-1} - w_{i-1}\circ\varphi_2^i}{\tau}, v\right).\end{aligned} \tag{4.15}$$

Applying the a priori estimates of Lemmas 3.3, 4.2 and 4.3, using the formula (4.3) and the estimates (3.3) we conclude that

$$\|\delta(b(u_i)w_i)\|_* \le C(L)(\|\nabla u_i\|_0 + \|\nabla w_i\|_0 + \|\nabla w_{i-1}\|_0).$$

Consequently, $\{\delta(b(\bar{u}^n)\bar{w}^n)\}$ is bounded in $L_2(I, V^*)$. There exists $\Phi \in L_2(I, V_2^*)$ so that $\delta(b(\bar{u}^n)\bar{w}^n) \rightharpoonup \Phi$ in $L_2(I, V^*)$. On the other hand, $b(\bar{u}^n)\bar{w}^n \to b(u)w$ in $L_2(I, L_2)$. This implies that $\Phi \equiv \partial_t(b(u)w)$. The rest of the proof follows from Lemma 4.2 and Lemma 4.3. □

4.4. Convergence of $\{\overline{u}^n\}$ and $\{\overline{w}^n\}$. Now we can formulate the main result of this section.

Theorem 4.5. *Let the assumptions H_1-H_6 be satisfied. Then,*

$$\bar{u}^{\bar{n}} \to u \quad in\ L_2(I, V_1), \qquad \bar{w}^{\bar{n}} \to w^L \quad in\ L_2(I, V_2)\ for\ n \to \infty.$$

Here, $\{\bar{u}^n\}$ and $\{\bar{w}^n\}$ are defined by (4.1) and by (3.1) and (3.2), while $\{u, w^L\}$ is a variational solution to Problem P^L in the sense of Definition 2.1. If the variational solution $u \in L_\infty(I, W^1_\infty)$, then $\{u, w^L\}$ is also a variational solution to Problem P. If $L \to \infty$, then $w^L \rightharpoonup w$ where $\{u, w\}$ is a variational solution to the Problem P in the sense of Definition 2.2, provided that $G(t, x, w)$ is linear in w (i.e., $G(t, x, w) = g_1(t, x) + g_2(t, x)w$). If the variational solution $\{u, w^L\}$ is unique (see, e.g., Theorem 2.5), then the original sequences $\{\bar{u}^n\}$ and $\{\bar{w}^n\}$ are convergent.

Proof. We rewrite (3.1) and (3.2) in terms of $\bar{u}^n, u^n$ and $\bar{w}^n$, defined in (4.1). We use (4.2) and (4.11). Integrating the resulting equations over I, we obtain

$$\begin{aligned} &\int_I \big(\delta b(\bar{u}^n), v\big) dt + \int_I \big(\bar{A}\nabla u_n, \nabla v\big) dt \\ &+ \int_I \big(\bar{\chi}^n(\bar{u}^n - \bar{u}^n_\tau \circ \bar{\varphi}^n_1), v\big) dt + \frac{1}{\tau}\int_I \big(b(\bar{u}^n_\tau) - b(\bar{u}^n_\tau \circ \bar{\varphi}^n_1), v\big) dt \\ &= \int_I \big(\bar{f}^n - \mathrm{div}_x \bar{F}(x, \bar{u}^n_\tau), v\big) dt \quad \forall v \in L_2(I, V_1) \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &\int_I \big(\delta(b(\bar{u}^n)\bar{w}^n), \eta\big) dt + \int_I \big(D^L_n(\bar{u}^n, \nabla\bar{u}^n)\nabla\bar{w}^n, \nabla\eta\big) dt \\ &= \int_I \big(G(t, x, \bar{w}^n_\tau), \eta\big) dt + \int_I \big(\delta b(\bar{u}^n)\bar{w}^n_\tau, \eta\big) dt - \int_I \left(b(\bar{u}^n)\frac{\bar{w}^n_\tau - \bar{w}^n_\tau \circ \bar{\varphi}^n_2}{\tau}, \eta\right) dt \end{aligned} \tag{4.17}$$

for all $\eta \in L_2(I, V_2) \cap L_\infty(Q_T)$, where $\bar{u}^n_\tau := \bar{u}^n(t - \tau)$ and $\bar{w}^n_\tau := \bar{w}^n(t - \tau)$. We pass to the limit $n \to \infty$ in (4.16). We readily get

$$\int_I (\delta b(\bar{u}^n), v) dt \to \int_I (\partial_t b(u), v) dt, \quad \int_I (\bar{A}\nabla u_n, \nabla v) dt \to \int_I (\bar{A}\nabla u, \nabla v) dt$$

and

$$\int_I (\bar{f}^n + \mathrm{div}_x \bar{F}(x, \bar{u}^n_\tau), v) dt \to \int_I (f + \mathrm{div}_x \bar{F}(x, u), v) dt.$$

Moreover, we obtain

$$\int_I (\bar{\chi}^n(\bar{u}^n - \bar{u}^n_\tau \circ \bar{\varphi}^n_1), v) dt \to 0$$

since $\bar{u}^n - \bar{u}^n_\tau \to 0$ in $L_2(I, L_2)$ and since

$$\bar{u}^n_\tau - \bar{u}^n_\tau \circ \bar{\varphi}^n_1 = \tau \int_0^1 \nabla \bar{u}^n_\tau(x + s(\bar{\varphi}^n_1 - x))ds \cdot \omega_h * \frac{\bar{F}'_u(x, \bar{u}^n_\tau)}{b'(\bar{u}^n_\tau)} \to 0 \quad \text{in } L_2(I, L_2).$$

The crucial point is to prove

$$\frac{1}{\tau} \int_I \big(b(\bar{u}^n_\tau) - b(\bar{u}^n_\tau \circ \bar{\varphi}^n_1), v\big) dt \to \int_I \big(\partial_u \bar{F}(x, u) \cdot \nabla u, v\big) dt. \tag{4.18}$$

For this purpose we first notice that

$$\frac{b(\bar{u}^n) - b(\bar{u}^n_\tau \circ \bar{\varphi}^n_1)}{\tau} = \int_0^1 b'\big(\bar{u}^n + s(\bar{u}^n_\tau \circ \bar{\varphi}^n_1 - \bar{u}^n)\big) ds \, \frac{\bar{u}^n_\tau - \bar{u}^n_\tau \circ \bar{\varphi}^n_1}{\tau}.$$

Next, we use that $\int_0^1 b'(.)ds \to b'(u)$ a.e. in Q_T, $b'(s) \le M$, as well as (4.3), (3.3), Lemma 4.3 and Lemma 4.4. Moreover, we use $\big\|\omega_h * \frac{\bar{F}'_u(x,\bar{u}^n_\tau)}{b'(\bar{u}^n_\tau)}\big\|_\infty \le C$, $\omega_h * \frac{\bar{F}'_u(x,\bar{u}^n_\tau)}{b'(\bar{u}^n_\tau)} \to \frac{\partial_u \bar{F}(x,u)}{b'(u)}$, for a.e $(t, x) \in Q_T$ when $n \to \infty$. We denote by

$$Z^n_\tau := \int_0^1 \tilde{\bar{u}}^n_\tau(t, x + s(\bar{\varphi}^n_1 - x)) \, ds.$$

Then, (3.3) and Lemma 4.4 imply that $\nabla Z^n_\tau \rightharpoonup \Phi$ in $L_2(I, L_2)$. On the other hand, from

$$Z^n_\tau - \bar{u}^n_\tau = \tau \int_0^1 \int_0^1 s \nabla \tilde{\bar{u}}^n_\tau(t, x + sr(\bar{\varphi}^n_1 - x)) \, ds \, dr \cdot \omega_h * \frac{\bar{F}'_u(x, \bar{u}^n_\tau)}{b'(\bar{u}^n_\tau)},$$

we get $Z^n_\tau - \bar{u}^n_\tau \to 0$ in $L_2(I, L_2)$, which implies that $Z^n_\tau \to u$ in $L_2(I, L_2)$ for $n \to \infty$. Thus, $\Phi \equiv \nabla u$, from which (4.18) follows. Consequently, we obtain that the function u from Lemma 4.4 satisfies (2.1).

To prove the convergence $\bar{u}^n \to u$ in $L_2(I, V_1)$ for $n \to \infty$ we put $v = \bar{u}^n - u - (\bar{u}^D_n - u_D)$ in (4.16) and use Lemma 4.4. We first have

$$\int_I \big(\bar{A} \nabla \bar{u}^n, \nabla(\bar{u}^n - u)\big) dt \ge C_A \int_I \|\nabla(\bar{u}^n - u)\|_0^2 dt - C_n$$

where $C_n = \int_I (\bar{A} \nabla u, \nabla(\bar{u}^n - u)) dt \to 0$ for $n \to \infty$ since $\bar{u}^n \rightharpoonup u$ in $L_2(I, W^1_2)$. The remaining terms in (4.16) converge to 0. To argue this, notice that $\bar{u}^n \to u$ in $L_2(I, L_2)$ and $\bar{u}^D_n \to u^D$ in $L_2(I, V_1)$. Furthermore, use the estimate $\big\|\frac{b(\bar{u}^n_\tau) - b(\bar{u}^n_\tau \circ \bar{\varphi}^n_1)}{\tau}\big\|_0 \le C \|\nabla \bar{u}^n_\tau\|_0 \le C$. Next, remark that $\delta b(\bar{u}^n) \rightharpoonup \partial_t b(u)$ in $L_2(I, L_2)$ and consequently $\int_I (\delta b(\bar{u}^n), \bar{u}^n - u) dt \to 0$ for $n \to \infty$.

Now we prove that (2.2) holds. To this end we take the limit $n \to \infty$ in (4.17). We first have (see Lemma 4.4)

$$\int_I \big(\delta(b(\bar{u}^n) \bar{w}^n), \eta\big) dt \to \int_I \langle \partial_t(b(u) w), \eta \rangle dt,$$

and

$$\int_I (D_n^L(\bar{u}^n, \nabla\bar{u}^n)\nabla\bar{w}^n, \nabla\eta)dt \to \int_I (D^L(u, \nabla u)\nabla w, \nabla\eta)dt.$$

This follows from the convergences $\nabla\bar{w}^n \rightharpoonup \nabla w$ in $L_2(I, L_2)$ and $\bar{u}^n \to u$ and $\nabla\bar{u}^n \to \nabla u$ a.e. in Q_T and from the estimate $\|D_n^L(\bar{u}^n, \nabla\bar{u}^n)\|_\infty \le C(L)$. Next, from Lemma 4.4 we deduce

$$\int_I (\delta b(\bar{u}^n)\bar{w}_\tau^n, \eta)dt \to \int_I (\partial_t(b(u))w, \eta)dt$$

and

$$\int_I (G(t, x, \bar{w}_\tau^n), \eta)dt \to \int_I (G(t, x, w), \eta)dt \quad \text{for } n \to \infty.$$

The crucial point is to prove

$$\int_I \left(b(\bar{u}^n)\frac{\bar{w}^n - \bar{w}_\tau^n \circ \bar{\varphi}_2^n}{\tau}, \eta\right)dt \to \int_I (\bar{v} \cdot \nabla w, \eta)dt \quad \text{for } n \to \infty.$$

where $\bar{v} = -(\bar{A}\nabla u + \bar{F}(x, u))$. We proceed analogously as in (4.18). We use the convergence $\omega_h * \frac{1}{b(\bar{u}^n)}(\bar{A}\nabla\bar{u}^n + \bar{F}(x, \bar{u}_\tau^n)) \to \frac{1}{b(u)}(\bar{A}\nabla u + \bar{F}(x, u))$ in $L_2(I, L_2)$ on account of

$$\frac{1}{b(\bar{u}^n\tau)}(\bar{A}\nabla\bar{u}^n + \bar{F}(x, \bar{u}^n\tau)) \to \frac{1}{b(u)}(\bar{A}\nabla u + \bar{F}(x, u)) \quad \text{in } L_2(I, L_2) \text{ for } n \to \infty,$$

where $h = \tau^\rho, \rho \in (0, \frac{2}{3})$. Along the same lines as in (4.18) we prove that

$$\int_0^1 \nabla\tilde{\bar{w}}^n(t, x + s(\bar{\varphi}_2^n(x) - x))ds \rightharpoonup \nabla w \quad \text{in } L_2(I, L_2).$$

Inserting the obtained limit results for $n \to \infty$ in (4.17) we arrive at (2.2).

To prove that the identity (2.3) is satisfied, we use $\partial_t(b(u)w) \in L_2(I, V_2^*)$ and we take the limit $\tau \to 0$ in the equality

$$\begin{aligned}
&\int_I \left\langle \frac{b(u(t))w(t) - b(u(t-\tau))w(t-\tau)}{\tau}, \zeta \right\rangle dt \\
&= \frac{1}{\tau}\int_I (b(u(t))w(t) - b(u(t-\tau))w(t-\tau), \zeta)dt \\
&= \int_0^{T-\tau} \left(b(u(t)), \frac{\zeta(t-\tau) - \zeta(t)}{\tau}\right)dt - \int_{T-\tau}^T (b(u(t))w(t), \zeta(t))dt \\
&\quad - \frac{1}{\tau}\int_0^\tau (b(u_0)w_0, \zeta(t))dt,
\end{aligned}$$

where $b(u(t))w(t) \equiv b(u_0)w_0$ for $t \in (-\tau, 0)$. Noticing that

$$\frac{b(u(t))w(t) - b(u(t-\tau))w(t-\tau)}{\tau} \to \partial_t(b(u)w) \quad \text{in } L_2(I, V_2^*) \text{ for } \tau \to 0$$

we get (2.3). Summarizing, we have proved that $\{u, w^L\}$ is a variational solution to Problem P^L. Lemma 4.3 guarantees that $\|w^L\|_{L_2(I,V_2)} \le C$ independently on L. Taking the the limit $L \to \infty$ in (2.2) and using $w^L \rightharpoonup w$ in $L_2(I, V_2)$, we conclude that $\{u, w\}$ is a variational solution to Problem P. The uniqueness of the variational solution to Problem P^L guarantees that the original sequences $\{\bar{u}^n\}$ and $\{\bar{w}^n\}$ are convergent.

To prove the convergence of $\bar{w}^n \to w^L$ in $L_2(I, V_2)$ for $n \to \infty$ we follow the same idea as in the proof of the convergence $\bar{u}^n \to u$ in $L_2(I, V_1)$. First, we show that

$$\limsup \int_0^t \big(\delta(b(\bar{u}^n)\bar{w}^n), \bar{w}^n - w\big) dt \ge 0, \quad a.e.\ t \in I. \tag{4.19}$$

For this purpose we use Abel's summation to obtain the equality

$$\begin{aligned}\sum_{i=1}^{j} \big(b(u_i)w_i - b(u_{i-1})w_{i-1}, w_i\big) = \frac{1}{2}\int_\Omega b(u_j)w_j^2 dx - \frac{1}{2}\int_\Omega b(u_0)w_0^2 dx \\ + \frac{1}{2}\sum_{i=1}^{j} \tau \int_\Omega \delta b(u_i) w_i^2 dx \\ + \frac{1}{2}\sum_{i=1}^{j} \int_\Omega b(u_{i-1})(w_i - w_{i-1})^2 dx.\end{aligned}$$

We take the limit $n \to \infty$ and use the convergences $\delta b(\bar{u}^n) \rightharpoonup \partial_t b(u)$, $\bar{w}^n, \bar{w}^n_\tau \to w$ in $L_2(I, L_2)$. We get

$$\begin{aligned}&\limsup \int_0^t \big(\delta(b(\bar{u}^n)\bar{w}^n), \bar{w}^n\big) dt \\ &\ge \frac{1}{2}\int_\Omega b(u(t))w^2(t)\, dx - \frac{1}{2}\int_\Omega b(u_0)w_0^2\, dx + \frac{1}{2}\int_\Omega \partial_t b(u) w^2\, dx\, dt.\end{aligned} \tag{4.20}$$

Then, from Lemma 4.3 and Lemma 2.4 we obtain (4.19). Next, we put $\eta(s) = (\bar{w}^n(s) - w(s) - (\bar{w}^D_n - w^D))\chi_{(0,t)}$ in (4.17), where $\chi_{(0,t)}$ is the characteristic function of $(0, t)$, and pass to the limit for $n \to \infty$. The elliptic part gives

$$\begin{aligned}&\int_0^t \big(D^L_n(\bar{u}^n, \nabla\bar{u}^n)\nabla\bar{w}^n, \nabla(\bar{w}^n - w)\big) dt \\ &\ge C_A \int_0^t \|\bar{w}^n - w\|^2 dt - \int_I \big(D^L_n(\bar{u}^n, \nabla\bar{u}^n)\nabla w, \nabla(\bar{w}^n - w)\big) dt.\end{aligned} \tag{4.21}$$

Since $\bar{w}^n \rightharpoonup w$ in $L_2(I, V_2)$ and since $(D^L_n(\bar{u}^n, \nabla\bar{u}^n)\nabla w \to (D^L(u, \nabla u)\nabla w$ in $L_2(I, L_2)$, the last term in (4.21) converges to 0 as $n \to \infty$. As the remaining terms in (4.17) also converge to 0, the convergence of $\{\bar{w}^n\}$ in $L_2(I, V_2)$ follows and the proof is complete. □

5. Full discretization scheme

The convergence results obtained in Section 4 remain valid for a suitable full discretization scheme where (3.1)–(3.2) are approximated on finite dimensional spaces, e.g., by a FEM. We look for $u_i^\gamma \in V_{1,\gamma} \subset V_1$, $w_i^\gamma \subset V_{2,\gamma} \subset V_2$, where $\dim V_{1,\gamma} < \infty$, $\dim V_{2,\gamma} < \infty$ and $V_{1,\gamma} \to V_2$, $V_{2,\gamma} \to V_2$ for $\gamma \to 0$ in canonical sense. We determine u_i^γ, w_i^γ for $i = 1, \ldots, n$ from (see (3.1), (3.2))

$$\left(\lambda_i \frac{u_i^\gamma - u_{i-1}^\gamma \circ \varphi_1^i}{\tau}, v\right) + \big(\bar{A}\nabla u_i^\gamma, \nabla v\big) = \big(f_i - \mathrm{div}_x \bar{F}(x, u_{i-1}^\gamma), v\big), \ \forall v \in V_{1,\gamma} \quad (5.1)$$

and

$$\left(b(u_i) \frac{w_i^\gamma - w_{i-1}^\gamma \circ \varphi_2^i}{\tau}, \psi\right) + \big(D_i^L \nabla w_i^\gamma, \nabla \psi\big) = \big(G(t_i, x, w_{i-1}^\gamma), \psi\big), \ \forall \psi \in V_{2,\gamma}, \quad (5.2)$$

where $u_0^\gamma := P_\gamma^{(1)} u_0$ and $w_0^\gamma := P_\gamma^{(2)} w_0$, with $P_\gamma^{(l)} : V_l \to V_{l,\gamma}$, $(l = 1, 2)$, being orthogonal projections, and where $\lambda_i \in L_\infty(\Omega)$ satisfies

$$\left| \lambda_i - \frac{b(u_i^\gamma) - b(u_{i-1}^\gamma \circ \varphi_1^i)}{u_i^\gamma - u_{i-1}^\gamma \circ \varphi_1^i} \right| \le \tau \,. \quad (5.3)$$

Here, $D_i^L = D(\sigma_L(u_i^\gamma), \sigma_L(\nabla u_i^\gamma))$, with

$$\varphi_1^i(x) \equiv x - \tau \omega_h * \frac{\bar{F}_u'(x, u_{i-1}^\gamma)}{b'(u_{i-1}^\gamma)} \quad \text{and} \quad \varphi_2^i(x) \equiv x - \tau \omega_h * \frac{\bar{v}(u_i^\gamma, \nabla u_i^\gamma)}{b(u_i^\gamma)} \,.$$

Let $\alpha := (\tau, \gamma)$ represents the discretization parameter corresponding to the time and space discretization. By means of u_i^γ, w_i^γ we define the step functions

$$\bar{u}^\alpha(t) = u_i^\gamma \quad \text{and} \quad \bar{w}^\alpha(t) = w_i^\gamma \quad \text{for } t \in (t_{i-1}, t_i),\ i = 1, \ldots, n. \quad (5.4)$$

Following the arguments in Section 4 we prove the convergence $\bar{u}^\alpha \to u$ and $\bar{w}^\alpha \to w^L$ in corresponding functional spaces for $\alpha \to 0$, where $\{u, w^L\}$ is a variational solution to the problem (1.1)–(1.5) in the sense of Definition 2.1. By $\{\bar{\alpha}\}$ we denote a subsequence of $\{\alpha\}$. We shall assume that

$$\|P_\gamma^{(2)} v\|_\infty \le C \|v\|_\infty \quad \forall v \in V_{2,\gamma} \cap L_\infty(\Omega) \,. \quad (5.5)$$

Theorem 5.1. *Retain the assumptions of Theorem* 4.5. *Assume that* (5.5) *holds. Moreover, let* $V_{l,\gamma} \to V_l$ $(l = 1, 2)$ *in the canonical sense. Then, one has that* $\bar{u}^{\bar{\alpha}} \to u$ *in* $L_2(I, V_1)$ *and* $\bar{w}^{\bar{\alpha}} \to w^L$ *in* $L_2(I, W_2^1)$ *for* $\bar{\alpha} \to 0$, *where* $\{u, w^L\}$ *is a variational solution to Problem* P^L *and* $\bar{u}^\alpha$ *and* $\bar{w}^\alpha$ *are from* (5.1)–(5.4). *If Problem* P^L *has a unique solution (see, e.g., Theorem* 2.5*), then the original sequences* $\{\bar{u}^\alpha\}$ *and* $\{\bar{w}^\alpha\}$ *are convergent.*

Proof. We follow the arguments in Section 4 and obtain a priori estimates for u_i^γ as in Lemma 4.2. Consequently, we get the same a priori estimates for w_i^γ as in Lemma 4.3 and the compactness of $\{\bar{u}^\alpha\}$ as in Lemma 4.4. In the same way as in Lemma 4.4 we find that $\delta b(\bar{u}^\alpha) \rightharpoonup \partial_t b(u)$ in $L_2(I, L_2)$. However, we cannot verify that $\delta(b(\bar{u}^\alpha)\bar{w}^\alpha) \rightharpoonup \partial_t(b(u)w)$ in $L_2(I, V_2^*)$, since we do not obtain the uniform boundedness of the functionals $\delta(b(\bar{u}^\alpha)\bar{w}^\alpha) \in L_2(I, V_2^*)$, (uniformly for α). We only have

$$\|\delta(b(\bar{u}^\alpha)\bar{w}^\alpha)\|_{L_2(I,V_{2,\gamma}^*)} \le C.$$

We extend the functional $\delta(b(\bar{u}^\alpha)\bar{w}^\alpha) \in L_2(I, V_{2,\gamma}^*)$ to $\mathcal{F}^\alpha \in L_2(I, V_2^*)$, so that $\|\mathcal{F}^\alpha\|_{L_2(I,V_2^*)} \le \|\mathcal{F}^\alpha\|_{L_2(I,V_{2,\gamma}^*)}$ by the definition

$$\int_I \langle \mathcal{F}^\alpha, v\rangle dt := \int_I \big\langle \delta(b(\bar{u}^\alpha)\bar{w}^\alpha), P_\gamma^{(2)} v\big\rangle\, dt = \int_I\int_\Omega \delta(b(\bar{u}^\alpha)\bar{w}^\alpha) P_\gamma^{(2)} v\, dx\, dt. \quad (5.6)$$

Then, $\mathcal{F}^\alpha \rightharpoonup \mathcal{F}$ in $L_2(I, V_2^*)$ since $L_2(I, V_2^*)$ is reflexive. Due to the compactness of $\{\bar{u}^\alpha\}$ and $\{\bar{w}^\alpha\}$ in $L_2(I, L_2)$, we have that $b(\bar{u}^\alpha) \to b(u)$ and $b(\bar{u}^\alpha)\bar{w}^\alpha) \to b(u)w$ in $L_2(I, L_2)$ for $\alpha \to 0$. From these facts we find that $\mathcal{F} = \partial_t(b(u)w)$.

To prove that $\{u,\ w^L\}$ is the variational solution to P^L, we use a test function $v = \tilde{z}^\alpha$ in (5.1), where $\tilde{z}^\alpha \to v$ in $L_2(I, V_1)$ for $\alpha \to 0$, and rewrite the equation in a similar way as (4.16). By the same arguments as in Theorem 4.5, taking the limit $\alpha \to 0$, we conclude that u satisfies (2.1).

We write $\bar{u}^\alpha = \bar{z}^\alpha + \bar{u}^{D,\alpha}$ and $u = z + u^D$. Let $\tilde{z}^\alpha \in L_2(I, V_{1,\gamma})$ be such that $\tilde{z}^\alpha \to z$ in $L_2(I, V_1)$, i.e., $\tilde{z}^\alpha + \bar{u}^{D,\alpha} \to u$ for $\alpha \to 0$ in $L_2(I, W_2^1)$. To prove $\bar{u}^\alpha \to u$ for $\alpha \to 0$ in $L_2(I, W_2^1)$ we choose the test function $v = \bar{u}^\alpha - (\tilde{z}^\alpha + \bar{u}^{D,\alpha})$ in (5.1) and transform the remaining equation in a similar way as (4.16). Using the same arguments as in the proof of Theorem 4.5 and noticing that

$$\int_I \big(\bar{A}\nabla(z^\alpha + \bar{u}^{D,\alpha}), \nabla(\bar{u}^\alpha - (z^\alpha + \bar{u}^{D,\alpha}))\big)\, dt \to 0, \quad \text{for } \alpha \to 0,$$

we arrive at $\bar{u}^\alpha \to u$ in $L_2(I, V_1)$ for $\alpha \to 0$. To prove that (2.2) holds, we use $\delta(b(\bar{u}^\alpha)\bar{w}^\alpha) \rightharpoonup \partial_t(b(u)w)$ in $L_2(I, V_2^*)$ and $\bar{u}^\alpha \to u$ in $L_2(I, V_1)$. By the same arguments as in Lemma 4.4 we can prove that $\bar{w}^\alpha \to w^L$ in $L_2(I, L_2)$ and $\bar{w}^\alpha \rightharpoonup w^L$ in $L_2(I, W_2^1)$. For this purpose we first rewrite (5.2) in a similar way as (3.2), where the splitting (4.11) is taken into account. Then, we multiply by τ and sum up for $i = j+1, \dots, j+k$. Next, we choose the test function $\psi = b(u_{j+k}^\gamma)w_{j+k}^\gamma - b(u_j^\gamma)w_j^\gamma - (P_\gamma^{(2)}[b(u_{j+k}^\gamma)w_{j+k}^{D,\gamma}] - P_\gamma^{(2)}[b(u_j^\gamma)w_j^{D,\gamma}])$ and sum up the resulting equation for $j = 1, \dots, n-k$. Similarly as in the proof of Lemma 4.3 we obtain

$$J_{1,1}^0 \ge \sum_{j=1}^{n-k} \big\|b(u_{j+k}^\gamma)w_{j+k}^\gamma - b(u_j^\gamma)w_j^{D,\gamma}\big\|_0^2 \tau$$

and

$$
\begin{aligned}
|J_{1,1}^D| &\le \varepsilon \sum_{j=1}^{n-k} \left\| b(u_{j+k}^\gamma) w_{j+k}^\gamma - b(u_j^\gamma) w_j^{D,\gamma} \right\|_0^2 \tau \\
&\quad + C_\varepsilon \sum_{j=1}^{n-k} \sum_{i=j}^{j+k} \left\| P(2)_\gamma [\delta b(u_i^\gamma) w_i^{D,\gamma}] \right\|_0 \cdot \left\| P_\gamma^{(2)} [b(u_{i-1}^\gamma) \delta w_i^{D,\gamma}] \right\|_0 \tau^2 \\
&\le \varepsilon \sum_{j=1}^{n-k} \left\| b(u_{j+k}^\gamma) w_{j+k}^\gamma - b(u_j^\gamma) w_j^{D,\gamma} \right\|_0^2 \tau + C_\varepsilon k \tau
\end{aligned}
$$

since $\|P_\gamma^{(2)}\| \le C$. The remaining steps are similar as in the proof of Theorem 4.5 and we conclude that $\bar{w}^\alpha \to w^L$ in $L_2(I, L_2)$ and $\bar{w}^\alpha \rightharpoonup w^L$ in $L_2(I, W_2^1)$ for $\alpha \to 0$. To show that $w = w^L$ obeys (2.2) and that $\{u, w^L\}$ is a variational solution to the problem (1.1)–(1.5) in the sense of Definition 2.1, we proceed as follows. Take $v \in L_2(I, V_2) \cap L_\infty(Q_T)$. Use the test function $\psi = P_\gamma^{(2)} v$ in (5.2) and rewrite this equation similarly as (4.17). In the first (parabolic) term we use (5.5). The crucial point is to prove that

$$
\int_I \left(b(\bar{u}^\alpha) \frac{\bar{w}_\tau^\alpha - \bar{w}_\tau^\alpha \circ \bar{\varphi}_2^\alpha}{\tau}, P_\gamma^{(2)} v \right) dt \to \int_I (\bar{v} \nabla w, v) dt \quad \text{for } \alpha \to 0.
$$

To this end we use

$$
\frac{\bar{w}_\tau^\alpha - \bar{w}_\tau^\alpha \circ \bar{\varphi}_2^\alpha}{\tau} = \int_0^1 \nabla \tilde{\bar{w}}_\tau^\alpha (x + s(\varphi^\alpha(x) - x)\, ds \cdot \omega_h * \frac{\bar{v}(\bar{u}^\alpha, \nabla \bar{u}^\alpha)}{b(\bar{u}^\alpha)}.
$$

Similarly as in Theorem 4.5 we obtain $\int_0^1 \nabla \tilde{\bar{w}}_\tau^\alpha (x + s(\varphi^\alpha(x) - x) ds \rightharpoonup \nabla w$ in $L_2(I, L_2)$ and $\omega_h * \frac{\bar{v}(\bar{u}^\alpha, \nabla \bar{u}^\alpha)}{b(\bar{u}^\alpha)} \to \frac{\bar{v}(u, \nabla u)}{b(u)}$ in $L_2(I, L_2)$ for $\alpha \to 0$. Then, using (5.5) and (5.6) we conclude that $w = w^L$ satisfies (2.2) and $\{u, w^L\}$ is a variational solution to problem (1.1)–(1.5) in the sense of Definition 2.1.

It remains to prove the $L_2(I, W_2^1)$-convergence of $\{\bar{w}^\alpha\}$. We follow the idea used for the convergence of $\{u^\alpha\}$ in $L_2(I, W_2^1)$. Let $w^\alpha = \bar{y}^\alpha + \bar{w}^{D,\alpha}$ and $w = y + w^D$, where $\bar{y}^\alpha \in L_2(I, V_{2,\gamma})$ and $\bar{y}^\alpha \rightharpoonup y$ in $L_2(I, W_2^1)$. Let $\tilde{y}^\alpha \in L_2(I, V_{2,\gamma})$ be such that $\tilde{y}^\alpha \to y$ in $L_2(I, W_2^1)$. Then , we put $\psi = \bar{w}^\alpha - (\tilde{y}^\alpha + \bar{w}^{D,\alpha})$ in (5.2) and rewrite this equation in a similar way as (4.17). We have that

$$
\limsup_{\alpha \to 0} \int_0^t \big(\delta(b(\bar{u}^\alpha) \bar{w}^\alpha), \bar{w}^\alpha - (\tilde{y}^\alpha + \bar{w}^{D,\alpha}) \big) dt \ge 0 \quad \text{for a.e. } t \in I
$$

since

$$
\begin{aligned}
\int_0^t \big(\delta(b(\bar{u}^\alpha) \bar{w}^\alpha), \bar{w}^\alpha \big) dt &= \frac{1}{2} \big(b(\bar{u}^\alpha), \{\bar{w}^\alpha\}^2 \big) - \frac{1}{2} \big(b(u_0), w_0^2 \big) \\
&\quad + \frac{1}{2} \int_0^t \int_\Omega \delta b(\bar{u}^\alpha (\bar{w}^\alpha)^2 dx\, dt + \frac{1}{2} \int_0^t \int_\Omega b(\bar{u}^\alpha (\bar{w}^\alpha - \bar{w}_\tau^\alpha)^2 dx\, dt \\
&\equiv J_\alpha(t) \quad \text{for } t = t_i \ \big(t_i = t_i^{(n)} \big)
\end{aligned}
$$

and

$$\limsup_{\alpha\to 0} J_\alpha(t) \ge J(t) := \frac{1}{2}\int_\Omega b(u(t))w^2(t)\,dx - \frac{1}{2}\int_\Omega b(u_0)w_0^2\,dx + \frac{1}{2}\int_0^t\int_\Omega \partial_t b(u)w\;dx\,dt \quad \text{for a.e. } t\in I.$$

On the other hand, $\int_0^t (\delta(b(\bar u^\alpha)\bar w^\alpha), \tilde y^\alpha + \bar w^{D,\alpha})dt \to J(t)$ for a.e. $t \in I$ for $\alpha \to 0$ because of $\delta(b(\bar u^\alpha)\bar w^\alpha) \rightharpoonup \partial_t(b(u)w)$ and $\tilde y^\alpha + \bar w^{D,\alpha} \to w^L$ in $L_2(I, W_2^1)$ for $\alpha \to 0$. Similarly as in (4.21) we have

$$\begin{aligned}&\int_0^t \big(D^L(\bar u^\alpha, \nabla\bar u^\alpha), \nabla(\bar w^\alpha - (\tilde y^\alpha + \bar w^{D,\alpha}))\big)dt\\ &\ge C_A\int_0^t \|\nabla(\bar w^\alpha - w)\|_0^2 dt - \int_0^t \big(D^L(\bar u^\alpha, \nabla\bar u^\alpha)\nabla w, \nabla(\bar w^\alpha - w)\big)dt\\ &\quad + \int_0^t \big(D^L(\bar u^\alpha, \nabla\bar u^\alpha)\nabla\bar w^\alpha, \nabla(w - (\tilde y^\alpha + \bar w^{D,\alpha}))\big)dt,\end{aligned}$$

where the last two terms converge to 0 with $\alpha \to 0$. Then we obtain $\bar w^\alpha \to w \equiv w^L$ in $L_2(I, W_2^1)$ along the same lines as in the proof of Theorem 4.5. Thus the proof is complete. □

Remark 5.2. If the convective term generated by $\bar F(x,u)$ is not dominant (with respect to the diffusion), then a simplified approximation scheme can be used. Here, (1.8) and (1.10) are replaced by

$$\lambda_i \frac{u_i - u_{i-1}}{\tau} - \operatorname{div}(\bar A\nabla u_i) = f_i - \operatorname{div}_x \bar F(x, u_{i-1}) - \frac{u_i - u_{i-1}\circ\varphi_3^i}{\tau}$$

with the "convergence" condition $\big|\lambda_i - \frac{b(u_i)-b(u_{i-1})}{u_i - u_{i-1}}\big| < \tau$, where $\varphi_3^i(x) := x - \tau\omega_h * \bar F_u'(x, u_{i-1})$. In this case the only change in the convergence analysis concerns the uniform L_∞-boundedness of u_i, $i = 1, \ldots, n$. If this boundedness is shown, then all other results remain valid.

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