Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 28 (2009), 411–430

Asymptotic Behavior of Generalized Eigenvectors of Jacobi Matrices in the Critical ("Double Root") Case

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Abstract. This paper is concerned with asymptotic behavior of generalized eigenvectors of a class of Hermitian Jacobi matrices J in the critical case. The last means that the fraction $\frac{q_n}{\lambda_n}$ generated by the diagonal entries q_n of J and its subdiagonal elements λ_n has the limit ± 2. In other words, the limit transfer matrix as $n \to \infty$ contains a Jordan box (= double root in terms of Birkhoff–Adams theory). This is the situation where the asymptotic Levinson theorem does not work and one has to elaborate more special methods for asymptotic analysis. It should be mentioned that the critical case exactly corresponds to spectral phase transition phenomena, where the spectral structure changes dramatically (from discreet spectrum to pure absolutely continuous one) whenever the parameters in matrix entries cross singular surfaces, see J. Janas and S. Naboko [Spectral properties of selfadjoint Jacobi matrices coming from birth and death processes, *Oper. Theory Adv. Appl.* 127 (2001), pp. 387–397]. A Jordan box is the limit transfer matrix for all values of the spectral parameter λ simultaneously, it describes the "moment" of spectral phase transition. An application to the case of $\lambda_n = n^\alpha(1 + r_n)$, $q_n = -2n^\alpha(1 + p_n)$ with small perturbations r_n, p_n and $\alpha \in (0, 1]$ is studied.
Keywords. Generalized eigenvector, Jacobi matrix, asymptotic behavior of solutions, spectrum, subordinacy theory, WKB asymptotics

Mathematics Subject Classification (2000). 39A10, 47B25

1. Introduction

In the last ten years appeared several papers devoted to spectral analysis of unbounded, self-adjoint Jacobi matrices [5–11,14,17,19–23,27,28,33–36]. Given

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a sequence $\{\lambda_n\}$ of positive numbers and a sequence $\{q_n\}$ of real numbers the Jacobi operator J is defined in $l^2 = l^2(\mathbb{N})$ by

$$\begin{aligned}(Ju)_1 &= q_1u_1 + \lambda_1u_2\\ (Ju)_n &= \lambda_{n-1}u_{n-1} + q_nu_n + \lambda_nu_{n+1} \quad (n > 1).\end{aligned} \tag{1.1}$$

More precisely, on its maximal domain J is always symmetric and sometimes selfadjoint. In the case when there exist limits (as n tends to infinity) of $\frac{q_n}{\lambda_n}$ and $\frac{\lambda_{n-1}}{\lambda_n}$ or sequences $\{\lambda_n\}$ and (or) $\{q_n\}$ are periodically perturbed, spectral analysis of J has been partially done in [17, 21, 22]. This analysis was based on Gilbert–Pearson subordinacy theory [15] (due to Khan and Pearson [26]) combined with various variants of discrete versions of Levinson theorem, see [12, 17, 30, 32, 33]. An especially interesting situation appears if $\lim_n \frac{q_n}{\lambda_n} = \pm 2$ which corresponds to the phase transition phenomena. If $|\lim_n \frac{q_n}{\lambda_n}| < 2$, then (under some regularity assumptions on q_n, λ_n) the spectrum of J is absolutely continuous; and when $\lim_n \frac{|q_n|}{\lambda_n} > 2$ the spectrum of J is discrete [20]. In this work we consider a special class of sequences $\{\lambda_n\}$ and $\{q_n\}$ given by

$$\lambda_n = n^\alpha(1 + x_n), \qquad q_n = -2n^\alpha(1 + y_n), \tag{1.2}$$

where $\alpha \in (0, 1]$ and $n^{\frac{\alpha}{2}}x_n$, $n^{\frac{\alpha}{2}}y_n$ belong to l^1 the standard space of summable sequences. This corresponds to the critical situation (double root) where $\lim_{n\to\infty} \frac{q_n}{\lambda_n} = -2$ (the most difficult case for investigation). Generally speaking, formulae (1.2) represent a very special case of the critical situation mentioned above. However, the aim of our paper is to demonstrate a new technique for treating the spectral phase transition point. The Jacobi matrix (1.2) is one of the simplest non-trivial model satisfying this aim. Note that for $q_n = 2n^\alpha(1+y_n)$ one can make a change of variables (diagonal unitary transformation) reducing to the above mentioned situation. The critical case corresponds to the situation where the limit of the transfer matrix for the recurrent equation (1.1) is given by the Jordan box. In other words it means the appearance of the irregular singular point with double root of the characteristic equation related to (1.1) in Birkhoff–Adams theory [1,4]. Recall that the case $\lambda_n = n+a$, $q_n = -2n$ related to the birth and death processes [24] was already studied in [19]. However, even in that paper spectral analysis of J was carried out only partially. The reason for this was due to difficulties in the study of asymptotic behavior of generalized eigenvectors of J, i.e, the solutions of the infinite system of equations

$$(Ju)_n = \lambda u_n, \qquad n = 2, 3, \ldots \tag{1.3}$$

for $\lambda > -1$. Later this problem was solved in unpublished work [31].

It turns out that the analysis of asymptotic behavior of solutions of (1.3) depends on the sign of λ. Namely for $\lambda < 0$ so-called "Ansatz" idea is used, while for $\lambda > 0$ a combination of the WKB (Wentzel–Kramers–Brillouin) approach with detailed analysis of products of the transfer matrices is applied (see Section 2 for details). In order to avoid some cumbersome formulae we shall present our results only for $\alpha \in (\frac{1}{2}, \frac{2}{3})$. But the methods we propose work for arbitrary $\alpha \in (0, 1]$ (see for some comments below). Note that the case $\alpha = 1$ can be easily deduced from the Birkhoff–Adams theory [13]. Unfortunately, this theory does not apply for $\alpha < 1$. We think that the ideas used in this work can also be efficient for other sequences $\{\lambda_n\}$, $\{q_n\}$ corresponding to the critical case $\lim_n \frac{q_n}{\lambda_n} = \pm 2$. Finally, we mention about still another approach to the critical case given by the first name author in [16]. This approach relies on some ideas found by W. Kelley in [25].

The paper consists of four sections. Section 2 contains necessary notions and notations and explains the WKB approach to asymptotic analysis of solutions of (1.3). Section 3 presents an asymptotic formula for a basis of solutions of (1.3) with $\lambda > 0$ (hyperbolic case). In turn Section 4 does the same for $\lambda < 0$ (elliptic case). The last Section contains applications of asymptotic results to spectral analysis of J.

2. Preliminaries

First note that the operator J defined by the sequences $\lambda_n = n^\alpha(1 + x_n)$ and $q_n = -2n^\alpha(1 + y_n)$, $\alpha \in (0, 1]$, is self-adjoint provided it acts on the maximal domain $D(J) := \{f \in l^2 : \{(Jf)_n\} \in l^2\}$. This is clear by the Carleman condition $\sum_k \frac{1}{\lambda_k} = +\infty$ [3]. As usual we rewrite the system (1.3) in the form

$$\vec{u}_{n+1} = B_n(\lambda)\vec{u}_n, \qquad \text{where } \vec{u}_n = \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}, \ B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{\lambda_{n-1}}{\lambda_n} & \frac{\lambda - q_n}{\lambda_n} \end{pmatrix}. \tag{2.1}$$

The matrix $B_n(\lambda)$ is called the transfer matrix of J. Therefore asymptotic behavior of solutions of (1.3) is equivalent to asymptotic behavior of arbitrary long products of the $B_k(\lambda)$'s. This idea was frequently used in many works on spectral properties of Jacobi operators. However, we start with another approach to the problem of asymptotic behavior. Our approach is based on the idea of application of the WKB asymptotic formula for solutions of a suitable differential equation related to (1.3). It should be mentioned that the idea to replace a difference relation by a proper continuous differential equation has been already used in theory of orthogonal polynomials. We describe below how to find this differential equation. Dividing (1.3) by n^α and disregarding lower order terms: $\frac{x_n u_{n+1}}{n^\alpha}$, $\frac{x_{n-1} u_{n-1}}{n^\alpha}$, $\frac{y_n u_n}{n^\alpha}$ and $O\left(\frac{1}{n^2}\right) u_{n-1}$ we rewrite (1.3) approximately as

$$u_{n+1} + u_{n-1} - 2u_n - \frac{\alpha}{n} u_{n-1} - \frac{\lambda}{n^\alpha} u_n \approx 0 \tag{2.2}$$

for large n. Denoting $\Delta f(n) := f(n+1) - f(n)$ and changing u_n by a continuous function $u(n)$ for $n \in \mathbb{R}^+$ we have $\Delta^2 u(n-1) + \frac{\alpha}{n}\Delta u(n-1) - \left(\frac{\lambda}{n^\alpha} + \frac{\alpha}{n}\right) u(n) \approx 0$. Replacing $\Delta^2 u(n-1)$ and $\Delta u(n-1)$ by $u''(n)$ and $u'(n)$ respectively, we obtain

$$u''(n) + \frac{\alpha}{n}u'(n) - \left(\frac{\lambda}{n^\alpha} + \frac{\alpha}{n}\right) u(n) \approx 0 \tag{2.3}$$

for $n \gg 1$. The change of $u(n)$ by $n^{-\frac{\alpha}{2}}v(n)$ allows to rewrite (2.3) as

$$v''(n) + \left[\frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)\frac{1}{n^2} - \frac{\alpha^2}{2n^2} - \frac{\alpha}{n} - \frac{\lambda}{n^\alpha}\right] v(n) \approx 0.$$

Finally, the above heuresis leads to the equation

$$v''(n) - \left(\frac{\lambda}{n^\alpha} + \frac{\alpha}{n}\right) v(n) = 0. \tag{2.4}$$

Denote $Q(n) := \frac{\lambda}{n^\alpha} + \frac{\alpha}{n}$. Applying to (2.4) the standard WKB formula [29] we find that (2.4) has a base of linearly independent solutions $v_\pm(\cdot)$ with the asymptotic given by $v_\pm(n) \sim Q(n)^{-\frac{1}{4}} \exp\left[\pm \int_1^n Q(t)^{\frac{1}{2}} dt\right]$, $n \to \infty$. Therefore one could make a reasonable "Ansatz" on the asymptotic formula for solutions of (1.3),

$$\begin{aligned} u_n &= n^{-\frac{\alpha}{2}} v(n) \\ &\sim n^{-\frac{\alpha}{4}} \exp\left[\pm \int_1^n \left(\frac{\lambda}{t^\alpha} + \frac{\alpha}{t}\right)^{\frac{1}{2}} dt\right] \\ &\sim n^{-\frac{\alpha}{4}} \exp\left(\pm\left[a_1 n^{1-\frac{\alpha}{2}} + \lambda^{-\frac{1}{2}} n^{\frac{\alpha}{2}} - \int_1^n \left(a_2 t^{\frac{3\alpha}{2}-2} + O\left(t^{\frac{5\alpha}{2}-3}\right)\right) dt\right]\right), \end{aligned} \tag{2.5}$$

where $a_1 = \sqrt{\lambda}\left(1 - \frac{\alpha}{2}\right)^{-1}$, $a_2 = \frac{\alpha^2}{8\lambda^{\frac{3}{2}}}$. In particular for $\alpha < \frac{2}{3}$ both terms $a_2 t^{\frac{3\alpha}{2}-2}$ and $O\left(t^{\frac{5\alpha}{2}-3}\right)$ are integrable over $(t_0, +\infty)$, for any $t_0 > 0$. By the way, this fact is one of the reasons to put an additional condition $\alpha < \frac{2}{3}$. As we shall see below (Section 3, Theorem 3.2) formula (2.5) is not correct. This phenomenon is due to inaccuracy between continuous approximation and the differential equation. Nevertheless, the essential part of (2.5) remains valid! For example the leading term of (2.5) given by

$$n^{-\frac{\alpha}{4}} exp\left(\pm\sqrt{\lambda}\left(1 - \frac{\alpha}{2}\right)^{-1} n^{1-\frac{\alpha}{2}}\right) \tag{2.6}$$

is correct. As it was mentioned above if $\alpha = 1$ asymptotic behavior of solutions of (1.3) can be deduced by applying classical result of Birkhoff–Adams ([13, Theorem 8.36]). This is no longer possible for $\alpha \in (0,1)$ as one can easily verify by checking the assumption of Theorem 8.36 in [13].

3. Case of positive λ and $\alpha \in (\frac{1}{2}, \frac{2}{3})$: hyperbolic situation

In what follows l^1, to avoid tedious notations, will also denote the space of vectors or matrices whose norms are summable sequences and we hope that this will not lead to misunderstanding. Since

$$\lambda_n = n^\alpha(1 + x_n), \qquad q_n = -2n^\alpha(1 + y_n), \tag{3.1}$$

where $n^{\frac{\alpha}{2}} x_n$ and $n^{\frac{\alpha}{2}} y_n$ belong to l^1, it follows that $\frac{\lambda_{n-1}}{\lambda_n} = 1 - \frac{\alpha}{n} + r_n^{(1)}$, $\frac{\lambda}{\lambda_n} = \frac{\lambda}{n^\alpha} + r_n^{(2)}$, and $\frac{q_n}{\lambda_n} = -2 + r_n^{(3)}$, where $\{r_n^{(i)} n^{\frac{\alpha}{2}}\} \in l^1$, $i = 1, 2, 3$. Therefore the transfer matrix

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \frac{1}{n^\alpha}\begin{pmatrix} 0 & 0 \\ \phi_n & \lambda \end{pmatrix} + R_n \tag{3.2}$$

with $\phi_n = \alpha n^{\alpha-1}$ and $\|R_n\| = O\left(|x_{n-1}| + |x_n| + |y_n|\right) + O\left(\frac{1}{n^2}\right)$, as $n \to \infty$. The leading term of $B_n(\lambda)$, the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, is similar to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The Jordan box appearing in the leading term ($\lim_{n\to\infty} B_n(\lambda)$) is the main difficulty of the analysis. As we tried to explain in the Introduction the following sequences are essential below:

$$z_k := k^{-\frac{\alpha}{4}} \exp(\rho k^\beta), \qquad \tilde{z}_k := k^{-\frac{\alpha}{4}} \exp(-\rho k^\beta); \tag{3.3}$$

here $\rho = (1 - \frac{\alpha}{2})^{-1}\sqrt{\lambda}$, $\beta = 1 - \frac{\alpha}{2}$ ($\sqrt{\lambda} > 0$). Below we shall find an asymptotic formula for generalized eigenvectors for the above fixed $\lambda > 0$. This formula will be computed in four steps.

`Step 1`. *Introducing the Ansatz.*
Consider the matrix S_n given by

$$S_k = \begin{pmatrix} \tilde{z}_{k-1} & z_{k-1} \\ \tilde{z}_k & z_k \end{pmatrix} \tag{3.4}$$

The matrix S_n appears naturally for the system (1.3). This will become clear later. It is obvious that

$$\prod_{k=2}^{n} B_k(\lambda) = S_{n+1}\left\{\prod_{k=2}^{n}(S_{k+1}^{-1} B_k(\lambda) S_k)\right\} S_2^{-1}.$$

This allows us to reduce the product of transfer matrices $B_k(\lambda)$ to the product of matrices $(S_{k+1}^{-1} B_k(\lambda) S_k)$ which might be much simpler due to the proper choice of matrices S_k (3.4).

Proposition 3.1. *In the above notations we have for $\alpha \in (\frac{1}{2}, \frac{2}{3})$:*

$$S_{k+1}^{-1} B_k(\lambda) S_k = \big(A_{ij}(k)\big) + R_k^{(1)},$$

where $R_k^{(1)} = S_{k+1}^{-1} R_k S_k$, $\psi_k = e^{\rho k^\beta}$ *and for* k *large enough matrix elements* $A_{ij}(k)$ *satisfy the relations*

$$\begin{aligned}
A_{11}(k) &= 1 - \frac{\alpha}{2\sqrt{\lambda}} k^{\frac{\alpha}{2}-1} + \frac{(\sqrt{\lambda})^3}{4!} k^{-\frac{3\alpha}{2}} + O\left(k^{-\frac{2+\alpha}{2}}\right)\\
A_{12}(k) &= \psi_k^2 \left(- \frac{\alpha}{2\sqrt{\lambda}} k^{\frac{\alpha}{2}-1} + \frac{(\sqrt{\lambda})^3}{4!} k^{-\frac{3\alpha}{2}} + O\left(k^{-\frac{2+\alpha}{2}}\right)\right)\\
A_{21}(k) &= \psi_k^{-2} \left(\frac{\alpha}{2\sqrt{\lambda}} k^{\frac{\alpha}{2}-1} - \frac{(\sqrt{\lambda})^3}{4!} k^{-\frac{3\alpha}{2}} + O\left(k^{-\frac{2+\alpha}{2}}\right)\right)\\
A_{22}(k) &= 1 + \frac{\alpha}{2\sqrt{\lambda}} k^{\frac{\alpha}{2}-1} - \frac{(\sqrt{\lambda})^3}{4!} k^{-\frac{3\alpha}{2}} + O\left(k^{-\frac{2+\alpha}{2}}\right)
\end{aligned}$$

Proof. Using (3.3) we have

$$\begin{aligned}
&S_{k+1}^{-1} B_k(\lambda) S_k\\
&= (\det S_{k+1})^{-1} \begin{pmatrix} z_{k+1} & -z_k \\ -\tilde{z}_{k+1} & \tilde{z}_k \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \frac{1}{k^\alpha} \begin{pmatrix} 0 & 0 \\ \phi_k & \lambda \end{pmatrix} + R_k \right] \begin{pmatrix} \tilde{z}_{k-1} & z_{k-1} \\ \tilde{z}_k & z_k \end{pmatrix}\\
&= (\det S_{k+1} \cdot k^\alpha)^{-1} \begin{pmatrix} z_{k+1} & -z_k \\ -\tilde{z}_{k+1} & \tilde{z}_k \end{pmatrix} \left[k^\alpha \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \phi_k & \lambda \end{pmatrix} + k^\alpha R_k \right] \begin{pmatrix} \tilde{z}_{k-1} & z_{k-1} \\ \tilde{z}_k & z_k \end{pmatrix}.
\end{aligned}$$

Denote by

$$\big(a_{ij}(k)\big) := \begin{pmatrix} z_{k+1} & -z_k \\ -\tilde{z}_{k+1} & \tilde{z}_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \tilde{z}_{k-1} & z_{k-1} \\ \tilde{z}_k & z_k \end{pmatrix}.$$

Then tedious but straightforward computation based on the expansion of $e^x = \sum_0^5 \frac{x^k}{k!} + O(x^6)$ up to the fifth term shows that

$$\begin{aligned}
a_{11}(k) &= k^{-\alpha} \Big[2\sqrt{\lambda} + \lambda k^{-\frac{\alpha}{2}} + \frac{(\sqrt{\lambda})^3}{3} k^{-\alpha} + \frac{\lambda^2}{12} k^{-\frac{3\alpha}{2}} + \frac{2(\sqrt{\lambda})^5}{5!} k^{-2\alpha} + O\Big(k^{-\frac{2+\alpha}{2}}\Big) \Big]\\
a_{12}(k) &= k^{-\alpha} \psi_k^2 \Big[\lambda k^{-\frac{\alpha}{2}} + \frac{\lambda^2}{12} k^{-\frac{3\alpha}{2}} - \frac{\alpha\sqrt{\lambda}}{k} + O\Big(k^{-\frac{2+\alpha}{2}}\Big) \Big]\\
a_{21}(k) &= k^{-\alpha} \psi_k^{-2} \Big[- \lambda k^{-\frac{\alpha}{2}} - \frac{\lambda^2}{12} k^{-\frac{3\alpha}{2}} - \frac{\alpha\sqrt{\lambda}}{k} + O\Big(k^{-\frac{2+\alpha}{2}}\Big) \Big]\\
a_{22}(k) &= k^{-\alpha} \Big[2\sqrt{\lambda} - \lambda k^{-\frac{\alpha}{2}} + \frac{(\sqrt{\lambda})^3}{3} k^{-\alpha} - \frac{\lambda^2}{12} k^{-\frac{3\alpha}{2}} + \frac{2(\sqrt{\lambda})^5}{5!} k^{-2\alpha} + O\Big(k^{-\frac{2+\alpha}{2}}\Big) \Big].
\end{aligned}$$

Again direct calculation leads to the formula

$$\begin{aligned}
&\big(\det S_{k+1} \cdot k^\alpha\big)^{-1}\\
&= \frac{1}{2\sqrt{\lambda}} \left[1 + \frac{\alpha}{2k} - \frac{\lambda}{3!} k^{-\alpha} - \frac{\lambda^2}{5!} k^{-2\alpha} + \frac{\lambda^2}{36} k^{-2\alpha} \right] + O\Big(k^{-\frac{2+\alpha}{2}}\Big).
\end{aligned} \tag{3.5}$$

Let

$$(b_{ij}(k)) := \begin{pmatrix} z_{k+1} & -z_k \\ -\tilde{z}_{k+1} & \tilde{z}_k \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \phi_k & \lambda \end{pmatrix} \begin{pmatrix} \tilde{z}_{k-1} & z_{k-1} \\ \tilde{z}_k & z_k \end{pmatrix}.$$

We have (using definitions of z_k, $\tilde{z}_k$)

$$b_{11}(k) = -\lambda z_k \tilde{z}_k - z_k \tilde{z}_{k-1}\phi_k = -\lambda k^{-\frac{\alpha}{2}} - \alpha k^{\frac{\alpha}{2}-1} - \frac{\alpha\sqrt{\lambda}}{k} + O\left(\frac{1}{k^{1+\frac{\alpha}{2}}}\right)$$

$$b_{12}(k) = -\lambda z_k^2 - z_k z_{k-1}\phi_k = -\psi_k^2\left[\lambda k^{-\frac{\alpha}{2}} + \alpha k^{\frac{\alpha}{2}-1} - \frac{\alpha\sqrt{\lambda}}{k} + O\left(\frac{1}{k^{1+\frac{\alpha}{2}}}\right)\right]$$

$$b_{21}(k) = \psi_k^{-2}\left[\lambda k^{-\frac{\alpha}{2}} + \alpha k^{\frac{\alpha}{2}-1} + \frac{\alpha\sqrt{\lambda}}{k} + O\left(\frac{1}{k^{1+\frac{\alpha}{2}}}\right)\right]$$

$$b_{22}(k) = \lambda k^{-\frac{\alpha}{2}} + \alpha k^{\frac{\alpha}{2}-1} - \frac{\alpha\sqrt{\lambda}}{k} + O\left(\frac{1}{k^{1+\frac{\alpha}{2}}}\right).$$

Combining the above equalities (for $a_{ij}(k)$ and $b_{ij}(k)$) we obtain

$$k^\alpha a_{11}(k) + b_{11}(k) = 2\sqrt{\lambda}\left[1+ \frac{\lambda}{6}k^{-\alpha} - \frac{\alpha}{2\sqrt{\lambda}}k^{\frac{\alpha}{2}-1} - \frac{\alpha}{2k} + \frac{(\sqrt{\lambda})^3}{4!}k^{-\frac{3\alpha}{2}} + \frac{\lambda^2}{5!}k^{-2\alpha}\right] + O\left(k^{-\frac{2+\alpha}{2}}\right)$$

$$k^\alpha a_{12}(k) + b_{12}(k) = \psi_k^2\left[-\alpha k^{\frac{\alpha}{2}-1} + \frac{\lambda^2}{12}k^{-\frac{3\alpha}{2}} + O\left(k^{-\frac{2+\alpha}{2}}\right)\right]$$

$$k^\alpha a_{21}(k) + b_{21}(k) = \psi_k^{-2}\left[\alpha k^{\frac{\alpha}{2}-1} - \frac{\lambda^2}{12}k^{-\frac{3\alpha}{2}} + O\left(k^{-\frac{2+\alpha}{2}}\right)\right]$$

$$k^\alpha a_{22}(k) + b_{22}(k) = 2\sqrt{\lambda}\left[1 + \frac{\lambda}{6}k^{-\alpha} + \frac{\alpha}{2\sqrt{\lambda}}k^{\frac{\alpha}{2}-1} - \frac{\alpha}{2k} - \frac{(\sqrt{\lambda})^3}{4!}k^{-\frac{3\alpha}{2}} + \frac{\lambda^2}{5!}k^{-2\alpha}\right] + O\left(k^{-\frac{2+\alpha}{2}}\right).$$

Finally, using (3.5) and the above four equalities we verify the thesis of Proposition 3.1. The proof is complete. □

`Step 2.` *Estimate of the remainder: reducing to l^1 error terms.*
Observe that $S_{k+1}^{-1}B_k(\lambda)S_k$ has the form $I + \begin{pmatrix} a_k & \psi_k^2 a_k \\ -\psi_k^{-2}a_k & -a_k \end{pmatrix} + R_k^{(2)}$, where $a_k := -\frac{\alpha}{2\sqrt{\lambda}}k^{\frac{\alpha}{2}-1} + \frac{(\sqrt{\lambda})^3}{4!}k^{-\frac{3\alpha}{2}}$, and

$$R_k^{(2)} := R_k^{(1)} + \begin{pmatrix} O\left(k^{-1-\frac{\alpha}{2}}\right) & \psi_k^2 O\left(k^{-1-\frac{\alpha}{2}}\right) \\ \psi_k^{-2}O\left(k^{-1-\frac{\alpha}{2}}\right) & O\left(k^{-1-\frac{\alpha}{2}}\right) \end{pmatrix} + O\left(k^{-\frac{2+\alpha}{2}}\right)$$

as $k \to \infty$. Write $R_k^{(1)} := (r_{ij}^{(1)}(k))$. Due to definitions we check that

$$r_{11}^{(1)}(k) = O\left(k^{\frac{\alpha}{2}}\|R_k\|\right), \qquad r_{12}^{(1)}(k) = \psi_k^2 O\left(k^{\frac{\alpha}{2}}\|R_k\|\right)$$

$$r_{21}^{(1)}(k) = \psi_k^{-2}O\left(k^{\frac{\alpha}{2}}\|R_k\|\right), \qquad r_{22}^{(1)}(k) = O\left(k^{\frac{\alpha}{2}}\|R_k\|\right).$$

Therefore, for $r_{ij}^{(2)}(k)$ we have

$$\begin{aligned} r_{11}^{(2)}(k) &= \left(O\left(k^{-1-\frac{\alpha}{2}}\right) + O\left(k^{\frac{\alpha}{2}}\|R_k\|\right)\right) \in l^1 \\ r_{12}^{(2)}(k) &= \psi_k^2\left(O\left(k^{-1-\frac{\alpha}{2}}\right) + O\left(k^{\frac{\alpha}{2}}\|R_k\|\right)\right) \\ r_{21}^{(2)}(k) &= \psi_k^{-2}\left(O\left(k^{-1-\frac{\alpha}{2}}\right) + O\left(k^{\frac{\alpha}{2}}\|R_k\|\right)\right) \\ r_{22}^{(2)}(k) &= \left(O\left(k^{-1-\frac{\alpha}{2}}\right) + O\left(k^{\frac{\alpha}{2}}\|R_k\|\right)\right) \in l^1. \end{aligned}$$

Note that the elements $a_k\psi_k^2$ and probably $r_{12}^{(2)}(k)$ grow to infinity as $k \to \infty$, and this makes a serious problem in the analysis of the product $\prod_k (S_{k+1}^{-1}B_k(\lambda)S_k)$. We try to kill this growth by finding suitable diagonal matrices $X_k = \begin{pmatrix} x_k & 0 \\ 0 & y_k \end{pmatrix}$ such that $X_{k+1}^{-1}S_{k+1}^{-1}B_k(\lambda)S_kX_k$ (the reasoning for appearance of matrices X_{k+1}^{-1} and X_k is similar to one for S_k) will be a bounded sequence. The right choice of X_k is given by $X_k := \begin{pmatrix} 1 & 0 \\ 0 & \psi_k^{-2} \end{pmatrix}$. This choice of X_k is determined by the factorization

$$\begin{pmatrix} 1 & \psi_k^2 \\ -\psi_k^{-2} & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \psi_k^{-2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \psi_k^{-2} \end{pmatrix}^{-1}.$$

It follows that

$$X_{k+1}^{-1}a_k\begin{pmatrix} 1 & \psi_k^2 \\ -\psi_k^{-2} & -1 \end{pmatrix}X_k = a_k\begin{pmatrix} 1 & 1 \\ -\left(\frac{\psi_{k+1}}{\psi_k}\right)^2 & -\left(\frac{\psi_{k+1}}{\psi_k}\right)^2 \end{pmatrix} \tag{3.6}$$

and

$$R_k^{(3)} := X_{k+1}^{-1}R_k^{(2)}X_k = \begin{pmatrix} r_{11}^{(2)}(k) & r_{12}^{(2)}(k)\psi_k^{-2} \\ \psi_{k+1}^2 r_{21}^{(2)}(k) & r_{22}^{(2)}(k)\left(\frac{\psi_{k+1}}{\psi_k}\right)^2 \end{pmatrix} \in l^1.$$

In this way we have proved that

$$\begin{aligned} &X_{k+1}^{-1}S_{k+1}^{-1}B_k(\lambda)S_kX_k \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{\psi_{k+1}}{\psi_k}\right)^2 \end{pmatrix} + a_k\begin{pmatrix} 1 & 1 \\ -\left(\frac{\psi_{k+1}}{\psi_k}\right)^2 & -\left(\frac{\psi_{k+1}}{\psi_k}\right)^2 \end{pmatrix} + R_k^{(3)}, \end{aligned} \tag{3.7}$$

where $\|R_k^{(3)}\| \in l^1$.

`Step 3.` *Asymtotics of solutions for an auxiliary linear system.* Denote by $p_k := \left(\frac{\psi_{k+1}}{\psi_k}\right)^2 - 1$, where $p_k \sim 2\sqrt{\lambda}k^{-\frac{\alpha}{2}}$ as $k \to \infty$. Then we can rewrite (3.7) as follows:

$$X_{k+1}^{-1}S_{k+1}^{-1}B_k(\lambda)S_kX_k = I + p_kV(k) + R_k^{(3)}, \tag{3.8}$$

where

$$V(k) := \begin{pmatrix} a_kp_k^{-1} & a_kp_k^{-1} \\ -\left(\frac{\psi_{k+1}}{\psi_k}\right)^2 a_kp_k^{-1} & -\left(\frac{\psi_{k+1}}{\psi_k}\right)^2 a_kp_k^{-1} + 1 \end{pmatrix}.$$

Note that $a_k p_k^{-1} \sim \frac{\alpha}{4\lambda} k^{\alpha-1}$, $k \to \infty$. Therefore, the original system of equations can be written as

$$\vec{u}(n+1) = S_{n+1} X_{n+1} \left\{ \prod_{k=2}^{n} \left(I + p_k V(k) + R_k^{(3)} \right) \right\} X_2^{-1} S_2^{-1} \vec{u}_2 \,. \tag{3.9}$$

Consider the auxiliary linear system

$$\vec{w}(n+1) = \left(I + p_n V(n) + R_n^{(3)} \right) \vec{w}(n) \,. \tag{3.10}$$

Observe that the sequence $\{V(n)\}$ is of bounded variation, i.e., $\sum_k \|V(k+1) - V(k)\| < +\infty$ (this fact can be verified by using definition of $V(k)$; namely both $a_k p_k^{-1}$ and $\frac{\psi_{k+1}}{\psi_k}$ are of bounded variations). Let $\sigma(V(k)) = \{\mu_1(k), \mu_2(k)\}$ be the spectrum of $V(k)$, i.e.,

$$\mu_1(k) = \frac{\operatorname{tr} V(k) - \sqrt{\operatorname{discr} V(k)}}{2}, \qquad \mu_2(k) = \frac{\operatorname{tr} V(k) + \sqrt{\operatorname{discr} V(k)}}{2},$$

where $\operatorname{discr} V := (\operatorname{tr} V)^2 - 4 \det V$ is the discriminant of V. Hence

$$\mu_1(k) = a_k p_k^{-1} + O\left(\left(\frac{a_k}{p_k} \right)^2 \right) \tag{3.11}$$

$$\mu_2(k) = 1 - \frac{a_k}{p_k}(1 + p_k) + O\left(\left(\frac{a_k}{p_k} \right)^2 \right) \tag{3.12}$$

(by definition of $V(k)$). Since $p_k \sim 2\sqrt{\lambda} k^{-\frac{\alpha}{2}}$, $k \to \infty$, using (3.11) and (3.12) we have

$$p_k \mu_1(k) = a_k + O\left(\frac{1}{k^{2-3\frac{\alpha}{2}}} \right) \tag{3.13}$$

$$p_k \mu_2(k) = p_k - a_k(1 + p_k) + O\left(\frac{1}{k^{2-3\frac{\alpha}{2}}} \right). \tag{3.14}$$

Due to our assumption $\alpha < \frac{2}{3}$ all $O\big(\frac{1}{k^{2-3\frac{\alpha}{2}}}\big)$ terms in the above equations are summable. Note that $V(n) \xrightarrow[n\to\infty]{} V_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Trivially, $V_\infty \vec{e}_1 = 0\vec{e}_1$ and $V_\infty \vec{e}_2 = \vec{e}_2$, where $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$. Applying Theorem 1.7(b) of [17] we obtain a basis $\vec{w}_s$, $s = 1, 2$, of solutions of (3.10) having the asymptotic form

$$\vec{w}_1(n) = \left\{ \prod_{k=2}^{n-1} (1 + p_k \mu_1(k)) \right\} (\vec{e}_1 + o(1)) \tag{3.15}$$

$$\vec{w}_2(n) = \left\{ \prod_{k=2}^{n-1} (1 + p_k \mu_2(k)) \right\} (\vec{e}_2 + o(1)). \tag{3.16}$$

Using (3.13) and (3.14) we find

$$\prod_{k=2}^{n-1}(1+p_k\mu_1(k)) = F(n)\prod_{2}^{n-1}(1+a_k) \tag{3.17}$$

$$\prod_{k=2}^{n-1}(1+p_k\mu_2(k)) = \left\{\prod_{2}^{n-1}\left(\frac{\psi_{k+1}}{\psi_k}\right)^2(1-a_k)\right\}\cdot G(n)\,, \tag{3.18}$$

where $F(n)$, $G(n)$ converge to some positive constants. Formally speaking in the above products in formulae (3.15), (3.16) one has began to calculate the products not from $k=2$ but rather from $k=k_0 \gg 1$ to avoid any "occasional" zeros in the product factors. Let us ignore this inessential "problem" to avoid new tedious notations.

Step 4. *Returning to the original linear system: obtaining the asymptotics of the solutions.*
Combining (3.15), (3.16), (3.17) and (3.18) we find that (see (3.9)) there exists a basis $\vec{u}_s(n)$ of solutions of original system given by

$$\begin{aligned}\vec{u}_1(n+1) &= S_{n+1}X_{n+1}F(n)\left\{\prod_{2}^{n-1}(1+a_k)\right\}(\vec{e}_1+o(1))\\ &= \tilde{F}(n)\exp\left(\Sigma_2^{n-1}a_k\right)\left[\begin{pmatrix}\tilde{z}_n\\ \tilde{z}_{n+1}\end{pmatrix}+\begin{pmatrix}\tilde{z}_n & z_n\psi_{n+1}^{-2}\\ \tilde{z}_{n+1} & z_{n+1}\psi_{n+1}^{-2}\end{pmatrix}o(1)\right]\\ &= \tilde{F}(n)\exp\left(\Sigma_2^{n-1}a_k\right)\tilde{z}_{n+1}\left[\begin{pmatrix}1\\1\end{pmatrix}+o(1)\right],\end{aligned} \tag{3.19}$$

for some $\tilde{F}(n)$ convergent to a positive F. Similarly for the second solution $\vec{u}_2(.)$ we have

$$\begin{aligned}\vec{u}_2(n+1) &= \left(\frac{\psi_n}{\psi_2}\right)^2\left\{\prod_{2}^{n-1}(1-a_k)\right\}G(n)\begin{pmatrix}\tilde{z}_n & z_n\psi_{n+1}^{-2}\\ \tilde{z}_{n+1} & z_{n+1}\psi_{n+1}^{-2}\end{pmatrix}\left[\vec{e}_2+o(1)\right]\\ &= \tilde{G}(n)\left(\frac{\psi_n}{\psi_2}\right)^2\exp\left(-\Sigma_2^{n-1}a_k\right)\tilde{z}_n\left[\begin{pmatrix}1\\1\end{pmatrix}+o(1)\right],\end{aligned} \tag{3.20}$$

for some convergent $\tilde{G}(n)$ to a positive constant. Using the Euler summation formula we can rewrite (3.19) and (3.20) as

$$\begin{aligned}\vec{u}_1(n+1) &= F_1(n)n^{-\frac{\alpha}{4}}\exp\left[-\rho n^\beta - \frac{n^{\frac{\alpha}{2}}}{\sqrt{\lambda}} + \frac{(\sqrt{\lambda})^3}{4!(1-\frac{3\alpha}{2})}n^{1-\frac{3\alpha}{2}}\right]\left[\begin{pmatrix}1\\1\end{pmatrix}+o(1)\right]\\ \vec{u}_2(n+1) &= G_1(n)n^{-\frac{\alpha}{4}}\exp\left[\rho n^\beta + \frac{n^{\frac{\alpha}{2}}}{\sqrt{\lambda}} - \frac{(\sqrt{\lambda})^3}{4!(1-\frac{3\alpha}{2})}n^{1-\frac{3\alpha}{2}}\right]\left[\begin{pmatrix}1\\1\end{pmatrix}+o(1)\right],\end{aligned} \tag{3.21}$$

where $F_1(n)$, $G_1(n)$ are convergent to positive constants. Summing up we have proved

Theorem 3.2. *Let* $\lambda_n = n^\alpha(1+x_n)$, *and* $q_n = -2n^\alpha(1+y_n)$, *where* $\alpha \in (\frac{1}{2}, \frac{2}{3})$, $\{x_n n^{\frac{\alpha}{2}}\}$ *and* $\{y_n n^{\frac{\alpha}{2}}\}$ *belong to* l^1. *Fix* $\lambda > 0$. *Then the system of equations*

$$\lambda_{n-1}u_{n-1} + q_n u_n + \lambda_n u_{n+1} = \lambda u_n \quad (n > 1)$$

has two linearly independent solutions $u_1(\cdot)$ *and* $u_2(\cdot)$ *with the asymptotic given by*

$$u_1(n) \sim n^{-\frac{\alpha}{4}} \exp\left[-\rho n^{1-\frac{\alpha}{2}} - \frac{1}{\sqrt{\lambda}} n^{\frac{\alpha}{2}} + \eta n^{1-\frac{3\alpha}{2}}\right](1+o(1))$$
$$u_2(n) \sim n^{-\frac{\alpha}{4}} \exp\left[\rho n^{1-\frac{\alpha}{2}} + \frac{1}{\sqrt{\lambda}} n^{\frac{\alpha}{2}} - \eta n^{1-\frac{3\alpha}{2}}\right](1+o(1)),$$

with $\rho = \sqrt{\lambda}\left(1-\frac{\alpha}{2}\right)^{-1}$ *and* $\eta = \lambda^{\frac{3}{2}}\left[4!(1-\frac{3\alpha}{2})\right]^{-1}$.

Remark 3.3. One can formulate extensions of the above formulae to the whole interval $\alpha \in (0,1)$ but the form of them becomes more cumbersome as α tends to zero or one. Remind that the aim of the paper is just to demonstrate a new technique in critical situation (the Jordan box).

Note that the asymptotic formulae from Theorem 3.2 collapse as α approaches $\frac{2}{3}$. Therefore, in order to preserve the form of asymptotics our condition $\alpha < \frac{2}{3}$ is essential. Hence, for other regions of α our approach works but gives another form of asymptotics.

4. The case of negative λ: elliptic situation

It turns out that the analysis of asymptotic behavior of solutions of (1.3) for $\lambda < 0$ can be done in a similar way. Assumptions on x_n and y_n are also the same as for $\lambda > 0$.

The reasoning presented in the first two steps in the proof of Theorem 3.2 remains unchanged. On the other hand, the matrix $V(k)$ in equation (3.8) has now complex entries and the scalars p_k are complex as well. Therefore formally we can not use the above evoked Theorem 1.7 from [17] because this theorem concerns only real valued matrices $V(k)$ and real sequences p_k. One can extend Theorem 1.7 to the complex case and then verify that the matrices $V(k)$ from equation (3.8) satisfy assumption of the above mentioned extension of Theorem 1.7. We do not want to use this approach here for two reasons. Firstly, it would require presentation of the analysis of "the dichotomy condition" from discrete variant of the Levinson theorem (see [17]). Secondly (more essential reason), the "Ansatz" approach we will use below for negative λ may be of some interest for itself, as an alternative method in asymptotic analysis of generalized eigenvectors of Jacobi matrices. This method was alredy used in [16] but we decided to present it here again due to some clarifications made in this work (in comparison to [16]).

In what follows we assume that

$$\{x_n n^{\frac{\alpha}{2}}\} \text{ and } \{y_n n^{\frac{\alpha}{2}}\} \text{ belong to } l^1. \tag{4.1}$$

The idea of the proof is based on right "Ansatz" for the asymptotic form for solutions of (1.3). This approach has been successfully used in [19] (for negative λ and $\alpha = 1$) and in [6] for a different model. Surely the form of the Ansatz we make below is inspired by Theorem 3.2 and the WKB approach (see Section 2).

Theorem 4.1. *Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$. Suppose that λ_n and q_n are given by $\lambda_n = n^\alpha(1 + x_n)$, $q_n = -2n^\alpha(1 + y_n)$. If x_n and y_n satisfy (4.1), then for any $\lambda < 0$ there are two linearly independent solutions $\vec{u}_\pm(n)$ of*

$$\vec{u}(n+1) = B_n(\lambda)\vec{u}(n) \tag{4.2}$$

with the asymptotic given by

$$u_\pm(n) = n^{-\frac{\alpha}{4}} \exp\left[\pm i(Dn^{1-\frac{\alpha}{2}} + En^{\frac{\alpha}{2}} + Fn^{1-3\frac{\alpha}{2}})\right](1 + o(1))$$

as $n \to \infty$, where $\vec{u}(n) := \binom{u(n-1)}{u(n)}$, $D := \sqrt{-\lambda}(1 - \frac{\alpha}{2})^{-1}$, $E := -(\sqrt{-\lambda})^{-1}$, $F := \frac{(\sqrt{-\lambda})^3}{24}(1 - 3\frac{\alpha}{2})^{-1}$.

Proof. We make the Ansatz (its summation form is convenient for the calculations below):

$$z_n = n^\gamma \exp i\left[\sum_1^n \left(Ak^\delta + Bk^\epsilon + Ck^\theta\right)\right],$$

where $-1 \le \theta < \epsilon < \delta < 0$, and A, B, C, γ are some real numbers. Define the matrix corresponding to the Ansatz

$$S_n = \begin{pmatrix} \bar{z}_{n-1} & z_{n-1} \\ \bar{z}_n & z_n \end{pmatrix},$$

where $\bar{z}_n$ denotes the complex conjugate of z_n as usual.

We want to choose A, B, C, γ, ϵ, δ, θ such that

$$S_{n+1}^{-1} B_n(\lambda) S_n = I + R_n \tag{4.3}$$

for some matrices R_n with $\{\|R_n\|\} \in l^1$. The reason for the appearance of the product $S_{n+1}^{-1} B_n(\lambda) S_n$ is the same as in Section 3. It follows that an arbitrary solution of (4.2) has the form $\vec{u}_{n+1} = S_{n+1}\vec{w}_n$, where $\vec{w}_n$ is a sequence of vectors which tends to a non-zero vector. Therefore the form of the asymptotics of $\vec{u}_n$ will be determined by the matrix S_n, i.e., by the parameters A, B, C, γ, δ, ϵ, θ.

In what follows we will see that

$$A = \pm\sqrt{-\lambda}, \quad B = \mp\frac{\alpha}{2\sqrt{-\lambda}}, \quad C = \pm\frac{(\sqrt{-\lambda})^3}{24}$$
$$\delta = -\frac{\alpha}{2}, \quad \epsilon = \frac{\alpha}{2} - 1, \quad \theta = -\frac{3\alpha}{2}, \quad \gamma = -\frac{\alpha}{4},$$

where all signs should be chosen correspondingly. Note that the choice of parameters γ, δ, ϵ, θ can be easily deduced from the WKB asymptotic formula. However, we plan to derive the values from the explicit calculations on the basis of cancellation of terms.

Remark 4.2. Note that the parameters γ, D and E from the asymptotic formula in Theorem 4.1 coincide with ones appearing in (2.5) after formal substitution $\sqrt{\lambda} = i\sqrt{-\lambda}$. However, the "$F$-term" is different here. See the discussion of the situation in the Section 2.

Denote, for fixed $\lambda < 0$, $\varphi(n) := 1 - \lambda_{n-1}\lambda_n^{-1}$ and $\psi(n) := \lambda\lambda_n^{-1} + 2(1 + y_n)(1 + x_n)^{-1} - 2$, where both sequences are real. Then

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \varphi(n) & \psi(n) \end{pmatrix}.$$

We have

$$S_{n+1}^{-1}B_n(\lambda)S_n = (\det S_{n+1})^{-1}\left[\begin{pmatrix} \rho_n & \eta_n \\ -\bar{\eta}_n & -\bar{\rho}_n \end{pmatrix} + \begin{pmatrix} s_n & t_n \\ -\bar{t}_n & -\bar{s}_n \end{pmatrix}\right], \tag{4.4}$$

where

$$\begin{aligned}
\rho_n &:= |z_n|^2(\bar{z}_{n-1}(\bar{z}_n)^{-1} + z_{n+1}z_n^{-1} - 2)\\
\eta_n &:= z_n^2(z_{n-1}z_n^{-1} + z_{n+1}z_n^{-1} - 2)\\
s_n &:= |z_n|^2(-\psi(n) - \varphi(n)\bar{z}_{n-1}(\bar{z}_n)^{-1})\\
t_n &:= z_n^2(-\psi(n) - \varphi(n)z_{n-1}z_n^{-1}).
\end{aligned}$$

`Step 1.` *Calculation of the off-diagonal term.*
Below we shall estimate the off-diagonal element $(\det S_{n+1})^{-1}(\eta_n + t_n)$ of $S_{n+1}^{-1}B_n(\lambda)S_n$. We compute

$$\begin{aligned}
z_{n-1}z_n^{-1} = \left(1 - \frac{\gamma}{n} + O\left(\frac{1}{n^2}\right)\right)&\Big[1 - i\Big(An^\delta + Bn^\epsilon + Cn^\theta\Big)\\
&- \frac{1}{2}\Big(An^\delta + Bn^\epsilon + Cn^\theta\Big)^2 + \frac{i}{3!}\Big(An^\delta + Bn^\epsilon + Cn^\theta\Big)^3 +\\
&+ \frac{1}{4!}\Big(An^\delta + Bn^\epsilon + Cn^\theta\Big)^4 + O\left(n^{5\delta}\right)\Big]
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned} z_{n+1}z_n^{-1} = {} & \left(1+\frac{\gamma}{n}+O\left(\frac{1}{n^2}\right)\right)\Big[1+i\Big(A(n+1)^\delta+B(n+1)^\epsilon \\ & +C(n+1)^\theta\Big)-\frac{1}{2}\Big(A(n+1)^\delta+B(n+1)^\epsilon+C(n+1)^\theta\Big)^2 \\ & -\frac{i}{3!}\Big(A(n+1)^\delta+B(n+1)^\epsilon+C(n+1)^\theta\Big)^3 \\ & +\frac{1}{4!}\Big(A(n+1)^\delta+B(n+1)^\epsilon+C(n+1)^\theta\Big)^4+O\left(n^{5\delta}\right)\Big]. \end{aligned} \tag{4.6}$$

Hence, using (4.5), (4.6) and the form of λ_n we have (after straightforward calculations)

$$\begin{aligned} \eta_n+t_n = z_n^2 & \Big\{-2+\left(1-\frac{\gamma}{n}+O\left(\frac{1}{n^2}\right)\right)\Big[1-i\Big(An^\delta+Bn^\epsilon+Cn^\theta\Big) \\ & -\frac{1}{2}\Big(An^\delta+Bn^\epsilon+Cn^\theta\Big)^2+\frac{i}{3!}\Big(An^\delta+Bn^\epsilon+Cn^\theta\Big)^3 \\ & +\frac{1}{4!}\Big(An^\delta+Bn^\epsilon+Cn^\theta\Big)^4+O\left(n^{5\delta}\right)\Big] \\ & +\left(1+\frac{\gamma}{n}+O\left(\frac{1}{n^2}\right)\right)\Big[1+i\Big(A(n+1)^\delta+B(n+1)^\epsilon+C(n+1)^\theta\Big) \\ & -\frac{1}{2}\Big(A(n+1)^\delta+B(n+1)^\epsilon+C(n+1)^\theta\Big)^2 \\ & -\frac{i}{3!}\Big(A(n+1)^\delta+B(n+1)^\epsilon+C(n+1)^\theta\Big)^3 \\ & +\frac{1}{4!}\Big(A(n+1)^\delta+B(n+1)^\epsilon+C(n+1)^\theta\Big)^4+O\left(n^{5\delta}\right)\Big] \\ & -\left(\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)+O\left(|x_{n-1}|+|x_n|\right)\right) \\ & \times\left[1-iAn^\delta+O\left(\frac{1}{n^2}\right)+O\left(n^\epsilon\right)\right]-\left[\frac{\lambda}{n^\alpha}+O\left(|y_n|+|x_n|\right)\right]\Big\}. \end{aligned}$$

Elementary calculations shows that the last expression is equal to

$$\begin{aligned} \eta_n+t_n = z_n^2 & \left\{O\left(\frac{1}{n^2}\right)+O\left(n^{5\delta}\right)+O\left(|x_{n+1}|+|x_n|+|y_n|\right)+O\left(\frac{1}{n^{1-\epsilon}}\right)\right. \\ & +O\left(\frac{1}{n^{1-2\delta}}\right)+O\left(n^{2\epsilon}\right)+O\left(n^{3\delta+\epsilon}\right)+\frac{2i\gamma A}{n^{1-\delta}}+\frac{iA\delta}{n^{1-\delta}}-A^2n^{2\delta} \\ & \left.-2ABn^{\delta+\epsilon}-2ACn^{\delta+\theta}+\frac{1}{12}A^4n^{4\delta}-\frac{\alpha}{n}+\frac{iA\alpha}{n^{1-\delta}}-\frac{\lambda}{n^\alpha}\right\}. \end{aligned}$$

Thus

$$\eta_n + t_n = z_n^2 \left\{ O\left(\frac{1}{n^2}\right) + O\left(n^{5\delta}\right) + O\left(|x_{n+1}| + |x_n| + |y_n|\right) + O\left(\frac{1}{n^{1-\epsilon}}\right) \right.$$
$$+ O\left(\frac{1}{n^{1-2\delta}}\right) + O\left(n^{2\epsilon}\right) + O\left(n^{3\delta+\epsilon}\right) - \left(A^2 n^{2\delta} + \frac{\lambda}{n^\alpha}\right)$$
$$\left. + \left(\frac{1}{12} A^4 n^{4\delta} - 2ACn^{\delta+\theta}\right) - \left(\frac{\alpha}{n} + 2ABn^{\delta+\epsilon}\right) + \frac{iA}{n^{1-\delta}}\left(\delta + 2\gamma + \alpha\right) \right\}.$$

Grouping in pairs above (presumably the terms of the same order) was based on the expecting values (from WKB approach) of the parameters α, δ, θ and ϵ. Now put the condition that all four brackets in the formula below are equal to zero separately. It immediately gives the values of parameters:

$$\begin{aligned}
&\text{from 1st bracket:} \quad \delta = -\frac{\alpha}{2} && \text{and} \quad A = \pm\sqrt{-\lambda} \\
&\text{from 2nd bracket:} \quad \theta = -\frac{3\alpha}{2} && \text{and} \quad C = \pm\frac{\sqrt{-\lambda}^3}{24} \\
&\text{from 3rd bracket:} \quad \epsilon = \frac{\alpha}{2} - 1 && \text{and} \quad B = \mp\frac{\alpha}{2\sqrt{-\lambda}} \\
&\text{from 4th bracket:} \quad \gamma = -\frac{\alpha}{4}.
\end{aligned}$$

Hence, substituting the values of powers δ, θ and ϵ,

$$\begin{aligned}
|\eta_n + t_n| &= |z_n|^2 \left\{ O\left(\frac{1}{n^2}\right) + O\left(n^{-\frac{5\alpha}{2}}\right) + O\left(|x_{n+1}| + |x_n| + |y_n|\right) \right. \\
&\quad \left. + O\left(\frac{1}{n^{2-\frac{\alpha}{2}}}\right) + O\left(\frac{1}{n^{1+\alpha}}\right) + O\left(\frac{1}{n^{2-\alpha}}\right) \right\} \\
&= |z_n|^2 \left\{ O\left(n^{-\frac{5\alpha}{2}}\right) + O\left(|x_{n+1}| + |x_n| + |y_n|\right) \right\}.
\end{aligned}$$

`Step 2.` *Calculation of the determinant.*
Explicit calculation of $\det S_{n+1}$ gives $\det S_{n+1} = \bar{z}_n z_{n+1} - z_n \bar{z}_{n+1} = 2i\Im(\bar{z}_n z_{n+1}) = 2in^{-\frac{\alpha}{2}}\left(1 + O\left(\frac{1}{n^2}\right)\right)\sin\left(A(n+1)^\delta + B(n+1)^\epsilon + C(+1)n^\theta\right) = \pm n^{-\alpha} 2i\sqrt{-\lambda}\left(1 + O\left(\frac{1}{n^{1-\alpha}}\right)\right)$, since $\alpha > \frac{1}{2}$. Therefore

$$(\det S_{n+1})^{-1} = \mp\frac{in^\alpha}{2\sqrt{-\lambda}}\left(1 + O\left(\frac{1}{n^{1-\alpha}}\right)\right), \tag{4.7}$$

which gives extra-multiple of order n^α in formula (4.4). Therefore by (4.7) one gets

$$\begin{aligned}
\left|(\det S_{n+1})^{-1}(\eta_n + t_n)\right| &= O\left(n^\alpha\right)|z_n|^2\left\{O\left(n^{-\frac{5\alpha}{2}}\right) + O\left(|x_{n+1}| + |x_n| + |y_n|\right)\right\} \\
&= O\left(n^{\frac{\alpha}{2}}\right)\left\{O\left(n^{-\frac{5\alpha}{2}}\right) + O\left(|x_{n+1}| + |x_n| + |y_n|\right)\right\} \\
&= O\left(n^{-2\alpha}\right) + O\left(n^{\frac{\alpha}{2}}\left(|x_{n+1}| + |x_n| + |y_n|\right)\right),
\end{aligned} \tag{4.8}$$

since $z_n^2 = O\left(n^{-\frac{\alpha}{2}}\right)$ due to $\gamma = -\frac{\alpha}{4}$. Thanks to $\alpha > \frac{1}{2}$ and conditions (4.1) the right-hand side term in (4.8) belongs to l^1. Now it is clear that the off-diagonal elements of $S_{n+1}^{-1}B_n(\lambda)S_n$ are summable under conditions of the Theorem 4.1.

Step 3. *Estimate of the diagonal elements.*
Concerning the diagonal element $(\det S_{n+1})^{-1}(\rho_n + s_n)$ note that

$$\begin{aligned}
&|\rho_n + s_n - \det S_{n+1}| \\
&= |z_n|^2\big|\bar{z}_{n-1}(\bar{z}_n)^{-1} + z_{n+1}z_n^{-1} - 2 - \psi(n) - \varphi(n)\bar{z}_{n-1}(\bar{z}_n)^{-1} \\
&\quad - (z_{n+1}z_n^{-1} - \bar{z}_{n+1}(\bar{z}_n)^{-1})\big| \\
&= |z_n|^2\big|\bar{z}_{n-1}(\bar{z}_n)^{-1} + \bar{z}_{n+1}(\bar{z}_n)^{-1} - 2 - \psi(n) - \varphi(n)\bar{z}_{n-1}(\bar{z}_n)^{-1}\big| \\
&= |z_n|^2\big|(z_{n-1}z_n^{-1} + z_{n+1}z_n^{-1} - 2 - \psi(n) - \varphi(n)z_{n-1}z_n^{-1})\big| \\
&= |\eta_n + t_n|.
\end{aligned} \tag{4.9}$$

Combining (4.9) and (4.8) we conclude the proof of (4.3) and the statement of Theorem. It is enough to know that the second diagonal element (in formula (4.4)) estimate follows from the estimate of the first one because $\det S_{n+1}$ is pure imaginary. □

5. An application to a class of Jacobi matrices

We mentioned in the Introduction the class of Jacobi matrices (studied in [19]) given by $\lambda_n = n + a$ and $q_n = -2n$. Below we also consider a much more general class of Jacobi matrices related to the ones from the theory of birth and death processes [19, 24]. The entries of such matrices must satisfy the identity

$$q_n + \lambda_{n-1} + \lambda_n = 0, \quad n \geq 1\,. \tag{5.1}$$

To be precise the right-hand side term in formula (5.1) should be equal to 1, but standard shift of the spectral parameter brings zero instead of 1. If $\lambda_n = n^\alpha$, $\alpha \in (0,1)$, then using (5.1) we find $q_n = -2n^\alpha\left(1 - \frac{\alpha}{2n} + O\left(\frac{1}{n^2}\right)\right)$ for large n. It follows that $y_n = -\frac{\alpha}{2n} + O\left(\frac{1}{n^2}\right)$ does not satisfy our assumption $\{n^{\frac{\alpha}{2}}y_n\} \in l^1$. Therefore, to avoid extra tedious calculations, we modify slightly the above definitions to obtain the cancellation of terms. Using the technique of present paper one could be able to consider the above mentioned model without any corrections, but it would force us to consider asymptotic formulae in more details. Remind that the aim of our paper is just to demonstrate the technique in the critical case.

Let $\lambda_k = k^\alpha(1 + r_k)$, $\alpha \in (\frac{1}{2}, \frac{2}{3})$, $r_k = O\left(\frac{1}{k^{1+x}}\right)$, $x > \frac{\alpha}{2}$ ($r_k \neq -1$ for any k). We claim also that $\lambda_k > 0$ for any k. Define the diagonal q_k by

$$q_k + \lambda_k + \lambda_{k-1} = -\frac{\alpha}{k^{1-\alpha}} + d_k k^\alpha, \tag{5.2}$$

for arbitrary real sequence d_k satisfying the condition $d_k = O\left(k^{-1-x}\right)$, $x > \frac{\alpha}{2}$. Thus $q_k = -2k^\alpha\left[1 + \frac{1}{2}(r_k + r_{k-1} - d_k) - \alpha r_{k-1}k^{-1} + O\left(k^{-2}\right)\right]$. Note that the new $y_k := \frac{1}{2}(r_k + r_{k-1} - d_k) - \alpha r_{k-1}k^{-1} + O\left(k^{-2}\right)$ fulfills the assumption $\{k^{\frac{\alpha}{2}} y_k\} \in l^1$. Without lost of generality, let us put the assumption that all new $\lambda_k > 0$. Therefore our asymptotic formulae are applicable. Consequently, we obtain the following spectral picture of Jacobi matrix J_0 defined by the entries given in formula (5.2).

Theorem 5.1. *The half-line* $(-\infty, 0)$ *is contained in the pure absolutely continuous spectrum of* J_0 *and its local multiplicity is equal to one, a.e.* $\lambda \in (-\infty, 0)$. *The spectrum of* J_0 *in the interval* $(0, +\infty)$ *is discreet and finite. Moreover, the number of eigenvalues of* J_0 *is less or equal to* N *provided that* (*see* (5.2)) $\alpha k^{-1} - d_k \geq 0$, $k > N$, *and the corresponding eigenvectors decay exponentially.*

Proof. Fix $\lambda < 0$. Applying Theorem 4.1 we know that for any solutions $\vec{u}(n)$ of the system (4.2) we obtain the estimate $\|\vec{u}(n)\|^2 \leq Cn^{-\frac{\alpha}{2}} \leq \frac{C_1}{\lambda_n}$ for some constants C, C_1 (depending on λ) and all n. Applying standard result [3, 18] (generalized Behncke–Stolz lemma) we conclude that λ belongs to the support of the spectral measure of J_0 and so $(-\infty, 0) \subset \sigma_{ac}(J_0)$. Moreover, the spectrum on the interval $(-\infty, 0)$ is pure absolutely continuous and its local multiplicity (Lebesgue measure) a.e. is equal to one (i.e., non-zero). The last result follows from Gilbert–Pearson subordinacy theory [26]. Concerning the point spectrum of J_0 note that it may appear on the semi-axis $\lambda \geq 0$ only. Actually, by the subordinacy theory positive spectrum is pure point. Moreover, using the technique of the paper [33] one can prove its discreetness. However, in our special case the discreetness can be proved easily. Indeed, for any $f \in D(J_0)$ we have $(f_0 := 0)$

$$(Jf, f) = \sum_{k=1}^{\infty} -k^\alpha(\alpha k^{-1} - d_k)|f_k|^2 - \sum_{k=0}^{\infty} \lambda_k |f_k - f_{k+1}|^2 \leq 0.$$

Therefore (by the Glazman lemma [2])

$$\text{the cardinality of } \{\lambda \in \sigma_p(J_0)\} \leq N, \tag{5.3}$$

where N has been chosen to satisfy the inequality: $\alpha k^{-1} - d_k \geq 0$ for any $k > N$. The final statement of Theorem 5.1 follows from the asymptotic formula (3.23) which gives the precise form for the eigenvectors asymptotics. □

Remark 5.2. Another approach to a similar class of Jacobi matrices based on the generalization of ideas of W. Kelley [25] (whose paper concerns the "double root" (= the Jordan box) case for Jacobi matrices whose matrix entries are rational functions of n) was given by the first named author in [16]. Our approach and the one of the paper [16] are complementary and seem to have different areas of applications.

Remark 5.3. If the choice of the right-hand side terms in (5.2) (and therefore the choice of a few first values of q_k and λ_k) is so that for some integer N

$$\sum_{k=1}^{N} k^\alpha(-\alpha k^{-1} + d_k) > \lambda_N + \lambda_1 = N^\alpha(1 + r_N) + \lambda_1 \,,$$

then $\sigma_p(J_0) \neq \varnothing$. In fact, for $\tilde{f} := (1, \ldots, 1, 0, 0, \ldots)$ only the first N coordinates are equal to 1. Therefore we have

$$(J_0\tilde{f}, \tilde{f}) = \left(\sum_{k=1}^{N} k^\alpha(-\alpha k^{-1} + d_k) - N^\alpha(1 + r_N) - \lambda_1 \right) > 0\,.$$

Acknowledgement. J. Janas and S. Naboko were supported by INTAS 05-1000008-7883, S. Naboko was supported in part by the grant RFBR-06-01-00249. We are thankful to S. Simonov for attentive reading of the manuscript and useful remarks.

References

[1] Adams, C. R., On the irregular cases of the linear ordinary difference equation. *Trans. Amer. Math. Soc.* 30 (1928)(3), 507 – 541.

[2] Akhiezer, N. I. and Glazman, I. M., *Theory of Linear Operators in Hilbert Space.* Reprint of the 1961 and 1963 transl.; two volumes bound as one. New York: Dover 1993.

[3] Berezans'kiĭ, Ju. M., *Expansions in Eigenfunctions of Selfadjoint Operators.* Transl. Math. Monogr. 17. Providence (RI): Amer. Math. Soc. 1968.

[4] Birkhoff, G. D., General theory of linear difference equations. *Trans. Amer. Math. Soc.* 12 (1911)(2), 243 – 284.

[5] Boutet de Monvel, A., Naboko, S. and Silva, L. O., The asymptotic behavior of eigenvalues of a modified Jaynes–Cummings model. *Assymptot. Anal.* 47 (2006)(3–4), 291 – 315.

[6] Damanik, D. and Naboko, S., A first-order phase transition in a class of unbounded jacobi matrices: critical coupling *J. Approx. Theory* 145 (2007)(2), 221 – 236.

[7] Dombrowski, J., Absolutely continous measures for systems of orthogonal polynomials with unbounded reccurence coefficients. *Constr. Approx.* 8 (1992)(2), 161 – 167.

[8] Dombrowski, J., Eigenvalues and spectral gaps related to periodic perturbations of Jacobi matrices. In: *Spectral Meth. Oper. Math. Phys.* (Bedlewo 2002; eds.: J. Janas, P. Kurasov, S. Naboko). Oper. Theory Adv. Appl. 154. Basel: Birkhäuser 2004, pp. 91 –100.

[9] Dombrowski, J. and Pedersen, S., Absolute continuity for unbounded Jacobi matrices with constant row sum. *J. Math. Anal. Appl.* 267 (2002), 695 – 713.

[10] Dombrowski, J., Janas, J., Moszynski, M. and Pedersen, S., Spectral gaps resulting from periodic perturbations of a class of Jacobi operators. *Constr. Approx.* 20 (2004)(4), 585 – 601.

[11] Dombrowski, J. and Pedersen, S., Spectral transition parameters for a class of Jacobi matrices. *Studia Math.* 152 (2002)(3), 217 – 229.

[12] Eastham, M. S. P., *The Assymptotic Solution of Linear Differential Systems. Applications of the Levinson Theorem.* London Math. Soc. Monographs. New Ser. 4. New York: Clarendon Press, Oxford Univ. Press 1989.

[13] Elaydi, S., *An Introduction to Difference Equations* (2nd ed.). New York: Springer 1999.

[14] Geronimo, J. S., Bruno, O. and Van Assche, W., WKB and turning point theory for second-order difference equations. In: *Spectral Meth. Oper. Math. Phys.* (Bedlewo 2002; eds.: J. Janas, P. Kurasov, S. Naboko). Oper. Theory Adv. Appl. 154. Basel: Birkhäuser 2004, pp. 101 – 138,

[15] Gilbert, D. J. and Pearson, D. B., On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operator. *J. Math. Anal. Appl.* 128 (1987), 30 – 56.

[16] Janas, J., The asymptotic analysis of generalized eigenvectors of some Jacobi operators. Jordan box case. *J. Diff. Equ. Appl.* 12 (2006)(6), 597 – 618.

[17] Janas, J. and Moszynski, M., Spectral properties of Jacobi matrices by asymptotic analysis. *J. Approx. Theory* 120 (2003), 309 – 336.

[18] Janas, J. and Naboko, S., Jacobi matrices with power-like weights–grouping in blocks approach. *J. Funct. Anal.* 166 (1999)(2), 218 – 243.

[19] Janas, J. and Naboko, S., Spectral properties of selfadjoint Jacobi matrices coming from birth and death processes. In: *Recent Adv. Oper. Theory Rel. Topics* (Szeged 1999; ed.: L. Kerchy). Oper. Theory Adv. Appl. 127. Basel: Birkhäuser 2001, pp. 387 – 397.

[20] Janas, J. and Naboko, S., Spectral analysis of selfadjoint Jacobi matrices with periodically modulated entries. *J. Funct. Anal.* 191 (2002), 318 – 342.

[21] Janas, J. and Naboko, S., Infinite Jacobi matrices with unbounded entries: asymptotics of eigenvalues and the transformation operator approach. *SIAM J. Math. Anal.* 36 (2004)(2), 643 – 658.

[22] Janas, J., Naboko, S. and Stolz, G., Spectral theory for a class of periodically perturbed unbounded Jacobi matrices: elementary methods. *J. Comput. Appl. Math.* 171 (2004)(1–2), 265 – 276.

[23] Jitomirskaja, S. and Last, Y., Power-law subordinacy and singular spectra. I. Half-line operators. *Acta Math.* 183 (1999), 171 – 189.

[24] Karlin, S. and McGregor, J., Linear growth birth and death processes. *J. Math. Mech.* 7 (1958)(4), 643 – 662.

[25] Kelley, W., Asymptotic analysis of solutions in the "double root" case. *Comput. Math. Appl.* 28 (1994)(1–3), 167 – 173.

[26] Khan, S. and Pearson, D. B., Subordinacy and spectral theory for infinite matrices. *Helv. Phys. Acta* 65 (1992), 505 – 527.

[27] Laptev, A., Naboko, S. and Safronov, O., On new relations between spectral properties of Jacobi matrices and their coefficients. *Comm. Math. Phys.* 241 (2003), 91 – 110.

[28] Moszynski, M., Spectral properties of some Jacobi matrices with double weights. *J. Math. Anal. Appl.* 280 (2003), 400 – 412.

[29] Olver, F. W. J., *Asymptotics and Special Functions.* Reprint [New York: Academic Press 1974]. AKP Classics. Wellesley (MA): A. K. Peters 1997.

[30] Remling, Ch., Spectral analysis of high-order differential operators. II. Fourth-order equations. *J. London Math. Soc. (2)* 59 (1999)(1), 188 – 206.

[31] Sheronova, E., Asymptotics of eigenvectors of Jacobi matrices for a model of spectral phase transition. M.S. Thesis, St. Petersburg 2001.

[32] Silva, L. O., Uniform Levinson type theorems for discrete linear systems. In: *Spectral Meth. Oper. Math. Phys.* (Bedlewo 2004; eds.: J. Janas, P. Kurasov, S. Naboko). Oper. Theory Adv. Appl. 154. Basel: Birkhäuser 2004, pp. 203 – 218.

[33] Silva, L. O., Uniform and smooth Benzaid–Lutz type theorem and applications to Jacobi matrices. In: *Oper. Theory Adv. Appl.* 174 (Bedlewo 2004; eds.: J. Janas, P. Kurasov, S. Naboko). Basel: Birkhäuser 2007, pp. 173 – 186.

[34] Silva, L. O. and Weder, R., On the two spectra inverse problem for semi-infinite Jacobi matrices. *Math. Phys. Anal. Geom.* 9 (2007), 263 – 290.

[35] Simonov, S., An example of spectral phase transition phenomenon in a class of Jacobi matrices with periodically modulated weights. In: *Oper. Theory Adv. Appl.* 174 (Bedlewo 2004; eds.: J. Janas, P. Kurasov, S. Naboko). Basel: Birkhäuser 2007, pp. 187 – 203.

[36] Stolz, G., Spectral theory for slowly oscillating potentials. I. Jacobi matrices. *Manuscripta Math.* 84 (1994)(3–4), 245 – 260.

Received December 12, 2007