

# On a Class of Measures of Noncompactness in Banach Algebras and Their Application to Nonlinear Integral Equations

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**Abstract.** We introduce a class of measures of noncompactness in Banach algebras satisfying certain condition and we prove a fixed point theorem for the product of two operators being contractions with respect to such measures of noncompactness. We also indicate measures of noncompactness in some Banach function algebras which satisfy the mentioned condition. The obtained results are applied to prove a few theorems on the existence of solutions of nonlinear integral equations in Banach algebras. Some characterizations of solutions of considered integral equations are also derived.

**Keywords.** Measure of noncompactness, Banach algebra, product of two operators, fixed point theorem, nonlinear integral equation

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## 1. Introduction

The paper is devoted to the study of the solvability of some operator equations in Banach algebras. The main tool used in our investigations is the technique associated with measures of noncompactness.

Assuming that measures of noncompactness used in investigations satisfy certain condition we are able to prove that functional-operator equations considered in suitable Banach algebras are solvable under some imposed conditions. The results we are going to prove in this paper generalize a lot of ones obtained previously in other papers and monographs (cf. [8–10, 17–19], for example).

It is worthwhile mentioning that the approach presented in this paper permits us to generalize a lot of fixed point theorems proved earlier in some special

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cases. On the other hand applying the mentioned theorems to functional integral equations studied in Banach algebras we prove not only the existence of solutions of considered equations but we also obtain some characterizations of those solutions. Obviously those characterizations depend on measures of noncompactness which are used in our considerations.

The main idea of our investigations depends on the indication of a class of measures of noncompactness in Banach algebras satisfying certain condition called here the condition  $(m)$ . That condition was used first in the paper [8] in the case of the Hausdorff measure of noncompactness in the Banach algebra  $C(I)$  consisting of real continuous functions defined on a closed and bounded interval  $I$ .

In this paper we indicate a wide class of measures of noncompactness satisfying condition  $(m)$ . Those measures will be considered in the Banach algebra  $C(I)$  and in the Banach algebra  $BC(\mathbb{R}_+)$  defined further on.

## 2. Preliminary results concerning measures of noncompactness

Assume that  $E$  is a given real Banach space with the norm  $\|\cdot\|$  and the zero element  $\theta$ . Denote by  $B(x, r)$  the closed ball in  $E$  centered at  $x$  and with radius  $r$ . We will write  $B_r$  to denote the ball  $B(\theta, r)$ . If  $X$  is a subset of  $E$ , then the symbols  $\bar{X}$ ,  $\text{Conv} X$  stand for the closure and convex closure of  $X$ , respectively. Moreover, by the symbol  $\|X\|$  we will denote the norm of a bounded set  $X$ , i.e.,  $\|X\| = \sup\{\|x\| : x \in X\}$ .

Further, let us denote by  $\mathfrak{M}_E$  the family of all nonempty and bounded subsets of  $E$  and by  $\mathfrak{N}_E$  its subfamily consisting of all relatively compact sets.

In what follows we will accept the following definition of the concept of a measure of noncompactness [5].

**Definition 2.1.** A mapping  $\mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+ = [0, \infty)$  is said to be a measure of noncompactness in  $E$  if it satisfies the following conditions:

1. the family  $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$  is nonempty and  $\ker \mu \subset \mathfrak{N}_E$ ;
2.  $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$ ;
3.  $\mu(\bar{X}) = \mu(\text{Conv } X) = \mu(X)$ ;
4.  $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$  for  $\lambda \in [0, 1]$ ;
5. if  $(X_n)$  is a sequence of closed sets from  $\mathfrak{M}_E$  such that  $X_{n+1} \subset X_n$  for  $n = 1, 2, \dots$  and if  $\lim_{n \rightarrow \infty} \mu(X_n) = 0$ , then the set  $X_\infty = \bigcap_{n=1}^{\infty} X_n$  is nonempty.

The family  $\ker \mu$  described in 1<sup>o</sup> is called *the kernel of the measure of noncompactness  $\mu$* .

It can be shown that the set  $X_\infty$  from the axiom  $5^\circ$  is a member of the kernel  $\ker \mu$ . This fact will be important in our further considerations.

Now, let us assume that  $\Omega$  is a nonempty subset of a Banach space  $E$  and  $F : \Omega \rightarrow E$  is a continuous operator which transforms bounded subsets of  $\Omega$  onto bounded ones. Suppose that  $\mu$  is a measure of noncompactness given in  $E$ .

**Definition 2.2** (see [5]). We say that  $F$  satisfies the *Darbo condition* with a constant  $k$  with respect to a measure of noncompactness  $\mu$  provided  $\mu(FX) \leq k\mu(X)$  for each  $X \in \mathfrak{M}_E$  such that  $X \subset \Omega$ . If  $k < 1$ , then  $F$  is called a *contraction* with respect to  $\mu$ .

Starting from now on we assume that the space  $E$  has the structure of Banach algebra. For given subsets  $X, Y$  of a Banach algebra  $E$  let us denote

$$XY = \{xy : x \in X, y \in Y\}.$$

We will say that the measure of noncompactness  $\mu$  defined on the Banach algebra  $E$  satisfies condition (m) if for arbitrary sets  $X, Y \in \mathfrak{M}_E$  the following inequality is satisfied:

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X).$$

In what follows we indicate measures of noncompactness satisfying condition (m) in some Banach algebras.

First of all let us mention that this condition was used for the first time in the paper [8] for measures of noncompactness defined on the Banach algebra  $C[a, b]$  with help of a sequence of functionals [23]. Particularly, the so-called Hausdorff measure of noncompactness  $\chi$  (cf. [5]) satisfies condition (m). Let us recall some details.

Namely, denote by  $C[a, b]$  the Banach space of real functions defined and continuous on the interval  $[a, b]$  with the standard maximum norm. Obviously the space  $C[a, b]$  has the structure of Banach algebra with usual product of functions.

Further, fix arbitrarily  $\varepsilon > 0$  and a set  $X \in \mathfrak{M}_{C[a,b]}$ . For  $x \in X$  denote by  $\omega(x, \varepsilon)$  the modulus of continuity of  $x$ , i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [a, b], |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\} \quad \text{and} \quad \omega_0(X) = \lim_{\varepsilon \rightarrow 0} \omega(X, \varepsilon).$$

It is well-known that  $\omega_0(X)$  is a measure of noncompactness in  $C[a, b]$  such that the Hausdorff measure  $\chi$  may be expressed by the formula  $\chi(X) = \frac{1}{2}\omega_0(X)$

(see [5]). Apart from this it is easy to check that the measure  $\omega_0(X)$  satisfies condition (m) [8].

Now we indicate another measure of noncompactness in the Banach algebra  $C[a, b]$  which satisfies condition (m) on some subfamily of the family  $\mathfrak{M}_{C[a,b]}$ . To do this let us take a set  $X \in \mathfrak{M}_{C[a,b]}$ . For  $x \in X$  let us consider the following quantities (cf. [7]):

$$\begin{aligned} d(x) &= \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in [a, b], t \leq s\} \\ i(x) &= \sup\{|x(s) - x(t)| - [x(t) - x(s)] : t, s \in [a, b], t \leq s\}. \end{aligned}$$

The quantity  $d(x)$  represents the degree of decrease of the function  $x$  while  $i(x)$  represents the degree of increase. Moreover,  $d(x) = 0$  if and only if  $x$  is nondecreasing on  $[a, b]$  and analogous characterization holds for the quantity  $i(x)$ . Further, let us put

$$\begin{aligned} d(X) &= \sup\{d(x) : x \in X\} \\ i(X) &= \sup\{i(x) : x \in X\}. \end{aligned}$$

Obviously the mappings  $d(X)$  and  $i(X)$  can be characterized in the same way as the quantities  $d(x)$  and  $i(x)$ . Finally, let us denote

$$\mu_d(X) = \omega_0(X) + d(X) \tag{1}$$

$$\mu_i(X) = \omega_0(X) + i(X). \tag{2}$$

It can be shown that these mappings are measures of noncompactness in the space  $C[a, b]$  (cf. [7]). Moreover,  $\ker \mu_d$  is the family consisting of all sets  $X$  belonging to  $\mathfrak{M}_{C[a,b]}$  such that all functions from  $X$  are equicontinuous and nondecreasing on  $[a, b]$ . Similarly we may characterize the family  $\ker \mu_i$ .

Now, we show that measures of noncompactness  $\mu_d$  and  $\mu_i$  satisfy "partly" condition (m).

**Theorem 2.3.** *The measures of noncompactness  $\mu_d$  and  $\mu_i$  satisfy condition (m) on the subfamily of the family  $\mathfrak{M}_{C[a,b]}$  consisting of sets of functions being nonnegative on the interval  $[a, b]$ .*

*Proof.* Let us take arbitrary sets  $X, Y \in \mathfrak{M}_{C[a,b]}$  such that functions belonging to these sets are nonnegative on  $[a, b]$ . Further, fix arbitrarily  $x \in X$  and  $y \in Y$  and take  $t, s \in [a, b]$  with  $t < s$ . Then we have

$$\begin{aligned} &|x(s)y(s) - x(t)y(t)| - [x(s)y(s) - x(t)y(t)] \\ &\leq |x(s)y(s) - x(s)y(t)| + |x(s)y(t) - x(t)y(t)| - \{[x(s)y(s) - x(s)y(t)] \\ &\quad + [x(s)y(t) - x(t)y(t)]\} \\ &= |x(s)||y(s) - y(t)| + |y(t)||x(s) - x(t)| - x(s)[y(s) - y(t)] - y(t)[x(s) - x(t)] \\ &= |x(s)|\{|y(s) - y(t)| - [y(s) - y(t)]\} + |y(t)|\{|x(s) - x(t)| - [x(s) - x(t)]\} \\ &\leq \|x\|d(y) + \|y\|d(x). \end{aligned}$$

This implies that  $d(XY) \leq \|X\|d(Y) + \|Y\|d(X)$ . Linking the above obtained assertion with the fact that the measure of noncompactness  $\omega_0(X)$  satisfy condition (m), we get:

$$\mu_d(XY) \leq \|X\|\mu_d(Y) + \|Y\|\mu_d(X).$$

In the same way we may show that the measure of noncompactness  $\mu_i$  satisfies condition (m) for sets from the family  $\mathfrak{M}_{C[a,b]}$  consisting of functions being nonnegative on  $[a, b]$ . This completes the proof.  $\square$

Now, let us consider the Banach space  $BC(\mathbb{R}_+)$  consisting of all functions  $x : \mathbb{R}_+ \rightarrow \mathbb{R}$  which are continuous and bounded on  $\mathbb{R}_+$ . This space is furnished with the standard norm  $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$ . Obviously  $BC(\mathbb{R}_+)$  has also the structure of Banach algebra with the standard multiplication of functions.

Further, fix a set  $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$  and numbers  $\varepsilon > 0$  and  $T > 0$ . For an arbitrary function  $x \in X$  let us denote by  $\omega^T(x, \varepsilon)$  the modulus of continuity of  $x$  on the interval  $[0, T]$ , i.e.,

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \varepsilon\}.$$

Next, let us put

$$\begin{aligned} \omega^T(X, \varepsilon) &= \sup\{\omega^T(x, \varepsilon) : x \in X\} \\ \omega_0^T(X) &= \lim_{\varepsilon \rightarrow 0} \omega^T(X, \varepsilon) \\ \omega_0^\infty(X) &= \lim_{T \rightarrow \infty} \omega_0^T(X). \end{aligned}$$

Now, let us define the following set mappings:

$$\begin{aligned} a(X) &= \lim_{T \rightarrow \infty} \left\{ \sup_{x \in X} \left\{ \sup\{|x(t)| : t \geq T\} \right\} \right\} \\ b(X) &= \lim_{T \rightarrow \infty} \left\{ \sup_{x \in X} \left\{ \sup\{|x(t) - x(s)| : t, s \geq T\} \right\} \right\}. \end{aligned}$$

Moreover, if  $t \in \mathbb{R}_+$  is a fixed number, let us denote

$$\begin{aligned} X(t) &= \{x(t) : x \in X\} \\ \text{diam } X(t) &= \sup\{|x(t) - y(t)| : x, y \in X\} \\ c(X) &= \limsup_{t \rightarrow \infty} \text{diam } X(t). \end{aligned}$$

With help of the above mappings we define the following measures of noncompactness in  $BC(\mathbb{R}_+)$  (cf. [5, 6]):

$$\mu_a(X) = \omega_0^\infty(X) + a(X) \tag{3}$$

$$\mu_b(X) = \omega_0^\infty(X) + b(X) \tag{4}$$

$$\mu_c(X) = \omega_0^\infty(X) + c(X). \tag{5}$$

Let us mention that the kernel  $\ker \mu_a$  of the measure  $\mu_a$  consists of all sets  $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$  such that functions from  $X$  are locally equicontinuous on  $\mathbb{R}_+$  and vanish uniformly at infinity, i.e., for any  $\varepsilon > 0$  there exists  $T > 0$  such that  $|x(t)| \leq \varepsilon$  for all  $x \in X$  and for any  $t \geq T$ . For description of the kernels  $\ker \mu_b$  and  $\ker \mu_c$  we refer to [6].

In the sequel we prove the following theorem.

**Theorem 2.4.** *The measures of noncompactness  $\mu_a, \mu_b$  and  $\mu_c$  satisfy condition (m).*

*Proof.* In the light of the facts quoted earlier it is easy to infer that the term  $\omega_0^\infty(X)$  satisfies condition (m). Thus, it is sufficient to show that the quantities  $a(X)$ ,  $b(X)$  and  $c(X)$  satisfy this condition. We show that the quantity  $c(X)$  satisfies condition (m).

To do this fix arbitrarily sets  $X, Y \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ . Choose arbitrary functions  $z_1, z_2 \in XY$ . This means that there exist functions  $x_1, x_2 \in X$  and  $y_1, y_2 \in Y$  such that  $z_1 = x_1y_1, z_2 = x_2y_2$ .

Next, for  $t \in \mathbb{R}_+$  we get:

$$\begin{aligned} |z_1(t) - z_2(t)| &= |x_1(t)y_1(t) - x_2(t)y_2(t)| \\ &\leq |x_1(t)y_1(t) - x_1(t)y_2(t)| + |x_1(t)y_2(t) - x_2(t)y_2(t)| \\ &= |x_1(t)||y_1(t) - y_2(t)| + |y_2(t)||x_1(t) - x_2(t)| \\ &\leq \|X\| \operatorname{diam} Y(t) + \|Y\| \operatorname{diam} X(t). \end{aligned}$$

Hence we obtain  $\operatorname{diam}(X(t)Y(t)) \leq \|X\| \operatorname{diam} Y(t) + \|Y\| \operatorname{diam} X(t)$  and consequently  $c(XY) \leq \|X\|c(Y) + \|Y\|c(X)$ . This implies that the measure of noncompactness  $\mu_c$  satisfies condition (m).

The proof that the quantities  $a(X)$  and  $b(X)$  satisfy condition (m) is similar and is omitted. □

In what follows assume that  $E$  is a Banach algebra. Let  $\mu$  be a measure of noncompactness in  $E$  satisfying condition (m). Then we have following theorem.

**Theorem 2.5.** *Assume that  $\Omega$  is nonempty, bounded, closed and convex subset of the Banach algebra  $E$ , and the operators  $P$  and  $T$  transform continuously the set  $\Omega$  into  $E$  in such a way that  $P(\Omega)$  and  $T(\Omega)$  are bounded. Moreover, we assume that the operator  $S = P \cdot T$  transforms  $\Omega$  into itself. If the operators  $P$  and  $T$  satisfy on the set  $\Omega$  the Darbo condition with respect to the measure of noncompactness  $\mu$  with the constants  $k_1$  and  $k_2$  respectively, then the operator  $S$  satisfies on  $\Omega$  the Darbo condition with the constant*

$$\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1.$$

*Particularly, if  $\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1 < 1$ , then  $S$  is a contraction with respect to the measure of noncompactness  $\mu$  and has at least one fixed point in the set  $\Omega$ .*

*Proof.* Let us take an arbitrary nonempty subset  $X$  of the set  $\Omega$ . Then in view of the assumption that  $\mu$  satisfies condition (m) we obtain

$$\begin{aligned} \mu(S(X)) &\leq \mu(P(X) \cdot T(X)) \\ &\leq \|P(X)\| \mu(T(X)) + \|T(X)\| \mu(P(X)) \\ &\leq \|P(\Omega)\| \mu(T(X)) + \|T(\Omega)\| \mu(P(X)) \\ &\leq \|P(\Omega)\| k_2 \mu(X) + \|T(\Omega)\| k_1 \mu(X) \\ &= [\|P(\Omega)\| k_2 + \|T(\Omega)\| k_1] \mu(X). \end{aligned}$$

Hence it follows that the operator  $S$  satisfies the Darbo condition with the constant  $\|P(\Omega)\| k_2 + \|T(\Omega)\| k_1$ . Moreover, if  $\|P(\Omega)\| k_2 + \|T(\Omega)\| k_1 < 1$ , then in view of a modified version of Darbo fixed point theorem [5] we infer that the operator  $S$  has at least one fixed point on the set  $\Omega$ . This completes the proof.  $\square$

**Remark 2.6.** It may be shown [5] that the set  $\text{Fix } S$  of all fixed points of the operator  $S$  on the set  $\Omega$  is a member of the kernel  $\ker \mu$ .

### 3. Existence of monotonic solutions of a functional integral equation of fractional order in the Banach algebra $C(I)$

In this section we will work in the Banach algebra  $C(I)$ , where  $I$  is a bounded and closed interval. For simplicity we assume that  $I = [0, 1]$ . We will use the measure of noncompactness  $\mu_d$  defined previously by formula (1).

The object of our study is the following functional integral equation:

$$x(t) = f(t, x(t)) \left( p(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t, s, (Gx)(s))}{(t-s)^{1-\alpha}} ds \right), \quad (6)$$

where  $t \in I$ ,  $\alpha$  is a fixed number from the interval  $(0, 1)$  and  $\Gamma(\alpha)$  denotes the gamma function. Moreover,  $G$  is an operator acting from  $C(I)$  into itself. Observe that the above equation may be written in the product form  $x(t) = (Fx)(t)(Vx)(t)$ , where  $F$  is the so-called superposition operator [3] defined by the formula

$$(Fx)(t) = f(t, x(t)), \quad t \in I,$$

and  $V$  is the Volterra integral operator of fractional order having the form

$$(Vx)(t) = p(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t, s, (Gx)(s))}{(t-s)^{1-\alpha}} ds, \quad t \in I.$$

It is worthwhile mentioning that integral and differential equations of fractional order play recently a very important role in mathematical investigations. The

mathematical literature concerning mentioned equations is very extensive and includes both research papers [4, 10, 13, 15] and monographs [21, 25, 26].

In what follows we will consider equation (6) assuming that the following conditions are satisfied:

- (i)  $p \in C(I)$  and  $p$  is nondecreasing and nonnegative function on the interval  $I$ .
- (ii) The function  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous with  $f(I \times \mathbb{R}_+) \subseteq \mathbb{R}_+$ . Moreover, the function  $t \rightarrow f(t, x)$  is nondecreasing on  $I$  for any fixed  $x \in \mathbb{R}_+$  and the function  $x \rightarrow f(t, x)$  is nondecreasing on  $\mathbb{R}_+$  for any fixed  $t \in I$ .
- (iii) There exists a nondecreasing function  $k(r) = k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$|f(t, x_1) - f(t, x_2)| \leq k(r)|x_1 - x_2|$$

for  $t \in I$  and for all  $x_1, x_2 \in [-r, r]$ .

- (iv)  $v : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that  $v : I \times I \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and  $v(t, s, x)$  is nondecreasing with respect to each variable  $t, s$  and  $x$ , separately.
- (v) There exists a continuous and nondecreasing function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $|v(t, s, x)| \leq \Phi(|x|)$  for  $t, s \in I$  and for all  $x \in \mathbb{R}$ .
- (vi) The operator  $G$  transforms continuously the space  $C(I)$  into itself and there exists a nondecreasing function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\|Gx\| \leq \varphi(\|x\|)$  for any  $x \in C(I)$ . Moreover, for each function  $x \in C(I)$  which is nonnegative on  $I$ , the function  $Gx$  is nonnegative and nondecreasing on  $I$ .
- (vii) There exists a positive solution  $r_0$  of the inequality

$$(rk(r) + \bar{F}) \left( \|p\| + \frac{\Phi(\varphi(r))}{\Gamma(\alpha + 1)} \right) \leq r,$$

where  $\bar{F} = \max\{f(t, 0) : t \in I\}$ . Moreover, the number  $r_0$  is such that  $k(r_0) \left( \|p\| + \frac{\Phi(\varphi(r_0))}{\Gamma(\alpha + 1)} \right) < 1$ .

Now we can formulate the existence result concerning the functional integral equation (6).

**Theorem 3.1.** *Under assumptions (i)–(vii) equation (6) has at least one solution  $x(t) = x \in C(I)$  which is nonnegative and nondecreasing on  $I$ .*

*Proof.* At first let us observe that in view of assumption (ii) we have (cf. [3]) that the operator  $F$  transforms the Banach algebra  $C(I)$  into itself and is continuous. We show that the operator  $V$  has also the same properties.

To do this let us fix  $\varepsilon > 0$  and take arbitrarily  $t_1, t_2 \in I$  such that  $|t_2 - t_1| \leq \varepsilon$ . Without loss of generality we may assume that  $t_1 < t_2$ . Then, for arbitrarily



fixed  $x \in C(I)$  we obtain

$$\begin{aligned}
 & |(Vx)(t_2) - (Vx)(t_1)| \\
 & \leq |p(t_2) - p(t_1)| \\
 & \quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_2} \frac{v(t_1, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \right| \\
 & \quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} \frac{v(t_1, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{v(t_1, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \right| \\
 & \quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{v(t_1, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{v(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds \right| \\
 & \leq \omega(p, \varepsilon) + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_2, s, (Gx)(s)) - v(t_1, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \\
 & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{|v(t_1, s, (Gx)(s))|}{(t_2 - s)^{1-\alpha}} ds \\
 & \quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |v(t_1, s, (Gx)(s))| \left| \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} \right| ds \\
 & \leq \omega(p, \varepsilon) + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{\omega_{\varphi(\|x\|)}(v, \varepsilon)}{(t_2 - s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\Phi(\varphi(\|x\|))}{(t_2 - s)^{1-\alpha}} ds \\
 & \quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \Phi(\varphi(\|x\|)) \left[ \frac{1}{(t_1 - s)^{1-\alpha}} - \frac{1}{(t_2 - s)^{1-\alpha}} \right] ds \\
 & \leq \omega(p, \varepsilon) + \frac{\omega_{\varphi(\|x\|)}(v, \varepsilon)}{\alpha\Gamma(\alpha)} t_2^\alpha + \frac{\Phi(\varphi(\|x\|))}{\alpha\Gamma(\alpha)} (t_2 - t_1)^\alpha \\
 & \quad + \frac{\Phi(\varphi(\|x\|))}{\alpha\Gamma(\alpha)} [t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha] \\
 & \leq \omega(p, \varepsilon) + \frac{\omega_{\varphi(\|x\|)}(v, \varepsilon)}{\Gamma(\alpha + 1)} + \frac{2\Phi(\varphi(\|x\|))}{\Gamma(\alpha + 1)} \varepsilon^\alpha,
 \end{aligned} \tag{7}$$

where we denoted

$$\omega_d(v, \varepsilon) = \sup\{|v(t_2, s, y) - v(t_1, s, y)| : t_2, t_1, s \in I, |t_2 - t_1| \leq \varepsilon, y \in [-d, d]\}.$$

Hence, taking into account the uniform continuity of the function  $v(t, s, x)$  on the set  $I \times I \times [-\varphi(\|x\|), \varphi(\|x\|)]$  we infer that the function  $Vx$  is continuous on  $I$ . Thus  $V$  transforms the Banach algebra  $C(I)$  into itself.

On the other hand, for a fixed  $x \in C(I)$  and  $t \in I$  we get

$$\begin{aligned}
 |(Fx)(t)| & \leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\
 & \leq k(\|x\|)\|x\| + f(t, 0) \\
 & \leq \|x\|k(\|x\|) + \overline{F}.
 \end{aligned} \tag{8}$$

Moreover, we obtain

$$\begin{aligned} |(Vx)(t)| &\leq |p(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|v(t, s, (Gx)(s))|}{(t-s)^{1-\alpha}} ds \\ &\leq \|p\| + \frac{\Phi(\varphi(\|x\|))}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\ &\leq \|p\| + \frac{\Phi(\varphi(\|x\|))}{\Gamma(\alpha+1)}. \end{aligned} \quad (9)$$

In what follows let us observe that linking (8), (9) and assumption (vii) we can deduce that there exists a positive number  $r_0$  such that the operator  $W = F \cdot V$  transforms the ball  $B_{r_0}$  into itself. On the other hand let us notice that from estimates (8) and (9) and from the fact established above we infer that the following inequalities are satisfied:

$$\|FB_{r_0}\| \leq r_0k(r_0) + \bar{F} \quad (10)$$

$$\|VB_{r_0}\| \leq \|p\| + \frac{\Phi(\varphi(r_0))}{\Gamma(\alpha+1)}. \quad (11)$$

Further, let us consider the set  $Q$  consisting of all nonnegative functions  $x \in B_{r_0}$ . Then, keeping in mind our assumptions we deduce that the operator  $W$  transforms the set  $Q$  into itself. Moreover, from (10) and (11) we obtain:

$$\|FQ\| \leq r_0k(r_0) + \bar{F} \quad (12)$$

$$\|VQ\| \leq \|p\| + \frac{\Phi(\varphi(r_0))}{\Gamma(\alpha+1)}. \quad (13)$$

In the sequel we show that the operator  $W = F \cdot V$  is continuous on the set  $Q$ . To do this let us first observe that the continuity of the operator  $F$  is an easy consequence of the assumptions (ii), (iii) and a well known result concerning the continuity of the superposition operator [3].

Next, we show that the operator  $V$  is continuous on the set  $Q$ . Thus, let us fix arbitrarily  $\varepsilon > 0$  and  $x_0 \in Q$ . In view of assumption (vi) we can find  $\delta > 0$  such that for an arbitrary  $x \in Q$  such that  $\|x - x_0\| \leq \delta$  we have that  $\|Gx - Gx_0\| \leq \varepsilon$ . Hence, for arbitrarily fixed  $t \in I$  we get

$$\begin{aligned} |(Vx)(t) - (Vx_0)(t)| &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^t \frac{v(t, s, (Gx)(s))}{(t-s)^{1-\alpha}} ds - \int_0^t \frac{v(t, s, (Gx_0)(s))}{(t-s)^{1-\alpha}} ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|v(t, s, (Gx)(s)) - v(t, s, (Gx_0)(s))|}{(t-s)^{1-\alpha}} ds \\ &< \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\bar{\omega}(v, \varepsilon)}{(t-s)^{1-\alpha}} ds \\ &\leq \frac{\bar{\omega}(v, \varepsilon)}{\Gamma(\alpha+1)}, \end{aligned}$$

where we denoted

$$\bar{\omega}(v, \varepsilon) = \sup\{|v(t, s, a) - v(t, s, b)| : t, s \in I, a, b \in [0, \varphi(r_0)], |a - b| \leq \varepsilon\}.$$

Obviously, in view of assumption (iv) we have that  $\bar{\omega}(v, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . This implies the desired continuity of the operator  $V$  on the set  $Q$ . Finally we conclude that  $W$  is continuous on the set  $Q$ .

Now, let us fix a nonempty subset  $X$  of the set  $Q$ . Next, choose a number  $\varepsilon > 0$  and take  $t_1, t_2 \in I$  such that  $|t_2 - t_1| \leq \varepsilon$ . Without loss of generality we may assume that  $t_1 < t_2$ . Then we have

$$\begin{aligned} |(Fx)(t_2) - (Fx)(t_1)| &\leq |f(t_2, x(t_2)) - f(t_2, x(t_1))| + |f(t_2, x(t_1)) - f(t_1, x(t_1))| \\ &\leq k(r_0)|x(t_2) - x(t_1)| + \omega_{r_0}(f, \varepsilon) \\ &\leq k(r_0)\omega(x, \varepsilon) + \omega_{r_0}(f, \varepsilon), \end{aligned}$$

where we denoted

$$\omega_{r_0}(f, \varepsilon) = \sup\{|f(t_2, x) - f(t_1, x)| : t_1, t_2 \in I, |t_2 - t_1| \leq \varepsilon, x \in [-r_0, r_0]\}.$$

Hence we infer  $\omega(Fx, \varepsilon) \leq k(r_0)\omega(x, \varepsilon) + \omega_{r_0}(f, \varepsilon)$  and consequently

$$\omega_0(FX) \leq k(r_0)\omega_0(X). \tag{14}$$

Further, evaluating similarly as in estimate (7), we obtain

$$\begin{aligned} |(Vx)(t_2) - (Vx)(t_1)| &\leq \omega(p, \varepsilon) + \frac{\omega_{\varphi(r_0)}(v, \varepsilon)}{\alpha\Gamma(\alpha)}t_2^\alpha + \frac{\Phi(\varphi(\|x\|))}{\alpha\Gamma(\alpha)}(t_2 - t_1)^\alpha \\ &\quad + \frac{\Phi(\varphi(\|x\|))}{\alpha\Gamma(\alpha)}[t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha] \\ &\leq \omega(p, \varepsilon) + \frac{\omega_{\varphi(r_0)}(v, \varepsilon)}{\Gamma(\alpha + 1)} + \frac{\Phi(\varphi(r_0))}{\Gamma(\alpha + 1)}\varepsilon^\alpha + \frac{\Phi(\varphi(r_0))}{\Gamma(\alpha + 1)}\varepsilon^\alpha \\ &= \omega(p, \varepsilon) + \frac{1}{\Gamma(\alpha + 1)}[\omega_{\varphi(r_0)}(v, \varepsilon) + 2\Phi(\varphi(r_0))\varepsilon^\alpha]. \end{aligned}$$

This implies  $\omega(VX, \varepsilon) \leq \omega(p, \varepsilon) + \frac{1}{\Gamma(\alpha+1)}[\omega_{\varphi(r_0)}(v, \varepsilon) + 2\Phi(\varphi(r_0))\varepsilon^\alpha]$  and consequently

$$\omega_0(VX) = 0. \tag{15}$$

Further on, assume (similarly as above) that  $t_1, t_2 \in I$  and  $t_1 < t_2$ . Then, taking an arbitrary function  $x \in X$  we get

$$\begin{aligned} &|(Vx)(t_2) - (Vx)(t_1)| - [(Vx)(t_2) - (Vx)(t_1)] \\ &\leq |p(t_2) - p(t_1)| - [p(t_2) - p(t_1)] \\ &\quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{v(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds \right| \\ &\quad - \frac{1}{\Gamma(\alpha)} \left[ \int_0^{t_2} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{v(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds \right]. \end{aligned} \tag{16}$$

Observe that keeping in mind our assumptions we have

$$\begin{aligned} & \int_0^{t_2} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{v(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds \\ &= \int_0^{t_1} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{v(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds + \int_{t_1}^{t_2} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \\ &\geq \int_0^{t_1} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{v(t_2, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds + \int_{t_1}^{t_2} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \\ &= \int_0^{t_1} v(t_2, s, (Gx)(s)) \left[ \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} \right] ds + \int_{t_1}^{t_2} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds. \end{aligned}$$

Hence, taking into account the fact that the term  $\frac{1}{(t_2-s)^{1-\alpha}} - \frac{1}{(t_1-s)^{1-\alpha}}$  is negative, we get

$$\begin{aligned} & \int_0^{t_2} \frac{v(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{v(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds \\ &\geq \int_0^{t_1} v(t_2, t_1, (Gx)(s)) \left[ \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} \right] ds + \int_{t_1}^{t_2} \frac{v(t_2, t_1, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \\ &\geq \int_0^{t_1} v(t_2, t_1, (Gx)(t_1)) \left[ \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} \right] ds + \int_{t_1}^{t_2} \frac{v(t_2, t_1, (Gx)(t_1))}{(t_2 - s)^{1-\alpha}} ds \\ &= v(t_2, t_1, (Gx)(t_1)) \left\{ \int_0^{t_1} \left[ \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} \right] ds + \int_{t_1}^{t_2} \frac{ds}{(t_2 - s)^{1-\alpha}} \right\} \\ &= v(t_2, t_1, (Gx)(t_1)) \frac{t_2^\alpha - t_1^\alpha}{\alpha} \geq 0. \end{aligned}$$

The above obtained inequality in conjunction with (16) allows us to deduce that  $d(Vx) = 0$ . Hence

$$d(VX) = 0. \tag{17}$$

Further, let us notice that from a result established in [11] we have that

$$d(FX) \leq k(r_0)d(X). \tag{18}$$

Now, linking (14), (15), (17), (18) and the definition of the measure of noncompactness  $\mu_d$  given by formula (1), we get

$$\mu_d(FX) \leq k(r_0)\mu_d(X) \tag{19}$$

$$\mu_d(VX) = 0. \tag{20}$$

Now, let us notice that taking into account estimates (12), (13), (19), (20) and assumption (vii) we conclude in view of Theorem 2.5 that the operator  $W$  is a contraction with respect to the measure of noncompactness  $\mu_d$  on the set  $Q$ . Thus, the operator  $W$  has a fixed point  $x$  in the set  $Q$ . Observe that in virtue of Remark 2.6 the function  $x = x(t)$  is a nonnegative and nondecreasing solution of the functional integral equation (6). This completes the proof.  $\square$

In what follows we illustrate the result contained in Theorem 3.1 by the following example.

**Example 3.2.** Consider the functional integral equation of fractional order having the form

$$\begin{aligned}
 x(t) = & \left[ \frac{t}{t^2 + 15} + x^2(t) \right] \\
 & \times \left[ t^2 e^{-2t} + \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{ts + \max \{ \sqrt{|x(\tau)|} : 0 \leq \tau \leq s \}}{(1 + s^2 + t^2)(t - s)^{\frac{1}{2}}} ds \right], \tag{21}
 \end{aligned}$$

where  $t \in I = [0, 1]$ . Observe that this equation is a particular case of equation (6), where

$$p(t) = t^2 e^{-2t}, \quad f(t, x) = \frac{t}{t^2 + 15} + x^2, \quad v(t, s, x) = \frac{ts + x}{1 + s^2 + t^2}.$$

Moreover,  $\alpha = \frac{1}{2}$  and  $(Gx)(t) = \max \{ \sqrt{|x(\tau)|} : 0 \leq \tau \leq t \}$ . It is easily seen that for equation (21) there are satisfied the assumptions of Theorem 3.1. Indeed, we have  $k(r) = 2r$ ,  $f(t, 0) = \frac{t}{t^2 + 15}$ ,  $\bar{F} = \frac{1}{16}$ . Apart from this, we have

$$|v(t, s, x)| \leq \frac{ts + |x|}{1 + s^2 + t^2} \leq \frac{ts}{1 + s^2 + t^2} + \frac{|x|}{1 + s^2 + t^2} \leq \frac{1}{2} + |x|.$$

Hence we see that the function  $\Phi(r)$  appearing in assumption (v) may be accepted in the form  $\Phi(r) = \frac{1+2r}{2}$ . Moreover, we have that  $\|Gx\| \leq \sqrt{\|x\|}$  so we can put  $\varphi(r) = \sqrt{r}$ . Additionally we have that  $\|p\| = \frac{1}{2e}$ . Thus the inequality from assumption (vii) has the form

$$\left( 2r^2 + \frac{1}{16} \right) \left( \frac{1}{2e} + \frac{1 + 2\sqrt{r}}{2\Gamma(\frac{3}{2})} \right) \leq r.$$

Taking into account that  $\Gamma(\frac{3}{2}) \geq 0.8856\dots$  (cf. [20]) we can check that the number  $r_0 = \frac{1}{4}$  satisfies the above inequality. Moreover, we have that

$$k(r_0) \left( \frac{1}{2e} + \frac{1 + 2\sqrt{r_0}}{2\Gamma(\frac{3}{2})} \right) = \frac{1}{2} \left( \frac{1}{2e} + \frac{1}{\Gamma(\frac{3}{2})} \right) < 1,$$

so this shows that the second inequality from assumption (vii) is also satisfied. Finally we conclude that equation (21) has a solution belonging to the ball  $B_{\frac{1}{4}}$  and being nonnegative and nondecreasing on the interval  $I$ .

It is worthwhile mentioning that the functional integral equation (21) belongs to the important class of integral equations called the *equations with supremum* (or *with maximum*). Equations of that type were recently investigated in some research papers (see [2, 12, 22], for example).

#### 4. Existence of solutions of a functional integral equation in the Banach algebra $BC(\mathbb{R}_+)$

This section is devoted to the study of the following operator equation

$$x(t) = (Vx)(t)(Ux)(t), \quad t \in \mathbb{R}_+, \quad (22)$$

where the operators  $V$  and  $U$  are defined on the Banach algebra  $BC(\mathbb{R}_+)$  in the following way:

$$(Vx)(t) = p_1(t) + f_1(t, x(t)) \int_0^t v(t, s, x(s)) ds$$

$$(Ux)(t) = p_2(t) + f_2(t, x(t)) \int_0^\infty u(t, s, x(s)) ds.$$

Notice that  $V$  represents the so-called quadratic Volterra integral operator and  $U$  is the quadratic Urysohn integral operator. Thus, equation (22) is a nonlinear integral equation of product type which contains a lot of particular cases of functional, integral and functional integral equations.

We will investigate the nonlinear integral equation (22) assuming the following hypotheses:

- (i)  $p_i \in BC(\mathbb{R}_+)$  and  $p_i(t) \rightarrow 0$  as  $t \rightarrow \infty$  ( $i = 1, 2$ ).
- (ii)  $f_i : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and such that  $f_i(t, 0) \rightarrow 0$  as  $t \rightarrow \infty$ , for  $i = 1, 2$ .
- (iii) The functions  $f_i$  ( $i = 1, 2$ ) satisfy the Lipschitz condition with respect to the second variable, i.e., there exists a constant  $k_i > 0$  such that

$$|f_i(t, x) - f_i(t, y)| \leq k_i|x - y|$$

for  $x, y \in \mathbb{R}$  and for  $t \in \mathbb{R}_+$  ( $i = 1, 2$ ).

- (iv)  $v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there exist a continuous function  $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and a continuous and nondecreasing function  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$|v(t, s, x)| \leq g(t, s)G(|x|)$$

for all  $t, s \in \mathbb{R}_+$  and  $x \in \mathbb{R}$ .

- (v)  $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there exist a continuous function  $h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and a continuous and nondecreasing function  $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$|u(t, s, x)| \leq h(t, s)H(|x|)$$

for all  $t, s \in \mathbb{R}_+$  and  $x \in \mathbb{R}$ .

- (vi) The function  $t \rightarrow \int_0^t g(t, s)ds$  is bounded on  $\mathbb{R}_+$ .
- (vii) For each  $t \in \mathbb{R}_+$  the function  $s \rightarrow h(t, s)$  is integrable on  $\mathbb{R}_+$  and the function  $t \rightarrow \int_0^\infty h(t, s)ds$  is bounded on  $\mathbb{R}_+$ .
- (viii) The improper integral  $\int_0^\infty h(t, s)ds$  is uniformly convergent with respect to  $t \in \mathbb{R}_+$ , i.e.,

$$\lim_{T \rightarrow \infty} \left\{ \sup_{t \in \mathbb{R}_+} \int_T^\infty h(t, s)ds \right\} = 0.$$

Observe that based on assumptions (ii), (vi) and (vii) we may define the following finite constants:

$$\begin{aligned} \bar{F}_i &= \sup \{ |f_i(t, 0)| : t \in \mathbb{R}_+ \}, \quad i = 1, 2 \\ \bar{G} &= \sup \left\{ \int_0^t g(t, s)ds : t \in \mathbb{R}_+ \right\} \\ \bar{H} &= \sup \left\{ \int_0^\infty h(t, s)ds : t \in \mathbb{R}_+ \right\}. \end{aligned}$$

Now we can formulate the last assumption:

- (ix) There exists a positive solution  $r_0$  of the inequality

$$[p + k\bar{G}rG(r) + \bar{F} \bar{G}G(r)][p + k\bar{H}rH(r) + \bar{F} \bar{H}H(r)] \leq r$$

such that

$$pk(\bar{G}G(r_0) + \bar{H}H(r_0)) + 2k\bar{F} \bar{G} \bar{H}G(r_0)H(r_0) + 2k^2r_0\bar{G} \bar{H}G(r_0)H(r_0) < 1,$$

where  $p = \max\{\|p_1\|, \|p_2\|\}$ ,  $\bar{F} = \max\{\bar{F}_1, \bar{F}_2\}$ ,  $k = \max\{k_1, k_2\}$ .

**Remark 4.1.** The concept of *the uniform convergence* of the improper integral occurring in assumption (viii) is taken from the theory of improper Riemann integral with a parameter [20]. Namely, in order to adopt this concept to our situation assume that  $h(t, s) = h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is a given function such that the integral

$$\int_0^\infty h(t, s) ds \tag{23}$$

exists for any fixed  $t \in \mathbb{R}_+$ . We say that integral (23) is *uniformly convergent with respect to*  $t \in \mathbb{R}_+$  [20] if

$$\lim_{T \rightarrow \infty} \int_0^T h(t, s)ds = \int_0^\infty h(t, s) ds$$

uniformly with respect to  $t \in \mathbb{R}_+$ . It is easily seen that we may say equivalently that integral (23) is uniformly convergent with respect to  $t \in \mathbb{R}_+$  if the condition from assumption (viii) is satisfied.

**Remark 4.2.** It is easy to check that if  $r_0$  is a positive solution of the first inequality from assumption (ix), then the following inequality is satisfied

$$pk(\overline{G}G(r_0) + \overline{H}H(r_0)) + 2k\overline{F} \overline{G} \overline{H}G(r_0)H(r_0) + 2k^2r_0\overline{G} \overline{H}G(r_0)H(r_0) \leq 1.$$

Thus, if we assume that  $p \neq 0$ , then the second inequality from assumption (ix) is automatically satisfied.

Now we can formulate an existence result concerning the functional integral equation (22).

**Theorem 4.3.** *Under the assumptions (i)–(ix) equation (22) has at least one solution  $x = x(t)$  in the Banach algebra  $BC(\mathbb{R}_+)$  such that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .*

*Proof.* At first, let us consider the quadratic Volterra integral operator  $V$  defined above and appearing in equation (22).

Suppose  $x$  is a fixed function from  $BC(\mathbb{R}_+)$ . Obviously we have that  $Vx$  is a continuous function on  $\mathbb{R}_+$  which is a simple consequence of assumptions (i), (ii) and (iv). Further we show that  $Vx$  is bounded on  $\mathbb{R}_+$ . In fact, using our assumptions, for arbitrarily fixed  $t \in \mathbb{R}_+$  we obtain:

$$\begin{aligned} |(Vx)(t)| &\leq |p_1(t)| + |f_1(t, x(t))| \int_0^t |v(t, s, x(s))| ds \\ &\leq |p_1(t)| + [|f_1(t, x(t)) - f_1(t, 0)| + |f_1(t, 0)|] \int_0^t g(t, s)G(|x(s)|) ds \quad (24) \\ &\leq |p_1(t)| + [k_1|x(t)| + |f_1(t, 0)|]G(\|x\|) \int_0^t g(t, s) ds. \end{aligned}$$

Hence we get

$$|(Vx)(t)| \leq \|p_1\| + k_1\overline{G}\|x\|G(\|x\|) + \overline{F}_1\overline{G}G(\|x\|). \quad (25)$$

The above estimate yields that the function  $Vx$  is bounded on  $\mathbb{R}_+$ . Joining this fact with the continuity of  $Vx$  on  $\mathbb{R}_+$  we conclude that the operator  $V$  transforms the Banach algebra  $BC(\mathbb{R}_+)$  into itself. Moreover, from (25) we obtain

$$\|Vx\| \leq p + k\overline{G}\|x\|G(\|x\|) + \overline{F} \overline{G}G(\|x\|). \quad (26)$$

We now proceed to the investigation of the quadratic Urysohn integral operator  $U$  defined above on  $BC(\mathbb{R}_+)$ . Thus, fix a function  $x \in BC(\mathbb{R}_+)$ . We show that the function  $Ux$  is continuous on  $\mathbb{R}_+$ .

To prove this fact let us fix  $T > 0$  and  $\varepsilon > 0$ . Next, take arbitrary numbers  $t, s \in [0, T]$  such that  $|t - s| \leq \varepsilon$ . Then, keeping in mind our assumptions, we



get

$$\begin{aligned}
 & |(Ux)(t) - (Ux)(s)| \\
 & \leq |p_2(t) - p_2(s)| \\
 & \quad + \left| f_2(t, x(t)) \int_0^\infty u(t, \tau, x(\tau)) d\tau - f_2(s, x(s)) \int_0^\infty u(t, \tau, x(\tau)) d\tau \right| \\
 & \quad + \left| f_2(s, x(s)) \int_0^\infty u(t, \tau, x(\tau)) d\tau - f_2(s, x(s)) \int_0^\infty u(s, \tau, x(\tau)) d\tau \right| \\
 & \leq \omega^T(p_2, \varepsilon) + [|f_2(t, x(t)) - f_2(t, x(s))| \\
 & \quad + |f_2(t, x(s)) - f_2(s, x(s))|] \int_0^\infty h(t, \tau) H(|x(\tau)|) d\tau \\
 & \quad + [|f_2(s, x(s)) - f_2(s, 0)| + |f_2(s, 0)|] \int_0^\infty |u(t, \tau, x(\tau)) - u(s, \tau, x(\tau))| d\tau \\
 & \leq \omega^T(p_2, \varepsilon) + [k_2|x(t) - x(s)| + \omega_{\|x\|}^T(f_2, \varepsilon)] H(\|x\|) \int_0^\infty h(t, \tau) d\tau \\
 & \quad + [k_2|x(s)| + |f_2(s, 0)|] \int_0^\infty |u(t, \tau, x(\tau)) - u(s, \tau, x(\tau))| d\tau,
 \end{aligned} \tag{27}$$

where we denoted

$$\omega_d^T(f_2, \varepsilon) = \sup\{|f_2(t, y) - f_2(s, y)| : t, s \in [0, T], y \in [-d, d], |t - s| \leq \varepsilon\}.$$

Obviously, in the above calculations we should put  $\|x\|$  instead of  $d$ .

Now, from estimate (27) we obtain

$$\begin{aligned}
 & |(Ux)(t) - (Ux)(s)| \\
 & \leq \omega^T(p_2, \varepsilon) + k_2 \bar{H} H(\|x\|) \omega^T(x, \varepsilon) + \bar{H} H(\|x\|) \omega_{\|x\|}^T(f_2, \varepsilon) \\
 & \quad + (k_2 \|x\| + \bar{F}_2) \left\{ \int_0^T |u(t, \tau, x(\tau)) - u(s, \tau, x(\tau))| d\tau \right. \\
 & \quad \left. + \int_T^\infty [|u(t, \tau, x(\tau))| + |u(s, \tau, x(\tau))|] d\tau \right\} \\
 & \leq \omega^T(p_2, \varepsilon) + k_2 \bar{H} H(\|x\|) \omega^T(x, \varepsilon) + \bar{H} H(\|x\|) \omega_{\|x\|}^T(f_2, \varepsilon) \\
 & \quad + (k_2 \|x\| + \bar{F}_2) \left\{ \int_0^T \omega_{\|x\|}^T(u, \varepsilon) d\tau + \int_T^\infty (h(t, \tau) + h(s, \tau)) H(\|x\|) d\tau \right\},
 \end{aligned}$$

where we denoted

$$\omega_d^T(u, \varepsilon) = \sup\{|u(t, \tau, y) - u(s, \tau, y)| : t, s, \tau \in [0, T], y \in [-d, d], |t - s| \leq \varepsilon\}.$$

Observe that  $\omega_{\|x\|}^T(f_2, \varepsilon) \rightarrow 0$  and  $\omega_{\|x\|}^T(u, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , which is a simple consequence of the uniform continuity of the function  $f_2$  on the set  $[0, T] \times$

$[-\|x\|, \|x\|]$  and the function  $u$  on the set  $[0, T] \times [0, T] \times [-\|x\|, \|x\|]$ . Further, from the above estimate we derive

$$\begin{aligned} |(Ux)(t) - (Ux)(s)| &\leq \omega^T(p_2, \varepsilon) + k_2 \overline{H}H(\|x\|)\omega^T(x, \varepsilon) \\ &\quad + \overline{H}H(\|x\|)\omega_{\|x\|}^T(f_2, \varepsilon) + (k_2\|x\| + \overline{F}_2)T\omega_{\|x\|}^T(u, \varepsilon) \\ &\quad + (k_2\|x\| + \overline{F}_2)2H(\|x\|) \sup \left\{ \int_T^\infty h(t, \tau) d\tau : t \in \mathbb{R}_+ \right\}. \end{aligned} \tag{28}$$

Now, let us notice that in virtue of assumption (viii) we can choose a number  $T$  so big that the last term of estimate (28) is sufficiently small. Hence, taking into account the facts established above we deduce that the function  $Ux$  is continuous on the interval  $[0, T]$  for any  $T > 0$  which is sufficiently big. This allows us to conclude that  $Ux$  is continuous on  $\mathbb{R}_+$ .

In what follows we show that  $Ux$  is bounded on  $\mathbb{R}_+$ . Indeed, in view of assumptions for arbitrarily fixed  $t \in \mathbb{R}_+$  we have

$$\begin{aligned} |(Ux)(t)| &\leq |p_2(t)| + |f_2(t, x(t))| \int_0^\infty |u(t, s, x(s))| ds \\ &\leq |p_2(t)| + [|f_2(t, x(t)) - f_2(t, 0)| + |f_2(t, 0)|] \int_0^\infty h(t, s)H(|x(s)|) ds \\ &\leq |p_2(t)| + k_2 \overline{H}H(\|x\|)|x(t)| + \overline{H}H(\|x\|)|f_2(t, 0)|. \end{aligned} \tag{29}$$

This yields the estimate

$$|(Ux)(t)| \leq \|p_2\| + k_2 \overline{H}\|x\|H(\|x\|) + \overline{F} \overline{H}H(\|x\|), \tag{30}$$

which means that the function  $Ux$  is bounded on  $\mathbb{R}_+$ . Linking the obtained fact with the continuity of the function  $Ux$  on  $\mathbb{R}_+$  we conclude that the operator  $U$  transforms the space  $BC(\mathbb{R}_+)$  into itself. Further, from (30) we obtain

$$\|Ux\| \leq p + k\overline{H}\|x\|H(\|x\|) + \overline{F} \overline{H}H(\|x\|). \tag{31}$$

Now, let us observe that linking estimates (26), (31) and assumption (ix) we infer that there exists a number  $r_0 > 0$  such that the operator  $W$  transforms the ball  $B_{r_0}$  into itself, where  $W$  is defined as the product of the operators  $V$  and  $U$ , i.e.,  $(Wx)(t) = (Vx)(t)(Ux)(t)$  for  $x \in BC(\mathbb{R}_+)$  and for  $t \in \mathbb{R}_+$ . Moreover, the number  $r_0$  satisfies the second inequality from assumption (ix).

On the other hand, let us notice that from the above statement and from estimates (26) and (31) we derive

$$\|VB_{r_0}\| \leq p + k\overline{G}r_0G(r_0) + \overline{F} \overline{G}G(r_0) \tag{32}$$

$$\|UB_{r_0}\| \leq p + k\overline{H}r_0H(r_0) + \overline{F} \overline{H}H(r_0). \tag{33}$$

In what follows we will work with the measure of noncompactness  $\mu_a$  defined in the Banach algebra  $BC(\mathbb{R}_+)$  by formula (3). So, let us fix a nonempty subset  $X$  of the ball  $B_{r_0}$ . Next, choose arbitrary numbers  $T > 0$  and  $\varepsilon > 0$ . Then, for  $x \in X$  and for  $t, s \in [0, T]$  with  $|t - s| \leq \varepsilon$  we obtain

$$\begin{aligned}
 & |(Vx)(t) - (Vx)(s)| \\
 & \leq |p_1(t) - p_1(s)| + |f_1(t, x(t)) - f_1(s, x(s))| \int_0^t |v(t, \tau, x(\tau))| d\tau \\
 & \quad + |f_1(s, x(s))| \int_0^t |v(t, \tau, x(\tau)) - v(s, \tau, x(\tau))| d\tau \\
 & \leq |p_1(t) - p_1(s)| + [|f_1(t, x(t)) - f_1(t, x(s))| \\
 & \quad + |f_1(t, x(s)) - f_1(s, x(s))|] \int_0^t g(t, \tau)G(|x(\tau)|) d\tau \\
 & \quad + [|f_1(s, x(s)) - f_1(s, 0)| + |f_1(s, 0)|] \int_0^t |v(t, \tau, x(\tau)) - v(s, \tau, x(\tau))| d\tau \\
 & \leq \omega^T(p_1, \varepsilon) + [k_1|x(t) - x(s)| + \omega_{r_0}^T(f_1, \varepsilon)]G(r_0) \int_0^T g(t, \tau) d\tau \\
 & \quad + (k_1|x(s)| + \bar{F}_1) \int_0^T \omega_{r_0}^T(v, \varepsilon) d\tau \\
 & \leq \omega^T(p_1, \varepsilon) + [k_1\omega^T(x, \varepsilon) + \omega_{r_0}^T(f_1, \varepsilon)]G(r_0)\bar{G} + (k_1r_0 + \bar{F}_1)T\omega_{r_0}^T(v, \varepsilon),
 \end{aligned} \tag{34}$$

where we denoted

$$\begin{aligned}
 \omega_d^T(f_1, \varepsilon) &= \sup\{|f_1(t, y) - f_1(s, y)| : t, s \in [0, T], y \in [-d, d], |t - s| \leq \varepsilon\} \\
 \omega_d^T(v, \varepsilon) &= \sup\{|v(t, \tau, y) - v(s, \tau, y)| : t, s, \tau \in [0, T], y \in [-d, d], |t - s| \leq \varepsilon\}.
 \end{aligned}$$

Observe that  $\omega_{r_0}^T(f_1, \varepsilon) \rightarrow 0$  and  $\omega_{r_0}^T(v, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . This fact follows easily from the uniform continuity of the function  $f_1$  on the set  $[0, T] \times [-r_0, r_0]$  and the function  $v$  on the set  $[0, T] \times [0, T] \times [-r_0, r_0]$ . Thus, in view of (34) we obtain  $\omega_0^T(VX) \leq \bar{G}k_1G(r_0)\omega_0^T(X)$ , and consequently

$$\omega_0^\infty(VX) \leq k\bar{G}G(r_0)\omega_0^\infty(X). \tag{35}$$

Further, let us notice that in the similar way as above, from estimate (28) we obtain

$$\omega_0^T(UX) \leq \bar{H}k_2H(r_0)\omega_0^T(X) + (k_2r_0 + \bar{F}_2)2 \sup \left\{ \int_T^\infty h(t, \tau) d\tau : t \in \mathbb{R}_+ \right\}.$$

Hence, taking into account assumption (viii), we get

$$\omega_0^\infty(UX) \leq k\bar{H}H(r_0)\omega_0^\infty(X). \tag{36}$$

In what follows let us fix  $T > 0$ . Then, for an arbitrarily fixed  $t \geq T$  from estimate (24) we obtain

$$\sup\{|(Vx)(t)| : t \geq T\} \leq \sup\{|p_1(t)| : t \geq T\} + \overline{G}k_1G(r_0) \sup\{|x(t)| : t \geq T\} + \overline{G}G(r_0) \sup\{|f_1(t, 0)| : t \geq T\}.$$

Hence, in view of assumptions (i) and (ii) we get

$$a(VX) \leq k\overline{G}G(r_0)a(X). \tag{37}$$

In the same way, based on assumptions (i), (ii) and estimate (29) we obtain

$$a(UX) \leq k\overline{H}H(r_0)a(X). \tag{38}$$

Finally, taking into account estimates (35)–(38) we obtain

$$\mu_a(VX) \leq k\overline{G}G(r_0)\mu_a(X) \tag{39}$$

$$\mu_a(UX) \leq k\overline{H}H(r_0)\mu_a(X), \tag{40}$$

where  $\mu_a$  is the measure of noncompactness defined by formula (3).

Now, let us observe that joining estimates (32), (33), (39), (40) and taking into account assumption (ix), in view of Theorems 2.4 and 2.5 we deduce that the operator  $W = VU$  is a contraction with respect to the measure of noncompactness  $\mu_a$ , with the constant  $L$  expressed by the formula

$$L = pk(\overline{G}G(r_0) + \overline{H}H(r_0)) + 2k\overline{F} \overline{G} \overline{H}G(r_0)H(r_0) + 2k^2r_0\overline{G} \overline{H}G(r_0)H(r_0).$$

Obviously  $L < 1$ , in view of assumption (ix).

Further, consider the sequence of sets  $(B_{r_0}^n)$ , where  $B_{r_0}^1 = \text{Conv}W(B_{r_0})$ ,  $B_{r_0}^2 = \text{Conv}W(B_{r_0}^1)$  and so on. Observe that all sets of this sequence are nonempty, bounded, closed and convex. Moreover,  $B_{r_0}^{n+1} \subset B_{r_0}^n \subset B_{r_0}$  for  $n = 1, 2, \dots$ . Thus, keeping in mind the above established facts we have  $\mu_a(B_{r_0}^n) \leq L^n\mu_a(B_{r_0})$ . This implies that  $\lim_{n \rightarrow \infty} \mu_a(B_{r_0}^n) = 0$ . Hence we deduce that the set  $Y = \bigcap_{n=1}^{\infty} B_{r_0}^n$  is nonempty, bounded, closed and convex. Moreover, by remark made after Definition 2.1 we infer that  $Y \in \ker \mu_a$ . Let us also notice that the operator  $W$  maps the set  $Y$  into itself.

In the sequel we show that the operator  $W$  is continuous on the set  $Y$ . To do this let us fix  $\varepsilon > 0$  and take functions  $x, y \in Y$  such that  $\|x - y\| \leq \varepsilon$ . Keeping in mind the facts that  $Y \in \ker \mu_a$  and the structure of sets belonging to  $\ker \mu_a$  (cf. Section 2) we can find a number  $T > 0$  such that for each  $z \in Y$  and  $t \geq T$  we have that  $|z(t)| \leq \varepsilon$ . Since  $W : Y \rightarrow Y$  we have that  $Wx, Wy \in Y$ . Thus, for  $t \geq T$  we get

$$|(Wx)(t) - (Wy)(t)| \leq |(Wx)(t)| + |(Wy)(t)| \leq 2\varepsilon. \tag{41}$$

On the other hand, taking an arbitrary number  $t \in [0, T]$  we obtain

$$\begin{aligned} & |(Wx)(t) - (Wy)(t)| \\ & \leq |(Ux)(t)| |(Vx)(t) - (Vy)(t)| + |(Vy)(t)| |(Ux)(t) - (Uy)(t)| \\ & \leq \|UB_{r_0}\| |(Vx)(t) - (Vy)(t)| + \|VB_{r_0}\| |(Ux)(t) - (Uy)(t)|. \end{aligned} \tag{42}$$

Further, we get

$$\begin{aligned} |(Vx)(t) - (Vy)(t)| & \leq |f_1(t, x(t)) - f_1(t, y(t))| \int_0^t |v(t, s, x(s))| ds \\ & \quad + |f_1(t, y(t))| \int_0^t |v(t, s, x(s)) - v(t, s, y(s))| ds \\ & \leq k_1|x(t) - y(t)| \int_0^t g(t, s)G(|x(s)|) ds + [k_1|y(t)| \\ & \quad + |f_1(t, 0)|] \int_0^t \bar{\omega}_{r_0}^T(v, \varepsilon) ds \\ & \leq k\varepsilon\bar{G}G(r_0) + (kr_0 + \bar{F})T\bar{\omega}_{r_0}^T(v, \varepsilon), \end{aligned} \tag{43}$$

where we denoted

$$\bar{\omega}_d^T(v, \varepsilon) = \sup\{|v(t, s, x) - v(t, s, y)| : t, s \in [0, T], x, y \in [-d, d], |x - y| \leq \varepsilon\}.$$

Obviously we have that  $\bar{\omega}_{r_0}^T(v, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

On the other hand, we obtain

$$\begin{aligned} & |(Ux)(t) - (Uy)(t)| \\ & \leq |f_2(t, x(t)) - f_2(t, y(t))| \int_0^\infty |u(t, s, x(s))| ds \\ & \quad + |f_2(t, y(t))| \int_0^\infty |u(t, s, x(s)) - u(t, s, y(s))| ds \\ & \leq \varepsilon k_2 \int_0^\infty h(t, s)H(|x(s)|) ds \\ & \quad + (k_2|y(t)| + |f_2(t, 0)|) \int_0^\infty |u(t, s, x(s)) - u(t, s, y(s))| ds \\ & \leq \varepsilon k_2\bar{H}H(r_0) + (k_2r_0 + \bar{F}_2) \left\{ \int_0^T |u(t, s, x(s)) - u(t, s, y(s))| ds \right. \\ & \quad \left. + \int_T^\infty [|u(t, s, x(s))| + |u(t, s, y(s))|] ds \right\} \\ & \leq \varepsilon k\bar{H}H(r_0) + (kr_0 + \bar{F}) \left\{ \int_0^T \bar{\omega}_{r_0}^T(u, \varepsilon) ds + 2 \int_T^\infty h(t, s)H(r_0) ds \right\} \\ & \leq \varepsilon k\bar{H}H(r_0) + (kr_0 + \bar{F}) \left\{ T\bar{\omega}_{r_0}^T(u, \varepsilon) + 2H(r_0) \sup_{t \in \mathbb{R}_+} \int_T^\infty h(t, s) ds \right\}, \end{aligned} \tag{44}$$

where we denoted

$$\bar{\omega}_d^T(u, \varepsilon) = \sup\{|u(t, s, x) - u(t, s, y)| : t, s \in [0, T], x, y \in [-d, d], |x - y| \leq \varepsilon\}.$$

Observe that  $\bar{\omega}_{r_0}^T(u, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . This is a simple consequence of the uniform continuity of the function  $u(t, s, x)$  on the set  $[0, T] \times [0, T] \times [-r_0, r_0]$ . Moreover, we can choose  $T$  in such a way (cf. assumption (viii)) that the last term in estimate (44) is sufficiently small.

Now, let us notice that taking into account (41), (42), (43) and (44) we conclude that the operator  $W$  is continuous on the set  $Y$ .

Finally, linking all above obtained facts concerning the set  $Y$  and the operator  $W : Y \rightarrow Y$  and using the classical Schauder fixed point principle we infer that the operator  $W$  has at least one fixed point  $x$  in the set  $Y$ . Obviously the function  $x = x(t)$  solves the integral equation (22). Moreover, in view of the fact that  $Y \in \ker \mu_a$  we infer that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . This completes the proof.  $\square$

It is worthwhile mentioning that the result contained in Theorem 4.3 generalizes a lot of results concerning nonlinear integral equations of various types (cf. [1, 3, 14, 16, 24, 27]).

**Remark 4.4.** From the proof of the above theorem we deduce that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  for any solution  $x = x(t)$  of equation (22) such that  $x \in B_{r_0}$ .

Now we provide an example illustrating the above obtained result.

**Example 4.5.** Let us consider the integral equation of the form (22), where

$$\begin{aligned} (Vx)(t) &= \frac{t}{t^2 + 4} + (x(t) + e^{-t}) \int_0^t \frac{s\sqrt{|x(s)|}}{(t + 1)(s^2 + 1)} ds \\ (Ux)(t) &= te^{-2t} + \frac{1}{\sqrt{2\pi}} \operatorname{arctg}(t + x(t)) \int_0^\infty e^{-s(t+1)} x^2(s) ds. \end{aligned}$$

Observe that  $V$  is the quadratic Volterra integral operator appeared in equation (22), where  $p_1(t) = \frac{t}{t^2+4}$ ,  $f_1(t, x) = e^{-t} + x$  and  $v(t, s, x) = \frac{s\sqrt{|x|}}{(t+1)(s^2+1)}$ . It is easily seen that  $\|p_1\| = \frac{1}{4}$  and  $f_1(t, x)$  satisfies the Lipschitz condition with the constant  $k_1 = 1$ . Moreover,  $f_1(t, 0) = e^{-t}$  and  $\bar{F}_1 = 1$ . Apart from this notice that we may put  $g(t, s) = \frac{s}{(t+1)(s^2+1)}$  and  $G(r) = \sqrt{r}$ . Further, we obtain  $\int_0^t g(t, s) ds = \frac{1}{2} \frac{\ln(t^2+1)}{t+1}$ . Hence, in view of the elementary inequality  $\ln(t^2 + 1) \leq t$  for  $t \geq 0$ , we obtain that  $\bar{G} \leq \frac{1}{2}$ .

On the other hand we see that for the above defined operator  $U$  we have  $p_2(t) = te^{-2t}$  and  $f_2(t, x) = \frac{1}{\sqrt{2\pi}} \operatorname{arctg}(t + x)$ ,  $u(t, s, x) = e^{-s(t+1)} x^2$ . It is easily seen that  $\|p_2\| = \frac{1}{2e}$  and the function  $f_2(t, x)$  satisfies the Lipschitz condition

with the constant  $k_2 = \frac{1}{\sqrt{2\pi}}$ . Moreover,  $f_2(t, 0) = \frac{1}{\sqrt{2\pi}} \arctgt$ . Hence  $\overline{F}_2 = \frac{1}{2} \sqrt{\frac{\pi}{2}}$ . Further we obtain that the function  $u(t, s, x)$  satisfies assumptions (v) and (vii), where  $h(t, s) = e^{-s(t+1)}$ ,  $H(r) = r^2$ . It can be calculated that  $\int_0^\infty h(t, s) ds = \frac{1}{t+1}$ . This implies that assumption (viii) is satisfied with  $\overline{H} = 1$ .

In what follows observe that  $p = \max\{\|p_1\|, \|p_2\|\} = \frac{1}{4}$ ,  $k = \max\{k_1, k_2\} = 1$ ,  $\overline{F} = \max\{\overline{F}_1, \overline{F}_2\} = 1$ . Thus the inequality from assumption (ix) has the form  $(\frac{1}{4} + \overline{G}r\sqrt{r} + \overline{G}\sqrt{r})(\frac{1}{4} + r^3 + r^2) \leq r$ . It is easy to check that the number  $r_0 = \frac{1}{2}$  is a solution of the above inequality satisfying also the second inequality imposed in assumption (ix).

Finally we conclude that there are satisfied the assumptions of Theorem 4.3. This implies that the considered integral equation has a solution  $x = x(t)$  belonging to the ball  $B_{\frac{1}{2}}$ . Moreover,  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  for any solution  $x = x(t)$  of that equation such that  $x \in B_{\frac{1}{2}}$  (cf. Remark 4.4).

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