Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 29 (2010), 1–20
DOI: 10.4171/ZAA/1395

Dynamic Adhesive Contact of a Membrane

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Abstract. This work presents a dynamic model for adhesive contact between a stretched viscoelastic membrane and a reactive obstacle that lies beneath it. The adhesion is described by a bonding field, and the model allows for failure, that is complete debonding in finite time. It is two-dimensional, but retains the essential mathematical structure of the full three-dimensional model. It is set as a hyperbolic equation for the vibrations of the membrane coupled with a nonlinear ordinary differential equation for the evolution of the bonding field. Existence and uniqueness of regular solutions are established in the case of positive viscosity, and in the case of no viscosity, existence of weak solutions is obtained, while the uniqueness of the solutions remains unresolved.

Keywords. Dynamic contact, existence, deformable obstacle, membrane, adhesion, hyperbolic variational inequality

Mathematics Subject Classification (2000). Primary 74M15, secondary 74H20, 74H25, 74H45, 35L85, 74F99

1. Introduction

Adhesion processes are of considerable importance in industry since nonmetallic parts and components cannot be joined by welding and so an adhesive is often utilized, especially when bolting is impractical, or expensive. This is especially important in laminates and in the joining of plastic components. Recently, composite materials, made of layers of simple materials with different fiber orientations, reached prominence since they are strong and light weight and, therefore, of considerable usefulness in aviation, space exploration, and the automotive industry, among many other industries.

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In recent mathematical publications (see, e.g., [1–3, 5–7, 9, 11, 13, 14, 18, 20–23], and the monographs [12, 24, 27], and references therein) the adhesion process has been modeled by introducing the *bonding field*, which measures the surface fractional density of active bonds, as an additional internal variable. As a result of the forces acting in the system its mechanical state evolves in time; in particular, the bonds may break and reform, or permanently break. The mathematical formulation of contact problems with adhesion is in the form of a hyperbolic variational equality or inequality for the mechanical behavior of the system coupled with an ODE for the bonding field. When the mechanical process is slow, the quasistatic approximation may be employed and the equation of motion reduces to a parabolic-like equality (if viscosity is included) or an elliptic variational equality or inequality (if viscosity is excluded).

The connection between models of quasistatic contact and elliptic variational inequalities is well understood and existence, uniqueness and regularity results for their solutions can be found in such monographs as [15, 24] and references therein. By contrast, there are fewer results involving obstacle problems and hyperbolic variational inequalities with adhesion. To obtain insight into the behavior of such models we use in this work a simpler setting which consists of a membrane and a deformable obstacle beneath it. As a result of the forces in the system the bonding deteriorates and may reach complete failure. The new feature in the model studied here is the choice of the adhesion rate exponent which allows for complete debonding of the membrane from the support, see [18] for a related one-dimensional model of an adhesive rod. Mathematically, the rate function is not Lipschitz which makes it necessary to study the ODE for the bonding field more carefully. We use the normal compliance contact condition, since we assume that the foundation is reactive.

In view of the simplicity and relative ease of computing numerical approximations of the model of an adhesive membrane or the one with an adhesive rod ([18]), they may be used as benchmarks for engineers to calibrate the system parameters needed if one wishes to numerically simulate the full three-dimensional model.

Following this introduction, we present the classical formulation of the problem of adhesive contact between a viscoelastic membrane and a reactive foundation in Section 2, Problem P_{NC}. For the sake of completeness, we also present the model when the foundation is rigid, which uses a modified Signorini-type contact condition, Problem P_S. The weak formulation of the problem, Problem P_{NC}, is provided in Section 3, where the assumptions on the problem data are given, and the existence of the unique weak solution is stated in Theorem 3.1. The proof of the theorem is done in Section 4, and is based on the study of a truncated problem. Once the necessary a priori estimates are derived, the solution follows. In Section 5 we use these estimates to pass to the limit of vanishing viscosity, $\nu \to 0$, and obtain a week solution of the inviscid problem.

The paper concludes in Section 6, where some unresolved issues are described, too.

2. The model

We construct, following [2], a model for the dynamic process of adhesive contact between a membrane and an obstacle or foundation. We refer the reader there for additional modeling details, as well as the literature cited above. The membrane is attached to a rigid rim, and is in adhesive contact with a reactive or deformable obstacle. For the sake of generality, we assume that the membrane is viscoelastic, with viscosity coefficient ν, very small in practice.

Let Ω denote the projection of the membrane on the xy plane, let $z = \phi(x, y)$ represent the location of the obstacle, and let $\Omega_T = \Omega \times (0, T)$. The membrane is being acted upon by a vertical force or load f. Adhesion is assumed to take place over all of Ω. The case where the adhesive is spread only over a part of Ω is also described shortly below, as it is a straightforward modification of the model. The setting is depicted in Figure 1.

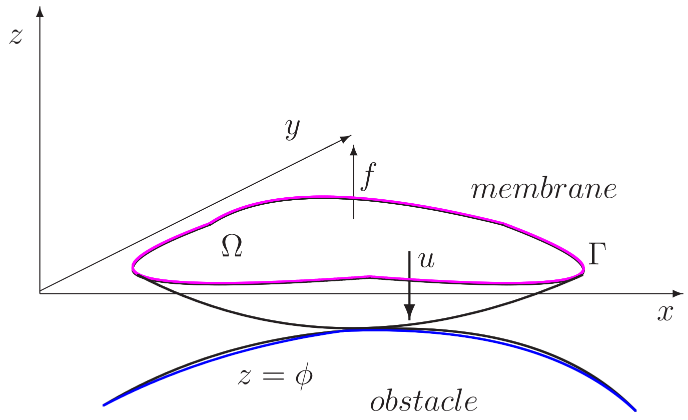

Fig. 1. The membrane and the obstacle

Let $u = u(x, y, t)$ represent the vertical displacement of the membrane, denote by $\xi = \xi(x, y, t)$ the reaction force of the obstacle, and let $\eta = \eta(x, y, t)$ represent the tensile adhesive force, which is described shortly. The problem is rescaled so that the membrane surface density ρ and its elastic constant k equal one. The evolution of the state of the membrane is governed by the equation of motion

$$u'' - \Delta u - \nu \Delta u' = f + \xi + \eta, \qquad \text{in } \Omega_T.$$

Here and below, a prime denotes differentiation with respect to time.

We describe the reaction force ξ of the deformable obstacle with the normal compliance condition, which allows for the interpenetration of the obstacle, $u < \phi$, but penalizes it,

$$\xi = c_N(\phi - u)_+,$$

where c_N is the normal compliance coefficient, very large when the obstacle is relatively rigid; $(r)_+ = \max\{r, 0\}$ is the positive part of r; and the reaction force is active, $\xi > 0$, only when $u < \phi$. Actually, we may choose a much more general condition, see, e.g., [24], however, for the sake of simplicity we use this one.

The adhesion process in this work is assumed to be either irreversible, i.e., severed bonds do not regenerate, or reversible, so that severed bonds do regenerate. The rebonding process can be found in many systems, such as those with velcro, where cycles of debonding and rebonding may go for long time periods (e.g., [14]). Irreversible adhesion processes were studied in [2, 13, 21], and references therein.

Let $\beta = \beta(x, y, t)$ denote the *bonding field*, which measures the pointwise fractional density of active bonds between the membrane and the obstacle. When $\beta = 1$ the bonding at the point is complete; when $\beta = 0$ there is no bonding and all the bonds are broken. Partial bonding is represented by $0 < \beta < 1$. Thus, the bonding field has to satisfy

$$0 \le \beta \le 1 \quad \text{in } \Omega_T.$$

The adhesive restoring force $\eta = \eta(x, y, t)$ is directed downward, trying to prevent the separation of the membrane from the obstacle, and is assumed proportional to the distance from the obstacle and to $\beta^{1+\alpha}$ (cf. [13]); thus,

$$\eta = -\kappa(u - \phi)_+\beta^{1+\alpha} \quad \text{in } \Omega_T,$$

where κ is the bonding coefficient or the adhesive stiffness and $\kappa\beta^{1+\alpha}$ is the system's 'spring constant'. We use $(u-\phi)_+$ since when there is interpenetration, the adhesive is assumed to be inert, not providing any opposing traction, i.e., $\eta = 0$.

The evolution of the bonding field is governed by the ordinary differential equation

$$\beta' = H_{ad}(u - \phi, \beta), \quad \text{in } \Omega_T.$$

Unlike the cases in [2, 18], here the *adhesive rate function* $H_{ad} = H_{ad}(\cdot, \cdot)$ is rather general. In particular, no assumption is made about its sign and so cycles of debonding and rebonding may occur. Moreover, it is a generalization of the irreversible condition

$$\beta' = -\gamma\beta_+^{\alpha}(u - \phi)_+^2.$$

We note that, as was explained in [20], complete failure or debonding can take place only when $0 \leq \alpha < 1$, since if $1 \leq \alpha$ there is no complete debonding in finite time (as can be seen for $\beta' = -\gamma\beta_+^\alpha$).

Therefore, we assume that H_{ad} satisfies the condition

$$|H_{ad}(a_2, b_2) - H_{ad}(a_1, b_1)| \leq L_H(|a_2 - a_1| + |b_2^\alpha - b_1^\alpha|),$$

for some constant L_H. Here, $0 \leq \alpha$, however, the interesting case is $0 \leq \alpha < 1$, and then H_{ad} is only Hölder continuous (with exponent α) in the second variable, and Lipschitz continuous in the first one.

Initially $\beta(x, y, 0) = \beta_0(x, y)$, where β_0 is a given adhesive intensity distribution.

To complete the model we assume that $u(x, y, t) = g(x, y, t)$ on Γ, for $0 \leq t \leq T$. However, for technical reasons it is convenient to formulate the problem so that the boundary condition on Γ is homogeneous. To that end assume that $g(x, y, t)$ is the restriction to Γ of a function in $\widetilde{g} \in H^2(\Omega \times [0, T])$, which we also denote by g, and $g \geq \phi$ in $\overline{\Omega}$. Now, we introduce the new variable $\widetilde{u} = u - g$, and then $\widetilde{u}(0) = u_0 - g(0)$ and $\widetilde{u'}(0) = v_0 - g'(0)$, where u_0 is the prescribed initial displacement and v_0 is the prescribed initial velocity. Finally, let $\tilde{f} = f + \Delta g + \nu\Delta g' - g''$ and $\widetilde{\phi} = \phi - g$, and below we omit the tildes.

Collecting the equations and conditions above, the classical formulation of *the problem of dynamic adhesive contact between a viscoelastic membrane and a reactive obstacle* is:

Problem P_{NC}**.** *Find a pair of functions* $\{u, \beta\}$ *such that*

$$u'' - \Delta u - \nu\Delta u' + \kappa(u - \phi)_+\beta^{1+\alpha} - c_N(u - \phi)_- = f \quad \text{in } \Omega_T \tag{1}$$

$$u = 0 \quad \text{on } \Gamma \times (0, T) \tag{2}$$

$$u(0) = u_0, \quad u'(0) = v_0 \quad \text{in } \Omega \tag{3}$$

$$\beta' - H_{ad}(u - \phi, \beta) = 0 \quad \text{in } \Omega_T \tag{4}$$

$$\beta(0) = \beta_0 \quad \text{in } \Omega. \tag{5}$$

The variational formulation of the problem is given in the next section.

We note that it is straightforward to modify the model to the case where the adhesive is spread only an open subset $\Omega_{ad} \subset \Omega$, assumed to have a regular boundary. Indeed, in this case equation (4) holds on $\Omega_{ad} \times [0, T]$, instead of Ω_T, condition (5) holds on Ω_{ad}, and if we denote by $\chi(\Omega_{ad})$ the indicator function of Ω_{ad}, then we have to replace the adhesive force term $\kappa(u - \phi)_+\beta^{1+\alpha}$ in (1) with $\kappa(u-\phi)_+\beta^{1+\alpha}\chi(\Omega_{ad})$. All the results below hold true in this case provided Ω_{ad} is open in Ω.

For the sake of completeness, we also describe a completely rigid obstacle. Then, the membrane is restricted to lie above it, so $u \geq \phi$ and the obstacle's reaction force ξ is directed upward and exactly cancels the applied force. When in contact $u = \phi$ implies $\xi \geq 0$; when there is no contact the reaction force vanishes, thus, $u > \phi$ implies $\xi = 0$. We combine these statements into the following linear complementarity condition:

$$\phi \leq u, \quad 0 \leq \xi, \quad \xi(u - \phi) = 0.$$

The last condition prevents both inequalities from being simultaneously strict since when contact takes place $\phi = u$ and in the absence of contact $\xi = 0$.

To describe the obstacle's reaction force ξ using the language of differential inclusions, we introduce the indicator function $\chi_{(-\infty,0]}$ of the interval $(-\infty, 0]$, so that $\chi_{(-\infty,0]}(r) = 0$ when $r \leq 0$ and $\chi_{(-\infty,0]}(r) = \infty$ when $r > 0$. Its subdifferential is

$$\partial\chi_{(-\infty,0]}(r) = \begin{cases} 0 & r < 0 \\ [0, \infty) & r = 0 \\ \emptyset & r > 0\,. \end{cases}$$

Then we may rewrite the condition in the concise form $\xi \in \partial\chi_{(-\infty,0]}(\phi - u)$, and ξ represents the obstacle's physical reaction that prevents $u < \phi$.

The classical formulation of *the problem of dynamic adhesive contact between a viscoelastic membrane and a rigid obstacle* is:

Problem P_S. *Find a pair of functions $\{u, \beta\}$ such that*

$$\begin{aligned} u'' - \Delta u - \nu\Delta u' - f + \kappa(u - \phi)\beta^{1+\alpha} &\in \partial\chi_{(-\infty,0]}(\phi - u) && \text{in } \Omega_T \\ u &\geq \phi, && \text{in } \Omega_T \\ u &= 0 && \text{on } \Gamma \times (0, T) \\ u(0) = u_0, \quad u'(0) &= v_0 && \text{in } \Omega \\ \beta' - H_{ad}(u - \phi, \beta) &= 0 && \text{in } \Omega_T \\ \beta(0) &= \beta_0 && \text{in } \Omega. \end{aligned}$$

Formally, Problem P_S is obtained from Problem P_{NC} in the limit $c_N \to \infty$, and we plan to study it in the future.

3. Weak formulation

We obtain the variational formulation of the problem, describe the assumption on the problem data, and state the existence result for Problem P_{NC}.

We assume that the boundary of the domain Ω is $C^{1,1}$ and hence the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is compact and, also, the usual elliptic regularity theorems are available. We use the notation $W = H_0^1(\Omega), H = L^2(\Omega)$, $\mathcal{W} = L^2(0, T; W), \mathcal{H} = L^2(0, T; H) = L^2(\Omega_T)$. For the sake of convenience, we choose the norm in W to be $\|u\|_W \equiv \int_\Omega |\nabla u|^2$. Here and below, we use the Lebesgue measure on Ω. The inner product on W is denoted by $(u, v)_W$, and the duality pairing between W and its dual $W' = H^{-1}(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle_W$, however, below we omit the subscript W, as it is clear from the context.

We also need the set of admissible adhesion functions

$$K_{ad} = \{w \in H : 0 \le w \le 1 \quad \text{a.e. in } \Omega\},$$

which is a closed and convex set in H.

A straightforward derivation leads to the following variational formulation of Problem P_{NC}, (1)–(5).

Problem P_{NCV}**.** *Find a pair* $\{u, \beta\}$ *such that*

$$u, u' \in \mathcal{W}, \qquad u'' \in \mathcal{W}' \tag{6}$$

$$\beta, \beta' \in \mathcal{H}, \quad \beta(t) \in K_{ad}, \tag{7}$$

and, for a.a. $t \in (0, T)$ *and for each* $w \in W$,

$$\begin{aligned} &\langle u''(t), w\rangle + \int_\Omega \nabla u(t) \cdot \nabla w + \nu \int_\Omega \nabla u'(t) \cdot \nabla w \\ &+\kappa \int_\Omega (u(t) - \phi)_+ \beta^{1+\alpha} w - c_N \int_\Omega (u(t) - \phi)_- w = \int_\Omega f(t) w \end{aligned} \tag{8}$$

$$u(0) = u_0, \quad u'(0) = v_0 \tag{9}$$

$$\beta'(t) = H_{ad}(u(t) - \phi,\, \beta(t)) \tag{10}$$

$$\beta(0) = \beta_0. \tag{11}$$

Next, we list the assumptions on the problem data:

$$f \in \mathcal{H} \tag{12}$$

$$\phi \in H, \qquad \phi \le 0 \ \text{ on } \partial\Omega \tag{13}$$

$$u_0 \in W, \qquad u_0 \ge \phi \ \text{ a.e. } \Omega, \quad v_0 \in H \tag{14}$$

$$\beta_0 \in K_{ad}. \tag{15}$$

For the sake of simplicity, c_N and κ are chosen as positive constants. Note that in the original variables, we must assume $g \in L^2(0, T; H^2(\Omega)) \cap H^2(0, T; H)$.

The adhesion rate function $H_{ad} : \mathbb{R} \times [0, 1] \to \mathbb{R}$ satisfies the following conditions: There is a constant L_H so that, for all $(a, b) \in \mathbb{R} \times [0, 1]$,

$$|H_{ad}(a_2, b_2) - H_{ad}(a_1, b_1)| \le L_H(|a_2 - a_1| + |b_2^\alpha - b_1^\alpha|) \tag{16}$$

$$H_{ad}(\cdot, b) \ge 0 \text{ for } b \le 0, \quad H_{ad}(\cdot, b) \le 0 \text{ for } b \ge 1. \tag{17}$$

There exists a positive constant c_H such that

$$(H_{ad}(a,b_1) - H_{ad}(a,b_2))\,(b_1 - b_2) \le c_H\,(b_1 - b_2)^2\,, \tag{18}$$

i.e., $-H_{ad}$ is monotone. Finally, for all $b \in [0,1]$,

$$H_{ad}(0,b) = 0. \tag{19}$$

These assumptions on H_{ad} guarantee that if $\beta\,(0) \in [0,1]$, the differential equation will cause β to remain in $[0,1]$. Moreover, the debonding rate function $H_{ad}(u-\phi,\beta) = -\gamma\beta_+^\alpha(u-\phi)_+^2$ locally satisfies them and to use it one must truncate the factor $(u-\phi)_+^2$.

The following is the main result in this work.

Theorem 3.1. *Under the assumptions* (12)–(19)*, there exists a unique solution* $\{u,\beta\}$ *of* (6)–(11)*, and*

$$u \in L^\infty(0,T;W) \cap C(0,T;H), \quad u' \in L^\infty(0,T;W), \quad u'' \in L^\infty(0,T;H) \tag{20}$$

$$\beta, \beta' \in \mathcal{H}, \quad \beta(t) \in K_{ad} \ \textit{for} \ 0 \le t \le T. \tag{21}$$

We conclude that Problem P_{NC} has a unique weak solution. The proof, based on truncation, follows.

We note, as is shown below, that when the data is assumed sufficiently regular, then the solutions have higher regularity. In fact, we first consider this case and obtain classical solutions, i.e., solutions which have classical derivatives. Then, we obtain the weak solutions described above as limits of the regular solutions.

4. Proof of Theorem 3.1

We use of the following fundamental theorem of Simon ([26]) that is an infinite dimensional version of the Arzela–Ascoli theorem.

Theorem 4.1. *Let $W \subseteq U \subseteq Y$ be three Banach spaces such that the inclusion map of W into U is compact and the inclusion map of U into Y is continuous and let*

$$S = \left\{ z : \|z(t)\|_W + \|z'\|_{L^q(0,T;Y)} \le R \ \textit{for} \ t \in [0,T] \right\}$$

for $q > 1$. Then S is precompact in $C([0,T];U)$.

Next, for $0 < R$, we introduce the truncation operator

$$\Phi_R(r) = \begin{cases} R, & r > R \\ r, & 0 \le r \le R \\ 0, & r \le 0. \end{cases}$$

We denote $\Phi_R(\cdot) = \Phi(\cdot)$ when $R = 1$.

The truncated problem is as follows, and we let $v = u'$.

Problem P_Φ. Find a pair $\{v, \beta\} \in \mathcal{W} \times C^1(0,T;H)$ such that

$$v' - \Delta u - \nu\Delta v + \kappa(u-\phi)_+\Phi^{1+\alpha}(\beta) - c_N(u-\phi)_- = f \qquad \text{in } \Omega_T \tag{22}$$

$$u(t) = u_0 + \int_0^t v\,(s)\,ds, \quad v(0) = v_0 \qquad \text{in } \Omega \tag{23}$$

$$\beta' - H_{ad}(u-\phi, \beta) = 0 \qquad \text{in } \Omega_T \tag{24}$$

$$\beta(0) = \beta_0 \qquad \text{in } \Omega. \tag{25}$$

We make a stronger assumption on the initial data:

$$u_0 \in H^2\,(\Omega) \cap H_0^1\,(\Omega)\,,\ v_0 \in H_0^1\,(\Omega) = W.$$

We begin the proof by establishing the solution to (22) and (23) for a given fixed β.

It is convenient to rewrite the problem in an abstract form, and to that end introduce the operators $J, S(\beta) : \mathcal{H} \longrightarrow \mathcal{H}$, defined by

$$\langle Jw, z\rangle = -\int_0^T\int_\Omega (w-\phi)_- z\,dt$$

$$\langle S(\beta)w, z\rangle = \kappa\int_0^T\int_\Omega \Phi^{1+\alpha}(\beta)(w-\phi)_+ z\,dt\,.$$

In terms of these operators, problem (22)–(25) may be written as:

Find a pair $\{u, v\}$ such that $v' \in \mathcal{H}$, $\Delta u, \Delta v \in \mathcal{H}$, $u(t), v(t) \in H^2\,(\Omega) \cap W$ for a.a $t \in [0, T)$, and

$$v' - \Delta u - \nu\Delta v + S(\beta)u + c_N Ju = f \quad \text{in } \mathcal{H} \tag{26}$$

$$v(0) = v_0 \tag{27}$$

$$u(t) = u_0 + \int_0^t v(s)\,ds. \tag{28}$$

To solve problem (26)–(28) we assume that only v is unknown. Thus, let v_1 be fixed in $L^2\,(0,T; H^2\,(\Omega) \cap W)$, set $u_1(t) = u_0 + \int_0^t v_1(s)\,ds$, and let w be the unique solution of the problem

$$w'(t) - \nu\Delta w(t) = f + \triangle u_0 + \int_0^t \Delta v_1 ds - S\,(\beta)\,u_1(t) - c_N Ju_1(t), \tag{29}$$

for a.a. $t \in (0, T)$, and $w\,(0) = v_0$. It is a parabolic problem and it follows from a well known result of Brezis [4] (see also [25]) that the unique solution w satisfies $w' \in \mathcal{H}$ and $\Delta w \in \mathcal{H}$ because the right-hand side is in $\mathcal{H}$.

Next, let w_i, for $i = 1, 2$, denote the solutions that correspond to $v_i \in L^2(0, T; H^2(\Omega) \cap W)$, and let $R_i = R_i(v_i)$ denote the right-hand side of (29) corresponding to v_i. Then, we subtract (29) for w_2 from (29) for w_1, multiply the result by $-(\Delta w_1(s) - \Delta w_2(s))$ and integrate, and straightforward computations yield

$$\begin{aligned} &\|w_1(t) - w_2(t)\|_W^2 + \nu \int_0^t |\Delta w_1(s) - \Delta w_2(s)|_H^2 \, ds \\ &= \int_0^t \big((R_1 - R_2), (-\Delta w_1(s) - (-\Delta) w_2(s))\big) ds. \end{aligned}$$

We proceed to estimate the right-hand side. First, the term with Δv_i is dominated by

$$\begin{aligned} &\left| \int_0^t \int_0^s (\Delta v_1(r) - \Delta v_2(r), \Delta w_1(s) - \Delta w_2(s)) \, dr \, ds \right| \\ &\quad \le \int_0^t \int_0^s |\Delta v_1(r) - \Delta v_2(r)|_H \, |\Delta w_1(s) - \Delta w_2(s)|_H \, dr \, ds \\ &\quad \le C_\nu \int_0^t \int_0^s |\Delta v_1(r) - \Delta v_2(r)|_H^2 \, dr \, ds + \frac{\nu}{8} \int_0^t |\Delta w_1(s) - \Delta w_2(s)|_H^2 \, ds. \end{aligned}$$

Next, consider the term with $S(\beta)$. It is dominated by an expression of the form

$$\begin{aligned} &\int_0^t \left| \left(\left(\int_0^s v_1(r) - v_2(r) \, dr \right), \Delta w_1(s) - \Delta w_2(s) \right)_H \right| ds \\ &\quad \le C_\nu \int_0^t \int_0^s |v_1(r) - v_2(r)|_H^2 \, dr \, ds + \frac{\nu}{8} \int_0^t |\Delta w_1(s) - \Delta w_2(s)|_H^2 \, ds. \end{aligned}$$

Similar considerations apply to the remaining term on the right (with $c_N J$), yielding a similar estimate.

Then, using the equivalence of the norm in $H^2(\Omega) \cap H_0^1(\Omega)$ with $|\Delta u|_H$, we obtain the following inequality:

$$\begin{aligned} &\|w_1(t) - w_2(t)\|_W^2 + \frac{\nu}{2} \int_0^t \|w_1(s) - w_2(s)\|_{H^2(\Omega)}^2 ds \\ &\qquad \le C_\nu \int_0^t \int_0^s \|v_1(r) - v_2(r)\|_{H^2(\Omega)}^2 dr \, ds, \end{aligned} \tag{30}$$

where C_ν depends on the problem data but not on v_1, v_2 or β.

Now, we define the mapping

$$\Theta : L^2\left(0, T; H^2(\Omega) \cap W\right) \to L^2\left(0, T; H^2(\Omega) \cap W\right)$$

by $\Theta(v_1) = w$, where $v_1 \in L^2(0, T; H^2(\Omega) \cap W))$ and w is the solution of (29) with $w(0) = v_0$. Estimate (30) shows that a sufficiently high power of Θ is a contraction map on $L^2(0, T; H^2(\Omega) \cap W)$, and so Θ has a unique fixed point which is the solution of the problem with given β. This has proved the following lemma.

Lemma 4.2. *For each $\beta \in \mathcal{H}$, there exists a unique solution (u, v) of (26)–(28) which satisfies $v' \in \mathcal{H}$, $u, v \in H^2(\Omega) \cap W$ pointwise for a.a. t, and $\Delta u, \Delta v \in \mathcal{H}$.*

Next, we consider the equation for β. The following useful lemma follows from the separability of $W = H_0^1(0, T)$.

Lemma 4.3. *Let $u, u' \in \mathcal{H}$. If $(x, y, t) \to u(x, y, t)$ is a measurable representative, then the mapping $t \to u(x, y, t)$ is continuous, on the complement of an exceptional set of zero measure in Ω.*

Everywhere below, we use such a measurable representative of u, whenever appropriate.

Lemma 4.4. *Let $u, u' \in \mathcal{H}$. Then, for a.e. $\boldsymbol{x} = (x, y) \in \Omega$, there exists a unique solution β to the initial value problem*

$$\begin{aligned} &\beta'(\boldsymbol{x}, t) = H_{ad}(u(\boldsymbol{x}, t) - \phi(\boldsymbol{x}), \beta(\boldsymbol{x}, t)) \\ &\beta(\boldsymbol{x}, 0) = \beta_0(\boldsymbol{x}). \end{aligned} \tag{31}$$

This solution satisfies $t \to \beta(\cdot, t)$ lies in $C([0, T]; H)$.

If u_1 and u_2 satisfy the above conditions, and β_i corresponds to u_i, for $i = 1, 2$, then on the complement of the union of the two exceptional sets,

$$(\beta_1(\boldsymbol{x}, t) - \beta_2(\boldsymbol{x}, t))^2 \leq C_T \int_0^t |u_1(\boldsymbol{x}, s) - u_2(\boldsymbol{x}, s)|^2 \, ds,$$

where C_T is a constant which is independent of β_i and u_i, for $i = 1, 2$. Furthermore,

$$|\beta_1(t) - \beta_2(t)|_H^2 \leq C_T \int_0^t |u_1(s) - u_2(s)|_H^2 \, ds, \tag{32}$$

and for u as above, $\beta : (0, T) \to H$ is the solution of the ordinary differential equation with values in H,

$$\beta' = H_{ad}(u - \phi, \beta), \quad \beta(0) = \beta_0.$$

Proof. The estimates follow from the anti-monotonicity assumption (18), which justifies the following manipulations, where β_i are solutions of (31) corresponding to u_i, for $i = 1, 2$, for $\boldsymbol{x}$ not in the two exceptional sets associated with either u_1 or u_2. Thus,

$$\begin{aligned}
&\frac{1}{2}\big(\beta_1(\boldsymbol{x},t)-\beta_2(\boldsymbol{x},t)\big)^2\\
&+\int_0^t \big(H_{ad}(u_1(\boldsymbol{x},s)-\phi(\boldsymbol{x}),\beta_1(s,\boldsymbol{x}))-H_{ad}(u_2(\boldsymbol{x},s)-\phi\left(\boldsymbol{x}\right),\beta_2(\boldsymbol{x},s))\big)\\
&\times\big(\beta_1(\boldsymbol{x},s)-\beta_2(s,\boldsymbol{x})\big)ds\\
&\le C\int_0^t \left(\beta_1(\boldsymbol{x},s)-\beta_2(\boldsymbol{x},s)\right)^2 ds\\
&\quad+\int_0^t \big|H_{ad}(u_1(\boldsymbol{x},s)-\phi(\boldsymbol{x}),\beta_2(s,\boldsymbol{x}))-H_{ad}(u_2(\boldsymbol{x},s)-\phi(\boldsymbol{x}),\beta_2(\boldsymbol{x},s)\big|\\
&\quad\times\big(\beta_1(\boldsymbol{x},s)-\beta_2(s,\boldsymbol{x})\big)ds\\
&\le C\int_0^t \big(\beta_1(\boldsymbol{x},s)-\beta_2(\boldsymbol{x},s)\big)^2 ds\\
&\quad+L_H\int_0^t |u_1(\boldsymbol{x},s)-u_2(\boldsymbol{x},s)|\,|\beta_1(\boldsymbol{x},s)-\beta_2(\boldsymbol{x},s)|\,ds\\
&\le C\int_0^t \big(\beta_1(\boldsymbol{x},s)-\beta_2(\boldsymbol{x},s)\big)^2 ds+L_H\int_0^t |u_1(\boldsymbol{x},s)-u_2(\boldsymbol{x},s)|^2\,ds,
\end{aligned}$$

and so by the Gronwall inequality

$$\left(\beta_1(\boldsymbol{x},t)-\beta_2(\boldsymbol{x},t)\right)^2 \le C_T\int_0^t |u_1(\boldsymbol{x},s)-u_2(\boldsymbol{x},s)|^2\,ds. \tag{33}$$

This also shows the uniqueness of the solution to (31).

The existence of a solution of (31), for a.e. $\boldsymbol{x}$, follows from the usual proof of the Peano existence theorem (see, e.g., [8]) along with the continuity of H_{ad} for $\boldsymbol{x}$ outside of the exceptional set of u. Thus, for a.e. $\boldsymbol{x}$,

$$\beta\left(\boldsymbol{x},t\right)=\beta_0\left(\boldsymbol{x}\right)+\int_0^t H_{ad}(u\left(\boldsymbol{x},s\right)-\phi(\boldsymbol{x}),\beta\left(\boldsymbol{x},s\right))ds.$$

The proof of the existence of β is based on retarding the second argument of H_{ad} and passing to the limit as the retardation parameter h vanishes. This is accomplished by using the Arzela–Ascoli theorem to get compactness of the β_h. The retardation operator is given by $\tau_h\beta\left(t\right)\equiv\beta\left(t-h\right)$ if $t\ge h$, and $\tau_h\beta\left(t\right)\equiv\beta_0$ for $t<h$, where h is small. We let β_h denote the solution of the retarded problem

$$\beta_h\left(\boldsymbol{x},t\right)=\beta_0\left(\boldsymbol{x}\right)+\int_0^t H_{ad}(u\left(\boldsymbol{x},s\right)-\phi(\boldsymbol{x}),\tau_h\beta_h\left(\boldsymbol{x},s\right))ds.$$

Then, β is the uniform limit $\beta_h\to\beta$ as $h\to 0$. The difficulty is that the subsequence might depend on $\boldsymbol{x}$.

However, since the solution of (31) is unique, we can choose a sequence, (say $h_n = \frac{1}{n}$), and obtain β as a uniform limit of $\beta_{h_n} = \beta_n$ for all $\boldsymbol{x}$ outside of the set of measure zero mentioned earlier. Thus, $\beta(\boldsymbol{x}, t)$ is the uniform limit in $C([0,T])$ of the functions β_n and $\boldsymbol{x} \to \beta_n(\boldsymbol{x}, t)$ is measurable for each n. Hence, $\beta(\boldsymbol{x}, t) = \lim_{n\to\infty} \beta_n(\boldsymbol{x}, t)$, uniformly for $t \in [0,T]$, and therefore, the limit function $\boldsymbol{x} \to \beta(\boldsymbol{x}, t)$ is also measurable. It follows from the properties of H_{ad} that $0 \le \beta \le 1$. Hence, for each t we have $\beta(\cdot, t) \in H$. Also, β is continuous with values in H since

$$\beta'(t) = (H_{ad}(u(\cdot, t) - \phi(\cdot), \beta(t))$$

and $t \to H_{ad}(u(\cdot, t) - \phi(\cdot), \beta(t))$ is in $\mathcal{H}$ so this shows $\beta \in C([0,T]; H)$, actually, $\beta \in H^1(0,T; H)$.

Since $\boldsymbol{x} \to \beta(\boldsymbol{x}, t)$ is measurable, integrating both sides of (33) over Ω yields

$$|\beta_1(t) - \beta_2(t)|_H^2 \le C_T \int_0^t |u_1(s) - u_2(s)|_H^2 \, ds.$$

This completes the proof of the lemma. □

Recalling that for a fixed β there exists a unique solution of (26)–(28), we now consider in more detail the term $c_N J(u)$. We define the function $\Psi(z, \phi)$ by

$$\Psi(z, \phi) = \frac{1}{2} \int_\Omega (z - \phi)_-^2 .$$

We let the term act on $v\chi_{(0,t)} = u'\chi_{(0,t)}$, where $\chi_{(0,t)}$ is the characteristic function of the interval $(0, t)$, and obtain

$$\langle c_N J u, v \rangle = \frac{1}{2} c_N \Psi(u(t), \phi) - \frac{1}{2} c_N \Psi(u_0, \phi) = \frac{1}{2} c_N \Psi(u(t), \phi),$$

since $u_0(x, y) \ge \phi(x, y)$, and so $\Psi(u_0, \phi) = 0$.

Multiplying (26) by v and integrating from 0 to t yields

$$\begin{aligned} &\frac{1}{2}|v(t)|_H^2 + \frac{1}{2}\|u(t)\|_W^2 + \nu \int_0^t \|v\|_W^2 ds + \frac{1}{2} c_N \Psi(u(t), \phi) \\ &\le \frac{1}{2}\|u_0\|_W^2 + \frac{1}{2}|v_0|_H^2 + \int_0^t |u - \phi|_H |v|_H \, ds + \frac{1}{2}\int_0^t |v|_H^2 \, ds + \frac{1}{2}\int_0^t |f|_H^2 \, ds. \end{aligned}$$

Using the Gronwall inequality and the definition of u leads to the estimate

$$|v(t)|_H^2 + \|u(t)\|_W^2 + 2\nu \int_0^t \|v\|_W^2 ds + c_N \Psi(u(t), \phi) \le C. \qquad (34)$$

Here, the constant $C = C(f, u_0, v_0)$ is independent of c_N, β or ν.

Next, we use another fixed point argument to establish the existence of the unique solution of Problem P_{NCV}. To that end we construct the mapping $\Lambda : \mathcal{H} \to \mathcal{H}$ as follows. Given $v \in \mathcal{H}$, we denote by u the function given in (28), and so $u, u' \in \mathcal{H}$. Now, let β_v denote the solution to the initial value problem (31) with these v and u. Then $\Lambda(v) = w_{\beta_v}$ denotes the solution of to (26)–(28) with β_v.

The following lemma is the main remaining step.

Lemma 4.5. *The operator $\Lambda : \mathcal{H} \to \mathcal{H}$ has a unique fixed point.*

Proof. Let v_i be two solutions of (26)–(28) corresponding to β_i, for $i = 1, 2$. Then,

$$\frac{1}{2}|v_1(t) - v_2(t)|_H^2 + \frac{1}{2}\|u_1(t) - u_2(t)\|_W^2 + \nu \int_0^t \|v_1 - v_2\|_W^2 \, ds$$
$$\leq \int_0^t |\langle S(\beta_1) u_1 - S(\beta_2) u_1, v_1 - v_2 \rangle| \, ds + \int_0^t |\langle S(\beta_2)(u_1 - u_2), (v_1 - v_2)\rangle| \, ds$$
$$+ c_N \int_0^t |u_1 - u_2|_H |v_1 - v_2|_H \, ds.$$

Since estimate (34) for $\|u(t)\|_W$ is independent of β, there exists a constant C, also independent of β, such that

$$\frac{1}{2}|v_1(t) - v_2(t)|_H^2 + \frac{1}{2}\|u_1(t) - u_2(t)\|_W^2 + \nu \int_0^t \|v_1 - v_2\|_W^2 \, ds$$
$$\leq C \int_0^t |\beta_1 - \beta_2|_H |v_1 - v_2|_H \, ds + (1 + c_N) \int_0^t |u_1 - u_2|_H |v_1 - v_2|_H \, ds.$$

Using routine manipulations, the Cauchy inequality with ε, and the Gronwall inequality lead to

$$|v_1(t) - v_2(t)|_H^2 + \|u_1(t) - u_2(t)\|_W^2 + \int_0^t \|v_1 - v_2\|_W^2 ds \leq C(\varepsilon, T) \int_0^t |\beta_1 - \beta_2|_H^2 \, ds.$$

This estimate and (32) yield

$$|\Lambda(v_1)(t) - \Lambda(v_2)(t)|_H^2 + \int_0^t \|\Lambda(v_1) - \Lambda(v_2)\|_W^2 ds$$
$$\leq C(\varepsilon, T) \int_0^t |\beta_{v_1}(s) - \beta_{v_2}(s)|_H^2 \, ds$$
$$\leq C(\varepsilon, T) \int_0^t \int_0^s |u_1(r) - u_2(r)|_H^2 \, dr \, ds$$
$$\leq C(\varepsilon, T) \int_0^t \int_0^s \int_0^r |v_1(\tau) - v_2(\tau)|_H^2 \, d\tau dr \, ds.$$

Therefore, by iterating this inequality, we obtain that the mapping Λ^n is a contraction, for a sufficiently high n, and so it has a unique fixed point in $\mathcal{H}$. This fixed point is also the fixed point of Λ. □

This proves the following theorem on the existence of the unique solution of P_Φ.

Theorem 4.6. *Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v_0 \in H_0^1(\Omega)$. Then there exists a unique solution (v, β) to* (22)*)–*(25)*).*

Proof. The result follows from the observation that in the definition of Φ the truncation is inactive when $R > 1$, and this is the case due to the properties of H_{ad} and the assumption $\beta_0 \in K_{ad}$. □

To complete the proof of Theorem 3.1 we need to generalize the result to the weaker initial conditions, namely $u_0 \in W$ and $v_0 \in H$. To that end, we first write this problem in an abstract form. Define the operator $A : W \to W'$ by

$$\langle Au, v \rangle \equiv \int_\Omega \nabla u \cdot \nabla v.$$

Then, Problem P_{NCV} is: Find $v \in \mathcal{W}$ and $\beta \in C^1([0,T] : H)$, such that $v' \in \mathcal{W}'$, and the following abstract evolution equation is satisfied:

$$v' + Au + \nu Av + S(\beta)u + c_N Ju = f \qquad \text{in } \mathcal{W}' \tag{35}$$

$$\beta' - H_{ad}(u - \phi, \beta) = 0 \qquad \text{in } \mathcal{H} \tag{36}$$

$$v(0) = v_0, \quad \beta(0) = \beta_0, \tag{37}$$

where $u(t) \equiv u_0 + \int_0^t v(s)\, ds$.

Since $\mathcal{H} = \mathcal{H}' \subseteq \mathcal{W}'$, the solution (v, β) to Problem P_{NCV} is also a solution to (35)–(37). Let $\{v_{0n}\}$ be a sequence in W which converges to v_0 in H and let $\{u_{0n}\}$ be a sequence in $H^2(\Omega) \cap W$ which converges to u_0 in W. Denote by (v_n, β_n) the solutions to (35)–(37) corresponding to these more regular initial data, which exists by Theorem 4.6. Estimate (34) implies that there is a constant C, independent of n, such that

$$|v_n(t)|_H^2 + \|u_n(t)\|_W^2 + \nu \int_0^t \|v_n\|_W^2 ds + c_N \Psi(u_n(t), \phi) \le C.$$

To obtain a bound on v' we apply (35) to $\psi \in \mathcal{W}$ and get

$$\begin{aligned} |\langle v_n', \psi \rangle| \le& \int_0^T \|\nabla u_n(t)\|_H \|\nabla \psi(t)\|_H \, dt + \nu \int_0^T \|\nabla v_n(t)\|_H \|\nabla \psi(t)\|_H \, dt \\ &+ \kappa \int_0^T \|u_n(t) - \phi\|_H \|\psi(t)\|_H \, dt + c_N \int_0^T \|u_n(t) - \phi\|_H \|\psi(t)\|_H \, dt \\ &+ \int_0^T \|f(t)\|_H \|\psi(t)\|_H \, dt, \end{aligned}$$

and hence

$$|\langle v_n', \psi \rangle| \leq \int_0^T (c_0 + c_1 \|u_n(t)\|_W + c_2 \|v_n(t)\|_W) \|\psi(t)\|_W \, dt.$$

Here, c_0, c_1 and c_2 depend on the problem data but are independent of n. Also, c_0, c_1 depend on c_N. Using the Hölder inequality and dividing both sides by $\|\psi(t)\|_{\mathcal{W}}$, we find that there is a constant C, independent of n, such that $\|v_n'\|_{\mathcal{W}'} \leq C$. Therefore, using Theorem 22, there is a subsequence, still indexed by n, such that the following convergences hold as $n \to \infty$:

$$v_n \to v \quad \text{weak} * \text{ in } L^\infty (0, T; H) \tag{38}$$

$$v_n' \to v' \quad \text{weakly in } \mathcal{W}' \tag{39}$$

$$v_n \to v \quad \text{weakly in } \mathcal{W} \tag{40}$$

$$u_n \to u \quad \text{weak} * \text{ in } L^\infty (0, T; W) \tag{41}$$

$$u_n \to u \quad \text{strongly in } C ([0, T] ; H^r (\Omega)), \tag{42}$$

where $r \in (\frac{1}{2}, 1)$ so that $H^r(\Omega)$ embeds compactly into H. From (32) we have that

$$\beta_n \to \beta \text{ strongly in } C ([0, T] ; H). \tag{43}$$

This, along with (42), implies there exists a further subsequence such that for each t,

$$u_n (x, y, t) \to u (x, y, t), \quad \beta_n (x, y, t) \to \beta (x, y, t) \text{ a.e. in } \Omega. \tag{44}$$

Also, from the differential equation satisfied by β_n and the properties of H_{ad}, the sequence β_n' is bounded in $\mathcal{H}$ and so there exists a subsequence such that, in addition,

$$\beta_n' \to \beta' \text{ weakly in } \mathcal{H}. \tag{45}$$

We have $\beta_n (t) = \beta_0 + \int_0^t H_{ad} (u_n (s) - \phi, \beta_n) \, ds$, and it follows now from (42)–(44), that it is possible to pass to the limit and obtain

$$\beta (t) = \beta_0 + \int_0^t H_{ad} (u (s) - \phi, \beta) \, ds.$$

This follows from the properties of H_{ad}, along with the observation that each β_n has values in $[0, 1]$.

Next, we pass to the limit in (35). First, we note that from the above convergences and standard results, since $v \in C([0, T]; H)$, $v_0 = \lim_{n \to \infty} v_n (0) = v (0)$, so the initial condition holds. Taking another subsequence if necessary, the terms Au_n and Av_n converge to Au and Av, respectively, since A is linear. The strong convergences above and the fact that β_n has values in $[0, 1]$ also imply that $c_N J u_n$ converges to $c_N J u$, and $S(\beta_n) u_n$ converges to $S(\beta) u$. We conclude that the limit (u, v, β) is a solution of Problem P_{NCV}.

The uniqueness of the solution follows in the same manner as above, since the regularity of the initial data was not used. This completes the proof of Theorem 3.1. □

5. The limit problem without viscosity

Although most materials exhibit a degree of viscosity, some do not, especially brittle materials. Also, on mathematical grounds, it is of interest to study the problem without viscosity, that is Problem P_{NC} in which $\nu = 0$. We do it by passing to the limit $\nu \to 0$. For mathematical convenience, we use the notation $\nu = \varepsilon$ in this section.

We assume that the initial data satisfies the conditions of Theorem 3.1. Thus, for each $\varepsilon > 0$ there exists a unique solution $(v_\varepsilon, u_\varepsilon, \beta_\varepsilon)$ to the problem

$$v' + Au + \varepsilon Av + S(\beta) u + c_N J u = f \qquad \text{in } \mathcal{W}' \tag{46}$$
$$u(t) = u_0 + \int_0^t v(s)\, ds, \quad v(0) = v_0$$
$$\beta' = H_{ad}(u - \phi, \beta), \quad \beta(0) = \beta_0.$$

It follows from (34) that the term $\varepsilon \langle Av_\varepsilon, v_\varepsilon \rangle$ is bounded independently of ε, and so

$$\varepsilon A v_\varepsilon \to 0 \text{ strongly in } \mathcal{W}'.$$

Also, it follows from estimate (34) and Theorem 4.1 that there is a sequence $\varepsilon \to 0$ such that (38)–(42) hold with the subscript n replaced by ε. It follows from (32) that $\{\beta_\varepsilon\}$ is a Cauchy sequence in $C([0,T]; H)$, thus,

$$\beta_\varepsilon \to \beta \text{ strongly in } C([0,T]; H).$$

Taking a further subsequence, if necessary, we can also assume (by using measurable representatives) that for all $t \in [0,T]$,

$$u_\varepsilon(\boldsymbol{x}, t) \to u(\boldsymbol{x}, t), \quad \beta_\varepsilon(\boldsymbol{x}, t) \to \beta(\boldsymbol{x}, t) \text{ pointwise a.e. } \boldsymbol{x} \in \Omega.$$

Now, let Q be the union of all the exceptional sets of measure zero for each ε in the sequence. Then, for each $\boldsymbol{x} \notin Q$, we can pass to the limit in

$$\beta_\varepsilon(\boldsymbol{x}, t) = \beta_0(\boldsymbol{x}) + \int_0^t H_{ad}\left(u_\varepsilon(\boldsymbol{x}, s) - \phi(\boldsymbol{x}), \beta_\varepsilon(\boldsymbol{x}, s)\right) ds,$$

using the Lebesgue dominated convergence theorem and obtain the limit expression above but without the ε. These convergences also make it possible to pass to the limit in (46), thus, the limit (v, β) satisfies

$$v' + Au + S(\beta)\,u + c_N Ju = f \quad \text{in } \mathcal{W}', \tag{47}$$

and, for $0 \leq t \leq T$,

$$u(t) = u_0 + \int_0^t v(s)\,ds, \qquad v(0) = v_0, \tag{48}$$

$$\frac{d\beta}{dt} = H_{ad}(u - \phi, \beta), \qquad \beta(0) = \beta_0. \tag{49}$$

The solution to the abstract system (47)–(49) is also the solution to Problem P_{NCV} with $\nu = 0$. This proves the following theorem.

Theorem 5.1. *Let the assumptions of Theorem* 3.1 *hold. Then, there exists a solution to Problem* P_{NCV}, (6)–(11), *with* $\nu = 0$.

We conclude that problem P_{NC}, (1)–(5), without the viscosity term has a weak solution. The uniqueness of the solution is unresolved, yet.

6. Conclusions

A model for the dynamics of a viscoelastic membrane in adhesive contact with a foundation or obstacle has been derived, and the existence of the unique weak solution was established for the problem with viscosity.

The novelty in this work is two-fold. The adhesion rate function H_{ad} is assumed to be only Hölder continuous in β, so that the exponent α may be smaller than one. From the modeling point of view, this allows for complete debonding, i.e., failure, in finite time. Mathematically, since H_{ad} is not Lipschitz, we needed to establish the existence in a non-routine way (Lemma 25), the difficulty being the measurability in the spacial variables, which was obtained from the uniqueness of the solutions and the usual proof of the Peano theorem.

The existence of a weak solution when the viscosity vanishes was obtained as the limit $\nu \to 0$.

The uniqueness of the solution of the problem without viscosity remains unresolved.

The case of a completely rigid obstacle, i.e., in the limit $c_N \to \infty$ remains open. The difficulty is in obtaining any estimate on the acceleration v' which is independent of c_N, without which one cannot pass to the limit in the equation.

Finally, since the model allows for failure, it may be of interest to obtain estimates on the time to failure, in terms of the problem data.

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Received April 1, 2008