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Approximate Differentiability Almost Everywhere

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Abstract. Let X be a bounded measurable subset of $\mathbb{R}^k$. We provide a characterization of the functions $f : X \to \mathbb{R}$ approximately differentiable almost everywhere. An important tool in the proof is a Saks' Theorem-type result.

Keywords. Parameter of regularity, asymptotically differentiable function, D-differentiable function, approximately differentiable function

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1. Introduction

Researches concerning the existence almost everywhere of classical, asymptotic derivatives were begun by Rademacher [18] and have been continued, among others, by Stepanoff ([23, 24]), Burkill [5], Haslam-Jones [14], Ward ([25, 26]) and Roger [19]. Several authors, among them de Lucia ([8, 9]), Oliveri [16], Bongiorno [1], Shmidov [22], and Guariglia ([11, 12]), have investigated characterizations of the functions asymptotically differentiable almost everywhere and have studied their properties. In particular, in [11], Guariglia provides a necessary and sufficient condition for a real valued measurable function of two variables be asymptotically differentiable.

Let X be a bounded measurable subset of $\mathbb{R}^k$. It is well-known that the three concepts, of asymptotic differentiability, of D-differentiability and of approximate differentiability, coincide for measurable functions (for a reference see, for instance, [7]). Therefore, in the measurable case, the standard terminology for asymptotic differentiability is approximate differentiability. For this reason, we use this terminology in this paper. Many important papers have been written on the subject during the last thirty years by mathematicians

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working in diverse areas such as real analysis, calculus of variations and PDE's. Some of them are: [2, 4, 10, 13, 17] and [21].

In this paper, we provide a characterization of the functions $f : X \to \mathbb{R}$ approximately differentiable almost everywhere. An important tool in the proof is a result analogous to Theorem 14.1 proved in ([20]: page 310) in the case when $k = 2$ (to which authors refer in the general case as well). We provide a proof in the general setting (Theorem 3.4).

The paper consists of four sections (this Introduction included). Section two contains preliminary definitions and results. In the third section we prove a Saks' Theorem-type result ([20]: page 310). In section four we recall the definition of a measurable function approximately differentiable almost everywhere, and we generalize the main result of [11] to the case of real valued measurable functions of any finite number of variables. Anyway, our technique is different and it is based on the Saks' Theorem -type result (of an independent interest) of Section three. As in [6], in this paper we make our analysis pointing our attention only on what happens inside certain "angles" of a fixed width and in this investigation an important role is played by the parameter of regularity.

2. Preliminaries

We denote by $\lambda^\star$ and λ the Lebesgue outer measure and the Lebesgue measure in $\mathbb{R}^k$, respectively. If X is a subset of $\mathbb{R}^k$, we denote by $\delta(X)$ its diameter (with respect to the Euclidean metric). By $\mathcal{I}$ we denote the collection of all closed non-degenerate intervals of $\mathbb{R}^k$ and, if $\mathbf{x}$ is a point of $\mathbb{R}^k$, by $\mathcal{I}(\mathbf{x})$ we mean the collections of the elements in $\mathcal{I}$ containing $\mathbf{x}$.

Definition 2.1 ([3, 15, 20])**.** Let X be a non-empty subset of $\mathbb{R}^k$ and $\mathbf{x}$ a point of $\mathbb{R}^k$. By writing

$$\lim_{I \Rightarrow \mathbf{x}} \frac{\lambda^*(X \cap I)}{\lambda(I)} = l, \quad l \in [0, 1],$$

we mean that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\frac{\lambda^\star(X \cap I)}{\lambda(I)} - l\right| < \epsilon$ for every $I \in \mathcal{I}(\mathbf{x})$ with $\delta(I) < \delta$. In particular, $\mathbf{x}$ is called *a density point* of X if $l = 1$, and $\mathbf{x}$ is called *a dispersion point* of X if $l = 0$.

Theorem 2.2 ([15, 20])**.** *Let X be a non-empty subset of $\mathbb{R}^k$. Then, almost every point of X is a density point of X. Moreover, if X is measurable, then almost all the points of the complement of X, $\mathbb{R}^k \setminus X$, are dispersion points of X.*

Given two points of $\mathbb{R}^k$, $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$, by $|\mathbf{x} - \mathbf{y}|$ we denote the Euclidean distance of $\mathbf{x}$ from $\mathbf{y}$. By $\pi(\mathbf{x})$ we denote the product of the co-ordinates of $\mathbf{x}$, i.e., $\pi(\mathbf{x}) = x_1 x_2 \cdots x_k$. We write $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$, for every

$i \in \{1, \ldots, k\}$. If $\mathbf{x} < \mathbf{y}$, by $r([\mathbf{x}, \mathbf{y}])$ we denote the *parameter of regularity* of the interval $[\mathbf{x}, \mathbf{y}]$, defined as $r([\mathbf{x}, \mathbf{y}]) = \frac{\lambda([\mathbf{x},\mathbf{y}])}{L^k}$, where $L = \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$. If $r([\mathbf{x}, \mathbf{y}]) = 1$, then $[\mathbf{x}, \mathbf{y}]$ is termed a *cube*.

3. A "Saks' Theorem"-type result

The following Theorem 3.4 is an important tool in the characterization of the functions $f : X \to \mathbb{R}$ approximately differentiable almost everywhere (Theorem 4.7). It is a Saks' Theorem-type result ([20, page 310]).

We now introduce the definition of $A(\tau)$ in $\mathbb{R}^k$.

Definition 3.1. Let $k \geq 2$. Given $\tau \in \mathbb{R}^k$ and $\alpha \in]0, 1[$, we define

$$A(\tau) = \big\{\mathbf{y} \in \mathbb{R}^k : \tau < \mathbf{y} \text{ and } r([\tau, \mathbf{y}]) > \alpha\big\}.$$

The point τ is termed *the vertex* of $A(\tau)$.

Remark 3.2. We mainly deal with subsets of $\mathbb{R}^k$ which are bounded and measurable and we focus our attention on their density points. First, we investigate what type of geometric objects these sets $A(\tau)$ are for $k = 2$ and $k = 3$.

`Case` $k = 2$: Fixed the parameter of regularity $\alpha \in]0, 1[$ and the point τ, an easy computation shows that $A(\tau)$ is the angle of semi-width $\frac{\pi}{4} - \arctan \alpha$, contained in the unbounded rectangle $]x_\tau, +\infty[\times]y_\tau, +\infty[$, with vertex $\tau = (x_\tau, y_\tau)$ and bisector the line $x - x_\tau = y - y_\tau$. The more α approaches 1 the more this angle gets small.

`Case` $k = 3$: Fixed the parameter of regularity $\alpha \in]0, 1[$ and the point τ, a simple computation shows that $A(\tau)$ is an unbounded solid, contained in the unbounded parallelepiped $]x_\tau, +\infty[\times]y_\tau, +\infty[\times]z_\tau, +\infty[$, and it is the union of the points inside three cones of vertex $\tau = (x_\tau, y_\tau, z_\tau)$ all containing inside the points of the line $x - x_\tau = y - y_\tau = z - z_\tau$. The more α approaches 1 the more these cones get small.

In order to prove the main result of this section (Theorem 3.4) we first need a simple preliminary Lemma.

Lemma 3.3. *Let $k \geq 2$, $\tau \in \mathbb{R}^k$ and $\alpha \in]0, 1[$. Then $\frac{\lambda(I(\tau)\setminus A(\tau))}{\lambda(I(\tau))}$ is constant, for every open neighbourhood $I(\tau)$ of τ of a fixed shape.*

Proof. Without loss of generality, we can assume that $\tau = \mathbf{0}$, i.e. we can restrict our attention to the case when τ is the origin of $\mathbb{R}^k$, and that every $I(\tau)$ is a ball centered at τ. For every $\rho > 0$, let $I_\rho(\mathbf{0})$ be the ball in $\mathbb{R}^k$ centered at $\mathbf{0}$ having radius ρ . Clearly, $I_\rho(\mathbf{0}) = \rho I_1(\mathbf{0})$ and, therefore, $\lambda(I_\rho(\mathbf{0})) = \rho^k \lambda(I_1(\mathbf{0}))$. We point out that, if we define

$$\rho(A(\mathbf{0})) = \{\rho\mathbf{y} : \mathbf{y} \in A(\mathbf{0})\},$$

then it is $\rho(A(\mathbf{0})) = A(\mathbf{0})$. In fact, $A(\mathbf{0}) = \{y \in \mathbb{R}^k : \mathbf{0} < \mathbf{y} \text{ and } r([\mathbf{0},\mathbf{y}]) > \alpha\} = \{\rho\mathbf{y} \in \mathbb{R}^k : \mathbf{0} < \rho\mathbf{y} \text{ and } r([\mathbf{0},\rho\mathbf{y}]) > \alpha\} = \rho(A(\mathbf{0}))$ (as, for every $\mathbf{y} > \mathbf{0}$, $r([\mathbf{0},\rho\mathbf{y}]) = r([\mathbf{0},\mathbf{y}])$ and "$\mathbf{0} < \rho\mathbf{y} \Leftrightarrow \mathbf{0} < \mathbf{y}$"). Then, for any $\rho > 0$,

$$\begin{aligned}\frac{\lambda(I_\rho(\mathbf{0}) \setminus A(\mathbf{0}))}{\lambda(I_\rho(\mathbf{0}))} &= \frac{\lambda(I_\rho(\mathbf{0}) \setminus \rho A(\mathbf{0}))}{\lambda(I_\rho(\mathbf{0}))} \\ &= \frac{\lambda(\rho(I_1(\mathbf{0}) \setminus A(\mathbf{0})))}{\lambda(\rho I_1(\mathbf{0}))} \\ &= \frac{\rho^k\lambda(I_1(\mathbf{0}) \setminus A(\mathbf{0}))}{\rho^k\lambda(I_1(\mathbf{0}))} \\ &= \frac{\lambda(I_1(\mathbf{0}) \setminus A(\mathbf{0}))}{\lambda(I_1(\mathbf{0}))} \\ &= c < 1,\end{aligned}$$

where c is the positive constant we are looking for. □

Theorem 3.4. *Let f be a finite function in $\mathbb{R}^k$ and let X be a subset of $\mathbb{R}^k$, for each point τ of which*

$$\limsup_{\mathbf{t}\to\tau,\mathbf{t}\in A(\tau)} f(\mathbf{t}) < \limsup_{\mathbf{t}\to\tau} f(\mathbf{t}).$$

Then, X has Lebesgue k-dimensional measure zero.

Proof. Let $\mathbb{Q}$ be the set of rational numbers. For any $r \in \mathbb{Q}$, we denote by X_r the set of the points τ of X such that

$$\limsup_{\mathbf{t}\to\tau,\mathbf{t}\in A(\tau)} f(\mathbf{t}) < r < \limsup_{\mathbf{t}\to\tau} f(\mathbf{t}).$$

Without loss of generality, by Theorem 2.2., we can assume that each $\tau \in X$ is a density point of X and that each $\tau \in X_r$ is a density point of X_r. For any fixed r, we observe that no point τ is an accumulation point for the part of the set X_r contained in the interior of the corresponding "angle" $A(\tau)$, i.e., $\tau \notin \overline{X_r \cap A(\tau)^o}$. This is claire as, fixed $\tau \in X_r$, by definition of limsup, there exists a neighbourhood of τ, $I(\tau)$, such that $f(\mathbf{t}) < r$, for every $\mathbf{t} \in A(\tau) \cap I(\tau)$.

Then, fixed $\tau_0 \in A(\tau)^o \cap I(\tau)$, there is a neighbourhood of τ_0, $I(\tau_0)$, contained in $A(\tau)^o \cap I(\tau)$ and, as τ_0 is a density point of X, it is $I(\tau_0) \cap X \neq \emptyset$. Therefore, $f(\mathbf{t}) < r$, for every $\mathbf{t}$ in $I(\tau_0) \cap X$, implies $\limsup_{\mathbf{t}\to\tau_0} f(\mathbf{t}) < r$, and so $\tau_0 \notin X_r$. Hence, no point of the set X_r can be a density point of this set as

$$\frac{\lambda(X_r \cap I(\tau))}{\lambda(I(\tau))} = \frac{\lambda(X_r \cap (I(\tau) \setminus A(\tau))}{\lambda(I(\tau))} \leq \frac{\lambda(I(\tau) \setminus A(\tau))}{\lambda(I(\tau))} = c < 1,$$

where c is the constant of Lemma 3.3. By Theorem 2.2., it must be $\lambda(X_r) = 0$. Finally, $X = \bigcup\{X_r : r \in \mathbb{Q}\}$ yields $\lambda(X) = 0$. □

4. Approximate differentiability almost everywhere

There are many notions in the literature which are generalizations of differentiability. For each of them, there are examples showing that a function can have this property without being differentiable in the ordinary sense.

Definition 4.1 ([11, page 13]). Let X be a measurable subset of $\mathbb{R}^k$ and let $f : X \to \mathbb{R}$ be measurable. Then, f is said to be *approximately differentiable* at a density point $\mathbf{x}$ of X if there exist k numbers $a_1, \ldots, a_k$ that satisfy

$$\lim as_{\mathbf{y}\to\mathbf{x}} \frac{f(\mathbf{y}) - f(\mathbf{x}) - \sum_{i=1}^{k} a_i(y_i - x_i)}{|\mathbf{y} - \mathbf{x}|} = 0,$$

i.e., if there exists a measurable subset U of X such that $\mathbf{x}$ is a density point of U and, denoted by f_U the restriction of f to U, it is

$$\lim_{\mathbf{y}\to\mathbf{x}} \frac{f_U(\mathbf{y}) - f_U(\mathbf{x}) - \sum_{i=1}^{k} a_i(y_i - x_i)}{|\mathbf{y} - \mathbf{x}|} = 0.$$

Our aim in this section is to provide a characterization of functions approximately differentiable almost everywhere in terms of what happens inside "angles" of a fixed width. To this aim, we introduce the sets $X_{n,\alpha}$ and $X'_{n,\alpha}$.

Definition 4.2. Let $k \geq 2$, let X be a subset of $\mathbb{R}^k$ and let $f : X \to \mathbb{R}$ be measurable. Then, fixed $\alpha \in]0, 1[$ and $n \in \mathbb{N}$, by $X_{n,\alpha}$ we denote the collection of all points $\mathbf{x} \in X$ with the following property:

For each $\mathbf{y} \in X$ satisfying $\mathbf{x} < \mathbf{y}$ and $r([\mathbf{x}, \mathbf{y}]) > \alpha$, from $|\mathbf{y} - \mathbf{x}| < \frac{1}{n}$ it follows that $|f(\mathbf{y}) - f(\mathbf{x})| \leq n|\mathbf{y} - \mathbf{x}|$.

We point out that if X is a measurable subset of $\mathbb{R}^k$ then $X_{n,\alpha}$ is measurable as well ([9, Proposition 2.1, page 60]).

Definition 4.3. Let $k \geq 2$, let X be a subset of $\mathbb{R}^k$ and let $f : X \to \mathbb{R}$ be measurable. If $\alpha \in]0, 1[$ and $n \in \mathbb{N}$, then by $X'_{n,\alpha}$ we denote the collection of all points $\mathbf{x} \in X$ with the following property:

For each $\mathbf{y} \in X$ satisfying $\mathbf{x} < \mathbf{y}$ and $r([\mathbf{x}, \mathbf{y}]) > \alpha$, from $|\mathbf{y} - \mathbf{x}| < \frac{1}{n}$ it follows that $f(\mathbf{y}) - f(\mathbf{x}) \leq n|\mathbf{y} - \mathbf{x}|$.

With an argument similar to [9, Proposition 2.1, page 60], we can show that if X is measurable, then $X'_{n,\alpha}$ is measurable as well.

Remark 4.4. Clearly, $X_{n,\alpha} \subseteq X'_{n,\alpha}$.

Proposition 4.5. *Let $k \geq 2$ and let X be a subset of $\mathbb{R}^k$. If X is closed and $f : X \to \mathbb{R}$ is continuous then, for every $\alpha \in]0, 1[$ and every $n \in \mathbb{N}$, $X'_{n,\alpha}$ is closed as well.*

Proof. We need to show that if $\mathbf{z}_0$ is the limit of a sequence of points of $X'_{n,\alpha}$, say $(\mathbf{z}_s)_{s\in\mathbb{N}}$, then $\mathbf{z}_0 \in X'_{n,\alpha}$.

To this aim, let $\mathbf{z}$ be a point of X with $\mathbf{z}_0 < \mathbf{z}$, $|\mathbf{z}-\mathbf{z}_0| < \frac{1}{n}$, $r([\mathbf{z}_0,\mathbf{z}]) > \alpha$. If $\mathbf{z}_0 = (z_{0,1},\dots,z_{0,k})$, $\mathbf{z} = (z_1,\dots,z_k)$ and, for $s \in \mathbb{N}$, $\mathbf{z}_s = (z_{s,1},\dots,z_{s,k})$, there obviously exists $s_1 \in \mathbb{N}$ such that, for all $s \geq s_1$, $\mathbf{z}_s < \mathbf{z}$ and $|\mathbf{z}-\mathbf{z}_s| < \frac{1}{n}$. Moreover, as for every $1 \leq j \leq k$ $\pi_{i\in\{1,\dots,k\}\setminus\{j\}}(z_i - z_{0,i}) > \alpha(z_j - z_{0,j})^{k-1}$, there exists $s_2 \in \mathbb{N}$ such that for every $1 \leq j \leq k$, for all $s \geq s_2$,

$$\pi_{i\in\{1,\dots,k\}\setminus\{j\}}(z_i - z_{s,i}) > \alpha(z_j - z_{s,j})^{k-1},$$

i.e., $r([\mathbf{z}_s,\mathbf{z}]) > \alpha$, for all $s \geq s_2$. Hence, if we set $s_0 = \max\{s_1, s_2\}$, then $\mathbf{z}_s < \mathbf{z}$, $|\mathbf{z}-\mathbf{z}_s| < \frac{1}{n}$ and $r([\mathbf{z}_s,\mathbf{z}]) > \alpha$, for all $s \geq s_0$. Therefore, for every $s \geq s_0$, $f(\mathbf{z}) - f(\mathbf{z_s}) \leq n|\mathbf{z}-\mathbf{z_s}|$ and so, because of the continuity of f in X, $f(\mathbf{z}) - f(\mathbf{z}_0) \leq n|\mathbf{z}-\mathbf{z}_0|$, and the claim is proved. □

Proposition 4.6. *Let $k \geq 2$, let X be a subset of $\mathbb{R}^k$ and let $f : X \to \mathbb{R}$ be measurable. Assume that there exist $\alpha \in]0,1[$ and $n \in \mathbb{N}$ such that $X'_{n,\alpha}$ is measurable. Then, for every $\epsilon > 0$, there is a closed subset C of $X'_{n,\alpha}$ with $\lambda(X'_{n,\alpha} \setminus C) < \epsilon$ and f_C, the restriction of f on C is almost everywhere differentiable (and, hence, almost everywhere approximately differentiable).*

Proof. From the definition of $X'_{n,\alpha}$ and from Theorem 3.4 it follows that, for almost every $\mathbf{x} \in X'_{n,\alpha}$,

$$\limsup_{\mathbf{y}\to\mathbf{x},\mathbf{y}\in A(\mathbf{x})} \frac{f(\mathbf{y}) - f(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} = \limsup_{\mathbf{y}\to\mathbf{x}} \frac{f(\mathbf{y}) - f(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} \leq n.$$

Let $X''_{n,\alpha}$ be the collection of all points of $X'_{n,\alpha}$ which are density points of $X'_{n,\alpha}$. Obviously, $X'_{n,\alpha}$ and $X''_{n,\alpha}$ have the same Lebesgue measure.

Given $\epsilon > 0$, by Lusin Theorem, there exists a closed subset $C = C(\epsilon)$ of $X''_{n,\alpha}$ such that $\lambda(X''_{n,\alpha} \setminus C) < \epsilon$ and f_C, the restriction of f to C is a continuous function. For every $j \in \mathbb{N}$, let

$$C_j = \Big\{\mathbf{x} \in C : \text{ for every } y \in C \text{ with } |\mathbf{x}-\mathbf{y}| < \tfrac{1}{j} \text{ it is } |f(\mathbf{x})-f(\mathbf{y})| \leq n\cdot|\mathbf{x}-\mathbf{y}|\Big\}.$$

Let, for every $i \in \mathcal{N}$,

$$C_{j,i} = C_j \cap \left[\frac{i}{n\sqrt{k}}, \frac{i+1}{n\sqrt{k}}\right]^k.$$

Clearly, each $C_{j,i}$ is closed and $C_j = \cup_i C_{j,i}$. Moreover, f is Lipschitz in each $C_{j,i}$ and hence it is almost everywhere differentiable in every $C_{j,i}$ and so it is almost everywhere differentiable in C, as $C = \cup C_j$. □

Theorem 4.7. *Let $k \geq 2$, let X be a bounded measurable subset of $\mathbb{R}^k$ and $f : X \to \mathbb{R}$ measurable. Then, the following assertions are equivalent:*

1. *f is approximately differentiable almost everywhere;*
2. *for every $\epsilon > 0$ there exists a measurable subset Y of X satisfying the conditions $\lambda(X \setminus Y) < \epsilon$ and "for each $\sigma > 0$ there exist $n \in \mathbb{N}$ and $\alpha \in]0, 1[$ such that $\lambda(Y \setminus Y'_{n,\alpha}) < \sigma$".*

Proof. ($\Rightarrow$) Given $\epsilon > 0$, by [27, Theorem 1, page 144], there exists a subset Y of X such that $\lambda(X \setminus Y) < \epsilon$ and the restriction of f to Y is almost everywhere differentiable in Y. By Proposition 2.3 of [9], for each $\sigma > 0$ there exist $n \in \mathbb{N}$ and $\alpha \in]0, 1[$ such that $\lambda(Y \setminus Y_{n,\alpha}) < \sigma$. Hence, as $Y_{n,\alpha} \subseteq Y'_{n,\alpha}$, for each $\sigma > 0$ there exist $n \in \mathbb{N}$ and $\alpha \in]0, 1[$ such that $\lambda(Y \setminus Y'_{n,\alpha}) < \sigma$.

($\Leftarrow$) In order to prove that the condition is sufficient, we observe that, without loss of generality, we can certainly assume that Y is closed and that the restriction of f to Y is continuous. Therefore, applying Proposition 4.5 and Proposition 4.6, we can deduce that the restriction of f to Y is approximately differentiable almost everywhere in Y. Hence, for any $\epsilon > 0$ there exists a subset Y of X such that $\lambda(X \setminus Y) < \epsilon$ and the restriction of f to Y is approximately differentiable almost everywhere in Y. This is enough to ensure that f is approximately differentiable almost everywhere in X. □

We end the paper with an application of Theorem 4.7 and an example. We start by showing that Whitney's characterization of an approximately differentiable almost everywhere function ([27, Theorem 1], $(a) \Leftrightarrow (c)$) is an application of Theorem 4.7.

Theorem 4.8 (Whitney's Theorem)**.** *Let X be a bounded measurable subset of $\mathbb{R}^k$ and $f : X \to \mathbb{R}$ measurable. Then, the following assertions are equivalent:*

(a) *The function f is approximately differentiable almost everywhere in X.*

(c) *For each $\epsilon > 0$ there is a closed set $Q \subseteq X$ such that $\lambda(X \setminus Q) < \epsilon$ and f is smooth (continuously differentiable) in Q.*

Proof. It is enough to observe that (c) is equivalent to (2) of Theorem 4.7 (apply Proposition 4.6 and Lusin Theorem). □

Example 4.9. Let $X = [0, 1] \times [0, 1]$, $A = \{(x, y) \in X : x \text{ and } y \text{ are rational}\}$ and $B = X \setminus A$. Consider the function $f : X \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) \in A \\ 1 & \text{if } (x, y) \in B. \end{cases}$$

Fix any natural number n and any $\alpha \in]0, 1[$. Observe that $X'_{n,\alpha} \supseteq B$ as, for any $\mathbf{x_0} = (x_{01}, x_{02})$ in B, the following holds:

"for each $\mathbf{x} = (x_1, x_2) \in X$ satisfying $\mathbf{x_0} < \mathbf{x}$ and $r([\mathbf{x_0}, \mathbf{x}]) > \alpha$, from $|\mathbf{x} - \mathbf{x_0}| < \frac{1}{n}$ it follows that $f(\mathbf{x}) - f(\mathbf{x_0})| \leq n|\mathbf{x} - \mathbf{x_0}|$".

This holds because

$$f(\mathbf{x}) - f(\mathbf{x_0}) = \begin{cases} -1 & \text{if } \mathbf{x} \in A \\ 0 & \text{if } \mathbf{x} \in B\,. \end{cases}$$

Therefore, $\lambda(X \setminus X'_{n,\alpha}) = 0$. Hence, and as we expected, by Theorem 4.7., f is approximately differentiable almost everywhere.

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