Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 29 (2010), 91–106
DOI: 10.4171/ZAA/1399

On Nonlinear Volterra Integral Equations with State Dependent Delays in Several Variables

Agata Gołaszewska and Jan Turo

Abstract. Lipschitz continuous solutions of nonlinear Volterra integral equations with state dependent delays in several variables are investigated. The results are based on a comparison method and the Banach fixed point principle.

Keywords. Initial problems, comparison method, neutral equations with state dependent delays, Volterra integral equations in several variables

Mathematics Subject Classification (2000). 34K40

1. Introduction

Let $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}^n$ be the real n-dimensional Euclidean space with the Euclidean norm $|\cdot|$. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and a, $r \in \mathbb{R}^n_+$. We denote by $C(U, V)$ the set of all continuous functions from U to V. Let $G = \{x \in \mathbb{R}^n : 0 \le x \le a\}$, $\Omega = \{x \in \mathbb{R}^n : -r \le x \le a\}$ and $B = \Omega \backslash G$ For $u : \Omega \to E$, where E is a Banach space with the norm $\|\cdot\|$, we define the function $u_x(\tau) = u(x + \tau)$, $\tau \in B$, $x \in G$. There are given the functions $F \in C(G \times E^m \times C(B, E), E)$, $f_i \in C(G \times G \times C(B, E), E)$, $i = 1, \dots, m$, $m \in \mathbb{N}$, $\theta, \Psi_i \in C(G \times C(B, E), G)$, $i = 1, \dots, m$, $m \in \mathbb{N}$, $\beta, \alpha_i \in C(G, G)$, $i = 1, \dots, m$, and $\varphi \in C(B, E)$.

Consider the problem

$$u(x) = F\left(x, \int_{H(x)} f\big(x, s, u_{\Psi(s, u_{\alpha(s)})}\big) ds, u_{\Theta(x, u_{\beta(x)})}\right), \quad x \in G \tag{1}$$

$$u(x) = \varphi(x), \qquad x \in B, \tag{2}$$

A. Gołaszewska: Department of Mathematics, Gdańsk University of Technology, 11/12 Narutowicza str., 80-952 Gdańsk, Poland; golaszewska@mif.pg.gda.pl
J. Turo: Department of Mathematics, Gdańsk University of Technology, 11/12 Narutowicza str., 80-952 Gdańsk, Poland; turo@mif.pg.gda.pl

where

$$\int_{H(x)} f\big(x,s,u_{\Psi(s,u_{\alpha(s)})}\big)ds$$
$$= \left(\int_{H_1(x)} f_1\big(x,s,u_{\Psi_1(s,u_{\alpha_1(s)})}\big)(ds)_{p_1},\dots,\int_{H_m(x)} f_m\big(x,s,u_{\Psi_m(s,u_{\alpha_m(s)})}\big)(ds)_{p_m}\right),$$

and $H_i(x) \subseteq \{s \in \mathbb{R}^n :\ 0 \le s \le x\}$, $x \in G$, $i = 1,\dots,m$.

Assume that $H_i(x)$ is contained in a p_i-dimensional hyperplane $(1 \le p_i \le n)$, where p_i does not depend on x, parallel to the coordinate axes, and that it is Lebesgue measurable, when considered as a p_i-dimensional set. Denote by $L_{p_i}(H_i(x))$ the p_i-dimensional measure of $H_i(x)$ and let Γ_i, $\bar{\Gamma}_i \subseteq \{1,\dots,n\}$, $i = 1,\dots,m$, be defined by

$$\Gamma_i = \big\{j :\ \text{the axis } Ox_j \text{ is parallel to the hyperplane containing } H_i(x)\big\},$$

and $\bar{\Gamma}_i = \{1,\dots,n\} \setminus \Gamma_i$.

Put $A = \{i : p_i = n\}$, $A' = \{i : 1 \le p_i < n\}$. Let a p_i-dimensional hyperplane containing $H_i(x)$ be defined by $x_{k_1} = x'_{k_1}, x_{k_2} = x'_{k_2}, \dots, x_{k_l} = x'_{k_l}$, $l = n - p_i$. Then $\int_{H_i(x)} z(x,s)(ds)_{p_i}$, $s = (s_1,\dots,s_n)$, denotes the p_i-dimensional Lebesgue integral with respect to the variables s_{k_j}, $k_j \in \Gamma_i$, and in the above integral we have $s_{k_j} = x'_{k_j}$ for $k_j \in \bar{\Gamma}_i$.

We define

$$G_i(x) = \left\{s \in \mathbb{R}^n : s_{k_j} = x'_{k_j} \text{ for } k_j \in \bar{\Gamma}_i,\ 0 \le x_{k_j} \le \phi^{(i)}_{k_j}(x) \text{ for } k_j \in \Gamma_i\right\},$$

where $\phi^{(i)}_{k_j} \in C(G,\mathbb{R})$, $k_j \in \Gamma_i$, and $H_i(x) \subseteq G_i(x) \subseteq \{s \in \mathbb{R}^n : 0 \le s \le x\}$. Then we have $L_{p_i}(G_i(x)) = \prod_{s\in\Gamma_i} \phi^{(i)}_s(x)$. To simplify we use the following notations:

$$L(G(x)) = \big(L_{p_1}(G_1(x)),\dots,L_{p_m}(G_m(x))\big),$$

and

$$K(x)\int_{H(x)} f(x,s,u_{\Psi(s,u_{\alpha(s)})})ds = \sum_{j=1}^m K_j(x)\int_{H_j(x)} f_j(x,s,u_{\Psi_j(s,u_{\alpha_j(s)})})(ds)_{p_j},$$
$$K(x)L(G(x)) = \sum_{j=1}^m K_j(x)L_{p_j}(G_j(x)),\quad x \in G,$$

where $x \in G$, $K = (K_1,\dots,K_m) \in C(G,\mathbb{R}^m)$.

Ordinary functional differential equations with state dependent delays have attracted the attention of many authors [1, 3, 5, 7–12, 20, 25], and [34]. The paper [19] initiated the study of the existence theory for first order functional partial differential equations with state dependent delays (see also [4,6,16]). A particular case of equation (1) with $n = 1$ was discussed in [3,5,15].

There are various problems for functional differential equations which lead to Volterra functional integral equations of type (1). One of the simplest problem of the form (1), (2) with $n = 1$ can be obtained from the initial value problem for the ordinary functional differential equations of the neutral type:

$$\begin{aligned} y'(t) &= F(t, y_{\alpha_1(t)}, \ldots, y_{\alpha_m(t)}, y'_{\beta(t)}), \quad && t \in [0, \bar{a}] \\ y(t) &= \varphi(t), && t \in [-r, 0]. \end{aligned}$$

In case $r = 0$ the above problem leads to the single equation of type (1) without the additional condition (2). Therefore equation (1) is a generalization of equations considered in [13,17,21,26,27,29].

Various initial value problems for the hyperbolic functional differential equations of the neutral type with two independent variables

$$\begin{aligned} &D_{xy} z(x, y) \\ &= F(x, y, z_{(\alpha_1^0(x,y)),(\alpha_2^0(x,y))} D_x z_{(\alpha_1^1(x,y)),(\alpha_2^1(x,y))} D_y z_{(\alpha_1^2(x,y)),(\alpha_2^2(x,y))} D_{xy} z_{(\beta_1(x,y)),(\beta_2(x,y))}), \end{aligned}$$

$(x, y) \in [0, \tilde{a}] \times [0, \tilde{b}]$, with the initial condition

$$z(x, y) = \varphi(x, y), \quad (x, y) \in [-r_1, \tilde{a}] \times [-r_2, \tilde{b}] \backslash [0, \tilde{a}] \times [0, \tilde{b}],$$

can also be transformed to the problem of type (1), (2). The Volterra functional integral equation corresponding to that problem takes the form

$$\begin{aligned} &u(x, y) \\ &= F\Bigg(x, y, -\varphi_{(0,0)} + \varphi_{(\alpha_1^0(x,y),0)} + \varphi_{(0,\alpha_2^0(x,y))} + \int_{H_0(x,y)} u_{(s,t)}\, ds\, dt, \\ &\quad D_x \varphi_{(\alpha_1^1(x,y),0)} + \int_{H_1(x,y)} u_{(s,t)} dt, D_y \varphi_{(0,\alpha_2^2(x,y))} + \int_{H_2(x,y)} u_{(s,t)}\, ds, u_{(\beta_1(x,y),\beta_2(x,y))}\Bigg), \end{aligned}$$

$(x, y) \in [-r_1, 0] \times [-r_2, 0]$, where

$$\begin{aligned} H_0(x, y) &= \Big\{(s, t) : s \in \big[0, \alpha_1^{(0)}(x, y)\big], t \in \big[0, \alpha_2^{(0)}(x, y)\big]\Big\} \\ H_1(x, y) &= \Big\{(s, t) : s = \alpha_1^{(1)}(x, y), t \in \big[0, \alpha_2^{(1)}(x, y)\big]\Big\} \\ H_2(x, y) &= \Big\{(s, t) : s \in \big[0, \alpha_1^{(2)}(x, y)\big], t = \alpha_2^{(2)}(x, y)\Big\}. \end{aligned}$$

For this reason, in case $r_1 = r_2 = 0$ equation (1) is a generalization of the equations investigated in [23,24,30,31].

The case where $\Psi(x,w) = \alpha(x)$, $\Theta(x,w) = \beta(x)$ was studied in [2] and [14]. The Cauchy problem and the Goursat problem for hyperbolic functional differential equations also lead to equations of type (1) (see [32]). Initial value problems for equations in more than two variables and problems for equations of higher order can be transformed in terms of Volterra functional integral equations. As a particular case of equation (1) we can obtain the system of Volterra integral equations which was discussed in [32, 33] or the functional integral equations considered in [18, 22, 28].

In this paper we prove a theorem of the existence and uniqueness of Lipschitz continuous solutions of the problem (1), (2). If we assume that the Lipschitz coefficient l of the function F with respect to the last variable satisfies the condition $l < 1$, then we have a theorem on the existence and uniqueness of solutions of (1), (2), which can be obtained by means of the Banach fixed point theorem. We relaxed this very restrictive condition. We proved that the integral operator defined by the right-hand side of (1) is a contraction with a weighted norm constructed with the help of a solution of a certain comparative integral equation.

2. Assumptions and lemmas

Suppose that for any $x \in G$ and $i \in A'$ the set $H_i(x)$ is contained in a p_i-dimensional hyperplane $S_i(x)$, parallel to the p_i coordinate axes, where $p_i = 1, \dots n-1$. Then for any $y \in \mathbb{R}^n$, such that $x + y \in G$, there exists a vector $v_i(x,y) \in \mathbb{R}^n$ perpendicular to $S_i(x)$ and $-v_i(x,y) + H_i(x+y) \subseteq S_i(x)$.

Assumption H_1. Suppose that

(i) there exist $\omega \in \mathbb{R}_+$: $L_n(H_i(x) \,\Delta\, H_i(\bar{x})) \le \omega|x - \bar{x}|$, $i \in A$ (the sign Δ denotes the symmetric difference of two sets);

(ii) $L_{p_i}(H_i(x) \,\Delta\, (-v_i(x, \bar{x} - x) + H_i(\bar{x}))) \le \omega|x - \bar{x}|$, $v_i(x,\bar{x}) \ge 0$, $\lim_{x\to\bar{x}} v_i(x, \bar{x} - x) = 0$, $i \in A'$, $x, \bar{x} \in G$, $x \le \bar{x}$;

(iii) $H_i(x) \subseteq H_i(\bar{x})$ for $x, \bar{x} \in G$, $x \le \bar{x}$, and $i \in A$;

(iv) $H_i(x) + v_i(x, \bar{x} - x) \subseteq H_i(\bar{x})$ for $x, \bar{x} \in G$, $x \le \bar{x}$, and $i \in A'$.

Assumption H_2. Suppose that

(i) l, $\bar{h} \in C(G, \mathbb{R}_+)$, $K \in C(G, \mathbb{R}^m_+)$, Θ, $\Psi_i : G \times C(B, E) \to G$, $i = 1, \dots, m$, are nondecreasing functions;

(ii) γ_i, $\zeta \in C(G, G)$, $i = 1, \dots, m$, are nondecreasing functions, and $\gamma_i(x) \le x$, $\zeta(x) \le x$ for $x \in G$, $i = 1, \dots, m$;

(iii) the function $\bar{m} : G \to \mathbb{R}_+$ is defined by

$$\bar{m}(x) = \sum_{i=0}^{+\infty} l_i(x)\bar{h}(\zeta_i(x)) < +\infty,$$

where $\zeta_0(x) = x, \zeta_{i+1}(x) = \zeta(\zeta_i(x))$, $l_0(x) = 1, l_{i+1} = l(x)l_i(\zeta(x))$, and $i = 0, 1, \ldots, x \in G$;

(iv) the function $M : G \to \mathbb{R}_+$ is given by

$$M(x) = \sum_{i=0}^{+\infty} l_i(x)K(\zeta_i(x))L\big(G(\zeta_i(x))\big) < +\infty, \quad x \in G;$$

(v) the function $\bar{M} : G \to \mathbb{R}_+$ given by

$$\bar{M}(x) = \sum_{i=0}^{+\infty} l_i(x)K(\zeta_i(x))L\big(G(\zeta_i(x))\big)\bigg(\prod_{s\in\Gamma_i} x_s\bigg)^{-1}$$

is bounded on G.

Further we will use the following notation:

$$\tilde{m}(x) = \sum_{i=0}^{+\infty} l_i(x)\tilde{h}(\zeta_i(x))$$

$$(Vu)(x) = \sum_{i=0}^{+\infty} l_i(x)(K(\zeta_i(x)) \int_{H(\zeta_i(x))} u(\gamma_i(s))ds).$$

Remark 1. Suppose that

(I) conditions (i)–(iv) of Assumption H_2 are satisfied;

(II) $\tilde{h} \in C(G, \mathbb{R}_+)$ and $\tilde{h}(x) \le \bar{h}(x)$ for $x \in G$;

(III) $g : G \to \mathbb{R}_+$ is upper semicontinuous.

Then $\tilde{m}$ and Vg are functions well defined for $x \in G$.

Lemma 2.1. *Suppose that Assumptions H_1, H_2 are satisfied, $\tilde{h} \in C(G, \mathbb{R}_+)$ is nondecreasing, and $\tilde{h}(x) \le \bar{h}(x)$ on G. Then*

(I) *there exists a nondecreasing solution $\bar{g} \in C(G, \mathbb{R}_+)$ of the equation*

$$g(x) = \tilde{m}(x) + (Vg)(x), \quad x \in G, \tag{3}$$

which is unique in the set $P(G, \mathbb{R}_+)$ of upper semicontinuous functions from G to $\mathbb{R}_+$;

(II) *the function $\bar{g}$ is a solution of the equation*

$$g(x) = K(x) \int_{H(x)} g(\gamma(s))ds + l(x)g(\zeta(x)) + \tilde{h}(x), \quad x \in G, \tag{4}$$

which is unique in the class $P(G, \mathbb{R}_+, \tilde{g})$ of all functions from the class $P(G, \mathbb{R}_+)$, such that $\inf\{\kappa \in \mathbb{R}_+ : g(x) \le \kappa\tilde{g}(x),\, x \in G\} < +\infty$, *where $\tilde{g}$ is a solution of* (3) *with $\bar{h} = \tilde{h}$;*

(III) *the function* $\tilde{g}$ *satisfies the condition*

$$\lim_{i\to+\infty} l_i(x)\tilde{g}(\zeta_i(x)) = 0 \tag{5}$$

uniformly on G.

Proof. First we show that equation (3) has a unique solution in the class $P(G,\mathbb{R}_+)$. We define the operator

$$(Tz)(x) = \bar{m}(x) + (Vz)(x), \quad x \in G.$$

We prove that $T : P(G,\mathbb{R}_+) \to P(G,\mathbb{R}_+)$. Let $z \in P(G,\mathbb{R}_+)$ and

$$v_{ij}(x) = \int_{H_j(\zeta_i(x))} z(\gamma_j(s))(ds)_{p_j},$$

where $x \in G$, $i = 1,\dots,n$, $j = 1,\dots,m$, $m,n \in \mathbb{N}$. Because of $z \in P(G,\mathbb{R}_+)$, so there exists the sequence $\{z_k\}_{k\in\mathbb{N}}$, such that $z_k \in C(G,\mathbb{R}_+)$ and $z_{k+1}(x) \le z_k(x)$, $z(x) = \lim_{k\to+\infty} z_k(x)$, $x \in G, k \in N$. Let

$$v_{ij}^{(k)}(x) = \int_{H_j(\zeta_i(x))} z_k(\gamma_j(s))(ds)_{p_j},$$

where $x \in G$, $i = 1,\dots,n$, $j = 1,\dots,m$, $k,m,n \in \mathbb{N}$. Functions $v_{ij}^{(k)}$ are continuous in G (see [32]) and $v_{ij}^{(k+1)}(x) \le v_{ij}^{(k)}(x)$. From Lebesgue's theorem (about the integration of the sequence of nonincreasing functions) we have

$$v_{ij}(x) = \lim_{k\to+\infty} v_{ij}^{(k)}(x), \quad x \in G,\ i = 1,\dots,n,\ j = 1,\dots,m,\ k,m,n \in N$$

and therefore $v_{ij} \in P(G,\mathbb{R}_+)$. From the Weierstrass criterium (elements of these series are continuous and nondecreasing) follows the uniform convergence of the series

$$\tilde{m}(x) = \sum_{i=0}^{+\infty} l_i(x)\bar{h}(\zeta_i(x)), \quad M(x) = \sum_{i=0}^{+\infty} l_i(x)K(\zeta_i(x))L\big(G(\zeta_i(x))\big).$$

Now we have $l_i(x)\bar{h}(\zeta_i(x)) \le l_i(x)\tilde{h}(\zeta_i(x))$, and

$$l_i(x)K(\zeta_i(x)) \int_{H(\zeta_i(x))} z(\gamma(s))(ds) \le \Big(\sup_{x\in G} z(x)\Big) l_i(x)K(\zeta_i(x))L\big(G(\zeta_i(x))\big),$$

where $i \in N$, $x \in G$. Hence we get that the series

$$\bar{m} = \sum_{i=0}^{+\infty} l_i(x)\bar{h}(\zeta_i(x)), \qquad \sum_{i=0}^{+\infty} l_i(x)K(\zeta_i(x)) \int_{H(\zeta_i(x))} z(\gamma(s))(ds)$$

are convergent. Now it is easily seen that $\bar{m} \in C(G, \mathbb{R}_+)$, $Vz \in P(G, \mathbb{R}_+)$ and consequently $T : P(G, \mathbb{R}_+) \to P(G, \mathbb{R}_+)$.

Now we show that operator T is a contraction. We define the norm

$$\|z\|_\lambda = \max_{x \in G} \left[|z(x)| \cdot \exp\left(-\lambda \sum_{p=1}^{n} x_p \right) \right], \quad z \in P(G, \mathbb{R}_+),$$

where $\lambda > \Lambda = \max\left\{1, \sup_{x \in G} \tilde{M}(x)\right\}$. For z, $w \in P(G, \mathbb{R}_+)$ we get

$$\begin{aligned} |(Tz)(x) - (Tw)(x)| &\le \sum_{i=0}^{+\infty} l_i(x) K(\zeta_i(x)) \int_{H(\zeta_i(x))} |z(\gamma(s)) - w(\gamma(s))|(ds) \\ &\le \|z - w\|_\lambda \sum_{i=0}^{+\infty} l_i(x) K(\zeta_i(x)) \int_{H(\zeta_i(x))} \exp\left(\lambda \sum_{p=1}^{n} s_p \right) ds. \end{aligned}$$

Using the estimation $\exp(\epsilon t) - 1 \le \epsilon \exp(t)$, $\epsilon \in [0, 1]$, $t \ge 0$, we have the following:

$$\begin{aligned} \int_{H_j(\zeta_i(x))} \exp\left(\lambda \sum_{p=1}^{n} s_p\right)(ds)_{p_j} &\le \exp\left(\lambda \sum_{p \in \bar{\Gamma}_j} x_p\right)(ds)_{p_j} \int_{G_j(\zeta_i(x))} \exp\left(\lambda \sum_{p \in \Gamma_j} s_p\right)(ds)_{p_j} \\ &\le \exp\left(\lambda \sum_{p \in \bar{\Gamma}_j} x_p\right) \prod_{p \in \Gamma_j} \left(\frac{1}{\lambda} \exp(\lambda \phi_p^{(j)}(\zeta_i(x)) - 1) \right) \\ &\le \frac{1}{\lambda} \exp\left(\lambda \sum_{p=1}^{n} x_p\right) L_{p_j}\big(G(\zeta_i(x))\big) \left(\prod_{p \in \Gamma_j} x_p \right)^{-1}. \end{aligned}$$

Further we have

$$\begin{aligned} &|(Tz)(x) - (Tw)(x)| \\ &\le \frac{1}{\lambda} \|z - w\|_\lambda \sum_{i=0}^{+\infty} l_i(x) \sum_{j=1}^{m} K_j(\zeta_i(x)) L_{p_j}\big(G(\zeta_i(x))\big) \left(\prod_{p \in \Gamma_j} x_p \right)^{-1} \exp\left(\lambda \sum_{p=1}^{n} x_p \right) \\ &\le \frac{\Lambda}{\lambda} \exp\left(\lambda \sum_{p=1}^{n} x_p \right) \|z - w\|_\lambda, \end{aligned}$$

and cosequently $\|Tz - Tw\|_\lambda \le \frac{\Lambda}{\lambda} \|z - w\|_\lambda$. From the Banach theorem we get that for $\lambda > \Lambda$ equation (3) has a unique solution $\bar{z} \in P(G, \mathbb{R}_+)$.

Now we show that $\bar{z} \in C(G, \mathbb{R}_+)$ and is nondecreasing. Indeed $\bar{z}(x) = \lim_{n \to +\infty} z_n(x)$, $x \in G$, where $z_0 \in P(G, \mathbb{R}_+)$, $z_0(x) =$const, and $z_{n+1}(x) = (Tz_n)(x) = \bar{m}(x) + (Vz_n)(x)$, $x \in G$, $n \in N$. The function $z_0 \in C(G, \mathbb{R}_+)$ is nondecreasing. Therefore we easily get that $z_n \in C(G, \mathbb{R}_+)$ for $n \in N$, and functions z_n are nondecreasing for $n \in N$. The point (I) is proved.

We prove point (II). Indeed, for $\tilde{z} \in C(G, \mathbb{R}_+)$ and $\bar{h} = \tilde{h}$ we get from (3) the following:

$$\begin{aligned} l(x)\tilde{z}(\zeta_i(x)) &= l_i(x) \sum_{j=0}^{+\infty} l_j(\zeta_i(x))\tilde{h}(\zeta_{i+j}(x)) \\ &\quad + l_i(x) \sum_{j=0}^{+\infty} l_j(\zeta_i(x)) \Big(K(\zeta_{i+j}(x)) \int_{H(\zeta_{i+j}(x))} \tilde{z}(\gamma(s))\, ds \Big) \\ &= \sum_{j=i}^{+\infty} l_j(x)\tilde{h}(\zeta_j(x)) + \sum_{j=i}^{+\infty} l_j(x) \Big(K(\zeta_j(x)) \int_{H(\zeta_j(x))} \tilde{z}(\gamma(s))\, ds \Big), \end{aligned}$$

where $x \in G$, $i \in N$.

Now we show that an arbitrary solution of (3) denoted by $\bar{z}$ is a solution of equation (4). If $\bar{z}$ is a solution of (3), then we have

$$\begin{aligned} &\bar{z}(x) - K(x) \int_{H(x)} \bar{z}(\gamma(s))ds - l(x)\bar{z}(\zeta(x)) \\ &= \sum_{i=0}^{+\infty} l_i(x)\bar{h}(\zeta_i(x)) + \sum_{i=0}^{+\infty} l_i(x) \Big(K(\zeta_i(x)) \int_{H(\zeta_i(x))} \bar{z}(\gamma(s))\, ds \Big) \\ &\quad - K(x) \int_{H(x)} \bar{z}(\gamma(s))ds - l(x) \sum_{i=0}^{+\infty} l_i(\zeta(x))\bar{h}\big(\zeta_i(\zeta(x))\big) \\ &\quad + \sum_{i=0}^{+\infty} l_i(\zeta(x)) \Big(K\big(\zeta_i(\zeta(x))\big) \int_{H(\zeta_i(\zeta(x)))} \bar{z}(\gamma(s))\, ds \Big) \\ &= \bar{h}(x). \end{aligned}$$

Now we prove that $\bar{z}$ is a unique solution of (4) in the class $P(G, \mathbb{R}_+, \tilde{g})$. Let $z \in P(G, \mathbb{R}_+, \tilde{g})$ be an arbitrary solution of (4). Then

$$\begin{aligned} z(x) &= \sum_{i=0}^{n-1} l_i(x) K(\zeta_i(x)) \int_{H(\zeta_i(x))} z(\gamma(s))ds \\ &\quad + \sum_{i=0}^{n-1} l_i(x)\bar{h}(\zeta_i(x)) + l_i(x) z(\zeta_i(x)), \quad x \in G,\, n \in N. \end{aligned} \tag{6}$$

Because $0 \le z(x) \le \kappa \tilde{z}(x)$ for a certain $\kappa \in \mathbb{R}_+$, then $\lim_{i\to+\infty} l_i(x) z(\zeta_i(x)) = 0$ uniformly on G. If in (6) $n \to +\infty$, then $z(x) = \bar{m}(x) + (Vz)(x)$ for $x \in G$, and it means that it is a solution of (3). Because equation (3) has the only one solution, then $z(x) = \bar{z}(x)$. The proof of Lemma 2.1 is finished. $\square$

In the space $C(B,E)$ we define the norm

$$\|u\|_0 = \sup_{\tau \in B} \|u(\tau)\|, \quad u \in C(B,E).$$

Assumption H_3. Suppose that there exist nondecreasing functions $\bar{p}_i$, $\bar{k}_i$, $\bar{l} \in C(G,\mathbb{R}_+)$, η, $\xi_i \in C(G,G)$, such that

$$\begin{aligned}
\|f_i(x,t,w) - f_i(x,t,\bar{w}\| &\le \bar{p}_i(x)\|w-\bar{w}\|_0, \quad i=1,\dots,m \\
\|F(x,v,w) - F(x,\bar{v},\bar{w}\| &\le \sum_{i=1}^{m} \bar{k}_i(x)\|v_i - \bar{v}_i\| + \bar{l}(x)\|w-\bar{w}\|_0 \\
\|\varphi(\tau)\| &\le \bar{g}(0) \quad \text{for } \tau \in B \\
\|\Theta(x,w)\| &\le \eta(x) \\
\|\Psi_i(x,w)\| &\le \xi_i(x) \quad \text{for } i=1,\dots,m,
\end{aligned}$$

where t, $x \in G$, v, $\bar{v} \in E^m$, w, $\bar{w} \in C(B,E)$, and for $x \in G$ we have $\eta(x) \le x$, $\xi_i(x) \le x$.

Remark 2. The consequence of Assumption H_2 is the fact that there exist functions δ_i, $\Delta : G \to \mathbb{R}_+$, $i=1,\dots,m$, such that

$$\begin{aligned}
\|f_i(x,t,w)\| &\le \bar{p}_i(x)\|w\|_0 + \delta_i(x), \quad i=1,\dots,m \\
\|F(x,v,w)\| &\le \sum_{i=1}^{m} \bar{k}_i(x)\|v_i\| + \bar{l}(x)\|w\|_0 + \Delta(x),
\end{aligned}$$

where t, $x \in G$, $\|w\|_0 \le \bar{g}(a)$, $\|v_i\|_0 \le \bar{p}_i(a)\bar{g}(a)L_{p_i}(G_i(a))$, and

$$\delta_i(x) = \max_{s\in[0,x]} \max_{t \in G} \|f_i(s,t,\theta)\|, \quad \Delta(x) = \max_{s \in [0,x]} \|F(s,\theta,\theta)\|.$$

θ means the zero in the space $C(B,E)$.

Lemma 2.2. *Suppose that the assumptions of Lemma* 2.1 *are satisfied with functions* $\gamma_i(s) = \xi_i(x)$, $i=1,\dots,m$, $\zeta(x) = \eta(x)$, $K(x) = \sum_{i=1}^m \bar{k}_i(x)\bar{p}_i(x)$, $\tilde{h}(x) = \Delta(x) + \sum_{i=1}^m \bar{k}_i(x)\delta_i(x)L_{p_i}(G_i(x))$, $l(x) = \bar{l}(x)$, *and Assumption* H_3 *holds. Then*

$$\mathcal{F} : B(\Omega,E,\bar{g}) \to B(\Omega,E,\bar{g}),$$

where $B(\Omega,E,\bar{g}) = \{u \in C(\Omega,E) : u|_B = \varphi, \|u(s)\| \le \bar{g}(t), s \in [-r,t], t \in G\}$, *and* $\mathcal{F}$ *is defined by right side of equation* (1).

Proof. Let $w \in B(\Omega, E, \bar{g})$. Then for $x \in G$ we have

$$\begin{aligned}
\|\mathcal{F}[u](x)\| &\le \sum_{i=1}^{m} \bar{k}_i(x) \int_{H_i(x)} \bar{p}_i(x) \|u_{\Psi_i(s,u_{\alpha_i(s)})}\|_0 (ds)_{p_i} \\
&\quad + \bar{l}(x)\bar{g}(\eta(x)) + \Delta(x) + \sum_{i=1}^{m} \bar{k}_i(x)\delta_i(x) L_{p_i}(G_i(x)) \\
&\le \Big[\sum_{i=1}^{m} \bar{k}_i(x)\bar{p}_i(x) \Big] \int_{H(x)} \bar{g}(\xi(s))ds + \bar{l}(x)\bar{g}(\eta(x)) \\
&\quad + \Delta(x) + \sum_{i=1}^{m} \bar{k}_i(x)\delta_i(x) L_{p_i}(G_i(x)) \\
&= \bar{g}(x).
\end{aligned}$$

Therefore $\|\mathcal{F}[u](x)\| \le \bar{g}(x)$ for $x \in G$. Hence it follows that $\mathcal{F}[u] \in B(\Omega, E, \bar{g})$. The lemma is proved. □

Assumption H_4. Suppose that there exist nondecreasing functions $\rho : G \to \mathbb{R}_+$, $\mu : G \to \mathbb{R}_+$, and constants d, q_i, ν, $\sigma \in \mathbb{R}_+$, such that

(i) $\|\Theta(x,w) - \Theta(x,\bar{w})\| \le \rho(x)\|w - \bar{w}\|_0$
(ii) $\|\Theta(x,w) - \Theta(\bar{x},w)\| \le d|x - \bar{x}|$
(iii) $\|f_i(x,t,w) - f_i(\bar{x},t,w)\| \le q_i|x - \bar{x}|$
(iv) $\|F_i(x,v,w) - F_i(\bar{x},v,w)\| \le \nu|x - \bar{x}|$
(v) $|\beta(x) - \beta(\bar{x})| \le \sigma|x - \bar{x}|$
(vi) $\|\Psi_i(x,w) - \Psi_i(x,\bar{w})\| \le \mu_i(x)\|w - \bar{w}\|_0$,

where (x,w), $(x,\bar{w}) \in G \times C(B,E)$.

We define functions M_1, M_2, $M_3 \in C(G, \mathbb{R}_+)$ as follows:

$$M_1(x) = \omega\Big[\bar{g}(x)\sum_{i=1}^{m} p_i(x) + \sum_{i=1}^{m}\delta_i(x)\Big] + \sum_{i=1}^{m} q_i L_{p_i}(G_i(x)) + \nu \tag{7}$$

$$M_2(x) = d\bar{l}(x) \tag{8}$$

$$M_3(x) = \sigma\bar{l}(x)\rho(x). \tag{9}$$

Suppose that $M_2(a) < 1$ and $[M_2(a) - 1]^2 - 4M_1(a)M_3(a) > 0$. Let λ_1, λ_2 be two different positive roots of the equation $M_3(a)\lambda^2 + [M_2(a) - 1]\lambda + M_1(a) = 0$. Now we define the following class of functions:

$$D([-r,a], E, \lambda) = \big\{u \in B([-r,a], E, \bar{g}) : \|u(x) - u(\bar{x})\| \le \lambda|x - \bar{x}|,\ x, \bar{x} \in G\big\},$$

where $\lambda \in [\lambda_1, \lambda_2]$, if $M_3(a) \ne 0$, and $\lambda \ge M_1(a)[1 - M_2(a)]^{-1}$, if $M_3(a) = 0$.

Lemma 2.3. *Suppose that the assumptions of Lemma* 2.2 *and Assumptions* (i)–(v) *of* H_4 *are satisfied and* $M_2(a) < 1$, $[M_2(a)-1]^2 - 4M_1(a)M_3(a) > 0$, *where the functions* M_1, M_2, M_3 *are defined by* (7)–(9). *Then* $\mathcal{F} : D([-r,a],E,\lambda) \to D([-r,a],E,\lambda)$.

Proof. From Assumptions H_3 and H_4 we have

$$\begin{aligned}
\|\mathcal{F}[u](x) - \mathcal{F}[u](\bar{x})\| &\le \nu|x-\bar{x}| + \bar{l}(x)\lambda\big[d|x-\bar{x}| + \rho(x)\|u_{\beta(x)} - u_{\beta(\bar{x})}\|_0\big] \\
&\quad + \sum_{i=1}^{m}\big[\omega(p_i(x)\bar{g}(\zeta_i(x)) + \delta_i(x)) + q_i L_{p_i}(G_i(x))\big]|x-\bar{x}| \\
&\le \nu|x-\bar{x}| + \lambda\bar{l}(x)[d|x-\bar{x}| + \rho(x)\lambda|\beta(x)-\beta(\bar{x})|] \\
&\quad + \sum_{i=1}^{m}\big[\omega p_i(x)\bar{g}(x) + \omega\delta_i(x) + q_i L_{p_i}(G_i(x))\big]|x-\bar{x}| \\
&\le \Bigg\{[\sigma\bar{l}(x)\rho(x)]\lambda^2 + [d\bar{l}(x)]\lambda + \Bigg[\nu + \bar{g}(x)\omega\sum_{i=1}^{m}p_i(x) \\
&\quad + \omega\sum_{i=1}^{m}\delta_i(x) + \sum_{i=1}^{m}\delta_i L_{p_i}(G_i(x))\Bigg]\Bigg\}|x-\bar{x}| \\
&\le \lambda|x-\bar{x}|,
\end{aligned}$$

where $x,\ \bar{x} \in G$. This means that $\mathcal{F} : D([-r,a],E,\lambda) \to D([-r,a],E,\lambda)$. The lemma is proved. □

3. The main theorem.

For $u \in D([-r,a],E,\lambda)$, where λ is defined in Lemma 2.3, we define the norm

$$\|u\|_x = \sup_{s\in[-r,x]}\|u(s)\|.$$

Theorem 3.1. *Let the assumptions of Lemma* 2.3 *and Assumption* (vi) *of* H_4 *hold. Then the Cauchy problem* (1), (2) *has the unique solution in the class* $D([-r,a],E,\lambda)$.

Proof. Because of Assumptions H_3, H_4, H_5 for $u,\ \bar{u} \in D([-r,a],E,\lambda)$ we have

$$\begin{aligned}
\|\mathcal{F}[u](x) - \mathcal{F}[\bar{u}](x)\| \le \sum_{i=1}^{m}\bar{k}_i(x)p_i(x)(1+\lambda\mu_i(x))\int_{H_i(x)}\|w-\bar{w}\|_{\bar{\alpha}_i(s)}(ds)_{p_i} \\
+ \bar{l}(x)(1+\lambda\rho(x))\|u-\bar{u}\|_{\bar{\beta}(x)},
\end{aligned}$$

where $x \in G$, and $\bar{\alpha}_i(s) = \max\{\zeta_i(s), \alpha_i(s)\}$, $\bar{\beta}(x) = \max\{\eta(x), \beta(x)\}$.

Let $\tilde{z} \in C(G, E)$ be a solution of equation (4) with $\bar{h} = \tilde{h}$. It is easily seen that $\tilde{z}(x) \geq \tilde{h}(x)$ for $x \in G$. If $\tilde{h}(x) > 0$ then $\tilde{z} > 0$. Suppose that $\bar{z}$ is any positive and nondecreasing extension of $\tilde{z}$ onto the set Ω. For $u \in D(\Omega, E, \lambda)$ we define the norm

$$\|u\|_\star = \max_{x\in G} \frac{1}{\bar{z}(x)}\|u\|_x.$$

We get

$$\begin{aligned}\|\mathcal{F}[u](x) - \mathcal{F}[\bar{u}](x)\| \leq \sum_{i=1}^{m} \bar{k}_i(x)p_i(x)[1+\lambda\mu_i(x)] \int_{H_i(x)} \|\bar{u} - u\|_{\bar{\alpha}_i(s)}(ds)_{p_i} \\ + \bar{l}(x)[1+\lambda\rho(x)]\|u-\bar{u}\|_{\bar{\beta}(x)}.\end{aligned}$$

Note that for $\tau \in B$ and $s \in G$ we have

$$\begin{aligned}&\|u(\bar{\alpha}_i(s)+\tau) - \bar{u}(\bar{\alpha}(s)+\tau)\| \\ &\leq \left[\frac{1}{\bar{z}(\bar{\alpha}_i(s)+\tau)}\|u(\bar{\alpha}_i(s)+\tau) - \bar{u}(\bar{\alpha}_i(s)+\tau)\|\right]\bar{z}(\bar{\alpha}_i(s)+\tau) \\ &\leq \|u-\bar{u}\|_\star \bar{z}(\bar{\alpha}_i(s)).\end{aligned}$$

Analogously we have a such estimation for $\|u(\bar{\beta}(x)+\tau) - \bar{u}(\bar{\beta}(x)+\tau)\|$. Therefore $\|u-\bar{u}\|_{\bar{\alpha}_i(s)} \leq \|u-\bar{u}\|_\star \bar{z}(\bar{\alpha}_i(s))$ and $\|u-\bar{u}\|_{\bar{\beta}(x)} \leq \|u-\bar{u}\|_\star \bar{z}(\bar{\beta}(x))$. Now we get

$$\begin{aligned}&\|\mathcal{F}[u](x) - \mathcal{F}[\bar{u}](x)\| \\ &\leq \left\{\sum_{i=1}^{m} \bar{k}_i(x)p_i(x)[1+\lambda\mu_i(x)] \int_{H(x)} \bar{z}(\bar{\alpha}(s))ds + \bar{l}(x)[1+\lambda\rho(x)]\bar{z}(\bar{\beta}(x))\right\}\|u-\bar{u}\|_\star \\ &\leq (\bar{z}(x) - \tilde{h}(x))\|u-\bar{u}\|_\star,\end{aligned}$$

where $x \in G$, and finally

$$\|\mathcal{F}[u](x) - \mathcal{F}[\bar{u}](x)\|_\star \leq \left(1 - \inf\frac{\tilde{h}(x)}{\bar{z}(x)}\right)\|u-\bar{u}\|_\star.$$

Thus by the Banach fixed point theorem the problem (1), (2) has a unique solution in the class $D([-r,a], E, \lambda])$, where λ is defined in Lemma 2.2. The main theorem is proved. $\square$

4. Some effective conditions

Now we give some examples of effective conditions for Assumptions (iii)–(v) of H_2 to be satisfied (see [18]).

Example 1. Suppose that there exist

(a) $\bar{K}_i$, $\bar{l}$, $\bar{\zeta}_j \in \mathbb{R}_+$, $i = 1, \dots, m$, $j = 1, \dots n$, such that

(i) $l(x) \le \bar{l}$, $K_i(x) \le \bar{K}_i$, $i = 1, \dots, m$

(ii) $\zeta_j(x) \le \bar{\zeta}_j x_j$, $\bar{\zeta} \le 1$, $j = 1, \dots, n$

(iii) $\bar{l} \prod_{s \in \bar{\Gamma}_i} \bar{\zeta}_s < 1$ for $i = 1, \dots, m$

(iv) $\sum_{N=0}^{+\infty} \bar{l}^N \tilde{h}\big(\bar{\zeta}_1^N x_1, \dots, \bar{\zeta}_n^N x_n\big) < +\infty$;

(b) $\bar{\gamma}_{k_j}^{(i)} \in \mathbb{R}_+$, such that $\phi_{k_j}^{(i)} \le \bar{\gamma}_{k_j}^{(i)} x_{k_j}$, and $\bar{\gamma}_{k_j}^{(i)} \le 1$, where $k_j \in \Gamma_i$.

Then the conditions (iii)–(v) of Assumption H_2 are satisfied.

Example 2. Suppose that

(a) there exist $\bar{l}$ and $\bar{K}_j \in \mathbb{R}_+$, $j = 1, \dots, n$, such that $l(x) \le \bar{l}$, $K_i(x) \le \sum_{j=1}^n \bar{K}_j x_j$, $i = 1, \dots, m$;

(b) conditions (ii), (iv) of Assumption (a) and Assumption (b) of Example 1 are satisfied.

Then the conditions (iii)–(v) of Assumption H_2 are fulfilled.

Example 3. Suppose that

(a) $G = [0, \bar{a}]$, $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$, $\bar{a}_j > 0$, $j = 1, \dots, n$;

(b) condition (ii) of Assumption (a), and Assumption (b) of Example 1 are satisfied;

(c) there exist $\bar{l} = (\bar{l}_1, \dots, \bar{l}_n)$, $\bar{K} = (\bar{K}_1, \dots, \bar{K}_m)$, such that $\bar{l}_j$, $\bar{K}_i \in \mathbb{R}_+$, $j = 1, \dots, n$, $i = 1, \dots, m$, and $l(x) \le \sum_{j=1}^n \bar{l}_j x_j$, $K_i(x) \le \bar{K}_i$, and the condition $\sum_{j=1}^n \bar{l}_j \bar{\zeta}_j \bar{a}_j < 1$ holds.

Then the conditions (iii)–(v) of Assumption H_2 are fulfilled.

Example 4. Suppose that

(a) conditions (ii), (iv) of Assumption (a) of Example 1 are satisfied;

(b) there exist $\bar{l}$, $\bar{K} = (\bar{K}_1, \dots, \bar{K}_n)$, such that $\bar{l}$, $\bar{K}_i \in \mathbb{R}_+$, $i = 1, \dots, n$, and $l(x) \le \bar{l}$, $K_j(x) \le \sum_{i=1}^n \bar{K}_i x_i$, $j = 1, \dots, m$, and $\bar{l}\bar{\zeta}_i (\prod_{s \in \bar{\Gamma}_j} \bar{\zeta}_s)^2 \le 1$, $i = 1, \dots, n$, $j = 1, \dots, m$,

(c) $\phi_i(x) \le (\bar{\gamma}_{k_1}^{(i)} x_{k_1}^2, \dots, \bar{\gamma}_{k_{p_i}}^{(i)} x_{k_{p_i}}^2)$, $k_j \in \bar{\Gamma}_i$, $\bar{\gamma}_{k_s}^{(i)} \in \mathbb{R}_+$.

Then the conditions (iii)–(v) of Assumption H_2 are fulfilled.

Example 5. Suppose that

(a) there exist $G = [0, \bar{a}]$, $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$, $0 < \bar{a}_i \leq 1$, $i = 1, \ldots n$, such that $\prod_{s \in \bar{\Gamma}_j} \bar{a}_s^2 < 1$, $j = 1, \ldots, m$;

(b) $\zeta(x) = (\zeta_1(x), \ldots, \zeta_n(x)) \leq (x_1^2, \ldots, x_n^2)$;

(c) condition (i) of Assumption (a), and Assumption (b) of Example 1 are fulfilled, and the condition $\sum_{N=0}^{+\infty} \bar{l}^N \tilde{h}(x_1^{2^N}, \ldots, x_n^{2^N}) < +\infty$ holds.

Then the conditions (iii)–(v) of Assumption H_2 are satisfied.

Example 6. Suppose that

(a) there exist $\bar{H}$, $P \in \mathbb{R}_+$, such that $\tilde{h}(x) = \tilde{h}(x_1, \ldots, x_n) \leq \bar{H}\big(\prod_{i=1}^n x_i\big)^P$;

(b) conditions (i), (ii) of Assumption (a) and Assumption (b) of Example 1 are fulfilled, and $\big(\prod_{s \in \bar{\Gamma}_j} \bar{\zeta}_s\big)^\nu \bar{l} \leq 1$. where $\nu = \min[1, P]$.

Then the conditions (iii)–(v) of Assumption H_2 are satisfied.

References

[1] Angelov, V. G. and Baĭnov, D. D., Existence and uniqueness of the global solution of the initial value problem for neutral type differential-functional equations in Banach space. *Nonlin. Anal.* 4 (1980), 93 – 107.

[2] Człapiński, T., On some nonlinear Volterra integral-functional equations in several variables. *Anal. Math.* 20 (1994), 241 – 253.

[3] Gołaszewska, A., Existence and uniqueness for neutral equations with delay dependent on a solution and its derivative. *Int. J. Pure Appl. Math.* 29 (2006)(3), 349 – 384.

[4] Gołaszewska, A., Generalized solutions of mixed problems for first order partial differential equations with state dependent delays. *Int. J. Pure Appl. Math.* 29 (2006)(3), 349 – 384.

[5] Gołaszewska, A. and Turo, J., Existence and uniqueness for neutral equations with state dependent delays. *Qual. Theory Diff. Equ. Appl.* (to appear).

[6] Gołaszewska, A. and Turo, J., Caratheodory solutions to quasi-linear hyperbolic systems of partial differential equations with state dependent delays. *Funct. Diff. Equ.* 14 (2007)(2–4), 257 – 278.

[7] Hale, J. K. and Cruz, M. A., Existence, uniqueness, and continuous dependence for hereditary systems. *Ann. Math. Pura. Appl.* 85 (1970), 63 – 81.

[8] Hartung, F., Herdman, T. L. and Turi, J., On existence, uniqueness and numerical approximation for neutral equations with state-dependent delays. *Appl. Numer. Math.* 24 (1997), 393 – 409.

[9] Hartung, F., Herdman, T. L. and Turi, J., Parameter indentification in classes of neutral differential equations with state-dependent delays. *Nonlin. Anal.* 39 (2000), 305 – 325.

[10] Hartung, F., Krisztin, T., Walther, H.-O. and Wu, J., Functional differential equations with state-dependent delay: theory and applications. In: *Handbook of Differential Equations:* Ordinary Differential Equations. Vol. III. Amsterdam: Elsevier/North-Holland 2006, pp. 435 – 545.

[11] Jackiewicz, Z., Existence and uniqueness of solutions of neutral delay-differential equations with state dependent delays. *Funkcial. Ekvac.* 30 (1987), 9 – 17.

[12] Jackiewicz, Z., A note on the existence and uniqueness of solutions of neutral functional-differential equations with state dependent delays. *Comment. Math. Univ. Carolin.* 36 (1995), 15 – 17.

[13] Jankowski, T. and Kwapisz, M., On the existence and uniqueness of solutions of systems of differential equations with a deviated argument. *Ann. Polon. Math.* 26 (1972), 253 – 277.

[14] Kamont, Z., *Hyperbolic Functional Differential Inequalities and Applications.* Dordrecht: Kluwer 1999.

[15] Kamont, Z., Initial problems for neutral functional differential equations with unbounded delay. *Studia Sci. Math. Hungar.* 40 (2003), 309 – 326.

[16] Kamont, Z., First order partial functional differential equations with state dependent delays. *Nonlinear Stud.* 12 (2005), 135 – 157.

[17] Kamont, Z. and Kwapisz, M., On the Cauchy problem for differential-delay equations in a Banach space. *Math. Nachr.* 74 (1976), 173 – 190.

[18] Kamont, Z. and Kwapisz, M., On nonlinear Volterra integral-functional equations in several variables. *Ann. Polon. Math.* 40 (1981), 1 – 29.

[19] Kamont, Z. and Turo, J., Caratheodory solutions to hyperbolic functional differential systems with state dependent delays. *Rocky Mountain J. Math.* 35 (2005), 1935 – 1952.

[20] Konstantinow, M. M. and Bainow, D. D., On the existence of solutions of systems of functional equations over neutral type with the iterative delay (in Russian). *Diff. Uravnenija* 10 (1974), 1988 – 1992.

[21] Kwapisz, M., On the existence and uniqueness of solutions of a certain integral-functional equations. *Ann. Polon. Math.* 31 (1975), 23 – 41.

[22] Kwapisz, M. and Turo, J., On some integral-functional equations with an unknown function of n variables. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 22 (1974), 1001 – 1009.

[23] Kwapisz, M. and Turo, J., On the existence and uniqueness of solutions of the Darboux problem for partial differential-functional equations. *Colloq. Math.* 29 (1974), 279 – 302.

[24] Kwapisz, M. and Turo, J., On the existence and uniqueness of solutions of the Darboux problem for partial differential-functional equations in a Banach space. *Ann. Polon. Math.* 29 (1974), 89 – 118.

[25] Kwapisz, M. and Turo, J., Existence and uniqueness of solutions for some integral-functional equation. *Comment. Math. Prace Mat.* 23 (1983), 259 – 267.

[26] Kwapisz, M. and Turo, J., On the existence and uniqueness of integrable solutions of functional equations in a Banach space. *Aequationes Math.* 19 (1979), 53 – 65.

[27] Kwapisz, M. and Turo, J., Existence, uniqueness and succesive approximations for a class of integral-functional equations. *Aequationes Math.* 14 (1976), 303 – 323.

[28] Kwapisz, M. and Turo, J., Some integral-functional equations. *Funkcial. Ekvac.* 18 (1975), 107 – 162.

[29] Kwapisz, M. and Turo, J., On the existence and convergence of succesive approximations for some functional equations in a Banach space. *J. Diff. Equ.* 16 (1973), 289 – 318.

[30] Pavel, P., Sur une équation aux dérivées partielles (in French). *Seminar on Differential Equations.* Preprint 88-8. Cluj-Napoca: Univ. "Babes-Bolyai" 1988, pp. 71 – 86

[31] Rzepecki, B., On hyperbolic partial differential equations in Banach spaces. *Atti Acad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei* (9) *Mat. Appl.* 80 (1986), 540 – 544.

[32] Walter, W., On nonlinear Volterra integral equations in several variables. *J. Math. Mech.* 16 (1967)(9), 967 – 985.

[33] Walter, W., *Differential and Integral Inequalities.* Berlin: Springer 1970.

[34] Żywotowskij, Ł. A., Existence and uniqueness of solutions for differential neutral equations with delay dependent on a solution and its derivative. *Diff. Equ.* 5 (1971), 1377 – 1384.

Received June 27, 2007; revised June 5, 2008