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Some Comparison Theorems on Eigenvalues for One Dimensional Schrödinger Operators with Applications

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Abstract. For one dimensional Schrödinger operators, in terms of local behavior of the potential, we give some comparison theorems for their eigenvalues. As an application, some monotone properties of the first eigenvalue of stepfunction potentials are given.

Keywords. One dimensional Schrödinger operator, comparison theorems on eigenvalues, stepfunction potentials, monotonicity, variational methods

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1. Introduction

Consider the Dirichlet problem for the one-dimensional Schrödinger operator on $(-L, L)$:

$$u'' - V(x)u + \lambda u = 0, \quad x \in (-L, L) \tag{1.1}$$

$$u(-L) = u(L) = 0, \tag{1.2}$$

with $V(x) \geq 0$, $L > 0$. Denote the n-th eigenvalue of (1.1) by $\lambda_n(V)$, the corresponding eigenfunction by $\phi_{n,V}$. It is well known (see [4]) that $\lambda_1(V) > 0$ and $\phi_{1,V}$ does not change the sign. We always assume that $\phi_{1,V} > 0$ in $(-L, L)$. In this article, mainly we give some comparison results of $\lambda_1(V)$ which depend on local behavior of V (Theorems 2.3, 2.4). As an application, some monotone properties of $\lambda_1(V)$ for V being step functions are given (Theorems 3.1, 3.2, 3.4 and 3.6). Such potentials are interesting in the study of the tunneling effects of the valence band problems (see [3, 8]). It follows that, in Corollary 3.7, the infimum of $\lambda_1(V)$ for those V with fixed integral can be obtained, this was not known before.

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Properties of eigenpairs related to geometric features of V were investigated by many authors before. For instance, about the optimum problem: in [6], it is investigated where to place the support of a central nonnegative potential so to optimize the principal Dirichlet eigenvalue. In [10,11], some optimization results of the principal eigenvalue are given (see Theorem A). Between the optimum, we investigate how $\lambda_1(V)$ would vary with respect to, say the (maximal) height of V (in this article), or the location of the support of V (in [2]). It turns out that some comparison theorems and monotone properties are obtained. It is interesting to observe that, in Theorems 3.1 and 3.2, although those V being considered are looked alike, their monotonic behaviors of $\lambda_1(V)$ are reversed. From the proofs, this phenomenon can be explained as: the shape of $\phi_{1,V}$ is also mattered. This point of view was mentioned in [9], too. Another important physical quantity "spectral gap" is also related to the geometric features of V. In [5] it is proved that a double-well potential which is large on its support will produce small gaps $\lambda_{i+1} - \lambda_i$ for $i = 1, 2$. In this aspect, some estimates for $\lambda_2(V)$ as well as for the position of nodal points of $\phi_{2,V}$ with V being step functions are given in [2]. With the comparison results in this article and the method in [2, Section 3.4], some comparison results on $\lambda_2(V)$, accordingly $\lambda_2(V) - \lambda_1(V)$, may also be obtained.

The organization of this article is the following: in Section 2 we present the main comparison results. In Section 3, stepfunction potentials are treated as mentioned above. The main tool we use is the variational method.

2. Main results

In the following, $\operatorname{supp} V$ denotes the support of V. To make the idea clear, Propositions 2.1 and 2.2 are stated specifically. More general results are stated in Theorems 2.3, 2.4. Remark 2.5 indicates the intrinsic idea of these results.

Proposition 2.1. *Let $0 < b_2 < b_1 \le L$. Suppose V_1, V_2 are two potentials in (1.1),(1.2). If*

1. *$\phi_{1,V_1}(x)$ is strictly decreasing on $[-b_1, 0]$, $V_2 \neq V_1$ only in $[-b_1, 0]$, $\operatorname{supp} V_2 \cap [-b_1, 0] \subset [-b_2, 0]$ and $V_2(x) \le \left(\frac{b_1}{b_2}\right) V_1\left(\frac{b_1 x}{b_2}\right)$ for $x \in (-b_2, 0)$;*
   or
2. *$\phi_{1,V_1}(x)$ is strictly increasing on $[0, b_1]$, $V_2 \neq V_1$ only in $[0, b_1]$, $\operatorname{supp} V_2 \cap [0, b_1] \subset [0, b_2]$ and $V_2(x) \le \left(\frac{b_1}{b_2}\right) V_1\left(\frac{b_1 x}{b_2}\right)$ for $x \in (0, b_2)$.*

Then $\lambda_1(V_2) < \lambda_1(V_1)$.

Proof. We prove 1. only, 2. can be similarly proved.

Let $\xi_1(x) = \phi_{1,V_1}(\frac{b_1 x}{b_2})$, $x \in [-b_2, 0]$, then from the decreasing property of ϕ_{1,V_1},

$$\phi_{1,V_1}(x) < \xi_1(x) \quad \text{for } x \in (-b_2, 0). \tag{2.1}$$

From the inequality between V_1 and V_2, a direct computation shows

$$\int_{-b_2}^{0} V_2(x)\xi_1^2(x)dx \le \int_{-b_1}^{0} V_1(x)[\phi_{1,V_1}(x)]^2(x)dx. \tag{2.2}$$

Using ϕ_{1,V_1} as a trial function in the Rayleigh quotient for $\lambda_1(V_2)$ on $[-L, L]$, by (2.1) (2.2) we have

$$\begin{aligned}
&\lambda_1(V_2) \\
&\le \frac{\int_{-L}^{L}[\phi'_{1,V_1}(x)]^2dx + \int_{[-L,L]\setminus[-b_2,0]} V_2(x)[\phi_{1,V_1}(x)]^2dx + \int_{-b_2}^{0} V_2(x)[\phi_{1,V_1}(x)]^2dx}{\int_{-L}^{L}[\phi_{1,V_1}(x)]^2dx} \\
&< \frac{\int_{-L}^{L}[\phi'_{1,V_1}(x)]^2dx + \int_{[-L,L]\setminus[-b_1,0]} V_2(x)[\phi_{1,V_1}(x)]^2dx + \int_{-b_2}^{0} V_2(x)\xi_1^2(x)dx}{\int_{-L}^{L}[\phi_{1,V_1}(x)]^2dx} \\
&\le \frac{\int_{-L}^{L}[\phi'_{1,V_1}(x)]^2dx + \int_{[-L,L]\setminus[-b_1,0]} V_1(x)[\phi_{1,V_1}(x)]^2dx + \int_{-b_1}^{0} V_1(x)[\phi_{1,V_1}(x)]^2dx}{\int_{-L}^{L}[\phi_{1,V_1}(x)]^2dx} \\
&= \lambda_1(V_1)
\end{aligned}$$

The proof is complete. □

Similarly we have

Proposition 2.2. *Let* $0 < b_2 < b_1 \le L$. *Suppose* V_1, V_2 *are two potentials in* (1.1), (1.2). *If*

1. $\phi_{1,V_2}(x)$ *is strictly increasing in* $[-b_1, 0]$. $V_2 \ne V_1$ *only in* $[-b_1, 0]$ *with* $V_1(x) \le \left(\frac{b_2}{b_1}\right)V_2\left(\frac{b_2x}{b_1}\right)$ *for* $x \in (-b_1, 0)$;
   or
2. $\phi_{1,V_2}(x)$ *is strictly decreasing in* $[0, b_1]$. $V_1 \ne V_2$ *only in* $[0, b_1]$ *with* $V_1(x) \le \left(\frac{b_2}{b_1}\right)V_2\left(\frac{b_2x}{b_1}\right)$ *for* $x \in (0, b_1)$.

Then $\lambda_1(V_1) < \lambda_1(V_2)$.

We put Propositions 2.1, 2.2 together in Theorem 2.3, which will be used in Section 3. A general case is stated in Theorem 2.4.

Theorem 2.3. *Let* $[c, d] \subset (a, b) \subset (-L, L)$. *In* (1.1), (1.2), *suppose* $V_1 \ne V_2$ *only in* $[a, b]$.

1. *If* ϕ_{1,V_1} *has only one local minimum at* $x_m \in (a, b)$, $\operatorname{supp} V_2 \cap (a, b) \subset [c, d]$ *and*

$$V_2(x) \le \left(\frac{x_m - a}{x_m - c}\right)V_1\left(\frac{(x_m - a)(x - x_m)}{x_m - c} + x_m\right) \quad \text{for } x \in (c, x_m)$$

$$V_2(x) \le \left(\frac{b - x_m}{d - x_m}\right)V_1\left(\frac{(b - x_m)(x - x_m)}{d - x_m} + x_m\right) \quad \text{for } x \in (x_m, d),$$

   then $\lambda_1(V_2) < \lambda_1(V_1)$.

2. *If ϕ_{1,V_2} has only one local maximum at $x_M \in (c,d)$ and*

$$V_1(x) \le \left(\frac{x_M - c}{x_M - a}\right) V_2\left(\frac{(x_M - c)(x - x_M)}{x_M - a} + x_M\right) \quad \textit{for } x \in (a, x_M)$$
$$V_1(x) \le \left(\frac{d - x_M}{c - x_M}\right) V_2\left(\frac{(d - x_M)(x - x_M)}{b - x_M} + x_M\right) \quad \textit{for } x \in (x_M, b),$$

then $\lambda_1(V_1) < \lambda_1(V_2)$.

Proof. 1. Apply Proposition 2.1-1. on $[a, x_m]$ and Proposition 2.1-2. on $[x_m, b]$. 2. is similarly treated. □

Theorem 2.4. *Let $[c,d] \subset (a,b) \subset (-L,L)$. In* (1.1), (1.2), *suppose $V_1 \neq V_2$ only in $[a,b]$.*

1. *If ϕ_{1,V_1} has only one local minimum at $x_m \in (a,b)$, $\operatorname{supp} V_2 \cap (a,b) \subset [c,d]$. Let $w(x)$ be a strictly increasing and differentiable (except possibly at $x = x_m$) function from $[c,d]$ onto $[a,b]$ with $w(x_m) = x_m$. Suppose $V_2(x) \le w'(x) V_1(w(x))$ for $x \in [c,d] \backslash \{x_m\}$. Then $\lambda_1(V_2) < \lambda_1(V_1)$.*
2. *If ϕ_{1,V_1} has only one local maximum at $x_M \in (c,d)$. Let $w(x)$ be a strictly increasing and differentiable (except possibly at $x = x_M$) function from $[a,b]$ onto $[c,d]$ with $w(x_M) = x_M$. Suppose $V_1(x) \le w'(x) V_2(w(x))$ for $x \in [a,b] \backslash \{x_M\}$. Then $\lambda_1(V_1) < \lambda_1(V_2)$.*

Proof. It suffices to prove 1. for the case in Proposition 2.1-1., that is, suppose $a = -b_1, b = 0 = d = x_m$, $c = -b_2 > -b_1$.

Let $\xi_2(x) = \phi_{1,V_1}(w(x))$, $x \in [-b_2, 0]$, then from the decreasing property of ϕ_{1,V_1},

$$\phi_{1,V_1}(x) < \xi_2(x) \quad \text{for } x \in (-b_2, 0). \tag{2.3}$$

From the inequality between V_1 and V_2, we have

$$\int_{-b_2}^{0} V_2(x) \xi_2^2(x) dx \le \int_{-b_1}^{0} [V_1(x)] \phi_{1,V_1}^2(x) dx. \tag{2.4}$$

Using ϕ_{1,V_1} as a trial function in the Rayleigh quotient for $\lambda_1(V_2)$ on $[-L, L]$, by (2.3) and (2.4), along the same lines as in the proof of Proposition 2.1-1., we have $\lambda_1(V_2) < \lambda_1(V_1)$. □

Remark 2.5. 1. Observe that, for instance, in the background of Proposition 2.1-2., we have $\int_0^{b_1} V(x)dx = \int_0^{b_2} \frac{b_1}{b_2} V\left(\frac{b_1 x}{b_2}\right) dx$ for any V, b_1, b_2. So in all comparison results in this section, V is deformed under the condition that its "total quantity" is not increased. However, to lower the first eigenvalue, according to Theorems 2.3, 2.4, the way to deform V depends on the (local) shape of $\phi_{1,V}$.

2. In Theorem 2.4-1., from the proof we see that the pointwise inequality relating V_1, V_2 can be replaced by the weaker condition (2.4). So do Propositions 2.1, 2.2 and Theorem 2.3.

3. Applications - monotone behavior of the first eigenvalues for stepfunction potentials

In this section, first in Theorems A-C and 3.5, the shape of $\phi_{1,V}$ for V being stepfunctions are revealed. Then through Theorem 2.3, we get some monotone properties of $\lambda_1(V)$ in Theorems 3.1, 3.2, 3.4, 3.6.

It is known that if V is a step function, then $\phi'_{1,V}$ is absolutely continuous in $[-L, L]$ (see [4]). Some terminology goes ahead: for fixed positive real numbers A, L, M, denote

$$\mathcal{A} = \left\{ V \geq 0 \,\middle|\, \int_{-L}^{L} V(x)dx = A \right\}; \quad \mathcal{A}_M = \left\{ V \in \mathcal{A} \,\middle|\, \max_{x\in[-L,L]} V(x) = M \right\}$$

and $\mu := \frac{\pi+\sqrt{\pi^2+8LA}}{4L}$. The even functions $V_{\max}(x) \in \mathcal{A}$ and $W_M \in \mathcal{A}_M$ are defined by

$$V_{\max}(x) = \begin{cases} \mu^2 & \text{for } x \in [0, L - \frac{\pi}{2\mu}] \\ 0 & \text{for } x \in (L - \frac{\pi}{2\mu}, L]; \end{cases} \quad W_M(x) = \begin{cases} 0 & \text{for } x \in [0, L - \frac{A}{2M}) \\ M & \text{for } x \in [L - \frac{A}{2M}, L]. \end{cases}$$

Some known results are quoted below. We remark that in contrast to Theorem A-1., $\lambda_{\min} = \inf_{V\in\mathcal{A}} \lambda_1(V)$ will be got in Corollary 3.7, which is unknown before.

Theorem A ([10, Theorem 3], [11, Theorem 1]).

1. *The unique potential that attains* $\lambda_{\max} := \max_{V\in\mathcal{A}} \lambda_1(V) = \mu^2 = \lambda_1(V_{\max})$ *is* $V_{\max}$ *with*
$$\phi_{1,V_{\max}} = \begin{cases} c > 0 & \text{in } [0, L - \frac{\pi}{2\mu}] \\ c \cdot \sin(\mu(L - x)) & \text{in } [L - \frac{\pi}{2\mu}, L]. \end{cases}$$
2. *The (unique) potential in* (1.1),(1.2) *that attains* $\lambda_{M,\min} = \min_{V\in\mathcal{A}_{\mathcal{M}}} \lambda_1(V)$ *is* W_M.

Since some results in [2] will be used, we use notations as in [2]. For $c > 0$, let $cF^{b_0(c)} \in \mathcal{A}$ be the step function which is supported on $[b_0(c),\ |b_0(c)|] \subset [-L, L]$ and with constant height $c\mu^2$ there (in fact, $0 > b_0(c) = \frac{1}{c}(-L + \frac{\pi}{2\mu})$). We see that $V_{\max} \equiv F^{-L+\frac{\pi}{2\mu}}$ (since $b_0(1) = -L + \frac{\pi}{2\mu}$). Theorem B describes the shape of $\phi_{1,cF^{b_0(c)}}$:

for $c > 1$, $\phi_{1,cF^{b_0(c)}}$ has exactly one minimum on $\text{supp}(cF^{b_0(c)})$;
for $c \leq 1$, $\phi_{1,cF^{b_0(c)}}$ has exactly one maximum on $\text{supp}(cF^{b_0(c)})$.

Theorem B ([2, Theorems 1–3]).

1. *For* $c > 1$, $\phi_{1,cF^{b_0(c)}}$ *is even and has exactly two peaks. Let* $p \in [-L, 0]$ *be one of them, then* $p \in (-L + \frac{\pi}{2\mu}, b_0(c))$.
2. *For* $c \leq 1$, $\phi_{1,cF^{b_0(c)}}$ *is even and has only one peak at* $x = 0$.

Moving the support of $cF^{b_0(c)}$ toward L (moving toward $-L$ can be similarly treated), denote the moved function by cF^b such that it is supported on $[b,\bar{b}]$ with the height still being $c\mu^2$ (we remark that $b \in [\frac{1}{c}(-L+\frac{\pi}{2\mu}),\ L(1-\frac{2}{c})+\frac{\pi}{c\mu}]$; $\bar{b} = b + \frac{2}{c}(L-\frac{\pi}{2\mu})$). Theorem C describes the shape of ϕ_{1,cF^b}:

for $c = 1$, ϕ_{1,F^b} is decreasing in $[b,\bar{b}]$;
for $c \neq 1$, the shape of ϕ_{1,cF^b} depends.

Theorem C ([2, Theorems 1–3])**.**

1. ϕ_{1,F^b} *has exactly one peak which is located in* $(-L+\frac{\pi}{2\mu}, b)$.
2. *For* $c < 1$, ϕ_{1,cF^b} *has exactly one peak which is located in* $(-L+\frac{\pi}{2\mu}, 0)$.
3. *For* $c > 1$, *let* $b_1(c) = b_0(c) + (1-\frac{1}{c})(L-\frac{\pi}{2\mu})$ (*so* $\bar{b}_1(c) = L - \frac{\pi}{2\mu}$). Then:
   If $b \geq b_1(c)$, ϕ_{1,cF^b} *has exactly one peak which lies in* $(-L+\frac{\pi}{2\mu}, b)$.
   If $b \in (b_0(c), b_1(c))$, *either* ϕ_{1,cF^b} *has exactly two peaks* p *and* $-p$ *with* $p \in (-L+\frac{\pi}{2\mu}, b)$; *or* ϕ_{1,cF^b} *has only one peak which is located in* $(-L+\frac{\pi}{2\mu}, b)$.

A more general situation is the following:

Theorem D ([1, Theorem 2; Theorem 6 (i)])**.**

1. *In* (1.1) (1.2), *suppose that* $V \in \mathcal{A}_M$ *and* $V^{-1}(\{M\})$ *contains an interval* J *with positive length. If* $M > \mu^2$, *then* $\phi_{1,V}$ *is convex in* J.
2. *If* $\phi_{1,V}$ *is concave down in* $[-L, L]$, *so is* $\phi_{1,cV}$ *for all* $0 < c < 1$.

Apply the comparison results in section 2, we get some "reversed" monotone behaviors of $\lambda_1(V)$ in Theorems 3.1, 3.2.

Theorem 3.1. *For* $c > 1$, $\lambda_1(cF^{b_0(c)})$ *is decreasing with respect to* c.

Proof. Suppose $1 < c_1 < c_2$, due to $c_1|b_0(c_1)| = c_2|b_0(c_2)|$, we see that $|b_0(c_1)| > |b_0(c_2)|$. Moreover, on $[b_0(c_2), -b_0(c_2)]$, $\left[c_2F^{b_0(c_2)}(x)\right] = \frac{b_0(c_1)}{b_0(c_2)}\left[c_1F^{b_0(c_1)}\left(\frac{b_0(c_1)x}{b_0(c_2)}\right)\right]$. From Theorems B-1. and 2.3-1., we know that the conclusion has to be true. □

Similarly, from Theorems B-2. and 2.3-2., we have

Theorem 3.2. *For* $c < 1$, $\lambda_1(cF^{b_0(c)})$ *is increasing with respect to* c.

Remark 3.3. In Theorem A-1., we see that $\phi_{1,V_{\max}}$ is constant on $\operatorname{supp} V_{\max}$. Theorems 3.1, 3.2 present a different view from that of [10]: $\phi_{1,V_{\max}}$ must be constant on $\operatorname{supp} V_{\max}$, or $\lambda_1(V_{\max})$ will not attain the maximum of $\lambda_1(V)$ in $\mathcal{A}$.

Numerical simulation shows that $\lambda_1(cF^b)$ do not have a simple monotonic property (with respect to c) as $\lambda_1(cF^{b_0(c)})$ does. Still, some interesting results exist. With similar arguments as in Theorems 3.1, 3.2, by Theorem C, we have

Theorem 3.4. *It holds:*

1. *For $b > b_0(1)$ and $c < 1$, $\lambda_1(cF^b) < \lambda_1(F^b)$. That is, in $\mathcal{A}$, fix the the left boundary of the support, as the height of F^b (uniformly) decreases, the first eigenvalue decreases as well.*
2. *For $b > b_0(1)$ and $c > 1$, $\lambda_1(cF^{\tilde{b}}) < \lambda_1(F^b)$, where $cF^{\tilde{b}}$ is supported in $[\tilde{b}, \bar{b}]$. That is, in $\mathcal{A}$, fix the right boundary of the support, as the height of F^b (uniformly) increases, the first eigenvalue will decrease.*
3. *Fix $\bar{c} > 1$ and $b > b_1(\bar{c})$, then for $c < \bar{c}$, $\lambda_1(cF^b) < \lambda_1(\bar{c}F^b)$.*

Proof. 1. By Proposition 2.2-2. (with $V_2 \equiv F^b$) and Theorem C-1., we know that the result has to be true.

2. By Proposition 2.1-1. (with $V_1 \equiv F^b$) and Theorem C-1., we know that the result has to be true.

3. By Proposition 2.2-2. (with $V_2 \equiv \bar{c}F^b$) and Theorem C-3., we know that the result has to be true. □

Now for $c > 0$, we consider another subfamily of stepfunction potentials. Let $cW^{d(c)} \in \mathcal{A}$ be supported in $[-L, -d(c)] \cup [d(c), L]$ and with constant height $c\mu^2$ there (in fact, $d(c) = L - |b_0(c)| > 0$, and $cW^{d(c)}$ is varied from $cF^{b_0(c)}$ by splitting the support evenly toward L and $-L$ respectively). We describe the shape of $\phi_{1,cW^{d(c)}}$ below.

Theorem 3.5. *For $c > 0$, each $\phi_{1,cW^{d(c)}}$ is even and has only one peak at $x = 0$.*

Proof. The evenness of $\phi_{1,cW^{d(c)}}(x)$ is owing to the nondegeneracy of the first eigenvalue. Observe that $\phi''_{1,cW^{d(c)}}(x) = -\lambda_1(cW^{d(c)})\phi_{1,cW^{d(c)}}(x) < 0$ in $[0, d(c))$. Being a C^1 even function, we have $\phi'_{1,cW^{d(c)}}(0) = 0$, which implies $\phi'_{1,cW^{d(c)}}(x) < 0$ in $[0, d(c)]$. From the Hopf maximum principle, we know that $\phi'_{1,cW^{d(c)}}(L) < 0$. Since $\phi''_{1,cW^{d(c)}} \equiv \text{constant} \cdot \phi_{1,cW^{d(c)}}$ on $[d(c), L]$, due to $\phi'_{1,cW^{d(c)}}(d(c)) < 0$, we see that no matter which sign $\phi''_{1,cW^{d(c)}}(x)$ has in $[d(c), L]$, we always have $\phi'_{1,cW^{d(c)}}(x) < 0$ in $[d(c), L]$. This completes the proof. □

By Theorems 2.3 1., 3.5, with similar arguments as before, lead to

Theorem 3.6. *For $c > 0$, $\lambda_1(cW^{d(c)})$ is decreasing with respect to c.*

From Theorem 3.6 and Theorem A-2., we have

Corollary 3.7. $\inf_{V \in \mathcal{A}} \lambda_1(V) = \lim_{c \to \infty} \lambda_1(cW^{d(c)})$.

Proof. Let $c\mu^2 = M$, then $W_M \equiv cW^{d(c)}$ on $[-L, L]$. From Theorem A-2., W_M attains the minimum of $\lambda_1(V)$ for $V \in \mathcal{A}_M$. From Theorem 3.6, we know that $\lambda_1(M) = \lambda_1(cW^{d(c)})$ is decreasing with respect to c, accordingly M. Notice that $\mathcal{A} = \bigcup_{M>0} \mathcal{A}_M$, and the result follows. □

Remark 3.8. As in Remark 3.3, by the comparison theorems, we may get some control on the shape of eigenfunctions as well. For instance, suppose $V_2 \neq V_1$ only in $[-b_1, 0]$ with $\operatorname{supp} V_2 \cap [-b_1, 0] \subset [-b_2, 0], 0 < b_2 < b_1$. If we know $\lambda_1(V_2) \geq \lambda_1(V_1)$, then by Proposition 2.1-1., ϕ_{1,V_1} is not decreasing in $[-b_1, 0]$.

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