Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 29 (2010), 125–148
DOI: 10.4171/ZAA/1402

Representations of Relaxations of Linear-Quadratic Optimal Control Problems for Elliptic Systems

Uldis Raitums

Abstract. The paper considers optimal control problems of the type

$$\begin{cases} I(h,\, u) = \displaystyle\int_\Omega [\langle B(x)\nabla u,\, \nabla u\rangle + \langle g(x,\, h(x)),\, \nabla u\rangle + F(x,\, h(x))]dx \to \min \\ \operatorname{div}[A(x)\nabla u - f(x,\, h(x))] = 0 \quad \text{in } \Omega \\ u = (u_1, \dots,\, u_m) \in H_0^1(\Omega;\, \mathbb{R}^m),\ h \in \mathcal{M}, \end{cases}$$

where the set $\mathcal{M}$ of admissible controls h consists of all measurable vector-functions $h(\cdot)$ with values from a given compact set $M \subset \mathbb{R}^r$. The functions f and g are affine with respect to h, but the matrix B can be negatively definite. It is shown that the relaxation of such problems can be represented as a joint passage from M to its convex hull and from F to a new function $\mathcal{F}$, which is lower semicontinuous with respect to h.
Keywords. Optimal control, elliptic system, weakly discontinuous functional, relaxation

Mathematics Subject Classification (2000). 49J45, 49J20

1. Introduction

We consider the optimal control problem

$$\begin{cases} I(h,u) \\ = \displaystyle\int_\Omega \big[\langle B(x)\nabla u(x),\nabla u(x)\rangle + 2\langle \nabla u(x), g(x,h(x))\rangle + F(x,h(x))\big]dx \to \min \\ \operatorname{div}[A(x)\nabla u(x) - 2f(x,h(x))] = 0 \ \text{ in } \Omega \\ u = (u_1, \dots,\, u_m) \in H_0^1(\Omega;\, \mathbb{R}^m) \\ h \in \mathcal{M} = \big\{h \in L_2(\Omega;\mathbb{R}^r) \,\big|\, h(x) \in \mathbb{M}(x) \text{ a.e. } x \in \Omega\big\}, \end{cases} \tag{1}$$

U. Raitums: Institute of Mathematics and Computer Science, University of Latvia, Rainis b. 29, LV-1459, Riga, Latvia; uldis.raitums@mii.lu.lv

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain homeomorphic to the unit ball; A and B are given measurable $nm \times nm$-matrices, A is positively definite; $\mathbb{M} : \mathbb{R}^n \mapsto 2^{\mathbb{R}^r}$ is a piecewise constant multivalued mapping with nonempty bounded and closed values; $f : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{nm}$, $g : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{nm}$ are given Carathéodory functions, affine with respect to h.

The main features of the problem (1) are the following:

(i) the sets $\mathbb{M}(x)$ are not, in general, convex;

(ii) the matrix B can be neither positively nor negatively definite;

(iii) we consider elliptic systems;

(iv) although the state equation in (1) defines an affine control-to-state mapping $h \to u(h)$, the resulting functional $h \to I(h, u(h))$ is, as a rule, weakly discontinuous and is not, in general, weakly lower semicontinuous.

Our interest in the problem (1) with mentioned above features is caused by two reasons. The first reason is that the problem (1) is, more or less, the simplest basic problem with weakly discontinuous functionals that involve elliptic systems. For such problems sufficient relaxations are known (or can be easily derived from known results) for the scalar case (more precisely, for $m < n$) with B nonnegative, F and f affine with respect to h, and g not depending on h, where the passage from $\mathcal{M}$ to its closed convex hull $\overline{\mathrm{co}}\mathcal{M}$ gives the relaxation of (1), see, for instance, Raitums [6, p. 104, Theorem 2.1]. As far as we know, for the general case of (1) with $m \geq n \geq 2$ and $r \geq 2$ even the possible functional type of the relaxed problem is not known.

The second reason is that for standard optimal design problems their second order approximations (in the L_∞ norm) lead to problems of the type (1). For instance, for the optimal design problem

$$\begin{cases} J(A) = \displaystyle\int_\Omega \langle g(x), \nabla u(x) \rangle dx \to \min \\ \operatorname{div}[A(x)\nabla u(x) - f(x)] = 0 \text{ in } \Omega, \quad u \in H_0^1(\Omega; \mathbb{R}^m), A(x) \in \mathcal{A}, \end{cases}$$

where $\mathcal{A}$ is a given bounded and closed set of positively definite symmetric $mn \times mn$-matrices, there is

$$J(A_0 + \delta A) = J(A_0) - \int_\Omega \langle \delta A \nabla u, \nabla \psi \rangle \, dx + \int_\Omega \langle A_0 \nabla \delta u, \nabla \delta \psi \rangle \, dx + O(\|\delta A\|^3_{L_\infty}), \quad (2)$$

where u and ψ from $H_0^1(\Omega; \mathbb{R}^m)$ are solutions of the state equation and the conjugate equation with $A = A_0$, respectively,

$$\operatorname{div}[A_0(x)\nabla u - f(x)] = 0 \text{ in } \Omega, \quad \operatorname{div}[A_0(x)\nabla \psi - g(x)] = 0 \text{ in } \Omega,$$

and δu and $\delta \psi$ satisfy

$$\begin{aligned} &\operatorname{div}[A_0(x)\nabla \delta u + \delta A(x) \nabla u] = 0 \text{ in } \Omega, \quad \delta u \in H_0^1(\Omega; \mathbb{R}^m) \\ &\operatorname{div}[A_0(x)\nabla \delta \psi + \delta A(x) \nabla \psi] = 0 \text{ in } \Omega, \quad \delta \psi \in H_0^1(\Omega; \mathbb{R}^m). \end{aligned} \quad (3)$$

In the representation (2) together with the system (3) we are in the framework of the problem (1) (with A_0 and, consequently, u and ψ fixed) with $\delta A = h$ and B as the block-matrix $\frac{1}{2}\begin{pmatrix} 0 & A_0 \\ A_0 & 0 \end{pmatrix}$. Obviously, such matrix B is neither positively definite nor negatively definite.

Therefore, for this case we have the second order approximation of the initial optimal design problem (in a neighbourhood of A_0). In order to understand intrinsic properties of optimal design problems, especially for the case of systems, one has to understand basic properties of problems of the type (1).

The specific property of the problem (1) is that the mapping $h \to u(h)$ is affine. Therefore, it is very natural to consider the relaxation of (1) as a passage from $\mathcal{M}$ to its closed convex hull $\overline{\text{co}}\mathcal{M}$ and from the functional $h \to I(h, u(h))$ to its sequentially weakly lower semicontinuous envelope. Further, because the cost functional I is a quadratic polynomial (if $F = 0$) in variables (h, u), then the impact of the weak convergence of a sequence of controls $\{h_k\}$ to an element h_0 on the value of the cost functional results in a functional depending only on the difference $h_k - h_0$ and not depending on the state $u(h_0)$. That together with the local character of G-convergence, see, for instance, Zhikov et al. [11, p. 155], and results on the relaxation of similar to (1) (with $B = 0$) problems, see, for instance, Raitums [6, p. 192], indicate that this resulting functional must be an integral functional whose integrand can be obtained by means of cell problems, which do not involve directly the state. This way, the relaxation procedure for the problem (1) preserves the state equation and consists of the joint passage from the initial set of controls $\mathcal{M}$ to its closed convex hull $\overline{\text{co}}\mathcal{M}$ and from the function F to a new function $\tilde{\mathcal{F}}$ in the integrand of the functional I. More precisely, we have the following result.

Theorem 1.1. *Let the hypotheses* H1–H5 *from Section* 2 *hold. Let the function* $\tilde{\mathcal{F}} : \Omega \times \mathbb{R}^r \to \mathbb{R}$ *be defined by*

$$\tilde{\mathcal{F}}(x,\hat{h}) := \begin{cases} \inf \Big\{ \displaystyle\int_K \Big[\langle B(x)\nabla v(y), \nabla v(y)\rangle \\ \qquad + 2\langle \nabla v(y), g(x,h(y)) - g(x,\hat{h})\rangle + F(x,h(y))\Big] dy \;\Big|\; \\ \qquad \operatorname{div}_y[A(x)\nabla v(y) - 2(f(x,h(y)) - f(x,\hat{h}))] = 0 \;\; in\ K; \\ \qquad v \in H^1_{loc}(\mathbb{R}^n;\mathbb{R}^m),\ v\ is\ K\text{-}periodic,\ K := (0,1)^n; \\ \qquad h \in L_2(K;\mathbb{R}^r),\ h(y) \in \mathbb{M}(x)\ \ a.e.\ y \in K,\ \displaystyle\int_K h(y)dy = \hat{h} \Big\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad if\ \hat{h} \in \overline{\text{co}}\mathbb{M}(x) \\ \sup \big\{ \tilde{\mathcal{F}}(x_*, \hat{h}_*) \mid x_* \in \Omega,\ \hat{h}_* \in \overline{\text{co}}\mathbb{M}(x_*) \big\}, \qquad otherwise, \end{cases} \tag{4}$$

and let $u(h)$ *denote, for a chosen* $h \in \overline{\text{co}}\mathcal{M}$*, the solution of the equation*

$$\operatorname{div}[A(x)\nabla u(x) - 2f(x,h(x))] = 0 \;\; in\ \Omega,\ u \in H^1_0(\Omega;\mathrm{R}^m). \tag{5}$$

Then:

(i) *the function* $\tilde{\mathcal{F}}$ *is a normal integrand of* $\Omega \times \mathbb{R}^r$*;*

(ii) *the mapping* $h \to \tilde{I}(h, u(h))$*, where*

$$\begin{aligned}\tilde{I}(h,u) \\ := \int_\Omega \big[\langle B(x)\nabla u(x), \nabla u(x)\rangle + 2\langle \nabla u(x), g(x,h(x))\rangle + \tilde{\mathcal{F}}(x,h(x))\big]dx\end{aligned} \tag{6}$$

is sequentially weakly lower semicontinuous on $\overline{\text{co}}\mathcal{M}$*;*

(iii) *for all* $h \in \mathcal{M}$ *there is* $\tilde{I}(h, u(h)) \leq I(h, u(h))$ *and for every* $h_0 \in \overline{\text{co}}\mathcal{M}$ *there exists a sequence* $\{h_k\} \subset \mathcal{M}$ *such that* $h_k \rightharpoonup h_0$ *weakly as* $k \to \infty$ *and* $I(h_k, u(h_k)) \to \tilde{I}(h_0, u(h_0))$ *as* $k \to \infty$*.*

Here by $\overline{\text{co}}\mathcal{S}$ we denote the closed convex hull of the set $\mathcal{S}$. A little bit unexpected feature here is that the function $\tilde{\mathcal{F}}$ can be only lower semicontinuous in $\hat{h}$. Nevertheless, we were able to show that the function $\tilde{\mathcal{F}}$ can be represented as $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2$, where the function $\tilde{\mathcal{F}}_1$ is Carathéodory and exactly defined by the initial data of the problem (1), but the function $\tilde{\mathcal{F}}_2(x, \cdot)$ is convex for $h \in \overline{\text{co}}\mathbb{M}(x)$.

The main ideas to justify these results are to transform the initial problem (1) to a problem of minimization of an integral functional $J = J(h, w), (h, w) \in \mathcal{M} \times W$, whose integrand L also is a quadratic polynomial (if $F = 0$) in variables (h, w). This property allows to "separate" variables h and w in an analogue of the standard cell problem for the Γ-limit integrand or for the quasi-convex envelope, see, for instance, Dal Maso [1, p. 248, formula (2.41)] or Fonseca and Müller [3, p. 1369], respectively, what leads to an analogue of (4) for the relaxed integrand $\tilde{L}$. After that, the obtained representation for $\tilde{L}$ is transformed back to initial terms of the problem (1).

The paper is organized as follows. In Section 2 we give precise formulations of assumptions on the data in the problem (1) and introduce the basic notations that will be used in the paper. In Sections 3 and 4 we introduce the transformed problem and our concept of the convexification, and in Sections 5-6 we obtain a representation of the relaxed problem for piecewise constant controls h and establish properties of the corresponding integrand $\tilde{L}$ of the relaxed problem. After that, in Section 7, we show that this representation holds true also for all controls h from the closed convex hull $\overline{\text{co}}\mathcal{M}$ of the initial set $\mathcal{M}$ of admissible controls. Finally, in Section 8, we derive the representation (4)–(6) for the relaxation of the problem (1) and discuss a simple illustrative example.

2. Preliminaries

Let m, n, r be positive integers, let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz domain homeomorphic to the unit cube $K = (0, 1)^n$, let $|Q|$ denote the Lebesgue measure of a set $Q \subset \mathbb{R}^n$ and let co$\mathcal{S}$ and $\overline{\text{co}}\mathcal{S}$ denote the convex hull and the closed convex hull of the set $\mathcal{S}$, respectively.

Throughout the paper we suppose that the following hypotheses hold:

H1: $A : \mathbb{R}^n \to \mathbb{R}^{nm\times nm}$ is a fixed $nm \times nm$-matrix function with entries from $L_\infty(\mathbb{R}^n)$, and there exist positive constants $0 < \nu \leq \mu$ such that $\langle A(x)\xi, \xi\rangle \geq \nu|\xi|^2$, $|A(x)\xi| \leq \mu|\xi|$ a.e. $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^{nm}$.

H2: $B : \mathbb{R}^n \to \mathbb{R}^{nm\times nm}$ is a fixed symmetric $nm \times nm$-matrix function with entries from $L_\infty(\mathbb{R}^n)$, and there exists a constant μ_1 such that $\mu_1 < \frac{\nu}{2}$ and $|B(x)\xi| \leq \mu_1|\xi|$ a.e. $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^{nm}$.

H3: $f : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{nm}$, $g : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{nm}$, $f = f(x,h)$, $g = g(x,h)$ are fixed Carathéodory functions, affine with respect to h, and there exists a constant μ_2 such that for a.e. $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^r$

$$|f(x,h)| \leq \mu_2(1 + |h|), \quad |g(x,h)| \leq \mu_2(1 + |h|).$$

H4: $\mathbb{M} : \mathbb{R}^n \mapsto 2^{\mathbb{R}^r}$ is a fixed multivalued mapping with nonempty bounded and closed values, and $\mathbb{M}$ is piecewise constant. More precisely, there exist a partition of Ω, $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{m_0}$, and bounded closed sets $M_0, M_1, \ldots, M_{m_0} \subset \mathbb{R}^r$ such that $\Omega_0, \ldots, \Omega_{m_0}$ are mutually disjoint sets and $\Omega_1, \ldots, \Omega_{m_0}$ are Lipschitz domains, $|\Omega_0| = 0$, and in every Ω_s

$$\mathbb{M}(x) = M_s \quad \text{provided } x \in \Omega_s,\ s = 0, 1, \ldots, m_0.$$

In addition, there exists a constant μ_3 such that

$$M_s \subset \{h \in \mathbb{R}^r \,|\, |h| \leq \mu_3\}, \quad s = 0, 1, \ldots, m_0.$$

For $x \in \mathbb{R}^n \setminus \Omega$ we put $\mathbb{M}(x) = M_1$.

H5: $F : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$, $F = F(x,h)$, is a given Carathéodory function, and there exists a constant μ_4 such that

$$|F(x,h)| \leq \mu_4 \quad \text{a.e. } x \in \mathbb{R}^n \text{ and all } h \in \mathbb{R}^r.$$

Together with standard Lebesgue and Sobolev spaces we shall use the following spaces:

$$\mathcal{V} = \big\{v \in L_2(\Omega; \mathbb{R}^{nm}) \;\big|\; v = \nabla u,\ u \in H_0^1(\Omega; \mathbb{R}^m)\big\}$$

$$\mathcal{N} = \big\{\eta \in L_2(\Omega; \mathbb{R}^{nm}) \;\big|\; \eta = (\eta^1, \ldots, \eta^m), \operatorname{div} \eta^i = 0 \text{ in the sense of distributions, } i = 1, \ldots, m\big\}$$

$$\mathcal{V}^\# = \big\{v \in L_2(K; \mathbb{R}^{nm}) \big|\, v = \nabla u,\ u \in H^1_{loc}(\mathbb{R}^n; \mathbb{R}^m),\ u \text{ is } K\text{-periodic}\big\}$$

$$\mathcal{N}^\# = \big\{\eta \in L_2(K; \mathbb{R}^{nm}) \big|\, \eta = (\eta^1, \ldots, \eta^m),\ \eta^i \in L_{2loc}(\mathbb{R}^n; \mathbb{R}^n)\ \eta^i \text{ is } K\text{-periodic, } \operatorname{div} \eta^i = 0 \text{ in the sense of distributions, } \textstyle\int_K \eta^i(x)\, dx = 0,\ i = 1, \ldots, m\big\}$$

$W = \mathcal{V} \times \mathcal{N}$ and $\mathcal{W}^\# = \mathcal{V}^\# \times \mathcal{N}^\#$ with elements $w = \binom{v}{\eta}$.

We recall, see, for instance, Zhikov et al. [11, p. 138, Lemma 4.4], that every vector-function $\phi \in L_2(\Omega;\mathbb{R}^n)$ with $\operatorname{div}\phi = 0$ in the sense of distributions has the representation

where $\operatorname{Div} = \big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\big)$, $\mathcal{U}$ is a skew-symmetric matrix with entries $\mathcal{U}_{ij}$ from $H^1(\Omega)$, and there exists a constant $c(\Omega,n)$ such that

$$\|\mathcal{U}_{ij}\|_{H^1(\Omega)} \le c(\Omega,n)\|\phi\|_{L_2(\Omega;\mathbb{R}^n)};\quad i,j=1,\dots,n.$$

For the case of the space $\mathcal{N}^\#$ the entries of $\mathcal{U}$ are K-periodic functions, for the case of elements ϕ with $\operatorname{supp}\phi \subset \overline{K}$ the entries of $\mathcal{U}$ are elements of $H^1_0(K)$.

By construction,

$$L_2(\Omega;\mathbb{R}^{nm}) = \mathcal{V}\oplus\mathcal{N},\quad L_2(K;\mathbb{R}^{nm}) = \mathcal{V}^\#\oplus\mathcal{N}^\#\oplus\mathbb{R}^{nm}.$$

From the existence of "potentials" u and $\mathcal{U}$ and well-known properties of Sobolev spaces we have that in $\mathcal{V}$, $\mathcal{N}$, $\mathcal{V}^\#$, $\mathcal{N}^\#$ are dense corresponding subsets of piecewise constant elements.

We shall use also the following notations:

$$\mathcal{M} = \{h\in L_2(\Omega;\mathbb{R}^r)|\, h(x)\in\mathbb{M}(x) \text{ a.e. } x\in\Omega\}$$
$$\overline{\text{co}}\mathcal{M} = \{h\in L_2(\Omega;\mathbb{R}^r)|\, h(x)\in\overline{\text{co}}\mathbb{M}(x) \text{ a.e. } x\in\Omega\}.$$

Obviously, $\overline{\text{co}}\mathcal{M}$ is the closure of $\mathcal{M}$ in the weak topology of $L_2(\Omega;\mathbb{R}^r)$. For properties of convex sets and convex functions we refer to Fonseca and Leoni [4] and Rockafellar [9]; and for the convenience of readers we present below some well-known properties of normal integrands and lower semicontinuous envelopes for the case of a bounded Lipshitz domain $\Omega\subset\mathbb{R}^n$.

Proposition 2.1 ([2, p. 232, Theorem 1.1]). *Let D be a Borel subset of $\mathbb{R}^r$. For $f:\Omega\times D\to\mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$ to be a normal integrand, it is necessary and sufficient that for every $\epsilon>0$ there exists a compact subset $K_\epsilon\subset\Omega$ such that $|\,\Omega\backslash K_\epsilon\,|<\epsilon$ for which the restriction of f to $K_\epsilon\times D$ is lower semicontinuous.*

Proposition 2.2 ([2, p. 239, Corollary 1.2]). *Let f be a positive normal integrand of $\Omega\times\mathbb{R}^r$. Then the function $h\to\int_\Omega f(x,h(x))dx$ is nonnegative and lower semicontinuous from $L_q(\Omega;\mathbb{R}^r)$ to $\mathbb{R}\cup\{+\infty\}$ for all $q, 1\le q\le\infty$.*

Proposition 2.3 ([2, p. 236, Theorem1.2]). *Let D be a compact subset of $\mathbb{R}^r$ and f a normal integrand of $\Omega\times D$. Then there exists a measurable mapping $\overline{h}:\Omega\to D$ such that for all $x\in\Omega$ there is $f(x,\overline{h}(x)) = \min_{h\in D} f(x,h)$.*

Proposition 2.4 ([4, p. 242, Proposition 3.16]). *Let $\mathcal{X}$ be a normed linear space with separable dual space and let $\mathcal{I}:\mathcal{X}\to\mathbb{R}\cup\{+\infty\}$ be coercive. Then for every $h\in\mathcal{X}$:*

$$\begin{aligned}&\inf\{\mathcal{S}(h)\mid \mathcal{S}\le\mathcal{I},\ \mathcal{S} \textit{ weakly lower semicontinuous}\}\\ &=\inf\{\mathcal{S}(h)\mid \mathcal{S}\le\mathcal{I},\ \mathcal{S} \textit{ sequentially weakly lower semicontinuous}\}\\ &=\inf\Big\{\liminf_{k\to\infty}\mathcal{I}(h_k)\,\Big|\{h_k\}\subset\mathcal{X},\ h_k\rightharpoonup h \textit{ weakly}\Big\}.\end{aligned}$$

All proofs in Sections below are, in essence, purely local or rely on global characteristics (such as constants $\nu, \mu, \dots$ in hypotheses H1–H5) and do not depend on specific properties of the partition $\Omega = \Omega_0 \cup \Omega_1 \cup \dots \cup \Omega_{m_0}$. All these proofs by means of obvious separating the reasoning to subsets $\Omega_1, \dots, \Omega_{m0}$, if necessary, can be reduced to the case, where the mapping $\mathbb{M}$ does not depend on $x \in \Omega$. Therefore, for the sake of simplicity of notations only, in what follows we shall assume that the mapping $\mathbb{M}$ is constant on $\mathbb{R}^n$ and all references to the sets $\Omega_0, \Omega_1, \dots, \Omega_{m_0}$, to the dependence on x of sets $\mathbb{M}(x)$ and so on will be omitted, i.e., we shall deal with a constant mapping $\mathbb{M}(x) \equiv M$. Because the set M is closed and bounded, then its closed convex hull $\overline{\text{co}}M$ coincides with its convex hull co$M$ and we shall use this notation whenever the closed convex hull of M is considered.

3. Transformed problem

Analogously as in Raitums and Schmidt [8, pp. 152–154] we shall transform our initial problem (1) to a variational problem depending on the control h as a parameter.

Let us define the vector functions $a : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{2nm}$, $b : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{2nm}$ and the block-matrix function $E : \mathbb{R}^n \to \mathbb{R}^{2nm \times 2nm}$ as

$$a(x,h) := \begin{pmatrix} 0 \\ g(x,h) + f(x,h) \end{pmatrix}, \quad b(x,h) := \begin{pmatrix} g(x,h) - f(x,h) \\ 0 \end{pmatrix}$$

$$E := \begin{pmatrix} A_s + B + (B + A_a)\,[A_s - B]^{-1}\,(B - A_a) & (B + A_a)\,[A_s - B]^{-1} \\ [A_s - B]^{-1}\,(B - A_a) & [A_s - B]^{-1} \end{pmatrix}.$$

Here, A_s and A_a are the symmetric and the antisymmetric part of A, respectively. By virtue of hypotheses H1 and H2 there exist positive constants $0 < \nu_1 < \mu_5$ such that for a.e. $x \in \mathbb{R}^n$ and all $z \in \mathbb{R}^{nm} \times \mathbb{R}^{nm}$

$$\langle E(x)z,\ z\rangle \ge \nu_1 |z|^2, \quad |E(x)z| \le \mu_5 |z|. \tag{7}$$

Let us denote, for $(x,h,z) \in \mathbb{R}^n \times \mathbb{R}^r \times (\mathbb{R}^{nm} \times \mathbb{R}^{nm})$,

$$L(x,h,z) := \langle E(x)(z + a(x,h)), z + a(x,h)\rangle + 2\langle z, b(x,h)\rangle + F(x,h),$$

and let us define the functional $J : \overline{\text{co}}\mathcal{M} \times W \to \mathbb{R}$ as

$$J(h,w) := \int_\Omega L(x,h(x),w(x))dx. \tag{8}$$

The construction of J ensures that for every $h \in \overline{co}\mathcal{M}$

$$I(h,u(h)) = \min_{w \in W} J(h,w).$$

Indeed, since the matrix E is uniformly bounded and positively definite (see (7)), for a fixed h the functional $w \to J(h,w)$ is coercive, continuous and strictly convex on W, and as such attains its minimum on an unique element $w(h)$. By using the duality principle with respect to the variable η we get

$$\begin{aligned}\inf_{w\in W} J(h,w) = \inf_{v_1\in\mathcal{V}}\sup_{v_2\in\mathcal{V}}\int_\Omega \big[\langle (A_s+B)v_1,v_1\rangle + 2\langle v_1, b(x,h)\rangle - \langle (A_s-B)v_2,v_2\rangle \\ + 2\langle v_2, a(x,h)\rangle + 2\langle (B-A_a)v_1,v_2\rangle + F(x,h)\big]dx \\ = \inf_{v_1\in\mathcal{V}}\sup_{v_2\in\mathcal{V}}\int_\omega \big[\langle A(v_1+v_2),(v_1-v_2)\rangle + \langle B(v_1+v_2),(v_1+v_2)\rangle \\ + 2\langle (v_1+v_2), g(x,h)\rangle - 2\langle v_1-v_2, f(x,h)\rangle + F(x,h)\big]dx,\end{aligned}$$

from where and from Euler equations for the saddle point elements it follows immediately that $J(h,w(h)) = I(h,u(h))$.

Here and what follows, we omit the reference to the spatial argument x if that does not cause misunderstanding.

This way, the original problem (1) is equivalent to the problem

$$\Big\{\, J(h,w) \to \min_{h\in\mathcal{M},\, w\in W} . \tag{9}$$

4. Convexification of the set of admissible controls

Let us consider the set $\mathcal{M}(K) := \{h \text{ measurable},\ h(y)\in M \text{ a.e. } y\in K\}$. This set can be represented as $\mathcal{M}(K) = \bigcup_{\hat h\in \mathrm{co}M} \mathcal{M}(\hat h)$ with

$$\mathcal{M}(\hat h) := \left\{ h\in L_2(K;\mathbb{R}^r) \,\middle|\, h(y)\in M \text{ a.e.} y\in K,\ \int_K h(y)dy = \hat h \right\}.$$

Consider the problem

$$\begin{cases} \tilde J(h,w) \to \min \\ h \in \overline{\mathrm{co}}\mathcal{M},\ w\in W \end{cases} \tag{10}$$

with

$$\begin{aligned}\tilde J(h,w) &:= \int_\Omega \tilde L(x,h(x),w(x))dx \\ \tilde L(x,h,z) &:= \langle E(x)(z+a(x,h)), z+a(x,h)\rangle + 2\langle z, b(x,h)\rangle + \mathcal{F}(x,h) \\ \mathcal{F}(x,\hat h) &:= \inf_{h\in\mathcal{M}(\hat h)}\inf_{w\in\mathcal{W}^\#}\int_K \big[\big\langle E(x)(w(y)+a(x,h(y))-a(x,\hat h)), \\ &\qquad w(y)+a(x,h(y))-a(x,\hat h)\big\rangle \\ &\qquad +2\big\langle w(y), b(x,h(y))-b(x,\hat h)\big\rangle + F(x,h(y))\big]dy.\end{aligned} \tag{11}$$

Let us denote, for a fixed $h \in \overline{\text{co}}\mathcal{M}$, by $w(h)$ the minimizer of $J(h,\cdot)$ (or $\tilde{J}(h,\cdot)$) over $w \in W$. Since the matrix E is bounded and positively definite and the functions a, b define affine continuous mappings from $\overline{\text{co}}\mathcal{M}$ to $L_2(\Omega;\mathbb{R}^{nm} \times \mathbb{R}^{nm})$, then from the Euler equation for minimizers $w(h)$ in (9) or (10) (these minimizers depend only on h and do not depend on the functions F and $\mathcal{F}$) it follows immediately that the mapping $h \to w(h)$ is continuous on $\overline{\text{co}}\mathcal{M}$. Hence, due to the hypothesis H5 the functional $h \to J(h, w(h))$ is continuous and bounded on $\overline{\text{co}}\mathcal{M}$ (we recall that we consider the set $\overline{\text{co}}\mathcal{M}$ as a subset of $L_2(\Omega;\mathbb{R}^r)$). Therefore, from Proposition 2.4 it follows that the functional J_0, defined on $\overline{\text{co}}\mathcal{M}$ as

$$J_0(h) := \inf\Big\{\liminf_{k\to\infty} J(h_k, w(h_k))\Big|\{h_k\}\subset\mathcal{M}, h_k \rightharpoonup h \text{ weakly as } k\to\infty\Big\}, \quad (12)$$

is sequentially weakly lower semicontinuous on $\overline{\text{co}}\mathcal{M}$.

This way, to justify that the problem (10)–(11) is a relaxation of (9) it sufficies to show that the functional $h \to \tilde{J}(h, w(h))$ on $\overline{\text{co}}\mathcal{M}$ coincides with the functional J_0.

5. Properties of solutions and approximations by continuous or piecewise constant data

We begin with Meyers' type estimates.

Lemma 5.1. *There exist constants $p > 2$ and $c(p)$ such that*

$$\|w(h)\|_{L_p(\Omega;\mathbb{R}^{nm}\times\mathbb{R}^{nm})} \le c(p) \quad \forall h \in \overline{\text{co}}\mathcal{M}.$$

Proof. From the duality principle and Euler equations for $w(h)$,

$$E(\cdot)(w(\cdot) + a(\cdot, h(\cdot))) + b(\cdot, h(\cdot)) \in L_2(\Omega;\mathbb{R}^{nm} \times \mathbb{R}^{nm}) \ominus W,$$

we have that every element $w(h) = (v, \eta)$ has the representation

$$\begin{aligned} v &= \tfrac{1}{2}(\nabla\phi + \nabla\psi) \\ \eta &= \tfrac{1}{2}(A_s - B)(\nabla\phi - \nabla\psi) - \tfrac{1}{2}(B - A_a)(\nabla\phi + \nabla\psi) - g(\cdot, h(\cdot)) + f(\cdot, h(\cdot)), \end{aligned} \quad (13)$$

where the pair (ϕ, ψ) is the solution of the elliptic system

$$\begin{cases} \operatorname{div}[A(x)\nabla\phi - 2f(x, h(x))] = 0 & \text{in } \Omega,\ \phi \in H_0^1(\Omega;\mathbb{R}^m) \\ \operatorname{div}[A^*(x)\nabla\psi + 2B(x)\nabla\phi + 2g(x, h(x))] = 0 & \text{in } \Omega,\ \psi \in H_0^1(\Omega;\mathbb{R}^m). \end{cases} \quad (14)$$

The elements f and g are uniformly bounded in the L_∞ norm, hence, by Meyers' type theorems, see, for instance, Meyers and Elcrat [5, p. 130, Theorem 2], it follows that the gradients $(\nabla\phi, \nabla\psi)$ of solutions of (14) belong to a bounded set in $L_p(\Omega;\mathbb{R}^{nm} \times \mathbb{R}^{nm})$ for some $p > 2$ uniformly with respect to $h \in \overline{\text{co}}\mathcal{M}$. From here and (13) follows the desired estimate. □

Lemma 5.2. *There exist constants $p > 2$ and $c(p)$ such that for a.e. $x \in \Omega$, all $\hat{h} \in \overline{\mathrm{co}}M$ and all $h \in \mathcal{M}(\hat{h})$ the minimizer $w(x, \hat{h}, h)$ for the inner infimum in the definition* (14) *of $\mathcal{F}$ satisfies $\|w(x, \hat{h}, h)\|_{L_p(K;\mathbb{R}^{nm}\times\mathbb{R}^{nm})} \le c(p)$.*

Proof. Due to the K-periodicity of elements $w \in \mathcal{W}^{\#}$ it is sufficient to use interior Meyers' type estimates. Analogously as in the proof of Lemma 5.1 we have for $w(x, \hat{h}, h)$ the representation (13) with vector functions (ϕ, ψ), which satisfy in K the same type of equations as in (14), but with periodic boundary conditions. These equations can be extended via K-periodicity to the cube $(-1, 2)^n$ and inner estimates from Meyers and Elcrat [5, p.123, Theorem 1] give the desired estimate. □

Theorem 5.3. *Let the hypotheses* H1–H5 *hold. Then the function $\mathcal{F} = \mathcal{F}(x, \hat{h})$, defined by* (11)*, is a bounded normal integrand of $\Omega \times \mathrm{co}M$, measurable in x and continuous in h on the relative interior* ri$\mathrm{co}M$ *of the convex hull of M.*

Proof. For $(x, \hat{h}) \in \Omega \times \mathrm{co}M$ the value $\mathcal{F}(x, \hat{h})$ is defined by (11) as

$$\begin{aligned}\mathcal{F}(x, \hat{h}) = \inf_{h\in\mathcal{M}(\hat{h})} \inf_{w\in\mathcal{W}^{\#}} \int_K \big[\big\langle E(x)(w(y) + a(x, h(y)) - a(x, \hat{h})),\\ w(y) + a(x, h(y)) - a(x, \hat{h})\big\rangle\\ + 2\big\langle w(y), b(x, h(y)) - b(x, \hat{h})\big\rangle + F(x, h(y))\big]\, dy.\end{aligned} \tag{15}$$

Let us denote by $w(x, \hat{h}, h)$ the minimizer for the inner infimum in (15) and let the vector functions a and b have the representation

$$a(x, h) = a_1(x)h + a_2(x), \quad b(x, h) = b_1(x)h + b_2(x),$$

where a_1, b_1 are $2nm \times r$-matrices. From hypotheses H1–H5, Lemma 5.2 and estimate (7) standard calculations give that for $h_1 \in \mathcal{M}(\hat{h}_1), h_2 \in \mathcal{M}(\hat{h}_2)$

$$\begin{aligned}&\|w(x_1, \hat{h}_1, h_1) - w(x_2, \hat{h}_2, h_2)\|_{L_2(K;\mathbb{R}^{nm}\times\mathbb{R}^{nm})}\\ &\le \frac{1}{\nu_1}(c(p) + 2\mu_2(1 + \mu_3))\|E(x_1) - E(x_2)\|\\ &\quad + \frac{1}{\nu_1}\mu_2(1 + \mu_5)\left(\int_K |h_1(y) - h_2(y)|^2\, dy\right)^{\frac{1}{2}}\\ &\quad + \frac{1}{\nu_1}\mu_3(1 + \mu_5)[\|a_1(x_1) - a_1(x_2)\| + \|b_1(x_1) - b_1(x_2)\|]\\ &\quad + \frac{1}{\nu_1}\mu_3(1 + \mu_5)[|a_2(x_1) - a_2(x_2)| + |b_2(x_1) - b_2(x_2)|],\end{aligned} \tag{16}$$

where the constant $c(p)$ is from Lemma 5.2 and by $\|C\|$ we denote the standard norm of a constant matrix C.

From (15) and the Euler equation for $w(x,\hat{h},h)$ we have

$$\mathcal{F}(x,\hat{h}) = \inf_{h\in\mathcal{M}(\hat{h})} \tilde{G}(x,\hat{h},h) \tag{17}$$

with

$$\tilde{G}(x,\hat{h},h) := \int_K \big[-\langle E(x)w(x,\hat{h},h)(y), w(x,\hat{h},h)(y)\rangle + \langle E(x)a(x,h(y)), a(x,h(y))\rangle \\ - \langle E(x)a(x,\hat{h}), a(x,\hat{h})\rangle + F(x,h(y))\big]\,dy.$$

In Lemma 9.1 (see Appendix) it has been shown that for every $h_s \in \mathcal{M}(\hat{h}_s)$ there exists an $h_{0s} \in \mathcal{M}(\hat{h}_0)$ such that

$$\|h_{0s} - h_s\|_{L_2(K;\mathbb{R}^r)} \to 0 \text{ as } s \to \infty \text{ provided that } \hat{h}_s \to \hat{h}_0 \text{ as } s \to \infty.$$

This property together with continuity of the mapping $h \to \tilde{G}(x,\hat{h},h)$ (estimate (16) and hypotheses H1–H5) ensure that the function $\mathcal{F}$ for a.e. $x \in \Omega$ is lower semicontinuous with respect to $\hat{h} \in \operatorname{co}M$. In turn, the representation (17) and estimate (16) together with measurability properties of E, a, b, F and Lemma 5.2 ensure that the function $\mathcal{F}$ is bounded on $\Omega \times \operatorname{co}M$ and measurable in x. Moreover, for every fixed $\varepsilon > 0$ there exists a compact $D_\varepsilon \subset \Omega$ with $|\ \Omega \setminus D_\varepsilon\ | < \varepsilon$ such that the function $\mathcal{F}$ is uniformly (for all $\hat{h} \in \operatorname{co}M$) continuous with respect to $x \in D_\varepsilon$. Consequently, $\mathcal{F}$ is a normal integrand of $\Omega \times \operatorname{co}M$.

Finally, continuity of the mapping $h \to \tilde{G}(x,\hat{h},h)$ together with Lemma 9.1 and Corollary 9.2 give that the function $\hat{h} \to \mathcal{F}(x,\hat{h})$ is continuous on $\operatorname{rico}M$. □

Let us denote, for a cube $Q \subset \mathbb{R}^n$ with edges parallel to the axis of coordinates, by $\mathcal{W}^0(Q)$ the set

$$\mathcal{W}^0(Q) = \left\{ (v,\eta) \,\middle|\, \begin{array}{l} v = \nabla u,\ u \in H_0^1(Q;\mathbb{R}^m),\ \eta = (\eta^1,\dots,\eta^m),\ \eta^i = \operatorname{Div}\mathcal{U}^i, \\ \mathcal{U}^i \text{ is skew-symmetric with entries from } H_0^1(Q),\ i=1,\dots,m \end{array} \right\}.$$

Obviously, $\mathcal{W}^0(K) \subset \mathcal{W}^\#$, and we shall denote this set simply by $\mathcal{W}^0$.

Lemma 5.4. *In definition* (11) *of the function* $\mathcal{F}$ *the space* $\mathcal{W}^\#$ *can be replaced by the set* $\mathcal{W}^0$.

Proof. Since $\mathcal{W}^0 \subset \mathcal{W}^\#$, then it is sufficient to show that for every fixed $h_0 \in$

$\mathcal{M}(\hat{h})$, $w_0 \in \mathcal{W}^{\#}$, $\epsilon > 0$, there are $h_k \in \mathcal{M}(\hat{h})$ and $w_k^0 \in \mathcal{W}^0$ such that

$$\begin{aligned}&\int_K \big[\langle E(x)(w_k^0(y)+a(x,h_k(y))-a(x,\hat{h})), w_k^0(y)+a(x,h_k(y))-a(x,\hat{h})\rangle \\ &+ 2\langle w_k^0(y), b(x,h_k(y))-b(x,\hat{h})\rangle + F(x,h_k(y))\big]\,dy \\ &\le \int_K \big[\langle E(x)(w_0(y)+a(x,h_0(y))-a(x,\hat{h})), w_0(y)+a(x,h_0(y))-a(x,\hat{h})\rangle \\ &\quad + 2\langle w_0(y), b(x,h_0(y))-b(x,\hat{h})\rangle + F(x,h_0(y))\big]\,dy + \epsilon.\end{aligned} \tag{18}$$

Let $(u,\mathcal{U})$ are the corresponding "potentials" for w_0, i.e. $w_0 = (\nabla u, \mathrm{Div}\mathcal{U})$. We extend u, $\mathcal{U}$ and h_0 via K-periodicity to the whole $\mathbb{R}^n$ and define

$$\begin{aligned} w_k(y) &= \big(\nabla\big[\tfrac{1}{k}u(ky)\big], \mathrm{Div}\big[\tfrac{1}{k}\mathcal{U}(ky)\big]\big) \\ h_k(y) &= h_0(ky) \\ w_k^0(y) &= \big(\nabla\big[\xi(y)\tfrac{1}{k}u(ky)\big], \mathrm{Div}\big[\xi(y)\tfrac{1}{k}\mathcal{U}(ky)\big]\big)\end{aligned}$$

for k large enough and with an appropriate cut-off function ξ,

$$\xi \in H_0^1(K), \quad |\nabla\xi(y)| \le \sqrt{k}, \quad \mathrm{meas}\{y \in K \,|\, \xi(y) \ne 1\} \le \tfrac{4n}{\sqrt{k}}.$$

By construction, $w_k^0 \in \mathcal{W}^0$, $k = 1, 2, \ldots$; the value of the integral in the right-hand side of (18) does not change if we replace (h_0, w_0) by (h_k, w_k); and the value of this integral after replacing w_k by w_k^0 and taking the limit as $k \to \infty$ is the same as with (h_0, w_0), what concludes the proof. □

Lemma 5.5. *Let* H1–H5 *hold.If* $J_0(h) = \tilde{J}(h, w(h))$ *for all* $h \in \overline{\mathrm{co}}\mathcal{M}$ *provided that the functions* A, B, f, g, F *are piecewise constant with respect to* $x \in \Omega$*, then this equality is true in the general case too.*

Proof. Due to H1–H5 the mappings E, B, f, g are measurable on Ω with bounded values and the function F is Carathéodory on $\Omega \times \mathrm{co}M$ and with bounded values too. Therefore, by virtue of Scorza–Dragoni theorem, see, for instance, Ekeland and Temam [2, p. 234], for every $\delta > 0$ there exists a closed subset $D_\varepsilon \subset \Omega$ such that the functions E, B, f, g, F are continuous on $D_\varepsilon \times \mathrm{co}M$ and the measure $|\Omega \setminus D_\varepsilon| < \varepsilon$. According to Stein [10, p. 272], for every continuous vector function $\varphi : D_\varepsilon \times \mathrm{co}M \to \mathbb{R}^s$ there exists an operator $\mathcal{E}$ of extension such that

(i) the function $\mathcal{E}\varphi$ is defined on the whole $\mathbb{R}^n \times \mathbb{R}^r$;
(ii) the function $\mathcal{E}\varphi$ coincides with φ on $D_\varepsilon \times \mathrm{co}M$;
(iii) for all arguments the values of $\mathcal{E}\varphi$ belong to the closed convex hull of the set of values of φ on the set $D_\varepsilon \times \mathrm{co}M$.

Further, continuous on $\overline{\Omega} \times \mathrm{co}M$ functions can be approximated in the maximum norm by means of piecewise constant (with respect to $x \in \overline{\Omega}$) functions. This way, for every $\delta > 0$ we have subsets $\Omega_0, \Omega_1, \ldots, \Omega_N$ and functions $E_\delta, B_\delta, g_\delta, f_\delta, F_\delta$ such that

(i) $\overline{\Omega} = \overline{\Omega_1} \cup \cdots \cup \overline{\Omega_N}$; Ω_s, $s = 1, \ldots, N$, are mutually disjoint Lipshitz domains, and $|\Omega_0| < \delta$;

(ii) in every Ω_s, $s = 1, \ldots, N$, the functions $E_\delta, B_\delta, g_\delta, f_\delta, F_\delta$ are constant with respect to the argument x;

(iii) the functions $E_\delta, B_\delta, f_\delta, g_\delta, F_\delta$ satisfy hypotheses H1–H5;

(iv) for all $x \in \Omega \backslash \Omega_0$ and all $h \in \mathrm{co}M$, $\|E(x) - E_\delta(x)\| < \delta$, $\|B(x) - B_\delta(x)\| < \delta$, $|f(x) - f_\delta(x)| < \delta$, $|g(x) - g_\delta(x)| < \delta$, $|F(x, h) - F_\delta(x, h)| < \delta$.

For every $\delta > 0$ and every collection $\{E_\delta, B_\delta, f_\delta, g_\delta, F_\delta\}$ that satisfies (i)–(iv) we define the function $\mathcal{F}_\delta$ and the functionals $J_\delta, J_{0\delta}, \tilde{J}_\delta$ by the same formulae as $\mathcal{F}, J, J_0, \tilde{J}$ with the functions $E_\delta, B_\delta, f_\delta, g_\delta, F_\delta$ instead of E, B, f, g, F, respectively.

Since H1–H5 hold, then the results of Lemma 5.1 and Lemma 5.2 apply to the minimizers $w_\delta(h)$ and $w_\delta(x, \hat{h}, h)$ defined by means of $E_\delta, B_\delta, f_\delta, g_\delta, F_\delta$ instead of E, B, f, g, F, respectively; and the constants $p, c(p)$ do not depend on δ. From here and the estimate (16) (the minimizer $w_\delta(x, \hat{h}, h)$ can be treated as $w(x_0, \hat{h}, h)$ with some fixed "virtual" x_0) we have the existence of a continuous function γ_1 with $\gamma_1(0) = 0$ and depending only on constants from H1–H5 such that for all $\delta \in (0, 1]$

$$|\mathcal{F}_\delta(x, \hat{h}) - \mathcal{F}(x, \hat{h})| < \gamma_1(\delta) \quad \forall x \in \Omega \setminus \Omega_0, \forall \hat{h} \in \mathrm{co}M.$$

In turn, hypotheses H1–H5 ensure that all functions $\mathcal{F}$ and $\mathcal{F}_\delta$ (with $\delta \in [0, 1]$) are uniformly bounded, what ensure that the integral over Ω_0 goes to zero as $\delta \to 0$. Hence, there exists another continuous function γ_2 with $\gamma_2(0) = 0$ such that for $\delta \in [0, 1]$

$$\int_\Omega |\mathcal{F}_\delta(x, h(x)) - \mathcal{F}(x, h(x))|\, dx \le \gamma_2(\delta) \quad \forall h \in \overline{\mathrm{co}}\mathcal{M}.$$

Analogous estimates for $J_\delta - J$, $J_{0\delta} - J_0$, $\tilde{J}_\delta - \tilde{J}$, for instance,

$$\sup_{h \in \overline{\mathrm{co}}\mathcal{M}} |\ \tilde{J}_\delta(h, w_\delta(h)) - \tilde{J}(h, w(h))\ | \le \gamma_3(\delta), \quad \gamma_3(\delta) \to 0 \text{ as } \delta \to 0,$$

are obvious due to the hypotheses H1–H5, estimate (5) and Lemmas 5.1 and 5.2. Here $w_\delta(h)$ denotes the minimizer of the functional $\tilde{J}_\delta(h, \cdot)$ on W.

Now, if $J_{0\delta}(h) = \tilde{J}_\delta(h, w_\delta(h))$ for all $h \in \overline{\mathrm{co}}\mathcal{M}$, then for every $h \in \overline{\mathrm{co}}\mathcal{M}$

$$\begin{aligned}
&|J_0(h) - \tilde{J}(h, w(h))| \\
&\le |J_0(h) - J_{0\delta}(h) + J_{0\delta}(h) - \tilde{J}_\delta(h, w_\delta(h)) + \tilde{J}_\delta(h, w_\delta(h)) - \tilde{J}(h, w(h))| \\
&\le |J_{0\delta}(h) - J_0(h)| + |\tilde{J}_\delta(h, w_\delta(h)) - \tilde{J}(h, w(h))| \to 0 \text{ as } \delta \to 0,
\end{aligned}$$

what completes the proof. □

6. Equivalence on piecewise constant controls

In this section, we prove the equality

$$J_0(h) = \tilde{J}(h, w(h)) \tag{19}$$

for piecewise constant elements $h \in \overline{\mathrm{co}}\mathcal{M}$. In the previous Section it was shown in Lemma 5.5 that it is sufficient to prove our equality (19) for the case where the functions E, B, f, g, F satisfy H1–H5 and, in addition, are piecewise constant with respect to $x \in \Omega$. In the sequel, we assume that this property holds.

Lemma 6.1. *If $h_0 \in \overline{\mathrm{co}}\mathcal{M}$ is piecewise constant, then $\tilde{J}(h_0, w(h_0)) \leq J_0(h_0)$.*

Proof. Let us suppose the contrary, i.e., that there exist a piecewise constant element $h_0 \in \overline{\mathrm{co}}\mathcal{M}$ and a constant $d > 0$ such that $\tilde{J}(h_0, w(h_0)) \geq J_0(h_0) + 2d$. Then from the definition of J_0 by (12) it follows the existence of a sequence $\{h_k\} \subset \mathcal{M}$ such that

$$h_k \rightharpoonup h_0 \text{ weakly as } k \to \infty, \quad w(h_k) \rightharpoonup w(h_0) \text{ weakly as } k \to \infty$$
$$\tilde{J}(h_0, w(h_0)) - \lim_{k\to\infty} J(h_k, w(h_k)) \geq d.$$

From here, Euler equations for $w(h_k)$ and $w(h_0)$ and definitions of J and $\tilde{J}$ by (8) and (11), respectively, we deduce (Euler equations are affine with respect to (h, w) and do not depend on F or $\mathcal{F}$)

$$\begin{aligned}\lim_{k\to\infty} \inf_{w\in W} \int_\Omega \big[\big\langle E(x)(w(x) + a(x, h_k(x)) - a(x, h_0(x))),\\ w(x) + a(x, h_k(x)) - a(x, h_0(x))\big\rangle\\ +2\big\langle w(x), b(x, h_k(x)) - b(x, h_0(x))\big\rangle + F(x, h_k(x))\big]\, dx\\ \leq \int_\Omega \mathcal{F}(x, h_0(x))dx - d.\end{aligned} \tag{20}$$

Because $h_k \rightharpoonup h_0$ weakly and Euler equations for the inner minimizers w_k in the left-hand side of (20) are affine with respect to (h, w), then $w_k \rightharpoonup 0$ weakly as $k \to \infty$. By virtue of Lemma 5.1 all w_k belong to a bounded set in $L_p(\Omega; \mathbb{R}^{nm} \times \mathbb{R}^{nm})$ with some $p > 2$, and (by virtue of embedding theorems), wihout loosing generality, we can assume that the corresponding "potentials" $(u_k, \mathcal{U}^k)$ converge to zero strongly in $[L_q(\Omega; \mathbb{R})]^m \times [L_q(\Omega; \mathbb{R}^{nn})]^m$ as $k \to \infty$ for some $q > 2$.

All functions $E, B, f, g, F, \mathcal{F}$ and h_0 are piecewise constant with respect to $x \in \Omega$. These properties are sufficient in order to guarantee the existence of a finite number of cubes $Q_s, s = 1, \dots, N$ with edges parallel to the coordinate axes and a set Ω_0 with the following properties: (i) $\Omega = \Omega_0 \cup Q_1 \cup \dots \cup Q_N$; (ii) in every Q_s the functions $E, B, f, g, F, \mathcal{F}$ do not depend on x; (iii) the measure $|\Omega_0|$ is small enough such that the contribution of integral over Ω_0 in the left-hand side of (20) with $w = w_k, k = 1, 2, \ldots$, and in the integral in the right-hand side of (20) is less than $\frac{d}{4}$ for all $k = 1, 2, \ldots$. From here it follows that there exists a cube Q with edges parallel to the coordinate axes such that

$$\begin{aligned}
\lim_{k\to\infty} \int_Q & \big[\big\langle E(x_0)(w_k(x) + a(x_0, h_k(x)) - a(x_0, h_0(x_0))), \\
& \quad w_k(x) + a(x_0, h_k(x)) - a(x_0, h_0(x_0))\big\rangle \\
& +2\big\langle w_k(x), b(x_0, h_k(x)) - b(x_0, h_0(x_0))\big\rangle + F(x_0, h_k(x))\big]\,dx \\
\leq \mathcal{F}&(x_0, h_0(x_0))|Q| - \frac{1}{2|\Omega|}|Q|d,
\end{aligned} \tag{21}$$

where x_0 is an arbitrary fixed point from Q.

Because the "potentials" $(u_k, \mathcal{U}^k)$ of minimizers w_k converge to zero strongly in the corresponding Lebesgue spaces $L_q(\Omega; \mathbb{R})$ and $L_q(\Omega; \mathbb{R}^{nn})$, respectively, then by means of appropriate cut-off functions, analogously as in the proof of Lemma 5.4, we can replace in (21) the sequence $\{w_k\}$ by a sequence $\{\tilde{w}_k\} \subset \mathcal{W}^0(Q)$, which also converges to zero weakly and is bounded in some Lebesgue space $L_{p_1}(Q; \mathbb{R}^{nm} \times \mathbb{R}^{nm})$, $p_1 > 2$. After obvious similarity transform $Q \mapsto K$, without loosing generality and for the sake of simplicity of notations only, we can assume that $Q = K$.

Since $h_k \rightharpoonup h_0$ weakly as $k \to \infty$, we have that $h_k \in \mathcal{M}(\hat{h}_k)$ with $\hat{h}_k \to \hat{h}_0 = h_0(x_0)$ as $k \to \infty$. By virtue of Lemma 9.1 (see Appendix) for every h_k there exists a corresponding element $h_k^{'} \in \mathcal{M}(\hat{h}_0)$, $k = 1, 2, \ldots$, such that

$$\|h_k^{'} - h_k\|_{L_2(K;\mathbb{R}^r)} \to 0 \text{ as } k \to \infty,$$

and, because the set M is bounded, the exponent 2 in this relationship can be replaced by an arbitrary exponent $q, 2 \leq q < \infty$. These properties together with Lemma 5.4 and boundedness of the set $\{\tilde{w}_k\} \subset L_{p_1}(K; \mathbb{R}^{nm} \times \mathbb{R}^{nm})$, after replacing h_k by $h_k^{'}$ in (21), give that the limit as $k \to \infty$ in the left-hand side of (21) is greater than or equal to $\mathcal{F}(x_0, h_0(x_0))$, what gives the contradiction $\mathcal{F}(x_0, h_0(x_0)) \leq \mathcal{F}(x_0, h_0(x_0)) - \frac{d}{2}$. □

Lemma 6.2. *Let $h_0 \in \overline{\text{co}}\mathcal{M}$ be piecewise constant, then $\tilde{J}(h_0, w(h_0)) \geq J_0(h_0)$.*

Proof. Let a piecewise constant $h_0 \in \overline{\text{co}}\mathcal{M}$ and $\varepsilon > 0$ be fixed. Piecewise constant elements are dense in W, hence there exist a partition $\Omega = Q_0 \cup Q_1 \cup \cdots \cup Q_{s_0}$ by means of mutually disjoint sets and an element $w_0 \in W$ such that:

(i) Q_s, $s = 1, \ldots, s_0$, are cubes with edges parallel to the coordinate axes and $|Q_s| \leq \varepsilon$, $s = 0, 1, \ldots, s_0$;

(ii) in every $Q_s, s = 1, \dots, s_0$, the functions E, a, b, F, h_0, $\mathcal{F}$, w_0 are constant with respect to x;

(iii) with some arbitrary fixed $x_s \in Q_s, s = 1, \dots, s_0$,

$$
\begin{aligned}
&J(h_0, w(h_0)) \\
&\geq \sum_{s=1}^{s_0} \int_{Q_s} \big[\big\langle E(x_s)(w_0(x_s) + a(x_s,\, h_0(x_s))), w_0(x_s) + a(x_s, h_0(x_s))\big\rangle \\
&\quad + 2\big\langle w_0(x_s), b(x_s, h_0(x_s))\big\rangle + \mathcal{F}(x_s, h_0(x_s))\big]\, dx - \varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{Q_0} \big|\big\langle E(x)(w_0(x) + a(x, h_0(x))), w_0(x) + a(x, h_0(x))\big\rangle \\
&+ 2\big\langle w_0(x), b(x, h_0(x))\big\rangle + \mathcal{F}(x, h_0(x))\big|\, dx < \frac{\varepsilon}{4}.
\end{aligned}
$$

The functions a and b are affine with respect to h, the mean values of elements $w \in \mathcal{W}^0(Q_s)$, $s = 1, \dots,\ s_0$; are equal to zero, hence, from the definition of $\mathcal{F}$ by (11) and Lemma 5.4 via simple calculations we get for $s = 1, \dots, s_0$

$$
\begin{aligned}
&\big[\big\langle E(x_s)(w_0(x_s) + a(x_s, h_0(x_s))), w_0(x_s) + a(x_s, h_0(x_s))\big\rangle \\
&+ 2\big\langle w_0(x_s), b(x_s, h_0(x_s))\big\rangle + \mathcal{F}(x_s, h_0(x_s))\big]|Q_s| \\
&\geq \int_{Q_s} \big[\big\langle E(x_s)(w_0(x_s) + w_s(y) + a(x_s, h_s(y))), w_0(x_s) + w_s(y) + a(x_s, h_s(y))\big\rangle \\
&\quad + 2\big\langle w_0(x_s) + w_s(y), b(x_s, h_s(y))\big\rangle + F(x_s, h_s(y))\big]\, dx - \frac{1}{4|\Omega|}\varepsilon|Q_s|,
\end{aligned}
$$

for some $w_s \in \mathcal{W}^0(Q_s)$, $h_s \in \mathcal{M}$, $\int_{Q_s} h_s(y)\, dy = h_0(x_s)|Q_s|$, $s = 1, \dots, s_0$.

Since for every $h \in \mathcal{M}$,

$$
w_* = w_0 + \sum_{s=1}^{s_0} \chi_{Q_s}(\,\cdot\,)w_s(\cdot) \in W, h_* = \sum_{s=1}^{s_0} \chi_{Q_s}(\cdot)h_s(\,\cdot\,) + \chi_{Q_0}(\cdot)h(\cdot) \in \mathcal{M},
$$

where χ_Q denotes the characteristic function of Q, then

$$
\begin{aligned}
&J(h_0, w(h_0)) \geq \int_\Omega L(x, h_*(x), w_*(x))\, dx - \int_{Q_0} L(x, h_*(x), w_0(x))\, dx - \frac{3}{2}\varepsilon \\
&\int_{Q_s} h_*(x)\, dx = \int_{Q_s} h_0(x)\, dx, \quad s = 1, \dots, s_0; \quad |Q_s| < \varepsilon, \quad s = 0, 1, \dots, s_0.
\end{aligned}
$$

After passing to the limit $\varepsilon \to 0$ in this relationship we get $J(h_0,\, w(h_0)) \geq J_0(h_0)$, what completes the proof. $\square$

Lemma 6.3. *The function* $G = G(x,h) := \langle E(x)a(x,h), a(x,h)\rangle + \mathcal{F}(x,h)$ *is a bounded normal integrant of* $\Omega \times \mathrm{co}M$*, convex in* $h \in \mathrm{co}M$ *and continuous on the relative interior* $\mathrm{rico}M$ *of* $\mathrm{co}M$*.*

Proof. Due to the hypotheses H1–H5 and Theorem 5.3 we have to prove only that the function $G(x,\cdot)$ is convex on $\mathrm{co}M$. The functional $\tilde{J}$ can be represented as

$$\tilde{J}(h,w(h)) := J_1(h) + J_2(h) \tag{22}$$

where

$$J_1(h) := \min_{w \in W} \int_\Omega \big[\langle E(x)w(x), w(x)\rangle + 2\langle w(x), b(x,h(x)) + E(x)a(x,h(x))\rangle\big]dx$$

$$J_2(h) := \int_\Omega \big[\langle E(x)a(x,h(x)), a(x,h(x))\rangle + \mathcal{F}(x,h(x))\big]dx.$$

Since the functions a and b are affine with respect to h and the mapping $h \to w(h)$ is continuous on $\overline{\mathrm{co}}\mathcal{M}$, then the functional J_1 is continuous and concave with respect to $h \in \overline{\mathrm{co}}\mathcal{M}$, and, as a consequence, weakly upper semicontinuous on $\overline{\mathrm{co}}\mathcal{M}$. Results of Lemmas 6.1 and 6.2 give that for piecewise elements $h \in \overline{\mathrm{co}}\mathcal{M}$ the functional $h \to \tilde{J}(h,w(h))$ coincides with J_0. By construction, the functional J_0 is sequentially weakly lower semicontinuous, hence

$$J_2(h_0) \le \lim_{k\to\infty} \inf J_2(h_k) \tag{23}$$

whenever $h_k \rightharpoonup h_0$ weakly as $k \to \infty$ and all h_0, h_k, $k = 1, 2, \ldots,$ are piecewise constant elements from $\overline{\mathrm{co}}\mathcal{M}$.

The integrand of J_2 is the function G and from (23) it follows that for every fixed $h_1, h_2 \in \mathrm{co}M, \lambda \in [0,1]$ and a fixed cube $Q \subset \Omega$ there is

$$\int_Q G(x, \lambda h_1 + (1-\lambda)h_2)dx$$
$$\le \lim_{k\to\infty} \inf \left[\int_Q \chi_k(x)G(x,h_1)dx + \int_Q (1-\chi_k(x))G(x,h_2)dx\right]$$

for every sequence $\{\chi_k\}$ of piecewise constant characteristic functions of subsets of Q such that $\chi_k \rightharpoonup \lambda$ weakly in $L_2(Q)$ as $k \to \infty$. This property is sufficient in order to guarantee that the function $h \to G(x,h)$ is convex on $\mathrm{co}M$ for a.e. $x \in \Omega$. □

Corollary 6.4. *The function* $\mathcal{F}$ *has the representation* $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$*, where*

$$\mathcal{F}_1 := -\langle E(x)a(x,h), a(x,h)\rangle, \quad \mathcal{F}_2 := \mathcal{F}(x,h) + \langle E(x)a(x,h), a(x,h)\rangle.$$

Here the function $\mathcal{F}_1$ *is explicitely defined by the initial data, but the function* $\mathcal{F}_2$ *is convex and lower semicontinuous in* $h \in \mathrm{co}M$*.*

7. Relaxation of the transformed problem

We recall that by virtue of Lemma 5.5 we have to consider only the case of piecewise constant with respect to x functions $E, B, f, g, F, \mathcal{F}$.

Lemma 7.1. *For every fixed* $h_0 \in \overline{\text{co}}\mathcal{M}$ *there exists a sequence* $\{h_k\} \subset \overline{\text{co}}\mathcal{M}$ *of piecewise constant elements with values in* $\text{rico}M$ *such that*

$$h_k \to h_0 \ \text{as}\ k \to \infty$$
$$\tilde{J}(h_0, w(h_0)) \geq \tilde{J}(h_k, w(h_k)) - \varepsilon_k, \ \ k = 1, 2, \ldots; \quad \varepsilon_k \to 0 \ \text{as}\ k \to \infty.$$

Proof. Let $h_0 \in \overline{\text{co}}\mathcal{M}$ be fixed. We shall use the functionals J_1 and J_2, introduced in (22). The functional J_1 does not depend on influence of specific properties of the function F, and hypotheses H1–H4 ensure that J_1 is continuous on $\overline{\text{co}}\mathcal{M}$ for a.e. $x \in \Omega$. Let us fix an element $h_* \in \text{rico}M$, which we shall consider also as a constant element of $\overline{\text{co}}\mathcal{M}$. From continuity of J_1 we have the existence of a continuous function γ_1 with $\gamma_1(0) = 0$ such that

$$J_1(h + \lambda(h_* - h)) - J_1(h) \leq \gamma_1(\lambda) \quad \forall h \in \overline{\text{co}}\mathcal{M}.$$

The function G (the integrand of the functional J_2) is convex in h and bounded on $\Omega \times \overline{co}M$. Therefore,

$$\begin{aligned} G(x, h + \lambda(h_* - h)) &\leq (1 - \lambda)G(x, h) + \lambda G(x, h_*) \\ &= G(x, h) + \lambda[G(x, h_*) - G(x, h)] \\ &\leq G(x, h) + c_1\lambda, \end{aligned}$$

where the constant c_1 does not depend on the choice of $h, h_* \in \overline{\text{co}}M$. From these estimates it follows

$$\begin{aligned} &\tilde{J}(h_0, w(h_0)) \\ &\geq J_1(h_0 + \lambda(h_* - h_0)) + J_2(h_0 + \lambda(h_* - h_0)) - \gamma_1(\lambda) - c_1\lambda|\Omega| \\ &= \tilde{J}(h_0 + \lambda(h_* - h_0)), w(h_0 + \lambda(h_* - h_0)) - \gamma_1(\lambda) - c_1\lambda|\Omega|. \end{aligned} \tag{24}$$

For every fixed $\lambda \in (0, 1)$ all elements h_λ, $h_\lambda(x) := h(x) + \lambda(h_* - h(x))$, $x \in \Omega$, with $h \in \overline{\text{co}}\mathcal{M}$ takes values from some convex closed set $M_\lambda \subset \text{rico}M$. By construction, $h_0 + \lambda(h_* - h_0) \to h_0$ as $\lambda \to 0$. Since the function $\mathcal{F}$ is continuous with respect to $h \in M_\lambda$, then the functional $h \to \tilde{J}(h, w(h))$ is continuous on $\mathcal{M}_\lambda := \{h \in \overline{\text{co}}\mathcal{M} \mid h(x) \in M_\lambda \text{ a.e. } x \in \Omega\}$. Obviously, the subset of piecewise constant elements is dense in $\mathcal{M}_\lambda$. From here and (24) immediately follows the assertion of the Lemma. □

Corollary 7.2. *For all* $h \in \overline{\text{co}}\mathcal{M}$ *there is*

$$J_0(h) \leq \tilde{J}(h, w(h)). \tag{25}$$

Proof. By virtue of Lemma 6.2 the inequality (25) holds for piecewise constant elements $h \in \overline{\text{co}}\mathcal{M}$. Let us fix an element $h_0 \in \overline{\text{co}}\mathcal{M}$. By virtue of Lemma 7.1 and lower semicontinuity of J_0 there exists a sequence of piecewise constant elements $\{h_k\} \subset \overline{\text{co}}\mathcal{M}$ such that

$$h_k \to h_0 \text{ strongly as } k \to \infty$$
$$\tilde{J}(h_0, w(h_0)) \geq \lim_{k\to\infty} \inf \tilde{J}(h_k, w(h_k)) \geq \lim_{k\to\infty} \inf J_0(h_k) \geq J_0(h_0),$$

what gives the assertion of Corollary. □

Theorem 7.3. *Let the hypotheses* H1–H5 *hold. Then the mapping* $h \to \tilde{J}(h, w(h))$ *is sequentially weakly lower semicontinuous on* $\overline{\text{co}}\mathcal{M}$*, and for all* $h \in \overline{\text{co}}\mathcal{M}$ *there is* $\tilde{J}(h, w(h)) = J_0(h)$*, where the functionals* $\tilde{J}$ *and* J_0 *are defined by* (11) *and* (12)*, respectively.*

Proof. Let $\{h_k\} \subset \overline{\text{co}}\mathcal{M}$, $h_k \rightharpoonup h_0$ weakly as $k \to \infty$, and let us suppose that there exists a $d > 0$ such that $\tilde{J}(h_0, w(h_0)) \geq \lim_{k\to\infty} \tilde{J}(h_k, w(h_k)) + d$, i.e., we suppose that the mapping $h \to \tilde{J}(h, w(h))$ is not sequentially weakly lower semicontinuous. Let $h_*, J_1, J_2, \gamma_1, c_1, M_\lambda, \mathcal{M}_\lambda$ are the same as in the proof of Lemma 7.1. The inequality (24) for h_k gives

$$\tilde{J}_1(h_k) + \tilde{J}_2(h_k) \geq \tilde{J}_1(h_k + \lambda(h_* - h_k)) + \tilde{J}_2(h_k + \lambda(h_* - h_k)) - \gamma_1(\lambda) - c_1\lambda|\Omega|.$$

On the other hand, the functionals $\tilde{J}_1$ and $\tilde{J}_2$ are lower semicontinuous on $\overline{\text{co}}\mathcal{M}$. Therefore $\tilde{J}_1(h_0) + \tilde{J}_2(h_0) \leq \lim_{\lambda\to 0} \big[\tilde{J}_1(h_0 + \lambda(h_* - h_0)) + \tilde{J}_2(h_0 + \lambda(h_* - h_0))\big]$. Our three inequalities give that for some $\lambda_0 > 0$

$$\begin{aligned}&\tilde{J}\big(h_0 + \lambda_0(h_* - h_0), w(h_0 + \lambda_0(h_* - h_0))\big)\\ &\geq \lim_{k\to\infty} \tilde{J}(h_k + \lambda_0(h_* - h_k), w(h_k + \lambda_0(h_* - h_k)) + \frac{d}{2}.\end{aligned}$$

In this inequality all arguments take values from a closed convex set $M_{\lambda_0} \subset$ rico$M$, and on the set $\mathcal{M}_{\lambda_0}$ the mapping $h \to \tilde{J}(h, w(h))$ is continuous. This mapping remains continuous (the function $\mathcal{F}$ is contionuous on rico$M$) on every subset $\mathcal{M}_* \subset \overline{\text{co}}\mathcal{M}$ of elements with values from a closed convex set $M_* \subset$ rico$M$. In particular, we can choose the set M_* so that there exists a constant $\delta > 0$ such that $\{h \in \text{co}M \mid \text{dist}(h; M_{\lambda_0}) \leq \delta\} \subset M_*$. That is sufficient in order to guarantee that there exist a piecewise constant element $\tilde{h}_0 \in \overline{\text{co}}\mathcal{M}$ and a sequence of piecewise constant elements $\{\tilde{h}_k\} \subset \overline{\text{co}}\mathcal{M}$ such that

$$\tilde{h}_k \rightharpoonup \tilde{h}_0 \text{ weakly as } k \to \infty, \quad \tilde{J}(\tilde{h}_0, w(\tilde{h}_0)) \geq \lim_{k\to\infty} \tilde{J}(\tilde{h}_k, w(\tilde{h}_k)) + \frac{d}{4}. \tag{26}$$

The obtained inequality in (26), however, contradicts to the facts that the functional J_0 is sequentially weakly lower semicontinuous on $\overline{\text{co}}\mathcal{M}$ and that

$J_0(h) = \tilde{J}(h, w(h))$ for piecewise constant elements $h \in \overline{\mathrm{co}}\mathcal{M}$. This way, we have obtained that the mapping $h \to \tilde{J}(h, w(h))$ is sequentially weakly lower semicontinuous on $\overline{\mathrm{co}}\mathcal{M}$.

Since $\overline{\mathrm{co}}\mathcal{M}$ is a convex, closed and bounded set in the separable Hilbert space $L_2(\Omega; \mathbb{R}^r)$, then from Proposition 2.4 (after an appropriate extension of mappings $h \to J_0(h)$ and $h \to \tilde{J}(h, w(h))$ outside $\overline{\mathrm{co}}\mathcal{M}$), from Corollary 7.2 and sequentially weak lower semicontinuity of the mapping $h \to \tilde{J}(h, w(h))$ on $\overline{\mathrm{co}}\mathcal{M}$ it follows immediately that $\tilde{J}(h, w(h)) = J_0(h)$ on $\overline{\mathrm{co}}\mathcal{M}$, what concludes the proof. □

8. Relaxation of the initial problem

By construction, see Section 3, $I(h, u(h)) = J(h, w(h))$ for all $h \in \mathcal{M}$. Exactly the same reasoning as in Section 3 gives that, for $h \in \overline{\mathrm{co}}\mathcal{M}$, $\tilde{J}(h, w(h))$ coincides with $\tilde{I}(h, u(h))$ provided that $\mathcal{F}(x, h) = \tilde{\mathcal{F}}(x, h)$ on $\Omega \times \mathrm{co}M$.

By using the duality principle with respect to the variable η in the relationship (11) that defines the function $\mathcal{F}$, we get

$$
\begin{aligned}
&\inf_{w \in \mathcal{W}^\#} \int_K \big[\big\langle E(x)(w(y)+a(x,h(y))-a(x,\hat{h})), w(y)+a(x,h(y))-a(x,\hat{h})\big\rangle \\
&+2\big\langle w(y), b(x,h(y)) - b(x,\hat{h})\big\rangle + F(x,h(y))\big]\,dy \\
&= \inf_{v_1 \in \mathcal{V}^\#} \sup_{v_2 \in \mathcal{V}^\# \oplus \mathbb{R}^{nm}} \int \big[\langle A(x)(v_1(y)+v_2(y)), v_1(y) - v_2(y)\rangle \\
&\quad + \big\langle B(x)(v_1(y)+v_2(y)), v_1(y)+v_2(y)\big\rangle \\
&\quad + 2\big\langle v_1(y)+v_2(y), g(x,h(y)) - g(x,\hat{h})\big\rangle \\
&\quad - 2\big\langle v_1(y)-v_2(y), f(x,h(y)) - f(x,\hat{h})\big\rangle + F(x,h(y))\big]\,dy.
\end{aligned}
\tag{27}
$$

Since the functions f and g are affine with respect to h, then the “right-hand sides” of Euler equations for the saddle point pair have zero mean values and we can consider elements $v_2 \in \mathcal{V}^\#$. Hence simple calculations and Euler equations for the saddle point in the right-hand side of (27) give that the value of the right-hand side in (27) is equal to the inner infimum over v in (4). That is sufficient in order to guarantee that the formulae (11) and (4) (after an appropriate extension of $\mathcal{F}$ to the whole $\Omega \times \mathbb{R}^r$) define one and the same function $\mathcal{F} = \tilde{\mathcal{F}}$.

Therefore, Theorem 5.3 gives the statement (i) of Theorem 1.1 and, together with Corollary 6.4, it gives additional properties of $\tilde{\mathcal{F}}$. Because $\tilde{I}(h, u(h))$ coincides with $\tilde{J}(h, w(h))$ on $\overline{\mathrm{co}}\mathcal{M}$, then the statements (ii) and (iii) of Theorem 1.1 now are straight consequences from Theorem 7.3. This way, we have proved the statements of Theorem 1.1.

We conclude this Section with a simple illustrative example. Consider, for $n = 2$, the problem

$$\begin{cases} I = \int_K \big[a(x)u_{x_1}^2 + b(x)u_{x_2}^2\big]dx \to \min \\ 4\Delta u = 2\frac{\partial}{\partial x_1}[f(x)h(x)], \ \ x \in K, \ u \in H_0^1(K) \\ h \in \mathcal{M} := \{h \,\text{measurable}, \ h(x) = 0 \text{ or } 1, \text{a.e. } x \in K\}, \end{cases} \tag{28}$$

where Δ denotes the Laplace operator. The relaxation of (28), according to Theorem 1.1, consists of the passage from $\mathcal{M}$ to its closed convex hull and introducing in the integrand of I an additional term $\tilde{\mathcal{F}}$. In turn, the function $\tilde{\mathcal{F}}$ is defined according to (4) as

$$\begin{aligned} \tilde{\mathcal{F}}(x,\hat{h}) := \inf \bigg\{ \int_K [a(x)v_{y_1}^2 + b(x)v_{y_2}^2]dy \ \bigg| \\ 4\Delta v(y) = 2\tfrac{\partial}{\partial y_1}[f(x)h(y)] \ \text{ in } K, \ v \text{ is } K\text{-periodic} \\ h(y) = 0 \text{ or } 1 \ \text{ a.e. } y \in K, \ \int_K h(y)dy = \hat{h}\bigg\}. \end{aligned} \tag{29}$$

The problem (28) is specific, i.e., the control variable h is a scalar function. It was shown in Raitums [7, pp. 81–83] that for problems of kind (29) the infimum over h can be obtained by means of rank-1 laminates, i.e., that it is sufficient to consider controls h in the form $h(y) = h(\langle l, y\rangle)$, $l := (l_1, l_2), l_1, l_2 \in \mathbb{Z}$. For such controls h the infimum in the right-hand side of (29) can be computed explicitely, what gives

$$\tilde{\mathcal{F}}(x,\hat{h}) = \begin{cases} 0 & \text{if } a(x) \geq 0, \ b(x) \geq 0 \\ \frac{a(x)}{4} f^2(x)\hat{h}(1-\hat{h}) & \text{if } a(x) \leq 0, \ a(x) \leq 2b(x) \\ -\frac{a^2(x)}{4(a(x)-b(x))} f^2(x)\hat{h}(1-\hat{h}) & \text{if } b(x) \leq 0, \ a(x) \geq 2b(x). \end{cases} \tag{30}$$

The formula (30) shows that the additional term (defined by $\tilde{\mathcal{F}}$) can be convex or concave depending on values $(a(x), b(x))$.

9. Appendix

In this section, we prove the basic properties of the multivalued mapping $\hat{h} \mapsto \mathcal{M}(\hat{h})$.

Lemma 9.1. *Let $M \subset \mathbb{R}^r$ be a nonempty bounded and closed set, let $\mathcal{M} := \{h \in L_2(K;\mathbb{R}^r) \,|\, h(y) \in M \text{ a.e. } y \in K\}$ and let $\mathcal{M}(\hat{h}) = \big\{h \in \mathcal{M} \ \big| \ \int_K h(y)\,dy = \hat{h}\big\}$ for $\hat{h} \in \overline{\text{co}}M$. Let $\hat{h}_0, \hat{h}_k \in \text{co}M$, $k = 1, 2, \ldots$, $\hat{h}_k \to \hat{h}_0$ as $k \to \infty$. Then for every $h_k \in \mathcal{M}(\hat{h}_k)$ there exists an element $h_{0k} \in \mathcal{M}(\hat{h}_0)$ such that*

$$\|h_k - h_{k0}\|_{L_2(K;\mathbb{R}^r)} \to 0 \quad \text{as } k \to \infty.$$

Proof. To begin with, we point out that due to the Carathéodory's theorem [4, p. 306, Theorem 4.95] the sets $\mathcal{M}(\hat{h})$ for $\hat{h} \in \mathrm{co}M$ are not empty. Let us denote by r_0 the dimension of co$M$.

`Step 1.` Let $\hat{h}_0 \in \mathrm{rico}M$. Then there exists $d > 0$ such that $\hat{h} \in \mathrm{rico}M$ whenever $\hat{h} \in \mathrm{co}M$ and $|\hat{h} - \hat{h}_0| \le d$. Let us fix $\epsilon > 0$, $0 < \epsilon < \frac{d}{4}$, and let $|\hat{h} - \hat{h}_0| \le \epsilon$. Then $\hat{h}_* := \hat{h} + \frac{d}{\epsilon(\hat{h}_0 - \hat{h})} \in \mathrm{rico}M$.

Let $h \in \mathcal{M}(\hat{h})$, $h_* \in \mathcal{M}(\hat{h}_*)$ be arbitrary chosen elements. By virtue of Lyapunov's theorem of the range of vectorial measures for every $\lambda \in [0,\, 1]$ there exists a measurable set $E_\lambda \subset K$ such that

$$|E_\lambda| = \lambda, \quad \int_{E_\lambda} h(y)\,dy + \int_{K\setminus E_\lambda} h_*(y)\,dy = \lambda\hat{h} + (1-\lambda)\hat{h}_*.$$

For a special choice $\lambda = \lambda_0 = 1 - \frac{\epsilon}{d}$ there is

$$h_0(\cdot) = \chi_{E_{\lambda_0}}(\cdot)h(\cdot) + (1 - \chi_{E_{\lambda_0}}(\cdot))h_*(\cdot) \in \mathcal{M}(\hat{h}_0)$$
$$\int_K (h(y) - h_0(y))^2\,dy = \int_{K\setminus E_{\lambda_0}} (h(y) - h_0(y))^2\,dy \le 4\mu_3^2\frac{\epsilon}{d},$$

where μ_3 is defined in H4. This way, the assertion of Lemma holds whenever $\hat{h}_0 \in \mathrm{rico}M$.

`Step 2.` Assume that $\hat{h}_0$ does not belong to rico$M$. Because rico$M$ is not empty (provided that M consists of more than one element) then there exist a vector $a \in \mathbb{R}^r$ and a constant c such that

$$|a| = 1, \quad \langle a, \hat{h}_0\rangle = c < \langle a, \hat{h}\rangle \text{ for all } \hat{h} \in \mathrm{rico}M.$$

Without loosing generality, we can assume that $c = 0$, otherwise we can use the transform $\hat{h} \mapsto \hat{h} - \hat{h}_0$.

Let $M_1 := \{h \in M \,|\, \langle a,\, h\rangle = 0\}$. Because the sets M and M_1 are compact, then there exists a continuous function γ, $\gamma(t) = 0$ if $t \le 0$, $\gamma(t) > 0$ if $t > 0$, such that

$$\langle a,\, h - \hat{h}_0\rangle \ge \gamma(\mathrm{dist}\{h; M_1\}) \quad \text{for all } h \in M. \tag{31}$$

Without loosing generality, we can assume that the function γ is convex, otherwise we can pass to the bipolar γ^{**}, which has the desired properties. By construction, for nonnegative τ there exists the inverse function γ^{-1}, $\gamma^{-1}(\gamma(t)) = t$ for $t \ge 0$, which is continuous and stritly increasing on $\{\tau \in \mathbb{R} \,|\, \tau \ge 0\}$. Now, from (31), Proposition 2.3 and convexity of γ it follows that for every chosen $h \in \mathcal{M}$ there exists an element h_*,

$$h_* \in \mathcal{M}_1 = \{h' \in L_2(K; \mathbb{R}^r) \,|\, h'(y) \in M_1 \text{ a.e. } y \in K\},$$

such that

$$\begin{aligned}\|h - h_*\|^2_{L_2(K;\mathbb{R}^r)} &\le c(r,\mu_3)\int_K |h(y) - h_*(y)|\,dy \\ &\le c(r,\mu_3)\gamma^{-1}\left(\gamma\left(\int_K |h(y) - h_*(y)|\,dy\right)\right) \\ &\le c(r,\mu_3)\gamma^{-1}\left(\int_K \langle a, h(y) - h_*(y)\rangle dy\right) \\ &\le c(r,\,\mu_3)\gamma^{-1}\left(\left|\int_K h(y)\,dy - \int_K h_*(y)\,dy\right|\right).\end{aligned}$$

This way, for our situation with a fixed $\hat{h}_0 \in \mathrm{co}M_1$, for every $\hat{h} \in \mathrm{co}M$ and arbitrary chosen $h \in \mathcal{M}(\hat{h})$ there exists a corresponding $h_* \in \mathcal{M}_1$ such that

$$\|h - h_*\|^2_{L_2(K;\,\mathbb{R}^r)} \le c(r,\,\mu_3)\gamma^{-1}(|\hat{h} - \hat{h}_0|).$$

By construction, $\int_K h_*(y)\,dy = \hat{h}_* \in \mathrm{co}M_1$, $\mathcal{M}(\hat{h}_0) \subset \mathcal{M}_1$, $\mathcal{M}(\hat{h}_*) \subset \mathcal{M}_1$ and the dimension of $\mathrm{co}M_1$ is less than r_0. From now on, we have to approximate the element $h_* \in \mathcal{M}(\hat{h}_*)$ by elements from $\mathcal{M}(\hat{h}_0)$, i.e., we have reduced the dimension r_0 of our problem to the problem with dimension less than or equal to $r_0 - 1$.

`Step 3.` To conclude our reasoning by induction over the dimension r_0 we have to prove our assertion for the case $r_0 = 1$.

Let $r_0 = 1$. If $\hat{h}_0 \in \mathrm{rico}M$, then we applay reasoning from Step 1. If $\hat{h}_0$ does not belong to $\mathrm{rico}M$, then the set M_1 from the Step 2 consists of only one element $\hat{h}_0$ and the set $\mathcal{M}_1$ consists of one constant function $h_0(y) = \hat{h}_0$ a.e. $y \in K$. For this case we can apply the same reasoning as in Step 2, what gives the assertion of Lemma for $r_0 = 1$. □

Corollary 9.2. *If $\hat{h}_0 \in \mathrm{rico}M$ and the sequence $\{\hat{h}_k\} \subset \mathrm{co}M$ converges to $\hat{h}_0$, then for every $h_0 \in \mathcal{M}(\hat{h}_0)$ there exist $h_k \in \mathcal{M}(\hat{h}_k)$, $k = 1, 2, \ldots$, such that $h_k \to h_0$ in $L_2(K;\,\mathbb{R}^r)$ as $k \to \infty$.*

Proof. The proof is the same as in Step 1. □

Remark 9.3. In general, the function $\mathcal{F}$, defined by (11), can be discontinuous with respect to $\hat{h} \in \mathrm{co}M$. We illustrate this property by a simple example. Let $n = 3$, $m = 1$, $F = 0$, A is the unit matrix,$B = -\frac{1}{4}A$, $g = 0$, $f(x,h) = h$ and

$$M := \{(-1,0,0); (1,0,0); (-1,0,1); (0,t,t^2)\}\,|\, t \in [0,1]\} \subset \mathbb{R}^3.$$

For this case $\mathcal{F}$ does not depend on x. Because $\hat{h}(t) = (0,t,t^2)$ with $t > 0$ is an extremal point of $\mathrm{co}M$ and for such $\hat{h}(t)$ the set $\mathcal{M}(\hat{h}(t))$ consists of one constant function $h(y) = \hat{h}(t)$, $y \in K$, then $\mathcal{F}((0,t,t^2)) = 0$ for all $0 < t \le 1$. On the other hand, simple calculations with laminated structures give $\mathcal{F}((0,0,0)) \le -1$.

Acknowledgements. This research was supported in part by Latvian Council of Sciences grant No. 05.1385.

The author would like to express his thanks to the anonymous referee for many constructive suggestions that considerebly improved the exposition of this paper.

References

[1] Dal Maso, G., *An Introduction to* Γ*-Convergence.* Boston: Birkhäuser 1993.

[2] Ekeland, I. and Temam, R., *Convex Analysis and Variational Problems.* Amsterdam: North-Holland 1976.

[3] Fonseca, I. and Müller, S., A-quasiconvexity, lower semicontinuity, and Young measures. *SIAM J. Math. Anal.* 30 (1999), 1355 – 1390.

[4] Fonseca, I. and Leoni, G., *Modern Methods in the Calculus of Variations:* L^p *spaces.* New York: Springer 2007.

[5] Meyers, N. G. and Elcrat, A., Some results on regularity for solutions of nonlinear elliptic systems and quasi-regular functions. *Duke Math. J.* 42 (1975), 121 – 136.

[6] Raitums, U., *Optimal Control Problems for Elliptic Equations* (in Russian). Riga: Zinatne 1981.

[7] Raitums, U., Relaxation of a weakly discontinuous functional depending on one control function. *Math. Model. Anal.* 13 (2008), 79 – 86.

[8] Raitums, U. and Schmidt, W. H., On necessary optimality conditions for optimal control problems governed by elliptic systems. *Optimization* 54 (2005), 149 – 160.

[9] Rockafellar, R. T., *Convex Analysis.* Princeton: Princeton Univ. Press 1997.

[10] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions.* Princeton: Princeton Univ. Press 1970.

[11] Zhikov, V. V., Kozlov, S. M. and Oleinik, O. A., *Homogenization of Differential Operators and Integral Functionals.* New York: Springer 1994.

Received February 19, 2008; revised April 8, 2009