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Regularity and Derivative Bounds for a Convection-Diffusion Problem with Neumann Boundary Conditions on Characteristic Boundaries

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Abstract. A convection-diffusion problem is considered on the unit square, with convection parallel to two of the square's sides. Dirichlet conditions are imposed on the inflow and outflow boundaries, with Neumann conditions on the other two sides. No assumption is made regarding the corner compatibility of the data. The regularity of the solution is expressed precisely in terms of the regularity and compatibility of the data. Pointwise bounds on all derivatives of the solution are derived and their dependence on the data regularity, its corner compatibility, and on the small diffusion parameter is made explicit. These results extend previous bounds of Jung and Temam [Int. J. Numer. Anal. Model. 2 (2005) 367–408] and of Clavero, Gracia, Lisbona and Shishkin [Z. Angew. Math. Mech. 82 (2002) 631–647].

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1. Introduction

Bounds on derivatives of solutions to singularly perturbed convection-diffusion boundary value problems are of importance for two main reasons: they reveal the fine structure of the solution, and they are also needed in the analysis of numerical methods for such problems. While many papers on this topic address ordinary differential equations, progress for problems posed in two dimensions has been much slower – relatively few papers, such as [1, 2, 6, 7, 9, 11, 12], rigorously prove pointwise derivative bounds for singularly perturbed problems posed in two-dimensional domains.

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The problem studied in this paper is posed on the unit square with Dirichlet conditions along $x = 0$ and $x = 1$ and Neumann conditions along $y = 0$ and $y = 1$. Related problems on the unit square but with different boundary conditions were considered in [6, 7, 11]. Some of this earlier work coincides with what is here and we shall exploit this overlap fully; nevertheless the alteration of the boundary conditions changes significantly the nature of the solution.

Problems similar to ours are considered by Clavero et al. [1] and by Jung and Temam [5], but in both these papers the analysis is simplified by an assumption that the data satisfies certain corner compatibility conditions that exclude corner singularities. We make no such assumption. Furthermore, unlike these earlier papers, we make precise the relationship between the given data and the regularity of the solution.

In [1] Robin conditions are imposed along $y = 0$ and $y = 1$ and pointwise bounds are proved under strong compatibility assumptions at the corners of the domain, but the arguments are not written down in full. Our pointwise bounds on derivatives of the solution agree with those of [1] when their corner compatibility conditions are satisfied. In [5] only L_2 and H^1-type bounds on derivatives are obtained, and our pointwise bounds imply these weaker estimates.

Shishkin [13] considers a convection-diffusion problem on a two-dimensional rectangle with Dirichlet boundary data, where the differential operator has variable coefficients, with small parameters ε_1 multiplying the second-order derivatives and ε_2 multiplying one of the first-order derivatives. When $\varepsilon_2 = 0$ this problem is related to our problem (1). The techniques of [13] are suitable for studying problems with stronger parabolic layers than those considered here.

The problem we shall consider is as follows. Let $u(x, y)$ be the solution to the boundary value problem

$$\begin{aligned} Lu(x,y) :&= -\varepsilon\Delta u(x,y) + pu_x(x,y) + qu(x,y) \\ &= f(x,y) \quad \forall\ (x,y) \in Q = (0,1)^2 \end{aligned} \tag{1a}$$

$$u_y(x,0) = h_s(x), \quad u_y(x,1) = h_n(x) \qquad \text{for } 0 < x < 1 \tag{1b}$$

$$u(0,y) = g_w(y), \quad u(1,y) = g_e(y) \qquad \text{for } 0 < y < 1. \tag{1c}$$

The constants p and q satisfy $p > 0$, $q > 0$. As we are interested in the singularly perturbed case, without loss of generality the diffusion parameter ε satisfies $0 < \varepsilon \le \min\left\{1, 12\frac{p^2}{q}\right\}$. Assume that $f(x, y)$ and the boundary data lie in certain Hölder spaces:

$$f \in C^{2\ell,\alpha}(\bar{Q}),\ g_w, g_e \in C^{2\ell,\alpha}[0,1] \tag{2a}$$

$$\int_0^x h_s(t)\,dt \in C^{2\ell,\alpha}[0,1],\ \int_0^x h_n(t)\,dt \in C^{2\ell,\alpha}[0,1], \tag{2b}$$

for some non-negative integer ℓ and $\alpha \in (0, 1)$. If $\ell > 0$, the condition on h_s and h_n is equivalent to requiring $h_s, h_n \in C^{2\ell-1,\alpha}[0, 1]$.

By a solution of (1) we mean a function $u \in C^{2,\alpha}(Q)$ that satisfies (1a) and can be extended up to the boundary to satisfy (1b)–(1c). Existence and uniqueness of this solution can be shown by combining techniques from [3, 14].

The purpose of this paper is to derive pointwise derivative bounds for the solution of (1), while making explicit their dependence on the parameter ε and on the corner compatibility and regularity of the data.

A typical solution to (1) will have an exponential boundary layer along $x = 1$ and weaker parabolic boundary layers along $y = 0$ and $y = 1$, with weak corner layers at the outflow corners $(1, 0)$ and $(1, 1)$. Furthermore, depending on the compatibility of the data at corners, the solution may contain corner singularities. We decompose the solution to (1) as

$$u = S + E + w_{00} + w_{01} + w_{10} + w_{11} + \breve{u} \quad \text{in } Q; \tag{3}$$

here each function (except the remainder $\breve{u}$) is the solution of a simpler half-plane or quarter-plane problem. The definitions of these functions will be given later, but for convenience it is summarized in Table 1, where the column labelled "L" gives the output when L is applied to each function in the first column and the other columns show the boundary conditions for each problem.

	L	Dirichlet cond. at $x = 0$	Dirichlet condition at $x = 1$	Neumann condition at $y = 0$	Neumann condition at $y = 1$
S	$f^*(x, y)$	$g_w^*(y)$			
E	0		$g_e^*(y) - S(1, y)$		
w_{00}	0	0		$h_s^*(x) - S_y(x, 0)$	
w_{01}	0	0			$h_n^*(x) - S_y(x, 1)$
w_{10}	0		$-\chi(1-y)w_{00}(1, y)$	$-\chi(x)E_y(x, 0)$	
w_{11}	0		$-\chi(y)w_{01}(1, y)$		$-\chi(x)E_y(x, 1)$
$\breve{u}$	0	$\breve{g}_w(y)$	$\breve{g}_e(y)$	$\breve{h}_s(x)$	$\breve{h}_n(x)$

where
$$\begin{aligned}
\breve{g}_w(y) &= -E(0, y) - w_{10}(0, y) - w_{11}(0, y) \\
\breve{g}_e(y) &= -[1 - \chi(1 - y)]w_{00}(1, y) - [1 - \chi(y)]w_{01}(1, y) \\
\breve{h}_s(x) &= -w_{01,y}(x, 0) - w_{11,y}(x, 0) - [1 - \chi(x)]E_y(x, 0) \\
\breve{h}_n(x) &= -w_{00,y}(x, 1) - w_{10,y}(x, 1) - [1 - \chi(x)]E_y(x, 1).
\end{aligned}$$

Table 1: The decomposition used for u.

The functions f^*, g_w^*, g_e^*, h_s^* and h_n^* are smooth extensions of f, g_w, g_e, h_s and h_n, respectively, that vanish outside some bounded set. In our notation the letter h is in general reserved for Neumann boundary conditions, while g is used for Dirichlet boundary conditions.

To ensure existence and uniqueness of the functions defined in (3) as solutions of problems on unbounded domains, one must impose certain growth restrictions at infinity. We do not state these explicitly in this current paper as their derivation is routine but tedious; see [6, Section 3.2] for an example of this. All the barrier functions that we shall employ on unbounded domains satisfy the requisite growth conditions. Some of these make use of the assumption that $q > 0$; cf. [6, Lemma 3.5].

To bound the derivatives of u, we shall derive bounds separately for each function in the decomposition (3). One of our estimates (Lemma 4.2) sharpens a similar bound obtained in [6].

The function S is the principal component in the solution of u. It provides a good approximation of u on Q except near the ordinary and parabolic layers and the corner singularities. The other terms in the decomposition handle these more difficult regions, as we now describe. The function E is a correction to S along $x = 1$ that yields the correct boundary condition there for u. The boundary data for u along $y = 0$ is provided by w_{00}; any corner singularity in u at (0,0) is also contained in w_{00}. The function w_{01} performs a role analogous to w_{00} along $y = 1$ and at (0,1). Any corner singularities at (1,0) and (1,1) are contained in w_{10} and w_{11}; these two functions also correct some of the boundary data of the earlier functions. Any boundary data not accounted for at this juncture is corrected by the remainder function $\breve{u}$.

1.1. Notation. Set $\Pi_x = \{(x, y) \in \mathbb{R}^2 : x > 0\}$, $\Pi_y = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $\mathbb{Q} = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$. For various measurable sets Ω, with integers $k \geq 0$ and $p \geq 1$, and $0 < \alpha < 1$, let $W^{k,p}(\Omega)$ denote the usual Sobolev space of functions on Ω whose weak derivatives of order at most k are in $L_p(\Omega)$, while $C^{k,\alpha}(\Omega)$ denotes the space of Hölder-continuous functions on Ω. Set $H^k(\Omega) = W^{k,2}(\Omega)$ and write $\|\cdot\|_{k,p,\Omega}$ for the norm in $W^{k,p}(\Omega)$.

Finally, we use C to denote a generic constant that is independent of the parameter ε but may depend on the remaining data of (1); note that C can take different values in different places during our analysis.

2. Compatibility conditions

To discuss the regularity of u on the closed domain $\bar{Q}$, one must consider the compatibility of the data of (1) at the corners of $\bar{Q}$. Number the corners (0,1), (0,0), (1,0) and (1,1) as 1, 2, 3 and 4, respectively. The data for the problem (1) is given by the 5-tuple $X = (g_w, g_e, h_s, h_n, f)$. Let $\ell \geq 1$ be an integer and let $0 < \alpha < 1$. For each integer $k \geq 2$, define the Banach space

$$\mathcal{D}_{k,\alpha} = \left(C^{k-1,\alpha}[0, 1]\right)^2 \times \left(C^{k-2,\alpha}[0, 1]\right)^2 \times C^{k-3,\alpha}(\bar{Q}).$$

For the Poisson equation $\Delta w = f$ with boundary conditions (1b) and (1c), necessary and sufficient conditions for the solution to belong to the space $C^{k,\alpha}(\bar{Q})$ were established by Volkov [14]. Corresponding conditions for the problem (1) are given in Theorem 2.1, which can be derived similarly to [11, Theorem 2.1]; see [10] for details.

Theorem 2.1. *Let $\ell \geq 1$ and ν be integers. Let $X \in \mathcal{D}_{2\ell,\alpha}$. Let u be the solution of* (1) *with data X. Then there are numbers $a^{(i)}_{\mu,\nu}$, $i = 1,\dots,4$, $\mu \geq 0$ and $b^{(i)}_{\mu_1,\mu_2,\nu}$, $i = 1,\dots,4$, $\mu_1 \geq 0$, $\mu_2 \geq 0$, which depend only on ε, p and q, such that when one sets* (*where each sum is interpreted as 0 if the upper limit is less than the lower limit*)

$$\Lambda^{(1)}_\nu(X) = g_w^{(2\nu+1)}(1) + \sum_{\mu=0}^{2\nu} a^{(1)}_{\mu,\nu} h_n^{(\mu)}(0) + \sum_{\mu_1+\mu_2\leq 2\nu-1} b^{(1)}_{\mu_1,\mu_2,\nu} D_x^{\mu_1} D_y^{\mu_2} f(0,1)$$

$$\Lambda^{(2)}_\nu(X) = g_w^{(2\nu+1)}(0) + \sum_{\mu=0}^{2\nu} a^{(2)}_{\mu,\nu} h_s^{(\mu)}(0) + \sum_{\mu_1+\mu_2\leq 2\nu-1} b^{(2)}_{\mu_1,\mu_2,\nu} D_x^{\mu_1} D_y^{\mu_2} f(0,0)$$

$$\Lambda^{(3)}_\nu(X) = g_e^{(2\nu+1)}(0) + \sum_{\mu=0}^{2\nu} a^{(3)}_{\mu,\nu} h_s^{(\mu)}(1) + \sum_{\mu_1+\mu_2\leq 2\nu-1} b^{(3)}_{\mu_1,\mu_2,\nu} D_x^{\mu_1} D_y^{\mu_2} f(1,0)$$

$$\Lambda^{(4)}_\nu(X) = g_e^{(2\nu+1)}(1) + \sum_{\mu=0}^{2\nu} a^{(4)}_{\mu,\nu} h_n^{(\mu)}(1) + \sum_{\mu_1+\mu_2\leq 2\nu-1} b^{(4)}_{\mu_1,\mu_2,\nu} D_x^{\mu_1} D_y^{\mu_2} f(1,1),$$

then $u \in C^{2\ell-1,\alpha}(\bar{Q})$ if and only if

$$\Lambda^{(i)}_\nu(X) = 0 \quad \textit{for } i = 1,2,3,4, \quad \textit{and } \nu = 0,1,\dots,\ell-1. \tag{4}$$

Furthermore, if (4) *holds and $X \in \mathcal{D}_{2\ell+1,\alpha}$, then $u \in C^{2\ell,\alpha}(\bar{Q})$. If $\nu \leq \ell - 1$, the expressions $\Lambda^{(i)}_\nu$ for $i = 1,2,3,4$ define bounded linear functionals on $\mathcal{D}_{2\ell,\alpha}$.*

By assumption (2) the data $X \in \mathcal{D}_{2\ell+1,\alpha}$ (in fact for the proof of Lemma 3.1 the assumption on f in (2) is stronger than this; see also the comment preceding Lemma 4.2).

2.1. Definition. Given a set of data $X \in \mathcal{D}_{2\ell+1,\alpha}$, define a compatibility index ν at each vertex as follows: set $jk = 10, 00, 01, 11$ if $i = 1,2,3,4$ respectively, then for each couple jk set $\nu_{jk}(X) = m$ if $\Lambda^{(i)}_\nu(X) = 0$ for $\nu = 0,\dots,m$ and $\Lambda^{(i)}_{m+1}(X) \neq 0$. If $\Lambda^{(i)}_0(X) \neq 0$, set $\nu_{jk}(X) = -1$.

3. Smooth component S

The first component in the decomposition (3) of u is the function S. Let f^* and g_w^* be smooth extensions of f and g_w to Π_x and $(-\infty,\infty)$ respectively that vanish outside some bounded neighbourhoods of $\bar{Q}$ and $[0,1]$ respectively. Similar extensions g_e^*, h_s^* and h_n^* of g_e, h_s and h_n are used later.

Define S to be the solution of the half-plane problem

$$LS = f^* \text{ for } (x,y) \in \Pi_x, \qquad S(0,y) = g_w^*(y) \text{ for } -\infty < y < \infty.$$

Then $S \in C^{2\ell,\alpha}(\bar{\Pi}_x)$. The same function appears in [11]; from there one has

Lemma 3.1 ([11, Theorem 3.2])**.** *There exists a constant C such that*

$$\|S\|_{m+n,\infty,\Pi_x} \le C\left(\|f^*\|_{m+n,\infty,\Pi_x} + \|g_w^*\|_{C^{m+n,\alpha}(\mathbb{R})}\right) \quad \textit{for } m+n \le 2\ell. \tag{5}$$

Recalling that the parameter ε may be close to zero and the constant C in (5) is independent of ε, we surmise that the regularity demanded of the data f in Lemma 3.1 is optimal while that demanded of g_w^* is slightly suboptimal.

4. Exponential layer component E

Let E be the solution to the half-plane problem

$$LE = 0 \qquad \text{for } x < 1, -\infty < y < \infty \tag{6a}$$
$$E(1,y) = -S(1,y) + g_e^*(y) \quad \text{for } -\infty < y < \infty. \tag{6b}$$

This function E is the same as that defined in [6, Section 5]. Here we use Fourier transforms to bound the derivatives of E as this requires less regularity than the approach followed in [6].

For convenience set $W(x,y) = E(1-x,y)$. Then

$$L^*W := -\varepsilon W_{xx} - \varepsilon W_{yy} - pW_x + qW = 0 \quad \text{on } \Pi_x \tag{7a}$$
$$W(0,y) = g(y) \qquad \text{for } -\infty < y < \infty, \tag{7b}$$

where $g(y) := -S(1,y) + g_e^*(y)$.

We shall need the following Mikhlin multiplier result.

Theorem 4.1 ([4, Theorem 6.2.3])**.** *Let $M \in C^1(\mathbb{R})$. Let K be a constant (independent of η) such that*

$$|M^{(j)}(\eta)| \le K(1+|\eta|)^{-j} \quad \textit{for } j = 0,1 \textit{ and } \eta \in \mathbb{R}. \tag{8}$$

Let $s \in C^{k,\alpha}(\mathbb{R})$ for some non-negative integer k. Define $h \in L_2(\mathbb{R})$ implicitly from its Fourier transform $\hat{h}$ by setting $\hat{h}(\eta) = M(\eta)\hat{s}(\eta)$. Then $h \in C^{k,\alpha}(\mathbb{R})$ and there exists a constant C such that

$$\|h\|_{C^{k,\alpha}(\mathbb{R})} \le CK\|s\|_{C^{k,\alpha}(\mathbb{R})}.$$

Theorem 4.1 is now used to establish bounds for derivatives of E. The bounds in the next lemma require more regularity of the data than one would expect from Theorem 2.1 because we need the constant C to be independent of ε. Recall that $f^* \in C^{2\ell,a}(\Pi_x)$ and $g_w^*,\ g_e^* \in C^{2\ell,\alpha}(\mathbb{R})$.

Lemma 4.2. *Let m and n be non-negative integers and let $\check{p} \in (0, p)$. Then for $m + n \leq 2\ell - 1$ one has*

$$|D_x^m D_y^n E(x,y)| \leq C \left[\|f^*\|_{m+n+1,\infty,\Pi_x} + \|g_w^*\|_{C^{m+n+1,\alpha}(\mathbb{R})} \right.$$
$$\left. + \|g_e^*\|_{C^{m+n,\alpha}(\mathbb{R})} \right] \varepsilon^{-m} e^{-\frac{\check{p}(1-x)}{\varepsilon}} \qquad \textit{for } (x,y) \in \Pi_x.$$

Proof. Define the Fourier transform of W with respect to y by

$$\widehat{W}(x,\eta) = (\mathcal{F}W)(x,\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(x,y) e^{-\imath y\eta} dy.$$

Similarly define the Fourier transform $\hat{g}$ of g. The Fourier transform of the problem satisfied by W is

$$\varepsilon\eta^2 \widehat{W} - \varepsilon \widehat{W}_{xx} - p\widehat{W}_x + q\widehat{W} = 0 \qquad \text{for } x > 0,\ -\infty < \eta < \infty \tag{9a}$$
$$\widehat{W}(0,\eta) = \hat{g}(\eta) \qquad \text{for } -\infty < \eta < \infty. \tag{9b}$$

Setting $r(\eta) = \frac{1}{2\varepsilon}\big(p + \sqrt{p^2 + 4\varepsilon^2\eta^2 + 4\varepsilon q}\big)$, one can verify that the solution to (9) is $\widehat{W}(x,\eta) = e^{-r(\eta)x}\hat{g}(\eta)$. Hence

$$\mathcal{F}(D_x^m W)(x,\eta) = D_x^m \widehat{W}(x,\eta) = (-r(\eta))^m e^{-r(\eta)x} \hat{g}(\eta). \tag{10}$$

Set $\hat{g}_m(\eta) = (1 + \imath\eta)^m \hat{g}(\eta)$. Then (10) can be rewritten as

$$D_x^m \widehat{W}(x,\eta) = (-r(\eta))^m (1 + \imath\eta)^{-m} e^{-r(\eta)x} \hat{g}_m(\eta). \tag{11}$$

Since $g_m(y) = (1 + \frac{d}{dy})^m g(y)$ is a linear combination of derivatives of g of order at most m, it is clear that

$$\|g_m\|_{C^{0,\alpha}(\mathbb{R})} \leq C\|g\|_{C^{m,\alpha}(\mathbb{R})} \quad \text{provided that } g \in C^{m,\alpha}(\mathbb{R}). \tag{12}$$

We shall use $M_m(\eta) = (-r(\eta))^m (1 + \imath\eta)^{-m} e^{-r(\eta)x}$ as the Mikhlin multiplier. In [11, Lemma 4.1] a related multiplier $M(\eta)$ is used; in fact $|M_m(\eta)| = |M(\eta)r(\eta)(1 + \imath\eta)^{-1}|$ and $M(\eta)$ satisfies (8) with $K = C\varepsilon^{1-m} e^{-\frac{\check{p}x}{\varepsilon}}$. Combining these facts with the inequalities $|r(\eta)| \leq C(|\eta| + \varepsilon^{-1})$ and $|r'(\eta)| \leq C$, one sees that $M_m(\eta)$ satisfies (8) with $K = C\varepsilon^{-m} e^{-\frac{\check{p}x}{\varepsilon}}$. Applying Theorem 4.1 with the multiplier $M_m(\eta)$ to (11) yields

$$\|D_x^m W(x,\cdot)\|_{C^{0,\alpha}(\mathbb{R})} \leq C\varepsilon^{-m} e^{-\frac{\check{p}x}{\varepsilon}} \|g_m\|_{C^{0,\alpha}(\mathbb{R})} \leq C\varepsilon^{-m} e^{-\frac{\check{p}x}{\varepsilon}} \|g\|_{C^{m,\alpha}(\mathbb{R})},$$

where we used (12). On recalling the definitions of W and g and invoking (5), the lemma is proved in the case $n = 0$.

Now suppose that $n \geq 1$. Set $v(x,y) = D_y^n W(x,y)$, apply D_y^n to (7), and apply the case $n = 0$ result to the function v. □

Remark 4.3. Lemma 4.2 requires less data regularity than the bound

$$|D_x^m D_y^n E(x,y)| \le C[\|g_e\|_{2\ell,\infty,(0,1)} + \|S(1,\cdot)\|_{2\ell,\infty,\mathbb{R}}]\varepsilon^{-m} e^{-\frac{p(1-x)}{\varepsilon}}$$

of [6, p.119].

The following result will be needed in the proof of Lemma 6.2.

Lemma 4.4. *For $m \le 2\ell - 2$ there is a constant C such that*

$$\left| D_x^m \left(e^{\frac{px}{\varepsilon}} E_y(1-x,y) \right) \right| \le C \left(\|f^*\|_{m+1,\infty,\Pi_x} + \|g_w^*\|_{m+2,\infty,\mathbb{R}} + \|g_e^*\|_{m+1,\infty,\mathbb{R}} \right)$$

on Π_x.

Proof. Apply D_y to (6) to get $LE_y = 0$ for $x < 1$, $E_y(1,y) = (g_e^*)'(y) - S_y(1,y)$. This is a problem similar to (6) but with one degree less regularity. Set $W_1(x,y) = e^{\frac{px}{\varepsilon}} E_y(1-x,y)$; then $LW_1 = 0$ on Π_x and $W_1(0,y) = -S_y(1,y) + (g_e^*)'(y)$. Now invoking Lemma 3.1 yields the desired result. $\square$

5. Incoming corner functions

At this stage of our construction the function $S + E$ essentially matches the boundary data for u in Q along the inflow ($x = 0$, where E is exponentially small) and outflow ($x = 1$) boundaries but may not agree with $\frac{\partial u}{\partial y}$ on the sides $y = 0, 1$. The incoming corner function w_{00} will handle the Neumann boundary data along the side $y = 0$ and any corner singularity at the point $(0,0)$. A related problem in [6] defines an incoming corner function z_{00} with Dirichlet boundary conditions; our analysis is based on [6, Section 2] but has many differences, especially in the construction of the function ζ below.

Define $w_{00} \in L_2(\mathbb{Q})$ to be the solution of the quarter-plane problem

$$\begin{aligned} Lw_{00} &= 0 && \text{on } \mathbb{Q} && \text{(13a)}\\ w_{00,y}(x,0) &= h_s^*(x) - S_y(x,0) && \text{for } x > 0 && \text{(13b)}\\ w_{00}(0,y) &= 0 && \text{for } y > 0. && \text{(13c)} \end{aligned}$$

By the Lax-Milgram theorem, this problem is well-posed in $H^1(\mathbb{Q})$. Then [8, Sections 10 & 12] imply that $w_{00} \in C^{2\ell,\alpha}(\mathbb{Q})$. The well-posedness of later boundary value problems can be handled similarly.

We shall present our arguments in the setting of a general quarter-plane problem so that the results can be applied to the later problem (32) as well as to (13). Set $\beta = \min\left\{\frac{p}{12}, \frac{q}{2p}, \sqrt{q}\right\}$. Assume that g and h are functions with

$g \in C^{2\ell,\alpha}(\mathbb{R}^+)$, $\int_0^x h(x)\, dx \in C^{2\ell,\alpha}(\mathbb{R}^+)$ for some integer $\ell \geq 0$ and $\alpha \in (0,1)$, and that for some constants $\bar{G}_{2\ell}$ and $\bar{H}_{2\ell}$ one has

$$|g^{(k)}(y)| \leq \bar{G}_{2\ell}\varepsilon^{\frac{1-k}{2}} e^{-\frac{\beta y}{2\sqrt{\varepsilon}}}, \quad \text{for } k = 0,\ldots,2\ell \tag{14a}$$

$$|h^{(k)}(x)| \leq \bar{H}_{2\ell}, \quad \text{for } k = 0,\ldots,2\ell-1 \tag{14b}$$

(for $\ell = 0$, take $|h(x)| \leq \bar{H}_0$). Define the function w by

$$\begin{aligned} Lw &= 0 && \text{on } \mathbb{Q} \\ w_y(x,0) &= h(x) && \text{for } x > 0 \\ w(0,y) &= g(y) && \text{for } y > 0. \end{aligned}$$

Assume that w has compatibility index ν at $(0,0)$.

The function w will be decomposed into a sum of solutions of half-plane problems. Extend $h(x)$ to a function $h_1(x)$ that vanishes for $x \leq -1$ with $\int_{-\infty}^x h_1(t)\, dt \in C^{2\ell,\alpha}(\mathbb{R})$. Let $w_1 \in C^{2\ell}(\bar{\Pi}_y)$ satisfy the grazing half-plane problem

$$Lw_1 = 0 \quad \text{on } \Pi_y \tag{15a}$$

$$w_{1,y}(x,0) = h_1(x) \quad \text{for } x \in \mathbb{R}. \tag{15b}$$

Write $\hat{w}_1(\xi,y)$ for the partial Fourier transform of w_1 with respect to x. Then transforming (15) yields

$$\begin{aligned} \varepsilon\xi^2\hat{w}_1 - \varepsilon\hat{w}_{1,yy} + \imath p\xi\hat{w}_1 + q\hat{w}_1 &= 0 && \text{for } -\infty < \xi < \infty,\ y > 0 \\ \hat{w}_{1,y}(\xi,0) &= \hat{h}_1(\xi) && \text{for } \xi \in \mathbb{R}. \end{aligned} \tag{16}$$

Let $r(\xi) = \sqrt{\xi^2 + q\varepsilon^{-1} + \imath p\xi\varepsilon^{-1}}$. One can verify that $r(\xi) = s + \imath t$ where $s = \frac{1}{\sqrt{2}}\left[\xi^2 + q\varepsilon^{-1} + \sqrt{(\xi^2+q\varepsilon^{-1})^2 + p^2\xi^2\varepsilon^{-2}}\,\right]^{\frac{1}{2}} > 0$ and $t = \frac{p\xi\varepsilon^{-1}}{2s}$. Thus

$$\hat{w}_1(\xi,y) = -\frac{\hat{h}_1(\xi)}{\sqrt{\xi^2 + q\varepsilon^{-1} + \imath p\xi\varepsilon^{-1}}}\, e^{-r(\xi)y} \tag{17}$$

is the solution of (16).

We bound the pure y-derivatives of w_1 in Lemma 5.1 and the remaining derivatives of w_1 in Lemma 5.2.

Lemma 5.1. *For $n \leq 2\ell$ there is a constant C such that*

$$\|w_1(\cdot,y)\|_{C^{n,\alpha}(\mathbb{R})} \leq Ce^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}}\varepsilon^{\frac{1-n}{2}}\|h_1\|_{C^{n,\alpha}(\mathbb{R})}.$$

Proof. Assume that $n \geq 1$ as the case $n = 0$ can be proved by a similar argument. From (17), using the same idea that took us from (10) to (11), for $n \geq 1$ we write

$$\mathcal{F}(D_y^n w_1)(\xi,y) = D_y^n\hat{w}_1(\xi,y) = e^{-r(\xi)y}[-r(\xi)]^{n-1}(1+\imath\xi)^{1-n}\hat{h}_{1,n-1}(\xi), \tag{18}$$

where $\hat{h}_{1,n-1}(\xi) = (1+i\xi)^{n-1}\hat{h}_1(\xi)$. Thus for $n = 1, \ldots, 2\ell$ one has $h_{1,n-1}(x) = (1+\frac{d}{dx})^{n-1}h_1(x)$ and $\|h_{1,n-1}\|_{C^{0,\alpha}(\mathbb{R})} \le C\|h_1\|_{C^{n-1,\alpha}(\mathbb{R})}$. We shall apply Theorem 4.1 to (18) with the multiplier

$$M(\xi) = [-r(\xi)]^{n-1}(1+\imath\xi)^{1-n}e^{-r(\xi)y} \quad \text{and} \quad K = C\varepsilon^{\frac{1-n}{2}}e^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}}.$$

Note that $|\Re(r(\xi))| = s \ge \sqrt{\xi^2 + q\varepsilon^{-1}} \ge \frac{|\xi|+\sqrt{q}\varepsilon^{-\frac{1}{2}}}{\sqrt{2}}$. Also $\left|\frac{r(\xi)}{1+\imath\xi}\right| \le C\varepsilon^{-\frac{1}{2}}$. Hence $|M(\xi)| \le K$. Now $r'(\xi) = \frac{2\varepsilon\xi+\imath p}{2\varepsilon\sqrt{\xi^2+q\varepsilon^{-1}+\imath p\xi\varepsilon^{-1}}}$ so

$$\begin{aligned} M'(\xi) = (-1)^{n+1}e^{-r(\xi)y}\big\{&(n-1)[r(\xi)]^{n-2}r'(\xi)(1+\imath\xi)^{1-n} \\ &-r'(\xi)y[r(\xi)]^{n-1}(1+\imath\xi)^{1-n} + \imath[r(\xi)]^{n-1}(1-n)(1+\imath\xi)^{-n}\big\}. \end{aligned}$$

It follows that

$$|M'(\xi)| \le C\left|[r(\xi)]^{n-1}(1+\imath\xi)^{1-n}e^{-r(\xi)y}\right|\left\{|[r(\xi)]^{-1}r'(\xi)| + |r'(\xi)|y + |(1+\imath\xi)^{-1}|\right\}.$$

Recall that $r = s + \imath t$; now $|t| = \frac{p|\xi|\varepsilon^{-1}s}{2s^2} \le s$ from the formula for s. Hence

$$|r(\xi)ye^{-r(\xi)y}| \le \sqrt{2}sye^{-sy} \le Ce^{-\frac{sy}{\sqrt{2}}} \le Ce^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}}.$$

Using this inequality to bound the second term in $\{\cdots\}$ above, we get

$$\begin{aligned} |M'(\xi)| &\le C\varepsilon^{\frac{1-n}{2}}e^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}}\left\{|[r(\xi)]^{-1}r'(\xi)| + |(1+\imath\xi)^{-1}|\right\} \\ &\le C\varepsilon^{\frac{1-n}{2}}e^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}}\left[\left|\frac{2\varepsilon\xi+\imath p}{2\varepsilon\xi^2+2q+\imath 2p\xi}\right| + \frac{1}{1+|\xi|}\right] \\ &\le \frac{K}{1+|\xi|}, \end{aligned}$$

on considering separately the cases $|\xi| \le 1$ and $|\xi| > 1$. Thus $M(\xi)$ is seen to satisfy (8) with $j = 1$. The lemma follows from Theorem 4.1. $\square$

We now proceed to bound all derivatives of w_1 in terms of the norm $\|\cdot\|_{\infty,\Pi_x}$.

Lemma 5.2. *For $m + n \le 2\ell - 1$ there is a constant C such that*

$$\|D_x^m D_y^n w_1\|_{\infty,\Pi_y} \le Ce^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}}\varepsilon^{\frac{1-n}{2}}\|h_1\|_{m+n+1,\infty,\mathbb{R}}. \tag{19}$$

Proof. Let $v(x,y) = D_x^m w_1(x,y)$. Applying D_x^m to (15) yields $Lv = 0$ on Π_y with $v(x,0) = D_x^m h_1(x)$ for $x \in \mathbb{R}$. Then an invocation of Lemma 5.1 gives $\|w_1(\cdot,y)\|_{C^{m+n,\alpha}(\mathbb{R})} \le Ce^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}}\varepsilon^{\frac{1-n}{2}}\|h_1\|_{C^{m+n,\alpha}(\mathbb{R})}$. The inequality (19) follows. $\square$

Continuing with the decomposition, set $w_2(x,y) = w(x,y) - w_1(x,y)$ and $g_2(y) = g(y) - w_1(0,y)$. Then w_2 satisfies the quarter-plane problem

$$\begin{aligned} Lw_2 &= 0 && \text{on } \mathbb{Q} \\ w_2(0,y) &= g_2(y) && \text{for } y > 0 \\ w_{2,y}(x,0) &= 0 && \text{for } x > 0. \end{aligned}$$

Like w, the function w_2 will have compatibility index ν at $(0,0)$, which implies $g_2^{(2k+1)}(+0) = 0$ for $k = 0,\dots,\nu$. Let $w_3(x,y)$ and $g_3(y)$ be even extensions of w_2 and g_2 for $y < 0$. The functions $w_{3,x}$, $w_{3,xx}$ and $w_{3,yy}$ are even functions of y and hence continuous across $y = 0$. Furthermore, $w_{3,y}$ is an odd function of y and $w_{3,y}(x,0) = 0$ so $w_{3,y}(x,y)$ is continuous across $y = 0$. Thus $w_3(x,y)$ is a classical solution of the boundary value problem $Lw_3 = 0$ on Π_x, $w_3(0,y) = g_3(y)$ for $y \in \mathbb{R}$.

Next, we deal with discontinuities in derivatives of $g_3(y)$. As g_3 is even, its even-order derivatives are automatically continuous on $\mathbb{R}$. Define the numbers $d_0, d_1, \dots, d_{\nu+1}$ by setting

$$\sum_{\mu=0}^{\nu+1} d_\mu 2^{2k\mu} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = 1,\dots,\nu+1. \end{cases} \tag{20}$$

(If $\nu = -1$, set $d_0 = 1$.) This Vandermonde system of linear equations has a unique solution $d_0,\dots,d_{\nu+1}$. Define the even function $\zeta(y) = \sum_{j=\nu+1}^{\ell} c_j \zeta_j(y)$ for all $y \in \mathbb{R}$, where

$$\zeta_j(y) = \sqrt{\varepsilon} \sum_{\mu=0}^{\nu+1} d_\mu (\sqrt{q}+j) 2^\mu \exp\left\{ -\frac{(\sqrt{q}+j)2^\mu |y|}{\sqrt{\varepsilon}} \right\} \quad \text{for } j = \nu+1,\dots,\ell \tag{21}$$

and the $\ell - \nu$ numbers c_j will be chosen shortly. For all k, clearly

$$\zeta_j^{(2k+1)}(\pm 0) = \mp \varepsilon^{-k} (\sqrt{q}+j)^{2k+2} \sum_{\mu=0}^{\nu+1} d_\mu 2^{(2k+2)\mu}. \tag{22}$$

From (20) and (22) we see that $\zeta_j^{(2k+1)}(y)$ is continuous at $y = 0$ for $k = 0,\dots,\nu$. All even-order derivatives of ζ_j are automatically continuous at $y = 0$. Thus $\zeta_j \in C^{2k+2}(\mathbb{R})$. To specify the $\ell - \nu$ numbers c_j, we first impose the $\ell - \nu - 1$ conditions

$$\zeta^{(2k+1)}(+0) = g_3^{(2k+1)}(+0) \quad \text{for } k = \nu+1, \cdots, \ell-1.$$

Equivalently, using (22) and $\zeta = \sum_{j=\nu+1}^{\ell} c_j \zeta_j$, we require

$$\varepsilon^k g_3^{(2k+1)}(+0) = -\left(\sum_{\mu=0}^{\nu+1} d_\mu 2^{(2k+2)\mu} \right) \sum_{j=\nu+1}^{\ell} c_j (\sqrt{q}+j)^{2k} \tag{23}$$

for $k = \nu+1,\dots,\ell-1$.

Recall that $g_3(y)$ is an even extension of $g_2(y)$ and $g_2(y) = g(y) - w_1(0, y)$, so $g_2 \in C^{2\ell,\alpha}(\mathbb{R})$. From (14) and Lemma 5.2 one sees that

$$\varepsilon^k |g_3^{(2k+1)}(y)| \leq C \quad \text{for } k = 0, \dots, \ell. \tag{24}$$

Now

$$\left| \int_0^\infty g_3(y)\, dy \right| = \left| \int_0^\infty g_2(y)\, dy \right| = \left| \int_0^\infty [g(y) - w_1(0, y)]\, dy \right| \leq C\varepsilon$$

by (14) and (19), while (21) and (20) yield $\int_0^\infty \zeta(y)\, dy = \varepsilon \sum_{j=\nu+1}^{l} c_j \sum_{\mu=0}^{\nu+1} d_\mu = \varepsilon \sum_{j=\nu+1}^{l} c_j$. We now impose the further condition

$$\int_0^\infty g_3(y)\, dy = \int_0^\infty \zeta(y)\, dy\,; \tag{25}$$

our calculations above show that this is equivalent to setting

$$\sum_{j=\nu+1}^{\ell} c_j = \phi, \quad \text{where } |\phi| \leq C. \tag{26}$$

As the d_μ are already determined, (23) and (26) together form a Vandermonde system for the c_j. Furthermore, (24) and $|\phi| \leq C$ imply that the c_j are bounded independently of ε. Finally, the construction of ζ ensures that $g_3 - \zeta$ is in $C^{2\ell,\alpha}(\mathbb{R})$.

To shorten the analysis we now aim to appeal to a result in [6], but this cannot be done directly because we have even boundary data for our half-plane problems while [6] has odd boundary data. This motivates the main idea in the proof of Lemma 5.3, which reveals the purpose of the condition (25).

Define the function $\Phi(x, y)$ by

$$L\Phi = 0 \;\text{ on } \Pi_x, \qquad \Phi(0, y) = \zeta(y) \;\text{ for } y \in \mathbb{R}.$$

Set $w_4(x, y) = w_3(x, y) - \Phi(x, y)$ and $g_4(y) = g_3(y) - \zeta(y)$. Then w_4 is the solution of the half-plane problem

$$Lw_4 = 0 \;\text{ on } \Pi_x, \qquad w_4(0, y) = g_4(y) \;\text{ for } y \in \mathbb{R}.$$

Lemma 5.3. *Let $\varepsilon < \frac{p^2}{q}$. Then for non-negative integers m and n satisfying $2m + n \leq 2\ell - 1$ there exists a constant C such that for all $(x, y) \in \bar{\Pi}_x$ one has*

$$|D_x^m D_y^n w_4(x, y)| \leq C\varepsilon^{\frac{1-n}{2}} e^{-\frac{\beta x}{2p}} e^{-\frac{\sqrt{q}|y|}{2\sqrt{\varepsilon}}}.$$

Proof. Our construction puts g_4 in $C^{2\ell,\alpha}(\mathbb{R})$. By (14), (21) and Lemma 5.2, the even function g_4 satisfies

$$|g_4^{(k)}(y)| \le C\varepsilon^{\frac{1-k}{2}} e^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}} \quad \text{for } y \in \mathbb{R} \ \text{ and } \ k = 0, 1, \ldots, 2\ell - 1. \tag{27}$$

Define the function G_4 by

$$G_4(y) = \begin{cases} \int_\infty^y g_4(t)\,dt & \text{if } y \ge 0 \\ \int_{-\infty}^y g_4(t)\,dt & \text{if } y < 0. \end{cases}$$

Then $G_4'(y) = g_4(y)$ for $y \ne 0$ and G_4 is an odd function. Furthermore, $G_4(0) = 0$ because (25) holds. Hence $G_4 \in C^{2\ell+1,\alpha}(\mathbb{R})$, and from (27) and the definition of G_4 we infer that

$$|G_4^{(k)}(y)| \le C\varepsilon^{\frac{2-k}{2}} e^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}} \quad \text{for } y \in \mathbb{R} \ \text{ and } \ k = 0, 1, \ldots, 2\ell. \tag{28}$$

Define the function W_4 by

$$LW_4 = 0 \ \text{ on } \Pi_x, \quad W_4(0, y) = G_4(y) \ \text{ for } y \in \mathbb{R}. \tag{29}$$

Bearing (28) in mind, we see that the problem (29) is identical to [6, (3.12)] except that the data has been multiplied by ε. Thus one can multiply the bound of [6, Theorem 3.2] by ε to get

$$|D_x^m D_y^n W_4(x, y)| \le C\varepsilon^{\frac{2-n}{2}} e^{-\frac{\beta x}{2p}} e^{-\frac{\sqrt{q}|y|}{2\sqrt{\varepsilon}}} \quad \text{on } \Pi_x \ \text{ for } 2m + n \le 2\ell;$$

this result is valid for $\varepsilon < \frac{p^2}{q}$. But $w_4 = W_{4,y}$, so we are done. □

Set $r = \sqrt{x^2 + y^2}$; this is the distance from (x, y) to, the corner $(0, 0)$. The following two definitions will recur frequently in subsequent bounds. Set

$$\phi_2(x, y) = \exp\left(-\frac{qx}{2p}\right) \exp\left(-\frac{\beta y}{2\sqrt{\varepsilon}}\right) \tag{30}$$

and

$$\psi(\nu, m, n, r) = \begin{cases} r^{2\nu+3-m-n}|\ln r| & \text{if } m + n \le 2\nu + 3 \\ r^{2\nu+3-m-n} & \text{if } m + n > 2\nu + 3. \end{cases} \tag{31}$$

For $\mu = 0, \ldots, \nu{+}1$ and $j = \nu{+}1, \ldots, \ell$, let $\Phi_{\mu,j}$ be the solution to the half-plane problem

$$L\Phi_{\mu,j} = 0 \ \text{ on } \Pi_x, \quad \Phi_{\mu,j}(0, y) = \sqrt{\varepsilon}\, e^{-(\sqrt{q}+j)2^\mu |y|/\sqrt{\varepsilon}} \ \text{ for } y \in \mathbb{R}.$$

Thus $\Phi(x, y) = \sum_{j=\nu+1}^{\ell} c_j \sum_{\mu=0}^{\nu+1} d_\mu (\sqrt{q} + j) 2^\mu \Phi_{\mu,j}(x, y)$ on Π_x.

Lemma 5.4. *Let $r^* \geq \varepsilon$ be given and n be a non-negative integer. Then there exists a constant C which depends on r^*, n and ν such that*

$$\left|D_y^n \Phi(x,y)\right| \leq C\left[\varepsilon^{\frac{1-n}{2}} + \varepsilon^{-\nu-1} r^{2\nu+3-n}\right] \qquad \text{for } r < \varepsilon$$

$$\left|D_y^n \Phi(x,y)\right| \leq C\varepsilon^{\frac{1-n}{2}}\left[1 + r^{\nu+\frac{3-n}{2}}\right]\phi_2(x,|y|) \quad \text{for } \varepsilon \leq r \leq r^*.$$

Proof. First consider the case $n = 0$. Use the barrier functions $W_1(x,y) = C\sqrt{\varepsilon}\, e^{-\frac{qx}{2p}} e^{-\frac{\beta y}{2\sqrt{\varepsilon}}}$ and $W_2(x,y) = C\sqrt{\varepsilon}\, e^{-\frac{qx}{2p}} e^{\frac{\beta y}{2\sqrt{\varepsilon}}}$. Then for $j = \nu+1,\ldots,\ell$ and $i = 1,2$ one has $LW_i \geq 0 = L\Phi_{\mu,j}$ in Π_x and $W_i(0,y) \geq |\Phi_{\mu,j}(0,y)|$. The growth conditions derived in [6, Section 3] show that the use of a maximum principle is justified; this establishes the bounds on $\Phi_{\mu,j}$. By linear superposition the same bounds hold for Φ.

Assume that $n \geq 1$. For $j = \nu+1,\ldots,\ell$, let θ_j be the solution of the half-plane problem $L\theta_j = 0$ on Π_x, $\theta_j(0,y) = \zeta_j(y)$ for $y \in \mathbb{R}$. Applying D_y to θ_j yields $L(D_y\theta_j) = 0$ on Π_x with

$$D_y\theta_j(0,y) = \zeta_j'(y) = -\sum_{\mu=0}^{\nu+1} d_\mu(\sqrt{q}+j)^2 2^{2\mu}(\operatorname{sgn} y)\exp\left\{-\frac{(\sqrt{q}+j)2^\mu|y|}{\sqrt{\varepsilon}}\right\}$$

for $y \in \mathbb{R}$. Equations (20) and (22) give $\zeta_j^{(2k+1)}(+0) = 0$ for $k = 0,\ldots,\nu$. Thus $D_y\theta_j$ has the same properties as $z(x,y)$ in [6, §4]. Consequently [6, Corollary 4.1, Lemmas 4.6 and 4.8] provide bounds for $D_y^n\theta_j(x,y)$ and by linearity for $D_y^n\Phi$. $\square$

Bounds for all derivatives of $\Phi(x,y)$ are given in Lemma 5.5.

Lemma 5.5. *Let $r^* \geq \varepsilon$ be given. Let m and n be non-negative integers. Then there exists a constant C which depends on r^*, m, n and ν such that*

$$\left|D_x^m D_y^n \Phi(x,y)\right| \leq C\left[\varepsilon^{\frac{1-n}{2}} + \varepsilon^{\nu+2-m-n} + \varepsilon^{-\nu-1}\psi(\nu,m,n,r)\right] \text{ for } r < \varepsilon$$

$$\left|D_x^m D_y^n \Phi(x,y)\right| \leq C\varepsilon^{\frac{1-n}{2}}\left[1 + r^{\nu-m+\frac{3-n}{2}}\right]\phi_2(x,|y|) \qquad \text{for } \varepsilon \leq r \leq r^*.$$

Proof. The case $m = 0$ is covered in Lemma 5.4. The inductive argument on m of [6, Theorem 4.1] gives the bounds for the higher-order x-derivatives. $\square$

Recall that $w = w_1 + w_4 + \Phi$. Combining Lemmas 5.2, 5.3 and 5.5 yields

Lemma 5.6. *Let m and n be non-negative integers satisfying $2m+n \leq 2\ell-1$. Fix $r^* \geq \varepsilon$. Then there is a constant C which depends on r^*, ℓ and ν, such that*

$$\left|D_x^m D_y^n w(x,y)\right| \leq C\left[\varepsilon^{\frac{1-n}{2}} + \varepsilon^{\nu+2-m-n} + \varepsilon^{-\nu-1}\psi(\nu,m,n,r)\right] \text{ for } r < \varepsilon$$

$$\left|D_x^m D_y^n w(x,y)\right| \leq C\varepsilon^{\frac{1-n}{2}}\left[1 + r^{\nu-m+\frac{3-n}{2}}\right]\phi_2(x,|y|) \qquad \text{for } \varepsilon \leq r \leq r^*,$$

where the functions ϕ_2 and ψ were defined in (30) *and* (31).

To close this section, we define the incoming corner function w_{01}, which handles the boundary data h_n^* on the side $y = 1$ of Q and any corner singularity at $(0, 1)$. It is the solution of the quarter-plane problem

$$\begin{aligned} Lw_{01}(x,y) &= 0 && \text{for } x > 0,\ y < 1 \\ w_{01,y}(x,1) &= h_n^*(x) - S_y(x,1) && \text{for } x > 0 \\ w_{01}(0,y) &= 0 && \text{for } y < 1. \end{aligned}$$

6. Outgoing corner functions

The next term in our decomposition of u is the outgoing corner function w_{10} which deals with the Dirichlet boundary data along the side $x = 1$ of Q as well as corner singularities that (1) may have at (1,0). The inadvertent introduction of any incompatibility at other corners of Q is avoided through a C^∞ cut-off function $\chi : \mathbb{R} \to [0, 1]$ that satisfies

$$\chi(t) = \begin{cases} 0 & \text{for } t \le \frac{1}{3} \\ 1 & \text{for } t \ge \frac{2}{3}. \end{cases}$$

Define w_{10} to be the solution of the quarter-plane problem

$$Lw_{10}(x,y) = 0 \qquad \text{for } x < 1,\ y > 0 \tag{32a}$$
$$w_{10}(1,y) = -\chi(1-y)w_{00}(1,y) \qquad \text{for } y > 0 \tag{32b}$$
$$w_{10,y}(x,0) = -\chi(x)E_y(x,0) \qquad \text{for } x < 1. \tag{32c}$$

The functions S, E, w_{01}, w_{10} and w_{11} are all smooth at (0,0) and – as we shall see in Section 7 – the function $\breve{u}$ is compatible to arbitrary order at (0,0). Thus, recalling the decomposition (3), it follows that w_{00} enjoys the same degree of compatibility ν_{00} at the corner $(0, 0)$ as the function u. A similar argument demonstrates that each of our four corner functions has the same degree of compatibility at its "home" corner as u has there.

Lemma 6.1. *Let m and n be nonnegative integers satisfying $2m + n \le 2\ell - 1$. Fix $r^* \ge \varepsilon$. Then there is a constant C, which depends on r^*, ℓ and ν_{00}, such that*

$$\begin{aligned} \left|D_x^m D_y^n\, w_{00}(x,y)\right| &\le C\left[\varepsilon^{\frac{1-n}{2}} + \varepsilon^{\nu_{00}+2-m-n}\right] && \text{for } m+n < 2\nu_{00}+3,\ r < \varepsilon \\ \left|D_x^m D_y^n\, w_{00}(x,y)\right| &\le C\left[\varepsilon^{\frac{1-n}{2}} + \varepsilon^{-\nu_{00}-1}\ln|r|\right] && \text{for } m+n = 2\nu_{00}+3,\ r < \varepsilon \\ \left|D_x^m D_y^n\, w_{00}(x,y)\right| &\le C\left[\varepsilon^{\frac{1-n}{2}} + \varepsilon^{-\nu_{00}-1}r^{2\nu_{00}+3-m-n}\right] && \text{for } m+n > 2\nu_{00}+3,\ r < \varepsilon \\ \left|D_x^m D_y^n\, w_{00}(x,y)\right| &\le C\varepsilon^{\frac{1-n}{2}}\left[1 + r^{\nu_{00}-m+\frac{3-n}{2}}\right]e^{\frac{-\beta y}{2\sqrt{\varepsilon}}} && \text{for } \varepsilon \le r \le r^*. \end{aligned}$$

Proof. We have just seen that w_{00} has degree of compatibility ν_{00} at the corner $(0,0)$. Furthermore, (2) and Lemma 3.1 imply that the boundary conditions (13b), (13c) satisfy (14). We can therefore invoke Lemma 5.6 to obtain the desired bounds for w_{00}. $\square$

Let

$$r_{ij} = \sqrt{(x-i)^2 + (y-j)^2} \tag{33}$$

denote the distance from (x,y) to the corner (i,j) of Q. Bounds on the derivatives of w_{01} follow from Lemma 6.1 on making the change of variable $y \mapsto 1-y$ with ν_{00} and r replaced by ν_{01} and r_{01}.

The next result resembles [7, Lemma 2].

Lemma 6.2. *Let m and n be nonnegative integers satisfying $2m+n \le 2\ell - 2$. Fix $r^* \ge \varepsilon$. Then there is a constant C, which depends on r^*, ℓ and ν_{10}, such that*

$$\left|D_x^m D_y^n\, w_{10}(x,y)\right| \le C\varepsilon^{-m+\frac{1-n}{2}} \qquad \text{for } m+n<2\nu_{10}+3,\, r_{10}<\varepsilon$$

$$\left|D_x^m D_y^n\, w_{10}(x,y)\right| \le C\left[\varepsilon^{-m+\frac{1-n}{2}} + \varepsilon^{-\nu_{10}-1}\ln|r_{10}|\right] \qquad \text{for } m+n=2\nu_{10}+3,\, r_{10}<\varepsilon$$

$$\left|D_x^m D_y^n\, w_{10}(x,y)\right| \le C\left[\varepsilon^{-m+\frac{1-n}{2}} + \varepsilon^{-\nu_{10}-1} r_{10}^{2\nu_{10}+3-m-n}\right] \qquad \text{for } m+n>2\nu_{10}+3,\, r_{10}<\varepsilon$$

$$\left|D_x^m D_y^n\, w_{10}(x,y)\right| \le C\varepsilon^{-m+\frac{1-n}{2}}\left[1+r_{10}^{\nu_{10}+\frac{3-n}{2}}\right] e^{-\frac{p(1-x)}{\varepsilon}} e^{-\frac{\beta y}{2\sqrt{\varepsilon}}} \qquad \text{for } \varepsilon \le r_{10} \le r^*.$$

Proof. Let $v(x,y) = e^{\frac{px}{\varepsilon}} w_{10}(1-x,y)$. Then

$$\begin{aligned} Lv(x,y) &= 0 && \text{for } x>0,\ y>0 \\ v(0,y) &= -\chi(1-y)w_{00}(1,y) && \text{for } y>0 \\ v_y(x,0) &= -e^{\frac{px}{\varepsilon}}\chi(1-x)E_y(1-x,0) && \text{for } x>0. \end{aligned}$$

From Lemmas 4.4 and 6.1 the boundary conditions for v satisfy (14) with 2ℓ changed to $2\ell-1$. Also the function v has the same compatibility ν_{10} at (0,0) as the function w_{10} had at (1,0). We can therefore apply Lemma 5.6 to v. Now

$$\begin{aligned} \left|D_x^m D_y^n w_{10}(x,y)\right| &\le C \sum_{i+j=m} \left| D_x^i \left(e^{-\frac{p(1-x)}{\varepsilon}}\right) D_x^j D_y^n v(1-x,y)\right| \\ &\le C e^{-\frac{p(1-x)}{\varepsilon}} \sum_{i+j=m} \varepsilon^{-i} \left|D_x^j D_y^n v(1-x,y)\right| \end{aligned}$$

and the desired result follows; note that in the case $m+n<2\nu_{10}+3$ and $r<\varepsilon$ we get the bound

$$\left|D_x^m D_y^n w_{10}(x,y)\right| \le C\left[\varepsilon^{\frac{-m+(1-n)}{2}} + \varepsilon^{\nu_{10}-m-n+2}\right] = C\varepsilon^{\frac{-m+(1-n)}{2}}\left[1+\varepsilon^{\frac{\nu_{10}+(3-n)}{2}}\right],$$

but $m+n<2\nu_{10}+3$ implies that $\varepsilon^{\frac{\nu_{10}+(3-n)}{2}} \le C$. $\square$

The outgoing corner function w_{11} is introduced to deal with the boundary conditions at the corner $(1,1)$. It is the solution of the quarter-plane problem

$$\begin{aligned} Lw_{11}(x,y) &= 0 && \text{for } x<1,\ y<1 \\ w_{11}(1,y) &= -\chi(y)w_{01}(1,y) && \text{for } y<1 \\ w_{11,y}(x,1) &= -\chi(x)E_y(x,1) && \text{for } x<1. \end{aligned}$$

Bounds on the derivatives of w_{11} follow from Lemma 6.2 on making the change of variable $y \mapsto 1-y$ with ν_{10} and r_{10} replaced respectively by ν_{11} and r_{11}.

7. The remainder term $\breve{u}$

Finally we come to the remainder term $\breve{u}$. This function satisfies

$$\begin{aligned} L\breve{u} &= 0 \qquad \text{on } Q \\ \breve{u}(0,y) = \breve{g}_w(y) &:= \ -E(0,y) - w_{10}(0,y) - w_{11}(0,y) \\ \breve{u}(1,y) = \breve{g}_e(y) &:= [\chi(1-y)-1]w_{00}(1,y) + [\chi(y)-1]w_{01}(1,y) \\ \breve{u}_y(x,0) = \breve{h}_s(x) &:= [\chi(x)-1]E_y(x,0) - w_{01,y}(x,0) - w_{11,y}(x,0) \\ \breve{u}_y(x,1) = \breve{h}_n(x) &:= [\chi(x)-1]E_y(x,1) - w_{00,y}(x,1) - w_{10,y}(x,1). \end{aligned}$$

We check compatibility at the origin; the other vertices are similar. Near (0,0) the boundary data for $\breve{u}$ can be written as

$$\begin{aligned} \breve{g}_w(y) &= -E(0,y) - w_{10}(0,y) - w_{11}(0,y) - w_{01}(0,y) \\ \breve{h}_s(x) &= -E_y(x,0) - w_{01,y}(x,0) - w_{11,y}(x,0) - w_{10,y}(x,0). \end{aligned}$$

Thus the boundary data for $\breve{u}$ is compatible to arbitrary order at (0,0) since E, w_{10}, w_{11} and w_{01} are all C^∞ functions in a neighbourhood of (0,0). As in [6, Theorem 5.1], an energy argument and Sobolev imbedding now show that $\|\breve{u}\|_{2\ell-2,\infty,Q} \le C$.

8. Bound on the derivatives of u

In Section 1 we outlined a decomposition for u into half-plane and quarter-plane problems. Subsequent sections proved derivative bounds for the solutions to each of these problems. In Theorem 8.1 the derivative bounds for the half-plane and quarter-plane problems are brought together to give derivative bounds for u.

Theorem 8.1. *Let m, n be non-negative integers satisfying $2m + n \leq 2\ell - 2$. Let r_{ij} be defined by* (33). *Let $\beta = \min\left\{\frac{p}{12}, \frac{q}{2p}, \sqrt{q}\right\}$. Let $\breve{p} \in (0, p)$. Let ν_{ij} give the compatibility of the data at the corner (i, j). Then for $(x, y) \in Q$, the solution u of* (1) *satisfies*

$$|D_x^m D_y^n u(x,y)| \leq C(1 + T_{00} + T_{01} + T_{10} + T_{11} + T_E)$$

with $T_E = \varepsilon^{-m} e^{-\frac{\breve{p}(1-x)}{\varepsilon}}$, where for $\mu = 0, 1$, one has

$$
\begin{aligned}
T_{0\mu} &= \varepsilon^{\frac{1-n}{2}} + \varepsilon^{\nu_{0\mu} - m - n + 2} && \text{for } m + n < 2\nu_{0\mu} + 3,\ r_{0\mu} < \varepsilon \\
T_{0\mu} &= \varepsilon^{\frac{1-n}{2}} + \varepsilon^{-\nu_{0\mu} - 1} |\ln r_{0\mu}| && \text{for } m + n = 2\nu_{0\mu} + 3,\ r_{0\mu} < \varepsilon \\
T_{0\mu} &= \varepsilon^{\frac{1-n}{2}} + \varepsilon^{-\nu_{0\mu} - 1} r_{0\mu}^{2\nu_{0\mu} + 3 - m - n} && \text{for } m + n > 2\nu_{0\mu} + 3,\ r_{0\mu} < \varepsilon \\
T_{0\mu} &= \varepsilon^{\frac{1-n}{2}} \left[1 + r_{0\mu}^{\nu_{0\mu} - m + \frac{3-n}{2}}\right] e^{(-1)^{\mu} \frac{\beta(\mu - y)}{2\sqrt{\varepsilon}}} && \text{for } r_{0\mu} \geq \varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
T_{1\mu} &= \varepsilon^{-m + \frac{1-n}{2}} && \text{for } m + n < 2\nu_{1\mu} + 3,\ r_{1\mu} < \varepsilon \\
T_{1\mu} &= \varepsilon^{-m + \frac{1-n}{2}} + \varepsilon^{-\nu_{1\mu} - 1} |\ln r_{1\mu}| && \text{for } m + n = 2\nu_{1\mu} + 3,\ r_{1\mu} < \varepsilon \\
T_{1\mu} &= \varepsilon^{-m + \frac{1-n}{2}} + \varepsilon^{-\nu_{1\mu} - 1} r_{1\mu}^{2\nu_{1\mu} + 3 - m - n} && \text{for } m + n > 2\nu_{1\mu} + 3,\ r_{1\mu} < \varepsilon \\
T_{1\mu} &= \varepsilon^{-m + \frac{1-n}{2}} \left[1 + r_{1\mu}^{\nu_{1\mu} + \frac{3-n}{2}}\right] e^{-\frac{p(1-x)}{\varepsilon}} e^{(-1)^{\mu} \frac{\beta(\mu - y)}{2\sqrt{\varepsilon}}} && \text{for } r_{1\mu} \geq \varepsilon.
\end{aligned}
$$

Proof. Use the decomposition (3) and add the bounds that we have proved for each of its terms. □

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