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Entropy Numbers of Limiting Embeddings of Logarithmic Sobolev Spaces into Exponential Spaces

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Abstract. It is known that in certain limiting cases, spaces of Sobolev type modelled upon Zygmund spaces are embedded in Orlicz spaces of exponential type. Estimates of the entropy numbers of such embeddings are studied.

Keywords. Entropy numbers, Lorentz–Zygmund space, Sobolev space, embeddings
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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, let $p, q \in (1, \infty)$ and suppose that $s \in \mathbb{N}$ is such that $s > n(\frac{1}{p} - \frac{1}{q})_+$; denote by $W^{s,p}(\Omega)$ the usual Sobolev space of order s, based on $L^p(\Omega)$. Then it is well known (see, for example, [14, Chapter 3]) that $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ and that this embedding Id has entropy numbers $e_k(Id)$ that satisfy

$$e_k(Id) \approx k^{-\frac{s}{n}} \quad (k \in \mathbb{N}),$$

by which we mean that $e_k(Id)\, k^{\frac{s}{n}}$ is bounded above and below by positive constants independent of k. If $s = n(\frac{1}{p} - \frac{1}{q})$, then Id is continuous but not compact. There is no embedding at all of $W^{\frac{n}{p},p}(\Omega)$ in $L^\infty(\Omega)$; an embedding can be obtained if the target space $L^\infty(\Omega)$ is replaced by the slightly larger Orlicz space $L_{\Phi_\nu}(\Omega)$ with Young function Φ_ν such that for large t, $\Phi_\nu(t)$ behaves like $\exp(t^\nu)$. In fact, if $0 < \nu \le p'$ (where $\frac{1}{p'} = 1 - \frac{1}{p}$), the embedding Id_ν: $W^{\frac{n}{p},p}(\Omega) \to L_{\Phi_\nu}(\Omega)$ exists and is continuous; Id_ν is compact if $0 < \nu < p'$. For results of this type we refer to [20, 21, 23, 25, 26]. The entropy numbers of Id_ν

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were estimated by Triebel [25], who showed in particular that if $0 < \nu < \frac{p}{2+p}$, then

$$e_k(Id_\nu) \approx k^{-\frac{1}{p}};$$

if $\frac{p}{2+p} \leq \nu < \frac{1}{p'}$, then upper and estimates of power type for $e_k(Id_\nu)$ are also available (see [25] and the later improvements, some involving logarithms, in [16–18]) but there is a gap between the exponents involved in the upper and lower bounds. These estimates also hold for spaces of fractional Sobolev and Besov type; indeed, settings of such generality were used in [25] and [16–18].

Entropy number estimates have also been obtained for embeddings between spaces of the type $W^{s,p}(\log W)^\alpha(\Omega)$, these being defined in the same way as $W^{s,p}(\Omega)$ but with the underlying space $L^p(\Omega)$ replaced by the Zygmund space $L^p(\log L)^\alpha(\Omega)$. A detailed account of this is given in [14], which also deals with the case of fractional Sobolev spaces modelled on Zygmund spaces; see also [12]. However, the limiting case in which the embedding is from such a space to an Orlicz space of exponential type does not seem to have been studied from the entropy number point of view, even though the existence and compactness of such an embedding is known from the work of [7]. The object of our paper is to address this question. We remark that knowledge of the behaviour of the entropy numbers of embeddings between function spaces may be used to gain information about the eigenvalues of (possibly degenerate) elliptic operators. This stems from the observation due to Bernd Carl (see, for example, [14, p. 20]) that links the entropy numbers $e_n(T)$ of a compact linear map T from a Banach space X to itself with its eigenvalues $\lambda_n(T)$, arranged by decreasing modulus and repeated according to algebraic multiplicity: his result is that

$$|\lambda_n(T)| \leq \sqrt{2} e_n(T) \quad (n \in \mathbb{N}).$$

The process of reduction of an elliptic boundary-value problem to an operator equation often gives rise to an operator that is the composition of an embedding map T_1 and a continuous map T_2, and since the entropy numbers are submultiplicative in the sense that

$$e_{m+n-1}(T_1 \circ T_2) \leq e_m(T_1) e_n(T_2) \quad (m, n \in \mathbb{N}),$$

we have $e_n(T_1 \circ T_2) \leq e_n(T_1) \, \|T_2\| \; (n \in \mathbb{N})$. Use of this together with Carl's inequality gives upper estimates for the eigenvalues of $T_1 \circ T_2$ in terms of the entropy numbers of the embedding map T_1; these estimates can then be translated into lower estimates for the eigenvalues of the elliptic problem. A full discussion of this procedure and the results obtainable by such means is given in [14, Chapter 5], which also contains an entropy version of the Birman–Schwinger principle that is useful in the study of the negative spectrum of certain self-adjoint elliptic operators.

For simplicity we deal only with the situation in which the domain of the embeddings is a first-order space of the form $W^{1,n}(\log W)^{\alpha}(\Omega)$. The case $\alpha = 0$ corresponds to the work of Triebel and Kühn and Schonbek already mentioned. When $0 \le \alpha < \frac{1}{n'}$, $W^{1,n}(\log W)^{\alpha}(\Omega)$ is compactly embedded in $E_{\nu}(\Omega)$, the Orlicz space with Young function having values behaving like $\exp(t^{\nu})$ for large t, provided that $0 < \nu < q_{\alpha}$, where $\frac{1}{q_{\alpha}} = \frac{1}{n'} - \alpha$. We give in Theorem 3.2 upper and lower estimates of power type for the entropy numbers of this embedding. The upper and lower rates of decay that we obtain coincide when $0 < \nu < \frac{n}{n+2}$, but for other values of ν there is a gap between these rates, as there is in the work of Triebel and Kühn and Schonbek dealing with the case $\alpha = 0$. The finer tuning provided by the index α also enables embeddings into other types of exponential spaces to be obtained. Thus, corresponding to the limiting case when $\alpha = \frac{1}{n'}$, we know from [7] that $W^{1,n}(\log W)^{1/n'}(\Omega)$ is compactly embedded into an Orlicz space with Young function having values behaving like $\exp(\exp(t^{\nu}))$ for large t, provided that $0 < \nu < n'$. We show that the k^{th} entropy number of this embedding is $O\big((\log k)^{-(\frac{1}{\nu} - \frac{1}{n'})}\big)$. We do not know whether or not the rate of decay of the entropy numbers in this case is really of logarithmic type, for the only lower bound we are able to prove is of the form $k^{-\frac{1}{n}}$. It seems very desirable to settle this question, and also to eliminate the gap present between the upper and lower estimates in the single exponential case, when $\nu \ge \frac{n}{n+2}$.

2. Notation

2.1. Basic notation. If $p \in [1, \infty]$, the conjugate number p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$, with the understanding that $1' = \infty$ and $\infty' = 1$.

For non-negative expressions (i.e., functions or functionals) F_1, F_2 we use the symbol $F_1 \lesssim F_2$ to mean that $F_1 \le C F_2$ for some constant $C \in (0, \infty)$ independent of the variables in the expressions F_1, F_2. If $F_1 \lesssim F_2$ and $F_2 \lesssim F_1$, we write $F_1 \approx F_2$.

If Ω is a measurable subset of $\mathbb{R}^n$ (with respect to n-dimensional Lebesgue measure), then by $|\Omega|_n$ we denote its n-volume.

Let $q \in (0, \infty]$. By the symbol $\|f\|_{q;\Omega}$ we denote the L^q-(quasi-)norm of a measurable function f on the measurable set $\Omega \subset \mathbb{R}^n$.

2.2. Lorentz-Karamata spaces. A nonnegative function b measurable on $(0, \infty)$, $0 \not\equiv b \not\equiv \infty$, is said to be *slowly varying* on $(0, \infty)$, written $b \in SV := SV(0, \infty)$, if, for each $\varepsilon > 0$, there are a nondecreasing nonnegative function g_{ε} and a nonincreasing nonnegative function $g_{-\varepsilon}$ which are measurable on $(0, \infty)$ and satisfy

$$t^{\varepsilon} b(t) \approx g_{\varepsilon}(t) \quad \text{and} \quad t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t) \quad \text{for all} \quad t \in (0, \infty).$$

Let $p, q \in (0,\infty]$, $b \in SV(0,\infty)$ and let Ω be a measurable subset of $\mathbb{R}^n$ (with respect to n-dimensional Lebesgue measure). The *Lorentz–Karamata space* $L_{p,q;b}(\Omega)$ consists of all measurable (real or complex) functions f on Ω such that the quantity

$$\|f\|_{p,q;b;\Omega} = \|f\|_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}}\, b(t)\, f^*(t) \right\|_{q;(0,\infty)}$$

is finite. Here f^* denotes the *non-increasing rearrangement* of f given by

$$f^*(t) = \inf \big\{ \lambda > 0;\, \big| \{x \in \Omega;\ |f(x)| > \lambda\} \big|_n \le t \big\}, \quad t \ge 0.$$

Particular choices of b give well-known spaces. Obviously, when b is the function identically equal to 1, the corresponding Lorentz–Karamata space coincides with the Lorentz space $L^{p,q}(\Omega)$. Moreover, if $m \in \mathbb{N}$ and

$$b(t) = \prod_{i=1}^{m} \ell_i^{\alpha_i}(t) \quad \text{for } t > 0, \text{ where } \ \alpha_1, \dots, \alpha_m \in \mathbb{R},$$

and, for $t > 0$,

$$\ell_1(t) = 1 + |\log t|, \quad \ell_i(t) = \ell_1(\ell_{i-1}(t)) \ \text{ if } i > 1,$$

then the Lorentz–Karamata space $L_{p,q;b}(\Omega)$ is the *generalized Lorentz–Zygmund space*$L_{p,q;\alpha_1,\dots,\alpha_m}(\Omega)$ of [9], which in turn becomes the *Lorentz–Zygmund space* $L^{p,q}(\log L)^{\alpha_1}(\Omega)$ of Bennett and Rudnick [2] when $m = 1$. If, moreover, $p = q$, it becomes the well-known *Zygmund space* $L^p(\log L)^{\alpha_1}(\Omega)$. We refer to [5] for more details of Lorentz–Karamata spaces.

2.3. Orlicz spaces. Let Φ be a *Young function* (that is, a continuous, non-negative, strictly increasing, convex function on $[0,\infty)$ such that $\lim_{t\to 0+} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0$) and Ω be a measurable subset of $\mathbb{R}^n$. By $L_\Phi(\Omega)$ we shall denote the corresponding *Orlicz space*, equipped with the Luxemburg norm $\|\cdot\|_\Phi$; for details of such spaces we refer to [1, 3, 19].

2.4. Relationship between Orlicz and Lorentz-Karamata spaces. Orlicz spaces and Lorentz-Karamata spaces are two different classes of function spaces having a nontrivial intersection.

We need some particular results.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with n-dimensional Lebesgue measure $|\Omega|_n < \infty$.*

(i) *Suppose that $p \in (1,\infty)$ and $a \in \mathbb{R}$. Then the space $L^p(\log L)^a(\Omega)$ coincides with the Orlicz space $L_\Psi(\Omega)$ with the Young function $\Psi(t) \approx t^p\, \ell_1(t)^{ap}$, $t > 0$, and the corresponding norms are equivalent.*

(ii) *Suppose that* $\nu > 0$. *Then the space* $L_{\infty,\infty;-1/\nu}(\Omega)$ *coincides with the Orlicz space* $L_\Phi(\Omega)$, *with Young function* $\Phi(t) = \exp(t^\nu) - 1$, $t \geq 0$, *and the corresponding norms are equivalent.*

(iii) *Suppose that* $\nu > 0$. *Then the space* $L_{\infty,\infty;0,-1/\nu}(\Omega)$ *coincides with the Orlicz space* $L_\Phi(\Omega)$, *with the Young function* $\Phi(t) = \exp\exp(t^\nu) - e$, $t \geq 0$, *and the corresponding norms are equivalent.*

Proof. For the proof of (i) see [7, Lemma 2.1 (ii)] and note that $L^p(\log L)^a(\Omega) = L_p(\log L)_{ap}(\Omega)$. Statements (ii) and (iii) can be proved much as [2, Theorem D] (cf. [6, Lemma 3.9]). The assertion about equivalent norms immediately follows from [3, Chapter 1, Theorem 1.8] and from the fact that all the spaces are (equivalent to) Banach function spaces. □

In the light of the previous lemma we introduce some notation.

Notation 2.2. Let $\Omega \subset \mathbb{R}^n$ be a domain such that $|\Omega|_n < \infty$ and let $\nu > 0$. Then we put

$$E_\nu(\Omega) := L_{\infty,\infty;-1/\nu}(\Omega), \qquad \|\cdot\|_{E_\nu(\Omega)} := \|\cdot\|_{\infty,\infty;\ell_1^{-1/\nu};\Omega}$$
$$EE_\nu(\Omega) := L_{\infty,\infty;0,-1/\nu}(\Omega), \qquad \|\cdot\|_{EE_\nu(\Omega)} := \|\cdot\|_{\infty,\infty;\ell_2^{-1/\nu};\Omega}\,.$$

2.5. Extrapolation results. For the exponential spaces $E_\nu(\Omega)$ and $EE_\nu(\Omega)$ introduced in Notation 2.2 we have the following extrapolation result.

Lemma 2.3. *Let the domain* $\Omega \subset \mathbb{R}^n$ *satisfy* $|\Omega|_n < \infty$ *and let* $\nu > 0$. *If* $j_0 \in \mathbb{N}$ *and* $q_0 \geq 1$, *then*

(i) *for all* $f \in E_\nu(\Omega)$,

$$\|f\|_{E_\nu(\Omega)} \approx \sup_{j\in\mathbb{N}, j\geq j_0} j^{-\frac{1}{\nu}}\, \|f\|_{j;\Omega} \approx \sup_{q\geq q_0} q^{-\frac{1}{\nu}}\, \|f\|_{q;\Omega};$$

(ii) *for all* $f \in EE_\nu(\Omega)$,

$$\|f\|_{EE_\nu(\Omega)} \approx \sup_{j\in\mathbb{N}, j\geq j_0} (\log j)^{-\frac{1}{\nu}}\, \|f\|_{j;\Omega} \approx \sup_{q\geq q_0} (\log q)^{-\frac{1}{\nu}}\, \|f\|_{q;\Omega}.$$

Proof. See [10, Corollary 3.2]. □

2.6. Sobolev-type spaces. Let Ω be a domain in $\mathbb{R}^n$. Let $k \in \mathbb{N}$ and let $X(\Omega)$ be a Banach function space. We define the *Sobolev space* modelled on $X(\Omega)$ to be the set

$$W^k X(\Omega) := \{u;\ D^\beta u \in X(\Omega) \text{ if } |\beta| \leq k\}$$

(where $D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}}$, $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_1, \ldots, \beta_n$ nonnegative integers and $|\beta| = \beta_1 + \cdots + \beta_n$), equipped with the norm

$$\|u\|_{W^k X(\Omega)} := \sum_{|\beta|\leq k} \|D^\beta u\|_{X(\Omega)}.$$

We denote $\|\cdot\|_{k;p,q;b;\Omega} := \|\cdot\|_{W^k L_{p,q;b}(\Omega)}$ and, for the special case $X(\Omega) = L^p(\log L)^\alpha(\Omega)$, we put $W^{k,p}(\log W)^\alpha(\Omega) := W^k L^p(\log L)^\alpha(\Omega)$ and $\|\cdot\|_{k;p;\alpha;\Omega} := \|\cdot\|_{W^{k,p}(\log W)^\alpha(\Omega)}$. When $\alpha = 0$ we write $W^{k,p}(\Omega)$ instead of $W^{k,p}(\log W)^0(\Omega)$.

2.7. Finite sequence spaces. Let $m \in \mathbb{N}$ and let X be a Banach function space on $\mathbb{R}$. Then we define the sequence space ℓ_X^m to be $\mathbb{C}^m$ furnished with the norm

$$\left\|\{a_i\}_{i=1}^m\right\|_{\ell_X^m} = \left\|\sum_{i=1}^m a_i\,\chi_{[i-1,i)}\right\|_X.$$

If $X = L^p(\mathbb{R})$ we use the notation ℓ_p^m instead of $\ell_{L^p(\mathbb{R})}^m$ and

$$\left\|\{a_i\}_{i=1}^m\right\|_p := \left\|\{a_i\}_{i=1}^m\right\|_{\ell_p^m} = \left(\sum_{i=1}^m |a_i|^p\right)^{\frac{1}{p}}$$

(with obvious modification when $p = \infty$).

2.8. Entropy numbers, embeddings. For a (quasi-) Banach space X denote by the symbol $\mathcal{U}_X$ its closed *unit ball*, that is, $\mathcal{U}_X = \{\, x \in X;\ \|x\|_X \le 1\}$, where $\|\cdot\|_X$ denotes the (quasi-)norm in the space X.

Let $T \in \mathcal{L}(X, Y)$, the *space of all bounded linear maps from X to Y*. Then we define the norm of T,

$$\|T\|_{X\to Y} = \sup_{f\in\mathcal{U}_X} \|Tf\|_Y,$$

and for each $k \in \mathbb{N}$ the k^{th} *entropy number* $e_k(T)$ is defined by

$$e_k(T) = \inf\left\{\varepsilon > 0;\ T(\mathcal{U}_X) \subset \bigcup_{j=1}^{2^{k-1}} (y_j + \varepsilon\,\mathcal{U}_Y) \text{ for some } y_1, \dots, y_{2^{k-1}} \in Y\right\}.$$

It is easy to verify (cf. [4, pp. 47–48]) that if X, Y, Z are Banach spaces, $R, S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(Y, Z)$, then

$$\|T\|_{X\to Y} = e_1(T) \ge e_2(T) \ge \dots \ge 0 \tag{1}$$

$$e_{k+l-1}(T \circ S) \le e_k(T)\, e_l(S) \qquad \text{for all } k, l \in \mathbb{N} \tag{2}$$

$$e_{k+l-1}(R + S) \le e_k(R) + e_l(S) \quad \text{for all } k, l \in \mathbb{N}$$

(similar properties also hold in quasi-Banach spaces, see [14, pp. 7–8] for the details).

Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ or $X \hookrightarrow\hookrightarrow Y$ if $X \subset Y$ and the natural *embedding* $Id : X \to Y$ is bounded or compact, respectively.

If $X \hookrightarrow\hookrightarrow Y$, then $e_k(Id : X \to Y) \to 0$ when $k \to \infty$.

Lemma 2.4. *Let $1 \le p < q \le \infty$ and for each $k \in \mathbb{N}$ let e_k be the k^{th} entropy number of the natural embedding $\ell_p^m \to \ell_q^m$. If $2^k \le m$, then*

$$e_k \approx 1,$$

where constants in this equivalence are independent of k, m, p and q.

Proof. For the *proof* see [22]. □

3. Entropy numbers of the single-exponential embedding

Throughout this section let us suppose that Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Together with B. Opic we have proved the following result (cf. [11] and the references therein).

Lemma 3.1. *Let $n \in \mathbb{N}$, $n > 1$, $\alpha < \frac{1}{n'}$ and $0 < \nu < q_\alpha$, where $\frac{1}{q_\alpha} = \frac{1}{n'} - \alpha$. Then*

$$W^{1,n}(\log W)^\alpha(\Omega) \hookrightarrow E_{q_\alpha}(\Omega) \tag{3}$$

and

$$W^{1,n}(\log W)^\alpha(\Omega) \hookrightarrow\hookrightarrow E_\nu(\Omega). \tag{4}$$

Recall that the embedding (3) is not compact.

In this section we derive upper and lower asymptotic estimates of entropy numbers of the embedding (4). We formulate the main result whose proof will be given in the following subsections.

Theorem 3.2. *Let $n \in \mathbb{N}$, $n \ge 2$, $0 \le \alpha < \frac{1}{n'}$ and $0 < \nu < q_\alpha$, where $\frac{1}{q_\alpha} = \frac{1}{n'} - \alpha$. Denote by e_k the k^{th} entropy number of the natural embedding $Id : W^{1,n}(\log W)^\alpha(\Omega) \to E_\nu(\Omega)$. Put $D := 1 + n\alpha \in [1, n)$.*

(i) *If $0 < \nu < \frac{n}{n+2}$, then $e_k \approx k^{-\frac{1}{n}}$.*
(ii) *If $\frac{n}{n+2} \le \nu < 1$, then $k^{-\frac{1}{n}} \lesssim e_k \lesssim k^{-\frac{1}{n}} (\log k)^{\frac{1}{n}}$.*
(iii) *If $\nu = 1$, then $k^{-\frac{1}{n}} \lesssim e_k \lesssim k^{-\frac{1}{n}} (\log k)^{\frac{2}{n}}$.*
(iv) *If $1 < \nu \le \frac{1}{1-\alpha}$, then $k^{-\frac{1}{n}} \lesssim e_k \lesssim k^{-\frac{1}{D}(\frac{1}{\nu}-\frac{1}{q_\alpha})} (\log k)^{\frac{1}{D}(\frac{1}{\nu}-\frac{1}{q_\alpha})}$*
(v) *If $\frac{1}{1-\alpha} < \nu < q_\alpha$, then $k^{-(\frac{1}{\nu}-\frac{1}{q_\alpha})} \lesssim e_k \lesssim k^{-\frac{1}{D}(\frac{1}{\nu}-\frac{1}{q_\alpha})} (\log k)^{\frac{1}{D}(\frac{1}{\nu}-\frac{1}{q_\alpha})}$.*

Remark 3.3. When $\alpha = 0$ so that $D = 1$, $q_\alpha = n'$, $0 < \nu < n'$, and Id is the natural embedding $W^{1,n}(\Omega)$ in $E_\nu(\Omega)$ (note that in this case (iv) is no longer relevant), the two-sided estimate (i) and the lower estimates in (ii),(iii) and (v) follow from the paper of Triebel [25] together with the upper estimate $e_k \lesssim k^{-\frac{1}{3}(\frac{1}{\nu}-\frac{1}{n'}))+\varepsilon}$ if $\frac{n}{n+2} \le \nu < n'$. This upper estimate was improved by Kühn in [16] by the estimates $e_k \lesssim k^{-\frac{1}{n}}(\log k)^{\frac{1}{n}+1}$ if $\frac{n}{n+2} \le \nu \le \frac{n}{n+1}$ and $e_k \lesssim k^{-\frac{1}{2}(\frac{1}{\nu}-\frac{1}{n'})}$ if $\frac{n}{n+1} < \nu < n'$. Further improvements of upper estimates in (ii),(iii) and (v) were obtained by Kühn and Schonbek in [18].

3.1. Upper estimates. We shall need some auxiliary results.

Proposition 3.4. *Let $n \ge 2$ and $\alpha \ge 0$. Then there is a positive constant C such that, for any $q \in (1,\infty)$ and all $k \in \mathbb{N}$,*

$$e_k\big(Id : W^{1,n}(\log W)^\alpha(\Omega) \to L^q(\Omega)\big) \le C \min\left\{ q\left(\frac{k}{\log k}\right)^{-\frac{1}{n}}, q^{1+\frac{2}{n}}\, k^{-\frac{1}{n}} \right\}. \tag{5}$$

Proof. As $|\Omega|_n < \infty$ and $\alpha \ge 0$ we have $L^n(\log L)^\alpha(\Omega) \hookrightarrow L^n(\Omega)$, and so the identity mapping $Id : W^{1,n}(\log W)^\alpha(\Omega) \to W^{1,n}(\Omega)$ is bounded. Together with (2) and the estimate

$$e_k\big(Id : W^{1,n}(\Omega) \to L^q(\Omega)\big) \lesssim q^{1+\frac{2}{n}}\, k^{-\frac{1}{n}}, \quad k \in \mathbb{N},\ q \in (1,\infty),$$

of Triebel [25, Section 4.3.4] this gives the second estimate of (5).

The first estimate in (5) we obtain by the same argument using the embedding $W^{1,n}(\Omega) = F^1_{n,2}(\Omega) \hookrightarrow B^1_{n,n}(\Omega)$ and the estimate

$$e_k\big(Id : B^1_{n,n}(\Omega) \to L^q(\Omega)\big) \lesssim q\left(\frac{k}{\log k}\right)^{-\frac{1}{n}}, \quad n \ge 2,\ k \in \mathbb{N},\ q \in (1,\infty),$$

of Kühn and Schonbek [18, Theorem 3.3]. (Here $F^1_{n,2}(\Omega)$ denotes a Triebel–Lizorkin space and $B^1_{n,n}(\Omega)$ denotes a Besov space. More details about such spaces and their properties can be found in [24].) □

We recall an extrapolation result of Kühn and Schonbek [18]. Let X be a Banach space and Y_θ, $\theta \in (0,1)$, be a family of Banach spaces such that

$$Y_\theta \hookrightarrow Y_{\theta'}, \quad \text{if } 0 < \theta \le \theta' < 1,$$

with uniformly bounded inclusion operators. Let η be a positive real number. Following [15] we consider the space $Y := \Delta\big(\theta^\eta Y_\theta\big)$,

$$f \in Y \iff f \in \bigcap_{\theta\in(0,1)} Y_\theta \quad \text{and} \quad \|f\|_Y := \sup_{\theta\in(0,1)} \theta^\eta \|f\|_{Y_\theta} < \infty.$$

Suppose that for a linear operator $T : X \to \bigcap_{\theta\in(0,1)} Y_\theta$ there is a constant $C > 0$ and parameters $\sigma_1, \sigma_2 > 0$ such that, for all $\theta \in (0,1)$ and all $k \in \mathbb{N}$,

$$\|T\|_{X\to Y_\theta} \le C\theta^{-\sigma_1} \tag{6}$$

$$e_k\big(T : X \to Y_\theta\big) \le C\theta^{-\sigma_2}\,\varphi(k), \tag{7}$$

where $\varphi : [1,\infty) \to (0,\infty)$ is a decreasing function such that $\lim_{x\to\infty}\varphi(x) = 0$ and, with some constant $C_0 > 1$, $\varphi(x) \le C_0\varphi(2x)$ for all $x \ge 1$.

Proposition 3.5 ([18, Theorem 2.1]). *Let T, X and Y_θ, $\theta \in (0,1)$, have the same meaning as above and conditions* (6) *and* (7) *be satisfied. Suppose that $\sigma_1 < \min\big(\eta, \sigma_2\big)$. Then*

$$e_k\big(T : X \to Y\big) \lesssim \begin{cases} \varphi(k), & \text{if } \eta > \sigma_2 \\ \varphi\big(\frac{k}{\log k}\big), & \text{if } \eta = \sigma_2 \\ \varphi(k)^\lambda, & \text{if } \sigma_1 < \eta < \sigma_2, \end{cases}$$

where $\lambda := \frac{\eta-\sigma_1}{\sigma_2-\sigma_1}$.

The main result of this subsection is the following lemma.

Lemma 3.6. *Let the assumptions of Theorem* 3.2 *be satisfied.*
- (i) *If $0 < \nu < \frac{n}{n+2}$, then $e_k \lesssim k^{-\frac{1}{n}}$ $(k \in \mathbb{N})$.*
- (ii) *If $\frac{n}{n+2} \le \nu < 1$, then $e_k \lesssim k^{-\frac{1}{n}}(\log k)^{\frac{1}{n}}$ $(k \in \mathbb{N})$.*
- (iii) *If $\nu = 1$, then $e_k \lesssim k^{-\frac{1}{n}}(\log k)^{\frac{2}{n}}$ $(k \in \mathbb{N})$.*
- (iv) *If $1 < \nu < q_\alpha$, then $e_k \lesssim k^{-\frac{1}{D}(\frac{1}{\nu}-\frac{1}{q_\alpha})}(\log k)^{\frac{1}{D}(\frac{1}{\nu}-\frac{1}{q_\alpha})}$ $(k \in \mathbb{N})$, where $D = 1 + n\alpha$.*

Proof. Put $X = W^{1,n}(\log W)^\alpha(\Omega)$, $Y_\theta = L^{\frac{1}{\theta}}(\Omega)$, $\eta = \frac{1}{\nu}$ and $T = Id$. Then, by Lemma 2.3 (i), we have $Y = E_\nu(\Omega)$. The estimate (i) follows by Proposition 3.5 on putting $\sigma_1 = \frac{1}{q_\alpha}$, $\sigma_2 = 1 + \frac{2}{n}$ and $\varphi(k) = k^{-\frac{1}{n}}$ and using Proposition 3.4. To obtain estimates (ii)–(iv) we put $\sigma_1 = \frac{1}{q_\alpha}$, $\sigma_2 = 1$ and $\varphi(k) = (\frac{k}{\log k})^{-\frac{1}{n}}$ and use again Propositions 3.5 and 3.4. □

3.2. Lower estimate. In this subsection we prove the following lemma.

Lemma 3.7. *Let the assumptions of Theorem* 3.2 *be satisfied. Then*

$$e_k \gtrsim \max\Big\{k^{-\frac{1}{n}}, k^{-\left(\frac{1}{\nu}-\frac{1}{q_\alpha}\right)}\Big\}.$$

Proof. At first we prove that

$$e_k \gtrsim k^{-\frac{1}{n}}. \tag{8}$$

Since Ω is bounded we have the embeddings

$$W^{1,p}(\Omega) \hookrightarrow W^{1,n}(\log W)^\alpha(\Omega) \hookrightarrow E_\nu(\Omega) \hookrightarrow L^q(\Omega), \tag{9}$$

where $p > n$ and $q > 1$ are fixed numbers. By the result of Edmunds and Triebel [13],

$$e_k\big(Id : W^{1,p}(\Omega) \to L^q(\Omega)\big) \approx k^{-\frac{1}{n}}. \tag{10}$$

Together with (9), (2) and (1), the last estimate immediately implies (8).

Observe that for proving the estimate

$$e_k \gtrsim k^{-(\frac{1}{\nu}-\frac{1}{q\alpha})} \tag{11}$$

we can assume, without loss of generality, that $\Omega = Q$, where Q is a cube. We shall prove (11) in three steps.

STEP 1. First we shall do some preliminary work. Let the cube $Q = (-\frac{1}{2}, \frac{1}{2})^n$ (centered at the origin) be subdivided into 2^{nk}, $k \in \mathbb{N}$, congruent (mutually disjoint and open) subcubes Q_i with centres $x^{[i]}$, $i = 1, \dots, 2^{nk}$.

Let $g \in W^{1,n}(\log W)^\alpha(Q)$ be a fixed nonnegative function which is positive on a subset of Q of positive measure and supported in Q. Fix the number $j \in \mathbb{N}$. We introduce the mappings

$$A : \ell_1^{2^{nk}} \to W^{1,n}(\log W)^\alpha(Q), \quad B : L^j(Q) \to \ell_j^{2^{nk}},$$

in the following way:

$$A : \{a_i\}_{i=1}^{2^{nk}} \mapsto \sum_{i=1}^{2^{nk}} a_i\, g\big(2^k(x - x^{[i]})\big)$$

and

$$B : f \mapsto \left\{ \frac{2^{nk}}{\|g\|_{j;Q}^j} \int_{Q_i} f(x) \left|g\big(2^k(x - x^{[i]})\big)\right|^{j-1} dx \right\}_{i=1}^{2^{nk}}$$

(note that $g \in L^j(Q)$ due to the embedding $W^{1,n}(\log W)^\alpha(Q) \hookrightarrow L^j(Q)$ for any $j \in \mathbb{N}$).

Let us verify that $(B \circ A)$ is the identity mapping $\ell_1^{2^{nk}} \to \ell_j^{2^{nk}}$. Having in mind that the cubes Q_i are mutually disjoint, $g \geq 0$ and $\operatorname{supp} g\big(2^k(\cdot - x^{[i]})\big) \subset Q_i$, $i = 1, \dots, 2^{nk}$, we obtain

$$\begin{aligned}
B\big(A\,\{a_i\}_{i=1}^{2^{nk}}\big) &= \left\{ \frac{2^{nk}}{\|g\|_{j;Q}^j} \int_{Q_i} \left[\sum_{l=1}^{2^{nk}} a_l\, g\big(2^k(x - x^{[l]})\big) \right] \left|g\big(2^k(x - x^{[i]})\big)\right|^{j-1} dx \right\}_{i=1}^{2^{nk}} \\
&= \left\{ \frac{2^{nk}}{\|g\|_{j;Q}^j} \int_{Q_i} a_i \left|g\big(2^k(x - x^{[i]})\big)\right|^{j} dx \right\}_{i=1}^{2^{nk}} \\
&= \left\{ \frac{2^{nk}}{\|g\|_{j;Q}^j}\, a_i \int_{Q} 2^{-nk} \left|g(y)\right|^{j} dy \right\}_{i=1}^{2^{nk}} \\
&= \{a_i\}_{i=1}^{2^{nk}}.
\end{aligned}$$

We have the following commutative diagram:

$$\begin{array}{ccc}
\ell_1^{2^{nk}} & \xrightarrow{Id_1} & \ell_j^{2^{nk}} \\
\Big\downarrow & & \Big\uparrow B \\
A \Big\downarrow & & L^j(Q) \\
\Big\downarrow & & \Big\uparrow Id_2 \\
W^{1,n}(\log W)^\alpha(Q) & \xrightarrow{Id} & E_\nu(Q)
\end{array}$$

Using the properties of entropy numbers given by (2) and (1) we have $e_k(Id_1) \le \|A\| \, \|B\| \, \| \, Id_2 \, \| \, e_k(Id)$, which implies that

$$e_k(Id) \ge \frac{e_k(Id_1)}{\|A\| \, \|B\| \, \| \, Id_2 \, \|}. \tag{12}$$

<u>`STEP 2.`</u> We estimate the entropy number $e_k(Id_1)$ from below and norms of the mappings A, B and Id_2 from above.

Lower estimate of $e_k(Id_1)$. By Lemma 2.4 we have

$$e_k(Id_1) \approx 1. \tag{13}$$

Norm of A. Let $\big\|\{a_i\}_{i=1}^{2^{nk}}\big\|_{\ell_1^{2^{nk}}} \le 1$. Using the triangle inequality and the translation invariance of the norm in $W^{1,n}(\log W)^\alpha(Q)$ we have

$$\big\|A\{a_i\}_{i=1}^{2^{nk}}\big\|_{1;n;\alpha;Q} \le \sup_{i \in \{1,\dots,2^{nk}\}} \big\|g\big(2^k(\cdot - x^{[i]})\big)\big\|_{1;n;\alpha;Q_i} \bigg(\sum_{i=1}^{2^{nk}} |a_i|\bigg) \le \big\|g\big(2^k \cdot\big)\big\|_{1;n;\alpha;Q}$$

(observe that $\operatorname{supp} g\big(2^k \cdot\big) \subset 2^{-k}Q$), and consequently,

$$\|A\|_{W^{1,n}(\log W)^\alpha(Q) \to \ell_1^{2^{nk}}} \le \big\|g\big(2^k \cdot\big)\big\|_{1;n;\alpha;Q}. \tag{14}$$

Norm of B. Let $f \in L^j(Q)$ be such that $\|f\|_{j;Q} \le 1$. Using the Hölder inequality, change of variables and the fact that the cubes Q_i, $i = 1, \dots, 2^{nk}$, are mutually disjoint, we arrive at

$$\begin{aligned}
\|B(f)\|_{\ell_j^{2^{nk}}} &= \frac{2^{nk}}{\|g\|_{j;Q}^j} \left(\sum_{i=1}^{2^{nk}} \left| \int_{Q_i} f(x) \left| g\big(2^k(x - x^{[i]})\big) \right|^{j-1} dx \right|^j \right)^{\frac{1}{j}} \\
&\le \frac{2^{nk}}{\|g\|_{j;Q}^j} \left(\sum_{i=1}^{2^{nk}} \int_{Q_i} |f(x)|^j \, dx \left[\int_{Q_i} \left| g\big(2^k(x - x^{[i]})\big) \right|^j dx \right]^{j-1} \right)^{\frac{1}{j}} \\
&\le \frac{2^{nk}}{\|g\|_{j;Q}^j} \left(2^{-nk} \int_Q |g(y)|^j \, dy \right)^{\frac{j-1}{j}} \\
&= 2^{\frac{nk}{j}} \, \|g\|_{j;Q}^{-1},
\end{aligned}$$

and so

$$\|B\|_{L^j(Q)\to\ell_j^{2^{nk}}} \le 2^{n\frac{k}{j}} \|g\|_{j;Q}^{-1}. \tag{15}$$

Norm of Id_2. From Lemma 2.3 we immediately obtain

$$\| \, Id_2 \, \|_{E_\nu(Q)\to L^j(Q)} \lesssim j^{\frac{1}{\nu}}. \tag{16}$$

Putting $j = k$, we get from (12)–(16) that

$$e_k(Id) \gtrsim k^{-\frac{1}{\nu}} \, \frac{\|g\|_{k;Q}}{\|g(2^k\cdot)\|_{1;n;\alpha;Q}}. \tag{17}$$

STEP 3. We choose a suitable function g to obtain the desired estimate from (17). For a sufficiently small τ belonging to $(0, \frac{1}{4})$ we put

$$h_\tau(t) = \ell_1^{-\frac{1}{n}-\alpha}(\tau)\, \chi_{[\tau,\frac{1}{4})}(t)\, t^{-1}, \quad t \ge 0 \tag{18}$$

$$\mathcal{H}_\tau(t) = \int_t^\infty h_\tau(y)\, dy, \qquad\qquad t > 0. \tag{19}$$

We immediately see that $\operatorname{supp} \mathcal{H}_\tau \subset [0, \frac{1}{4}]$ and

$$\mathcal{H}_\tau(t) \approx \ell_1^{\frac{1}{n'}-\alpha}(\tau) = \ell_1^{\frac{1}{q_\alpha}}(\tau), \quad \text{for all } t \in [0,\tau]. \tag{20}$$

Set $g(x) = \mathcal{H}_\tau\big(|x|\big)$, $x \in \mathbb{R}^n$, with $\tau = e^{-k}$ and $k \in \mathbb{N}$. Obviously, $\operatorname{supp} g \subset Q$, and due to (20) we have, for large $k \in \mathbb{N}$,

$$\|g\|_{k;Q} \gtrsim \ell_1^{\frac{1}{q_\alpha}}(\tau)\, \tau^{\frac{n}{k}} \approx k^{\frac{1}{q_\alpha}}. \tag{21}$$

With respect to (17) and (21) it is sufficient to show that this function g satisfies the estimate

$$\|g(2^k\cdot)\|_{1;n;\alpha;Q} \le C \tag{22}$$

with a positive constant independent of k.

By (19) we have

$$\big|\nabla\big(g(2^k x)\big)\big| = 2^k\, |\nabla g|(2^k x) = 2^k\, h_\tau(2^k|x|), \quad \text{for a.e. } x \in \mathbb{R}^n,$$

which together with (18) gives

$$\big|\nabla\big(g(2^k x)\big)\big| = \ell_1^{-\frac{1}{n}-\alpha}(\tau)\, \chi_{[\tau,\frac{1}{4})}(2^k|x|)\, |x|^{-1} \lesssim h_\eta(|x|) \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{23}$$

where $\eta = (e+2)^{-k}$. We can easily compute (cf. [8, (4.8)]) that

$$h_\eta^*(s) = h_\eta\bigg(\Big[\frac{s}{\kappa_n} + \eta^n \Big]^{\frac{1}{n}} \bigg), \quad 0 \le s \le \kappa_n(1-\eta^n), \tag{24}$$

where κ_n denotes the volume of the unit ball in $\mathbb{R}^n$. Let us estimate from above the norm of the function g in the logarithmic Sobolev space. By the assumption, $\alpha \geq 0 > -\frac{1}{n}$. With the aid of (23), (24), (18) and the substitution $\frac{s}{\kappa_n} + \eta^n = t^n$ this condition implies

$$\begin{aligned}
&\big\|\nabla\big(g(2^k\cdot)\big)\big\|_{n;\alpha;Q} \\
&\lesssim \left(\int_0^1 \big(h_\eta^*(s)\,\ell_1^\alpha(s)\big)^n\,ds\right)^{\frac{1}{n}} \\
&\approx \left(\int_\eta^1 \big[h_\eta(t)\,\ell_1^\alpha\big(\kappa_n(t^n-\eta^n)\big)\big]^n t^{n-1}\,dt\right)^{\frac{1}{n}} \\
&\approx \left(\int_{2\eta}^1 \big[h_\eta(t)\,\ell_1^\alpha(t)\big]^n t^{n-1}\,dt\right)^{\frac{1}{n}} + h_\eta(\eta)\left(\int_\eta^{2\eta} \ell_1^{n\alpha}\big(\kappa_n(t^n-\eta^n)\big)\,t^{n-1}\,dt\right)^{\frac{1}{n}} \\
&\approx 1, \quad \text{when } k \to \infty
\end{aligned}$$

(cf. [8, the proof of Lemma 4.1]), verifying (22) for the gradient part of the norm of $g(2^k\cdot)$. The estimate of the remaining part is even simpler. Since $\operatorname{supp} g(2^k\cdot) \subset 2^{-k}Q$ and by monotonicity of $\mathcal{H}_\tau$ plus (20), $\mathcal{H}_\tau(t) \lesssim \ell_1^{\frac{1}{n'}-\alpha}(\tau)$ for all $t \geq 0$, we have

$$\begin{aligned}
\|g(2^k\cdot)\|_{n;\alpha;Q} &\lesssim \ell_1^{\frac{1}{n'}-\alpha}(\tau)\left(\int_0^{2^{-nk}} \ell_1^{n\alpha}(s)\,ds\right)^{\frac{1}{n}} \\
&\lesssim \ell_1^{\frac{1}{n'}-\alpha}(e^{-k})\,\ell_1^\alpha(2^{-nk})\,2^{-k} \\
&\approx 2^{-k}\,k^{\frac{1}{n'}} \lesssim 1.
\end{aligned}$$

The lemma is proved. □

3.3. Proof of Theorem 3.2. The estimates follow from Lemma 3.6 and Lemma 3.7 observing that $k^{-\frac{1}{n}} \gtrsim k^{-(\frac{1}{\nu}-\frac{1}{q\alpha})}$, $k \in \mathbb{N}$, if and only if $\nu \leq \frac{1}{1-\alpha}$. □

4. Concluding remarks

It is a natural question whether it would be possible to obtain asymptotic estimates of entropy numbers of other extremal embeddings in the sense of [6–11]. Let us look at the model case

$$W^{1,n}(\log W)^{1/n'}(\Omega) \hookrightarrow\hookrightarrow EE_\nu(\Omega), \quad 0 < \nu < n'.$$

Using a method of extrapolation (either a modification of the method of Kühn and Schonbek [18] similar to that employed in the proof of Lemma 3.6 or the method of Triebel [25, Paragraph 4.5.2]) together with Lemma 2.3 (ii) and Proposition 3.4 we obtain the following upper estimate.

Claim 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$, Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary and let $0 < \nu < n'$. Denote by e_k the k^{th} entropy number of the natural embedding $Id : W^{1,n}(\log W)^{1/n'}(\Omega) \to EE_\nu(\Omega)$. Then*

$$e_k \lesssim \big(\log k\big)^{-\left(\frac{1}{\nu}-\frac{1}{n'}\right)}. \tag{25}$$

Unfortunately, the best lower asymptotic estimate of e_k (under the assumptions of Claim 4.1) which we can obtain is

$$e_k \gtrsim k^{-\frac{1}{n}}. \tag{26}$$

To see this, we use the chain of embeddings

$$W^{1,p}(\Omega) \hookrightarrow W^{1,n}(\log W)^{\frac{1}{n'}}(\Omega) \hookrightarrow EE_\nu(\Omega) \hookrightarrow L^q(\Omega),$$

(where $p > n$ and $q > 1$ are fixed numbers) and (10). It is surprising that using a method analogous to deriving (11) (with a suitable extremal function h_τ) we obtain a lower estimate $e_k \gtrsim k^{-\frac{1}{n}}(\log k)^{-(\frac{1}{\nu}-\frac{1}{n'})}$ which is worse than (26).

It seems likely that the upper estimate (25) is not optimal and the reason why it is so probably lies in the estimate (5) where the dependence on q on the right hand side is not precise (we have used the rough estimate $\| Id \|_{W^{1,n}(\log W)^{1/n'}(\Omega)\to W^{1,n}(\Omega)} \lesssim 1$). At this moment, it is not clear how to improve this estimate. Both estimates (from the proof of Proposition 3.4) are based on the upper estimates of entropy numbers of embeddings of the type

$$Id : B \to L^q,$$

where B is a Besov space, and we are not aware of any suitable analogue of it in our case. This question is also interesting in the single-exponential case, since better estimates can improve the constant D in Theorem 3.2.

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