

# Measures of Noncompactness in the Study of Asymptotically Stable and Ultimately Nondecreasing Solutions of Integral Equations

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**Abstract.** We introduce a new class of measures of noncompactness related to asymptotic stability and ultimate monotonicity in the space of continuous and bounded functions on an unbounded interval. With help of those measures of noncompactness and a fixed point theorem of Darbo type we investigate the existence of asymptotically stable and ultimately nondecreasing solutions of some quadratic functional integral equations of Hammerstein–Volterra type.

**Keywords.** Measure of noncompactness, Darbo fixed point theorem, superposition operator, quadratic functional integral equation, asymptotic stability, ultimate monotonicity

**Mathematics Subject Classification (2000).** Primary 47H09, secondary 26A48, 45G10, 47H10

## 1. Introduction

The theory of integral equations is an important branch of mathematical analysis. This theory applies to many real world problems, especially in mathematical physics, mechanics, engineering, biology, economics and so on (cf. [1, 13, 16, 18, 21, 22, 23]). It is worthwhile mentioning that, building on the theory of integral equations one can answer many questions arising in the kinetic theory of gases, the theory of radiative transfer and the theory of neutron transport [14, 15, 19, 20].

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The principal aim of this paper is to study the existence of solutions of quadratic functional integral equations in some special classes of functions. Namely, we will look for such solutions of integral equations which are simultaneously asymptotically stable and ultimately nondecreasing in the sense defined below. The main tools which will be used are the technique of measures of noncompactness and a fixed point theorem of Darbo type.

In order to realize our purposes we define a class of measures of noncompactness related to asymptotic stability and ultimate monotonicity. The use of those measures of noncompactness enables us to prove that some quadratic functional integral equations of Hammerstein–Volterra type have solutions in the space of bounded and continuous functions on the real half axis which are nonnegative, asymptotically stable and ultimately nondecreasing.

The results of this paper generalize several results obtained earlier. Moreover, our main theorem obtained below seems to apply as well to other classes of functional and functional integral equations.

## 2. Notation, definitions and some auxiliary facts

In this section we introduce some notation and collect some basic facts needed in what follows. By  $\mathbb{R}$  we denote the set of real numbers, by  $\mathbb{R}_+ = [0, \infty)$  the set of all nonnegative real numbers.

Let  $E$  be an infinite dimensional real Banach space with norm  $\|\cdot\|$ . We denote by  $B(x, r)$  the closed ball centered at  $x$  and with radius  $r$ . If  $X$  is a subset of  $E$  then the symbols  $\overline{X}$  and  $\text{Conv}X$  denote the closure and convex closure of  $X$ , respectively. If  $X$  and  $Y$  are subsets of  $E$  and  $\lambda \in \mathbb{R}$  then we write  $X + Y$  and  $\lambda X$  to denote the usual algebraic operations on sets.

Throughout this paper, we denote by  $\mathfrak{M}_E$  the family of all nonempty bounded subsets of  $E$  and by  $\mathfrak{N}_E$  its subfamily consisting of all nonempty relatively compact sets.

Now we recall a few definitions which turn out to be useful in nonlinear analysis and its applications (see [2, 3, 5, 7], for example). Given a Banach space  $E$  and a function  $\mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+$ , in what follows we call the set

$$\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\} \quad (2.1)$$

the *kernel* of  $\mu$ . In mathematical analysis such set functions  $\mu$  are used over and over, but in the sequel we will use only set functions which are measures of noncompactness in the axiomatic sense of [7]. Let us recall the necessary definition.

**Definition 2.1.** A set function  $\mu$  is said to be a *measure of noncompactness* in  $E$  if it satisfies the following conditions:

- 1° The kernel  $\ker \mu$  is nonempty and  $\ker \mu \subseteq \mathfrak{N}_E$ .  
 2°  $X \subseteq Y$  implies  $\mu(X) \leq \mu(Y)$ .  
 3°  $\mu(\overline{X}) = \mu(X)$ .  
 4°  $\mu(\text{Conv}X) = \mu(X)$ .  
 5°  $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$  for  $\lambda \in [0, 1]$ .  
 6° If  $(X_n)$  is a sequence of closed sets from  $\mathfrak{M}_E$  such that  $X_{n+1} \subseteq X_n$  ( $n = 1, 2, \dots$ ) and if  $\lim_{n \rightarrow \infty} \mu(X_n) = 0$ , then the intersection set  $X_\infty = \bigcap_{n=1}^{\infty} X_n$  is nonempty.

Measures of noncompactness are closely related to the following fixed point theorem of Darbo type (cf. [17,7]) which will be used further on.

**Theorem 2.1.** *Let  $\Omega$  be a nonempty, bounded, closed and convex subset of a Banach space  $E$ , and let  $V : \Omega \rightarrow \Omega$  be a continuous mapping. Assume that there exists a constant  $k \in [0, 1)$  such that  $\mu(VX) \leq k\mu(X)$  for any nonempty subset  $X$  of  $\Omega$ . Then  $V$  has a fixed point in the set  $\Omega$ .*

We remark that, under the hypotheses of Theorem 2.1, it can be shown [7] that the set of all fixed points of the operator  $V$  in  $\Omega$  always belongs to the kernel  $\ker \mu$ . This observation, although being quite simple, will be crucial in our further considerations.

Now let us introduce two Banach function spaces which we will use in our existence theorems for functional integral equations in Section 5. Namely, denote by  $B(\mathbb{R}_+)$  the space consisting of all bounded real functions on the half-line  $\mathbb{R}_+$ , equipped with the standard supremum norm

$$\|x\| = \sup \{|x(t)| : t \in \mathbb{R}_+\}. \quad (2.2)$$

Together with  $B(\mathbb{R}_+)$  we will also consider the closed subspace  $BC(\mathbb{R}_+)$  of all continuous functions  $x \in B(\mathbb{R}_+)$ . Obviously, both  $B(\mathbb{R}_+)$  and  $BC(\mathbb{R}_+)$  are Banach spaces with respect to the norm (2.2).

In what follows we assume that  $\Omega$  is a nonempty subset of the space  $BC(\mathbb{R}_+)$  and  $V$  is an operator defined on  $\Omega$  with values in  $BC(\mathbb{R}_+)$ . Consider the operator equation

$$x(t) = (Vx)(t), \quad (t \geq 0). \quad (2.3)$$

**Definition 2.2.** We say that solutions of equation (2.3) are *asymptotically stable* if there exists a ball  $B(x_0, r)$  in the space  $BC(\mathbb{R}_+)$  such that  $B(x_0, r) \cap \Omega \neq \emptyset$  and for each  $\varepsilon > 0$  there exists  $T > 0$  such that  $|x(t) - y(t)| \leq \varepsilon$  for all solutions  $x, y \in B(x_0, r) \cap \Omega$  of (2.3) and for  $t \geq T$ .

We mention that the concept of asymptotic stability in the sense of Definition 2.2 was introduced in [9, 10]. Clearly, a *unique* solution of equation (2.3) is always asymptotically stable in the sense of Definition 2.2, so only in the case of multiple solutions this notion becomes interesting.

### 3. Some set functions in the spaces $B(\mathbb{R}_+)$ and $BC(\mathbb{R}_+)$

In this section we will work in the space  $B(\mathbb{R}_+)$  or in its closed subspace  $BC(\mathbb{R}_+)$  described in the previous section. Let us recall that both spaces are endowed with the standard supremum norm (2.2).

For a function  $x \in B(\mathbb{R}_+)$  and fixed  $T > 0$  we define the quantities

$$\begin{aligned} d_T(x) &= \sup \{|x(s) - x(t)| - [x(s) - x(t)] : T \leq t < s\} \\ i_T(x) &= \sup \{|x(s) - x(t)| - [x(t) - x(s)] : T \leq t < s\}. \end{aligned}$$

The quantity  $d_T(x)$  represents the so-called *modulus of decrease* of the function  $x$  on the interval  $[T, \infty)$ , while  $i_T(x)$  represents the *modulus of increase* of  $x$  on the interval  $[T, \infty)$ . These quantities have been introduced in [8] in the case of a bounded interval  $[a, b]$  (cf. also [11]).

Next, for  $X \in \mathfrak{M}_{B(\mathbb{R}_+)}$  we put

$$d_T(X) = \sup \{d_T(x) : x \in X\}, \quad i_T(X) = \sup \{i_T(x) : x \in X\}. \quad (3.1)$$

Observe that the functions  $T \mapsto d_T(x)$  and  $T \mapsto i_T(x)$  are nonincreasing on the half-axis  $\mathbb{R}_+$ . This implies that the limits

$$d_\infty(x) = \lim_{T \rightarrow \infty} d_T(x), \quad i_\infty(x) = \lim_{T \rightarrow \infty} i_T(x)$$

exist. Similarly, since  $T \mapsto d_T(X)$  and  $T \mapsto i_T(X)$  are also nonincreasing on  $\mathbb{R}_+$ , the limits

$$d_\infty(X) = \lim_{T \rightarrow \infty} d_T(X), \quad i_\infty(X) = \lim_{T \rightarrow \infty} i_T(X) \quad (3.2)$$

exist as well. In what follows, we will say that a function  $x \in B(\mathbb{R}_+)$  is *ultimately nondecreasing* if  $d_\infty(x) = 0$ , and *ultimately nonincreasing* if  $i_\infty(x) = 0$ .

Consider now the families  $\ker d_\infty$  and  $\ker i_\infty$  in the sense of (2.1). If a set  $X$  contains only functions which are nondecreasing (resp. nonincreasing) on  $\mathbb{R}_+$  or on a subinterval  $[b, \infty)$  of  $\mathbb{R}_+$  then clearly  $X \in \ker d_\infty$  (resp.  $X \in \ker i_\infty$ ). This implies, in particular, that neither  $\ker d_\infty$  nor  $\ker i_\infty$  is the kernel of a measure of noncompactness in  $B(\mathbb{R}_+)$ , and so neither  $d_\infty$  nor  $i_\infty$  is a measure of noncompactness in the space  $B(\mathbb{R}_+)$ . However, as we will see later (see (3.10) and (3.11) below), one may use the set functions  $d_\infty$  and  $i_\infty$  to construct a natural measure of noncompactness in the subspace  $BC(\mathbb{R}_+)$ .

Suppose now that a set  $X$  consists entirely of functions having limits at infinity and tending to their limits uniformly with respect to  $X$ , i.e., the limit

$$\lim_{t \rightarrow \infty} x(t) = g_x \quad (3.3)$$

exists uniformly with respect to  $x \in X$ . We claim that in this case  $X \in \ker d_\infty$  and  $X \in \ker i_\infty$ . To see this, we first remark that such a set  $X$  can be characterized by the relation (see [6, 7])

$$\lim_{T \rightarrow \infty} \left\{ \sup_{x \in X} \left\{ \sup \{ |x(t) - x(s)| : t, s \geq T \} \right\} \right\} = 0. \quad (3.4)$$

Thus, for  $\varepsilon > 0$  we can find  $T > 0$  such that  $|x(t) - x(s)| \leq \varepsilon$  for all  $t, s \geq T$  and for each  $x \in X$ . Hence, taking  $s \geq t \geq T$  and choosing any  $x \in X$  we obtain the estimates

$$| |x(s) - x(t)| - [x(s) - x(t)] | \leq 2|x(s) - x(t)| \leq 2\varepsilon$$

and

$$| |x(s) - x(t)| - [x(t) - x(s)] | \leq 2|x(s) - x(t)| \leq 2\varepsilon.$$

This shows that  $d_\infty(X) \leq 2\varepsilon$  and  $i_\infty(X) \leq 2\varepsilon$ , by definition (3.2), and so  $X \in \ker d_\infty$  and  $X \in \ker i_\infty$ , since  $\varepsilon > 0$  was arbitrary.

We will show later that condition (3.4) is sufficient, but not necessary for a set  $X$  to belong to the families  $\ker d_\infty$  and  $\ker i_\infty$ . A condition which is both necessary and sufficient is contained in the following theorem.

**Theorem 3.1.** *A bounded and nonempty subset  $X$  of the space  $B(\mathbb{R}_+)$  belongs to the family  $\ker d_\infty$  if and only if for any  $\varepsilon > 0$  there exists  $T > 0$  such that for each  $x \in X$  and for all  $s \geq t \geq T$  the inequality*

$$x(t) \leq x(s) + \varepsilon \quad (3.5)$$

holds.

*Proof.* Suppose first that  $X \in \ker d_\infty$ , and let  $\varepsilon > 0$ . Then there exists  $T > 0$  such that for each  $x \in X$  and for all  $s \geq t \geq T$  we have

$$|x(s) - x(t)| - [x(s) - x(t)] \leq \varepsilon. \quad (3.6)$$

If  $x(s) \geq x(t)$  for  $s \geq t \geq T$ , then (3.5) is satisfied. Conversely, if  $x(s) < x(t)$ , then  $|x(s) - x(t)| = -[x(s) - x(t)]$ , and from (3.6) we get

$$|x(s) - x(t)| - [x(s) - x(t)] = -2[x(s) - x(t)] \leq \varepsilon,$$

hence  $x(t) \leq x(s) + \frac{\varepsilon}{2} \leq x(s) + \varepsilon$ . This shows that for any  $\varepsilon > 0$  there exists  $T > 0$  such that for every  $x \in X$  and for all  $s \geq t \geq T$  the inequality (3.5) is satisfied.

Conversely, assume now that the condition (3.5) holds. Fix  $\varepsilon > 0$  and choose  $T > 0$  according to (3.5). Further, let  $x \in X$  and  $t, s$  be such that  $T \leq t \leq s$ . If  $x(s) \geq x(t)$ , then we have

$$|x(s) - x(t)| - [x(s) - x(s)] = 0 \leq \varepsilon.$$

On the other hand, if  $x(s) < x(t)$ , then in view of (3.5) we obtain  $x(t) - x(s) \leq \varepsilon$ , hence

$$|x(s) - x(t)| - [x(s) - x(t)] = -2[x(s) - x(t)] = 2[x(t) - x(s)] \leq 2\varepsilon.$$

In both cases we have shown that  $d_T(x) \leq 2\varepsilon$  for any function  $x \in X$ , and so  $d_T(X) \leq 2\varepsilon$ , by (3.1). Since  $\varepsilon$  was arbitrary we have  $d_\infty(X) = 0$  as claimed. This completes the proof.  $\square$

Of course, in exactly the same way we can prove the following parallel result for  $\ker i_\infty$ .

**Theorem 3.2.** *A bounded and nonempty subset  $X$  of the space  $B(\mathbb{R}_+)$  belongs to the family  $\ker d_\infty$  if and only if for any  $\varepsilon > 0$  there exists  $T > 0$  such that for each  $x \in X$  and for all  $s \geq t \geq T$  the inequality  $x(s) \leq x(t) + \varepsilon$  holds.*

In the following theorem we give a necessary condition for a bounded set  $X \subset B(\mathbb{R}_+)$  to belong to  $\ker d_\infty$  or  $\ker i_\infty$ .

**Theorem 3.3.** *Let  $X \in \ker d_\infty$  or  $X \in \ker i_\infty$ . Then each function  $x \in X$  has a finite limit at infinity, i.e., the limit (3.3) exists and is finite.*

*Proof.* First let us notice that, in view of the boundedness of  $x$ , if the limit (3.3) exists then it is finite. Suppose that the limit (3.3) does not exist for some function  $x \in X$ . Then  $x$  does not satisfy the Cauchy condition, i.e., there exists  $\varepsilon_0 > 0$  such that for each  $T > 0$  we find  $t, s$  with  $s > t \geq T$  such that  $|x(s) - x(t)| \geq \varepsilon_0$ . This implies that there exists an increasing sequence  $\{t_n\}$  such that  $T \leq t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$|x(t_{n+1}) - x(t_n)| \geq \varepsilon_0. \quad (3.7)$$

In view of the boundedness of the function  $x$ , the sequence  $\{x(t_n)\}$  is bounded. By the classical Bolzano-Weierstrass theorem there exists a subsequence  $\{x(t_{k_n})\}$  of the sequence  $\{x(t_n)\}$  which is convergent to a finite limit, say  $x_g$ . Moreover, there exists a monotonic subsequence of the sequence  $\{x(t_{k_n})\}$  (which we denote by the same symbol  $\{x(t_{k_n})\}$ ) which converges to  $x_g$ . Without loss of generality, we may assume that  $\{x(t_{k_n})\}$  is nondecreasing.

Thus, keeping in mind (3.7) we get

$$|x(t_{k_{n+1}}) - x(t_{k_n})| = x(t_{k_{n+1}}) - x(t_{k_n}) \geq \varepsilon_0,$$

and so  $x(t_{k_{n+1}}) \geq x(t_{k_n}) + n\varepsilon_0$  for  $n = 1, 2, \dots$ . But this implies  $\lim_{n \rightarrow \infty} x(t_{k_n}) = \infty$ , contradicting the boundedness of the sequence  $\{x(t_{k_n})\}$ . This completes the proof.  $\square$

We point out again that Theorem 3.3 gives only a necessary condition (existence of the limit (3.3) for each  $x \in X$ ), while condition (3.4) is only sufficient (existence of this limit uniformly with respect to  $x \in X$ ) for a set  $X$  to belong to  $\ker d_\infty$  (or to  $\ker i_\infty$ ). This is illustrated by means of the following example.

**Example 3.1.** Consider the set  $X \subset B(\mathbb{R}_+)$  consisting of all functions  $x_n$  ( $n = 1, 2, \dots$ ) defined by

$$x_n(t) = \begin{cases} 0 & \text{for } t \in [0, n) \\ t - n & \text{for } t \in [n, n + 1) \\ 1 & \text{for } t \geq n + 1. \end{cases}$$

It is easily seen that  $d_T(x_n) = 0$  for each fixed  $T > 0$  and for any  $n = 1, 2, \dots$ , and so  $d_T(X) = 0$ , by (3.1). Consequently,  $d_\infty(X) = 0$  as well, by (3.2).

On the other hand, observe that all functions belonging to  $X$  tend to 1 at infinity, but not uniformly with respect to  $X$ . So (3.3) holds with  $g_{x_n} \equiv 1$ , but (3.4) fails. Let us also remark that  $i_\infty(X) = 1$  in this example.

Now we introduce and study some measures of noncompactness involving the quantities  $d_\infty$  and  $i_\infty$  defined above. To this end, we first recall the construction of a special measure of noncompactness for continuous functions from [7] (cf. also [6]).

For fixed  $\varepsilon > 0$  and  $T > 0$  we denote by

$$\omega^T(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \varepsilon\}$$

the usual *modulus of continuity* of a function  $x$  on the interval  $[0, T]$ . Next, for a nonempty bounded subset  $X$  of the space  $BC(\mathbb{R}_+)$  we put

$$\omega^T(X, \varepsilon) = \sup \{\omega^T(x, \varepsilon) : x \in X\}, \quad \omega_0^T(X) = \lim_{\varepsilon \rightarrow 0} \omega^T(X, \varepsilon),$$

and

$$\omega_0(X) = \lim_{T \rightarrow \infty} \omega_0^T(X). \quad (3.8)$$

In what follows, for  $t \in \mathbb{R}_+$  we will use the shortcut

$$X(t) = \{x(t) : x \in X\}, \quad \text{diam } X(t) = \sup \{|x(t) - y(t)| : x, y \in X\}.$$

Then the function  $\mu$  defined by the formula

$$\mu(X) = \omega_0(X) + \limsup_{t \rightarrow \infty} \text{diam } X(t) \quad (X \in \mathfrak{M}_{BC(\mathbb{R}_+)}) \quad (3.9)$$

is a measure of noncompactness in the space  $BC(\mathbb{R}_+)$  in the sense of Definition 2.1. This measure has also some additional properties [6]. Note that

the kernel (2.1) of the measure  $\mu$  can be characterized as follows [6]: A set  $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$  belongs to  $\ker \mu$  if and only if  $X$  is locally equicontinuous on  $\mathbb{R}_+$  and the “thickness” of the bundle  $X(t)$  formed by the elements of  $X$  tends to zero as  $t \rightarrow \infty$ .

In the sequel we consider the functions  $\mu_d$  and  $\mu_i$  defined on the family  $\mathfrak{M}_{BC(\mathbb{R}_+)}$  by

$$\mu_d(X) = \mu(X) + d_\infty(X) \tag{3.10}$$

and

$$\mu_i(X) = \mu(X) + i_\infty(X), \tag{3.11}$$

respectively, where  $\mu$  denotes the measure of noncompactness (3.9), while  $d_\infty$  and  $i_\infty$  are given by (3.2). It is not hard to see that  $\mu_d$  and  $\mu_i$  are in fact measures of noncompactness in the sense of Definition 2.1. For example, the condition 6° of Definition 2.1 follows easily from the inequalities  $\mu_d(X) \geq \mu(X)$  and  $\mu_i(X) \geq \mu(X)$  (for  $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ ) and the fact that  $\mu$  is a measure of noncompactness in the space  $BC(\mathbb{R}_+)$ .

To give a simple example, for the set  $X \in \mathfrak{M}_{B(\mathbb{R}_+)}$  (actually,  $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ ) from Example 3.1 we have

$$\begin{aligned} d_\infty(X) = 0, \quad i_\infty(X) = \omega_0(X) = \lim_{t \rightarrow \infty} \text{diam} X(t) = 1, \\ \mu_i(X) = 3, \quad \mu(X) = \mu_d(X) = 2. \end{aligned}$$

The kernel (2.1) of  $\mu_d$  and  $\mu_i$  may also be easily characterized: For example, a set  $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$  belongs to  $\ker \mu_d$  if and only if  $X$  is locally equicontinuous on  $\mathbb{R}_+$ , the thickness of the bundle  $X(t)$  formed by the elements of  $X$  tends to zero as  $t \rightarrow \infty$ , and all functions  $x \in X$  are ultimately nondecreasing on  $\mathbb{R}_+$ . Of course, a similar characterization holds for the kernel  $\ker \mu_i$  of the measure of noncompactness  $\mu_i$ . These characterizations of  $\ker \mu_d$  and  $\ker \mu_i$  will be crucial in our further considerations.

#### 4. Properties of superposition operators related to ultimate monotonicity

One of the simplest and most frequently used nonlinear operators is the so-called superposition operator (cf. [4]). Given an interval  $J \subseteq \mathbb{R}$ , we denote by  $X_J$  a set of functions  $x : \mathbb{R}_+ \rightarrow J$ . Furthermore, given a function of two variables  $f : \mathbb{R}_+ \times J \rightarrow \mathbb{R}$ , we may assign to every function  $x \in X_J$  the function  $Fx$  defined by

$$(Fx)(t) = f(t, x(t)) \quad (t \geq 0). \tag{4.1}$$

The operator  $F$  defined in this way is called *the superposition operator* generated by the function  $f$ . In the sequel we will suppose throughout that the following hypotheses are satisfied:

- ( $\alpha$ ) The function  $f$  is continuous on the set  $\mathbb{R}_+ \times J$ .  
 ( $\beta$ ) The function  $t \mapsto f(t, u)$  is ultimately nondecreasing uniformly with respect to  $u$  belonging to bounded subintervals of  $J$ , i.e.,

$$\lim_{T \rightarrow \infty} \left\{ \sup \{ |f(s, u) - f(t, u)| - [f(s, u) - f(t, u)] : s \geq t > T, u \in J_1 \} \right\} = 0$$

for any bounded subinterval  $J_1 \subseteq J$ .

- ( $\gamma$ ) For any fixed  $t \in \mathbb{R}_+$  the function  $u \mapsto f(t, u)$  is nondecreasing on  $J$ .  
 ( $\delta$ ) The function  $u \mapsto f(t, u)$  satisfies a Lipschitz condition, i.e., there exists a constant  $k > 0$  such that

$$|f(t, u) - f(t, v)| \leq k|u - v| \quad (4.2)$$

for all  $t \geq 0$  and all  $u, v \in J$ .

**Theorem 4.1.** *Suppose that the assumptions ( $\alpha$ ) – ( $\delta$ ) are satisfied. Then the inequality*

$$d_\infty(Fx) \leq kd_\infty(x) \quad (4.3)$$

holds for any  $x \in X_J \cap B(\mathbb{R}_+)$ , where  $k$  is the Lipschitz constant from (4.2).

*Proof.* Choose a bounded function  $x \in X_J$  and fix  $T > 0$ . We define a subset  $I_e^T \subseteq \mathbb{R}_+ \times J$  by

$$I_e^T = \{(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ : T \leq t < s \text{ and } x(t) = x(s)\}.$$

Then for  $(t, s) \in I_e^T$  we have

$$\begin{aligned} & |(Fx)(s) - (Fx)(t)| - [(Fx)(s) - (Fx)(t)] \\ &= |f(s, x(s)) - f(t, x(t))| - [f(s, x(s)) - f(t, x(t))] \\ &= |f(s, x(t)) - f(t, x(t))| - [f(s, x(t)) - f(t, x(t))]. \end{aligned}$$

Hence, keeping in mind the boundedness of  $x$  and assumption ( $\beta$ ) we get

$$\begin{aligned} & \lim_{T \rightarrow \infty} \left\{ \sup \{ |(Fx)(s) - (Fx)(t)| - [(Fx)(s) - (Fx)(t)] : s > t \geq T, (s, t) \in I_e^T \} \right\} \\ &= 0. \end{aligned} \quad (4.4)$$

Now assume that  $t, s \in \mathbb{R}_+$ ,  $T \leq t < s$  and  $(t, s) \notin I_e^T$  i.e.,  $x(t) \neq x(s)$ . Then

we obtain

$$\begin{aligned}
& |(Fx)(s) - (Fx)(t)| - [(Fx)(s) - (Fx)(t)] \\
&= |f(s, x(s)) - f(t, x(t))| - [f(s, x(s)) - f(t, x(t))] \\
&\leq |f(s, x(s)) - f(t, x(s))| + |f(t, x(s)) - f(t, x(t))| - [f(s, x(s)) - f(t, x(s))] \\
&\quad - [f(t, x(s)) - f(t, x(t))] \\
&= \{|f(t, x(s)) - f(t, x(t))| - [f(t, x(s)) - f(t, x(t))]\} \\
&\quad + \{|f(s, x(s)) - f(t, x(s))| - [f(s, x(s)) - f(t, x(s))]\} \\
&\leq \left\{ \left| \frac{f(t, x(s)) - f(t, x(t))}{x(s) - x(t)} \right| |x(s) - x(t)| \right. \\
&\quad \left. - \frac{f(t, x(s)) - f(t, x(t))}{x(s) - x(t)} [x(s) - x(t)] \right\} \\
&\quad + \{|f(s, x(s)) - f(t, x(s))| - [f(s, x(s)) - f(t, x(s))]\} \\
&= \left\{ \left| \frac{f(t, x(s)) - f(t, x(t))}{x(s) - x(t)} \right| |x(s) - x(t)| \right. \\
&\quad \left. - \left| \frac{f(t, x(s)) - f(t, x(t))}{x(s) - x(t)} \right| [x(s) - x(t)] \right\} \\
&\quad + \{|f(s, x(s)) - f(t, x(s))| - [f(s, x(s)) - f(t, x(s))]\} \\
&\leq \frac{k|x(s) - x(t)|}{|x(s) - x(t)|} \{|x(s) - x(t)| - [x(s) - x(t)]\} \\
&\quad + \{|f(s, x(s)) - f(t, x(s))| - [f(s, x(s)) - f(t, x(s))]\} \\
&\leq k \cdot \sup \{|x(s) - x(t)| - [x(s) - x(t)] : s > t \geq T, (t, s) \notin I_e^T\} \\
&\quad + \sup \{|f(s, x(s)) - f(t, x(s))| \\
&\quad - [f(s, x(s)) - f(t, x(s))]: s > t \geq T, (t, s) \notin I_e^T\}.
\end{aligned}$$

Observe that in the second equality sign we used the fact that the expression  $\frac{f(t, x(s)) - f(t, x(t))}{x(s) - x(t)}$  is nonnegative. From these estimates and assumption  $(\beta)$  we further obtain

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \left\{ \sup \{|(Fx)(s) - (Fx)(t)| - [(Fx)(s) - (Fx)(t)]: s > t \geq T, (t, s) \notin I_e^T\} \right\} \\
& \leq k \lim_{T \rightarrow \infty} \left\{ \sup \{|x(s) - x(t)| - [x(s) - x(t)]: s > t \geq T, (t, s) \notin I_e^T\} \right\}. \tag{4.5}
\end{aligned}$$

Combining now (4.4) and (4.5) we obtain (4.3), and the proof is complete.  $\square$

Theorem 4.1 generalizes a result given in [12]. Note that in the proof of Theorem 4.1 we tacitly assumed that both the set  $I_e^T$  and its complement  $\mathbb{R} \setminus I_e^T$  are unbounded. Obviously, in the case when one of these sets is bounded we can repeat the same reasoning referring only to an unbounded set.

A particularly interesting special case in Theorem 4.1 is when we may choose  $k < 1$  in (4.2). In this case (4.3) means that, loosely speaking, the superposition operator  $F$  generated by  $f$  strictly improves the degree of decrease of any bounded subset  $X$  of  $X_J$ .

In the following corollaries we consider two special cases where the hypotheses  $(\alpha)$ – $(\delta)$  are easy to verify. The first corollary is an immediate consequence of the mean value theorem, the second follows from a straightforward calculation. Afterwards we illustrate these corollaries with two simple examples.

**Corollary 4.1.** *Suppose that the function  $f : \mathbb{R}_+ \times J \rightarrow \mathbb{R}$  satisfies the assumptions  $(\alpha)$  and  $(\beta)$  of Theorem 4.1. Moreover, assume that the partial derivative  $f_u$  exists and is nonnegative and bounded on the set  $\mathbb{R}_+ \times J$ . Then  $f$  also satisfies the hypotheses  $(\gamma)$  and  $(\delta)$ , where the Lipschitz constant in (4.2) may be taken as*

$$k = \sup \{f_u(t, u) : (t, u) \in \mathbb{R}_+ \times J\}.$$

**Corollary 4.2.** *Let  $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be continuous, bounded and ultimately nondecreasing, and  $h : J \rightarrow \mathbb{R}$  differentiable with nonnegative and bounded derivative on  $J$ . Then both functions  $f(t, u) = a(t)h(u)$  and  $f(t, u) = a(t) + h(u)$  satisfy assumptions  $(\alpha)$ – $(\delta)$  of Theorem 4.1. The Lipschitz constant  $k$  appearing in (4.2) is given by*

$$k = \|a\|_{BC(\mathbb{R}_+)} \sup \{h'(u) : u \in J\}$$

in the first case, and by

$$k = \sup \{h'(u) : u \in J\}$$

in the second case.

**Example 4.1.** For fixed  $\alpha > 0$ , let  $f : \mathbb{R}_+ \times [\alpha, \infty) \rightarrow \mathbb{R}_+$  be defined by  $f(t, u) = a(t)\sqrt{u}$ , where

$$a(t) = e^{-\frac{t}{4}} \sin t + \frac{t}{t+1} \quad (t \geq 0).$$

It is easily seen that the function  $a$  takes positive values and is continuous and bounded on  $\mathbb{R}_+$  with  $\|a\|_{BC(\mathbb{R}_+)} \leq \frac{3}{2}$ , but is certainly not monotonic on  $\mathbb{R}_+$ . However, since  $a(t) \rightarrow 1$  as  $t \rightarrow \infty$ , from the properties of the quantity  $d_\infty$  established in Section 3 we conclude that  $d_\infty(a) = 0$  which means that  $a$  is ultimately nondecreasing on  $\mathbb{R}_+$ .

Further, putting  $J = [\alpha, \infty)$  and  $h(u) = \sqrt{u}$  we see that the derivative  $h'(u) = \frac{1}{2}\sqrt{u}$  is positive and bounded on  $J$ , and so we may apply the first case of Corollary 4.2. A trivial calculation shows that  $k = \frac{3}{4}\sqrt{\alpha}$  in this example; in

particular,  $k < 1$  if  $\alpha > \frac{9}{16}$ . Thus, according to Corollary 4.2 the function  $f$  satisfies all the assumptions  $(\alpha) - (\delta)$ , and so Theorem 4.1 applies.

**Example 4.2.** Let  $f : \mathbb{R}_+ \times [3, \infty) \rightarrow \mathbb{R}$  be defined by

$$f(t, u) = \sin \left[ \pi \frac{3t+1}{2t+1} \left( \frac{1}{2} + \frac{u}{9+u^2} \right) \right].$$

We claim that the function  $f$  satisfies all assumption  $(\alpha) - (\delta)$  of Theorem 4.1 on  $J = [3, \infty)$ . The hypothesis  $(\alpha)$  is obvious. To see that  $(\beta)$  holds as well, note that for fixed  $u \in J$  we have  $\lim_{t \rightarrow \infty} f(t, u) = \sin \left[ \frac{3}{2}\pi \left( \frac{1}{2} + \frac{u}{9+u^2} \right) \right]$ . Consider the set  $X = \{f(\cdot, u) : 3 \leq u < \infty\} \subset BC(\mathbb{R}_+)$ . The estimate

$$\left| \sin \left[ \pi \frac{3t+1}{2t+1} \left( \frac{1}{2} + \frac{u}{9+u^2} \right) \right] - \sin \left[ \frac{3}{2}\pi \left( \frac{1}{2} + \frac{u}{9+u^2} \right) \right] \right| \leq \frac{\pi}{3} \cdot \frac{1}{2t+1}$$

shows that all functions in  $X$  tend to their limits *uniformly* with respect to the set  $X$ . So from the facts established in Section 3 we conclude that the function  $f$  satisfies assumption  $(\beta)$ .

Now, combining the inequalities  $\pi \leq \pi \frac{3t+1}{2t+1} \leq \frac{3}{2}\pi$  and  $\frac{1}{2} \leq \frac{1}{2} + \frac{u}{9+u^2} \leq \frac{2}{3}$ , we see that

$$\frac{\pi}{2} \leq \pi \frac{3t+1}{2t+1} \left( \frac{1}{2} + \frac{u}{9+u^2} \right) \leq \pi \quad (4.6)$$

for all pairs  $(t, u) \in \mathbb{R}_+ \times J$ . For the partial derivative of  $f$  with respect to  $u$  we obtain

$$f_u(t, u) = \frac{9-u^2}{(9+u^2)^2} \cos \left[ \pi \frac{3t+1}{2t+1} \left( \frac{1}{2} + \frac{u}{9+u^2} \right) \right].$$

Consequently, taking into account the estimate (4.6) we conclude that  $0 \leq f_u(t, u) \leq \frac{1}{72}$  on  $\mathbb{R}_+ \times J$ , and so we may apply Corollary 4.1 with  $k = \frac{1}{72}$ . This shows that the function  $f$  satisfies all the assumptions  $(\alpha) - (\delta)$ , and so Theorem 4.1 applies.

## 5. Application to a functional integral equation

In this section we will consider the quadratic functional integral equation of Hammerstein–Volterra type

$$x(t) = m(t) + f(t, x(t)) \int_0^t g(t, \tau) h(\tau, x(\tau)) d\tau \quad (t \geq 0). \quad (5.1)$$

Integral equations of this type may be encountered in the mathematical modelling of real world problems arising in mathematical physics, engineering, economics, biology and so on (cf. [1, 13, 16, 18, 20, 21, 22, 23]).

Using the measure of noncompactness (3.10) and some results concerning the superposition operator (4.1) established in Section 4 we will show now that equation (5.1) has asymptotically stable and ultimately nondecreasing solutions in the space  $BC(\mathbb{R}_+)$ . To this end, throughout this section we impose the following hypotheses:

- (i) The function  $m : \mathbb{R}_+ \rightarrow \mathbb{R}$  is continuous, bounded and ultimately nondecreasing with

$$m_0 := \inf \{m(t) : t \in \mathbb{R}_+\} \geq 0.$$

- (ii) The function  $f : \mathbb{R}_+ \times J \rightarrow \mathbb{R}_+$  satisfies the assumptions  $(\alpha) - (\gamma)$  from Section 4 on the interval  $J = [m_0, \infty)$ .  
 (iii) There exists a nondecreasing function  $k : J \rightarrow \mathbb{R}_+$  such that

$$|f(t, u) - f(t, v)| \leq k(r)|u - v|$$

for all  $t \in \mathbb{R}_+$  and all  $u, v \in J$  with  $|u - v| \leq r$ .

- (iv) The function  $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous and satisfies the condition

$$\lim_{T \rightarrow \infty} \left\{ \sup \left\{ \int_0^s \{|g(s, \tau) - g(t, \tau)| - [g(s, \tau) - g(t, \tau)]\} d\tau : s > t \geq T \right\} \right\} = 0.$$

- (v) The function  $h : \mathbb{R}_+ \times J \rightarrow \mathbb{R}_+$  is continuous, and there exists a continuous nondecreasing function  $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $p(0) = 0$  and

$$|h(t, u) - h(t, v)| \leq p(|u - v|)$$

for all  $t \in \mathbb{R}_+$  and all  $u, v \in J$ .

- (vi) The functions  $t \mapsto f(t, m(t))$  and  $t \mapsto h(t, m(t))$  are bounded on  $\mathbb{R}_+$ .  
 (vii) The function  $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defined by the formula

$$a(t) = \int_0^t g(t, \tau) d\tau$$

vanishes at infinity, i.e.,  $\lim_{t \rightarrow \infty} a(t) = 0$ .

Apart from the function  $a$ , in what follows we will also use auxiliary functions  $b$ ,  $c$  and  $d$  defined by

$$\begin{aligned} b(t) &= f(t, m(t))a(t) \\ c(t) &= \int_0^t g(t, \tau)h(\tau, m(\tau))d\tau \\ d(t) &= f(t, m(t))c(t). \end{aligned} \tag{5.2}$$

By construction, it is clear from (vii) that the function  $a$  is bounded. We claim that the functions  $b$ ,  $c$  and  $d$  are bounded as well. In fact, from (vi) we deduce that the numbers

$$F_m = \sup \{f(t, m(t)) : t \in \mathbb{R}_+\}, \quad H_m = \sup \{h(t, m(t)) : t \in \mathbb{R}_+\} \quad (5.3)$$

are finite. This immediately implies that

$$b(t) \leq F_m a(t), \quad c(t) \leq H_m a(t), \quad d(t) \leq H_m f(t, m(t)) a(t) = H_m b(t).$$

So the four constants

$$\begin{aligned} A &= \sup \{a(t) : t \in \mathbb{R}_+\}, & B &= \sup \{b(t) : t \in \mathbb{R}_+\}, \\ C &= \sup \{c(t) : t \in \mathbb{R}_+\}, & D &= \sup \{d(t) : t \in \mathbb{R}_+\} \end{aligned} \quad (5.4)$$

are all finite. Using these constants, we impose, in addition to (i)–(vii), a final condition:

(viii) There exists a positive real number  $r_0$  which simultaneously satisfies the inequalities

$$\begin{cases} Ark(r)p(r) + Bp(r) + Crk(r) + D \leq r \\ (Ap(r) + C)k(r) < 1, \end{cases}$$

where  $k(r)$  is the local Lipschitz constant from assumption (iii) and  $p(r)$  is from assumption (v).

At first glance, the hypotheses (i)–(viii) may appear somewhat artificial. In the last section we will give a nontrivial illustrative example of equation (5.1) where all hypotheses (i)–(viii) are satisfied. Now we are in a position to formulate and prove our main existence result for the functional integral equation (5.1).

**Theorem 5.1.** *Under the assumptions (i)–(viii) equation (5.1) has at least one solution  $x \in BC(\mathbb{R}_+)$  which is nonnegative, ultimately nondecreasing, and asymptotically stable, and satisfies the two-sided estimate*

$$m(t) \leq x(t) \leq m(t) + r_0 \quad (t \geq 0),$$

where  $r_0$  is the number occurring in assumption (viii).

*Proof.* Consider the subset  $\Omega$  of the space  $BC(\mathbb{R}_+)$  defined by

$$\Omega = \{x \in BC(\mathbb{R}_+) : m(t) \leq x(t) \text{ for } t \in \mathbb{R}_+\}.$$

We define an operator  $V$  on  $\Omega$  by putting

$$(Vx)(t) = m(t) + f(t, x(t)) \int_0^t g(t, \tau) h(\tau, x(\tau)) d\tau \quad (t \geq 0).$$

This operator may be represented in the form

$$Vx = m + (Fx)(Hx), \tag{5.5}$$

where  $F$  is the superposition operator (4.1) and  $H$  is the Hammerstein integral operator defined by

$$(Hx)(t) = \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau \quad (t \geq 0).$$

With this notation, we may rewrite the functional integral equation (5.1) equivalently as fixed point equation (2.3) for the operator (5.5). In the remaining part of this section we will show how to apply Theorem 2.1 to this operator.

Let us show first that the operator  $V$  maps the set  $\Omega$  into itself. To prove this, fix  $x \in \Omega$ . In view of the assumptions (i), (ii), (iv) and (v) we see that the function  $Vx$  is continuous on  $\mathbb{R}_+$  and  $(Vx)(t) \geq m(t)$  for  $t \in \mathbb{R}_+$ , since both  $(Fx)(t) \geq 0$  and  $(Hx)(t) \geq 0$ . To show that  $Vx \in \Omega$  it remains to prove that  $Vx$  is bounded on  $\mathbb{R}_+$ .

From (iii) we get, for arbitrary  $t \in \mathbb{R}_+$ ,

$$\begin{aligned} |(Fx)(t)| &\leq |f(t, x(t)) - f(t, m(t))| + f(t, m(t)) \\ &\leq k(\|x - m\|)|x(t) - m(t)| + f(t, m(t)) \\ &\leq \|x - m\|k(\|x - m\|) + f(t, m(t)), \end{aligned} \tag{5.6}$$

while (v) implies that

$$\begin{aligned} |(Hx)(t)| &\leq \int_0^t g(t, \tau)[|h(\tau, x(\tau)) - h(\tau, m(\tau))| + h(\tau, m(\tau))] d\tau \\ &\leq \int_0^t g(t, \tau)[p(|x(\tau) - m(\tau)|) + h(\tau, m(\tau))] d\tau \\ &\leq p(\|x - m\|) \int_0^t g(t, \tau) d\tau + \int_0^t g(t, \tau)h(\tau, m(\tau)) d\tau \\ &\leq p(\|x - m\|)a(t) + c(t). \end{aligned} \tag{5.7}$$

Thus, using the representation (5.5) and multiplying (5.6) and (5.7), by definition (5.4) of the numbers  $B, C$  and  $D$  we obtain

$$\begin{aligned} |(Vx)(t)| &\leq \|m\| + \|x - m\|k(\|x - m\|)p(\|x - m\|)a(t) \\ &\quad + \|x - m\|k(\|x - m\|)c(t) + p(\|x - m\|)a(t)f(t, m(t)) \\ &\quad + c(t)f(t, m(t)) \\ &\leq \|m\| + \|x - m\|k(\|x - m\|)p(\|x - m\|) \\ &\quad + \|x - m\|k(\|x - m\|)C + p(\|x - m\|)B + D. \end{aligned} \tag{5.8}$$

Since the last expression in (5.8) does not depend on  $t$ , we conclude that  $Vx$  is indeed a bounded function on the interval  $\mathbb{R}_+$ , and so  $Vx \in \Omega$ .

A similar computation as in (5.8) shows that, for arbitrary  $x \in \Omega$ , the inequality

$$\begin{aligned} \|Vx - m\| &\leq \|x - m\|k(\|x - m\|)p(\|x - m\|)A \\ &\quad + \|x - m\|k(\|x - m\|)C + p(\|x - m\|)B + D \end{aligned}$$

holds, where again we have used the upper bounds (5.4). Hence, taking into account the first inequality in assumption (viii) we conclude that there exists a positive real number  $r_0$  such that  $V$  transforms the set

$$\Omega_{r_0} = \Omega \cap B(m, r_0) = \{x \in BC(\mathbb{R}_+) : m(t) \leq x(t) \leq m(t) + r_0 \text{ for } t \in \mathbb{R}_+\} \quad (5.9)$$

into itself. Clearly, the set  $\Omega_{r_0}$  is bounded, convex, and closed. In the sequel we will consider the operator  $V$  on the set  $\Omega_{r_0}$  and apply Theorem 2.1 to  $V$  on this set.

We show now that the operator  $V$  is continuous on the set  $\Omega_{r_0}$ . To do this fix a number  $\varepsilon > 0$  and take  $x, y \in \Omega_{r_0}$  such that  $\|x - y\| \leq \varepsilon$ . Then, by our assumptions (ii), (iii) and (v) we have

$$\begin{aligned} & |(Vx)(t) - (Vy)(t)| \\ & \leq \left| f(t, x(t)) \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau - f(t, y(t)) \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau \right| \\ & \quad + \left| f(t, y(t)) \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau - f(t, y(t)) \int_0^t g(t, \tau)h(\tau, y(\tau))d\tau \right| \\ & \leq |f(t, x(t)) - f(t, y(t))| \\ & \quad \times \int_0^t g(t, \tau)[|h(\tau, x(\tau)) - h(\tau, m(\tau))| + h(\tau, m(\tau))]d\tau \\ & \quad + [|f(t, y(t)) - f(t, m(t))| \\ & \quad + f(t, m(t))] \int_0^t g(t, \tau)|h(\tau, x(\tau)) - h(\tau, y(\tau))|d\tau \tag{5.10} \\ & \leq k(r_0)|x(t) - y(t)| \int_0^t g(t, \tau)[p(|x(\tau) - m(\tau)|) + h(\tau, m(\tau))]d\tau \\ & \quad + [k(r_0)|y(t) - m(t)| + f(t, m(t))] \int_0^t g(t, \tau)p(|x(\tau) - y(\tau)|)d\tau \\ & \leq k(r_0)p(r_0)a(t)|x(t) - y(t)| + k(r_0)c(t)|x(t) - y(t)| \\ & \quad + r_0k(r_0) \int_0^t g(t, \tau)p(|x(\tau) - y(\tau)|)d\tau \\ & \quad + f(t, m(t)) \int_0^t g(t, \tau)p(|x(\tau) - y(\tau)|)d\tau. \end{aligned}$$

By the definition (5.4) of the numbers  $A$ ,  $B$  and  $C$ , this yields

$$|(Vx)(t) - (Vy)(t)| \leq k(r_0)p(r_0)A\varepsilon + k(r_0)C\varepsilon + r_0k(r_0)Ap(\varepsilon) + Bp(\varepsilon),$$

and so the continuity of the operator  $V$  on the set  $\Omega_{r_0}$  follows from the continuity of the function  $p$  assumed in (v).

In the next step we show that

$$\limsup_{t \rightarrow \infty} \text{diam}(VX)(t) = 0 \quad (5.11)$$

for any nonempty subset  $X$  of  $\Omega_{r_0}$ . In fact, let  $x, y \in X$ . Then, for a fixed  $t \in \mathbb{R}_+$ , from the estimate (5.10) we obtain

$$\begin{aligned} \text{diam}(VX)(t) &\leq a(t)k(r_0)p(r_0)\text{diam}X(t) + c(t)k(r_0)\text{diam}X(t) \\ &\quad + r_0k(r_0) \int_0^t g(t, \tau)p(\text{diam}X(\tau))d\tau \\ &\quad + f(t, m(t)) \int_0^t g(t, \tau)p(\text{diam}X(\tau))d\tau \\ &\leq a(t)k(r_0)p(r_0)\text{diam}X(t) + c(t)k(r_0)\text{diam}X(t) \\ &\quad + a(t)r_0k(r_0)p(r_0) + b(t)p(r_0). \end{aligned}$$

Consequently, taking into account assumption (vii) we see that (5.11) is true. We will now combine this with estimates for the set functions (3.8) and (3.9), in order to derive an estimate for the measure of noncompactness (3.10).

Fix  $T > 0$  and  $\varepsilon > 0$ , and let  $x \in X$ . Next, take  $t, s \in [0, T]$  such that  $|t - s| \leq \varepsilon$ . Without loss of generality we may assume that  $t < s$ . Then, again by our assumptions (ii), (iii) and (v) and the representation (5.5) we obtain

$$\begin{aligned} &|(Vx)(s) - (Vx)(t)| \\ &\leq |m(s) - m(t)| + |(Fx)(s)(Hx)(s) - (Fx)(t)(Hx)(s)| \\ &\quad + |(Fx)(t)(Hx)(s) - (Fx)(t)(Hx)(t)| \leq \omega^T(m, \varepsilon) \\ &\quad + |(Fx)(s) - (Fx)(t)|(Hx)(s) + (Fx)(t)|(Hx)(s) - (Hx)(t)| \\ &\leq \omega^T(m, \varepsilon) + [|f(s, x(s)) - f(s, x(t))| \\ &\quad + |f(s, x(t)) - f(t, x(t))|] \int_0^s g(s, \tau)h(\tau, x(\tau))d\tau \\ &\quad + [|f(t, x(t)) - f(t, m(t))| + f(t, m(t))] \\ &\quad \times \left| \int_0^s g(s, \tau)h(\tau, x(\tau))d\tau - \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \omega^T(m, \varepsilon) + [k(r_0)|x(s) - x(t)| \\
 &\quad + \omega_1^t(f, \varepsilon)] \int_0^s g(s, \tau)[|h(\tau, x(\tau)) - h(\tau, m(\tau))| + h(\tau, m(\tau))]d\tau \\
 &\quad + [r_0k(r_0) + f(t, m(t))] \\
 &\quad \times \left[ \left| \int_0^s g(s, \tau)h(\tau, x(\tau))d\tau - \int_0^s g(t, \tau)h(\tau, x(\tau))d\tau \right| \right. \\
 &\quad \left. + \left| \int_0^s g(t, \tau)h(\tau, x(\tau))d\tau - \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau \right| \right] \\
 &\leq \omega^T(m, \varepsilon) + [k(r_0)\omega^T(x, \varepsilon) + \omega_1^t(f, \varepsilon)] [a(s)p(r_0) + c(s)] \\
 &\quad + [r_0k(r_0) + f(t, m(t))] \left[ \int_0^s |g(s, \tau) - g(t, \tau)|h(\tau, x(\tau))d\tau \right. \\
 &\quad \left. + \int_t^s g(t, \tau)h(\tau, x(\tau))d\tau \right] \tag{5.12} \\
 &\leq \omega^T(m, \varepsilon) + [k(r_0)\omega^T(x, \varepsilon) + \omega_1^T(f, \varepsilon)] (Ap(r_0) + C) \\
 &\quad + [r_0k(r_0) + \sup \{f(t, m(t)) : t \in [0, T]\}] \\
 &\quad \times \left[ \int_0^s \omega_1^t(g, \varepsilon)[p(r_0) + h(\tau, m(\tau))]d\tau \right. \\
 &\quad \left. + \varepsilon \sup \{g(t, \tau) : t, \tau \in [0, T]\}(p(r_0) + H_m) \right],
 \end{aligned}$$

where we have used (5.3) and the shortcuts

$$\begin{aligned}
 \omega_1^T(f, \varepsilon) &= \sup \{|f(s, \tau) - f(t, \tau)| : t, s, \tau \in [0, T], |t - s| \leq \varepsilon, \tau \in [m_0, \|m\| + r_0]\} \\
 \omega_1^T(g, \varepsilon) &= \sup \{|g(s, \tau) - g(t, \tau)| : t, s, \tau \in [0, T], |t - s| \leq \varepsilon\}.
 \end{aligned}$$

Now, from the uniform continuity of the function  $f$  on the set  $[0, T] \times [m_0, \|m\| + r_0]$  and of the function  $g$  on the set  $[0, T] \times [0, T]$  we infer that  $\omega_1^T(f, \varepsilon) \rightarrow 0$  and  $\omega_1^T(g, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Combining the above statements with (5.12) we get  $\omega_0^T(VX) \leq (Ap(r_0) + C)k(r_0)\omega_0^T(X)$ , and so, by (3.8), also

$$\omega_0(VX) \leq (Ap(r_0) + C)k(r_0)\omega_0(X). \tag{5.13}$$

Fix  $T > 0$  and take  $t, s$  such that  $s > t \geq T$ . Then, keeping in mind the representation (5.5), for any  $x \in X$  we obtain

$$\begin{aligned}
 &|(Vx)(s) - (Vx)(t)| - [(Vx)(s) - (Vx)(t)] \\
 &\leq |m(s) - m(t)| - [m(s) - m(t)] \\
 &\quad + |(Fx)(s)(Hx)(s) - (Fx)(t)(Hx)(s)| \\
 &\quad + |(Fx)(t)(Hx)(s) - (Fx)(t)(Hx)(t)| \\
 &\quad - [(Fx)(s)(Hx)(s) - (Fx)(t)(Hx)(s)] \\
 &\quad - [(Fx)(t)(Hx)(s) - (Fx)(t)(Hx)(t)]
 \end{aligned}$$

and hence

$$\begin{aligned}
 & |(Vx)(s) - (Vx)(t)| - [(Vx)(s) - (Vx)(t)] \\
 & \leq d_T(m) + (Hx)(s)\{|(Fx)(s) - (Fx)(t)| - [(Fx)(s) - (Fx)(t)]\} \\
 & \quad + (Fx)(t)\{|(Hx)(s) - (Hx)(t)| - [(Hx)(s) - (Hx)(t)]\} \\
 & \leq d_T(m) + (Hx)(s)d_T(Fx) \\
 & \quad + (Fx)(t)\{|(Hx)(s) - (Hx)(t)| - [(Hx)(s) - (Hx)(t)]\}.
 \end{aligned} \tag{5.14}$$

Thus, by (5.3) and our assumptions (ii), (iii) and (v) we obtain the estimates

$$(Hx)(s) \leq \int_0^s g(s, \tau)[p(r_0) + h(\tau, m(\tau))]d\tau \leq a(s)p(r_0) + c(s) \tag{5.15}$$

$$(Fx)(t) \leq |f(t, x(t)) - f(t, m(t))| + f(t, m(t)) \leq r_0k(r_0) + F_m, \tag{5.16}$$

and

$$\begin{aligned}
 & |(Hx)(s) - (Hx)(t)| - [(Hx)(s) - (Hx)(t)] \\
 & = \left| \int_0^s g(s, \tau)h(\tau, x(\tau))d\tau - \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau \right| \\
 & \quad - \left[ \int_0^s g(s, \tau)h(\tau, x(\tau))d\tau - \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau \right] \\
 & \leq \left| \int_0^s g(s, \tau)h(\tau, x(\tau))d\tau - \int_0^s g(t, \tau)h(\tau, x(\tau))d\tau \right| \\
 & \quad + \left| \int_0^s g(t, \tau)h(\tau, x(\tau))d\tau - \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau \right| \\
 & \quad - \left[ \int_0^s g(s, \tau)h(\tau, x(\tau))d\tau - \int_0^s g(t, \tau)h(\tau, x(\tau))d\tau \right] \\
 & \quad - \left[ \int_0^s g(t, \tau)h(\tau, x(\tau))d\tau - \int_0^t g(t, \tau)h(\tau, x(\tau))d\tau \right] \\
 & \leq \int_0^s |g(s, \tau) - g(t, \tau)|h(\tau, x(\tau))d\tau + \left| \int_t^s g(t, \tau)h(\tau, x(\tau))d\tau \right| \\
 & \quad - \int_0^s [g(s, \tau) - g(t, \tau)]h(\tau, x(\tau))d\tau - \int_t^s g(t, \tau)h(\tau, x(\tau))d\tau \\
 & = \int_0^s \{|g(s, \tau) - g(t, \tau)| - [g(s, \tau) - g(t, \tau)]\}h(\tau, x(\tau))d\tau \\
 & \leq \int_0^s \{|g(s, \tau) - g(t, \tau)| - [g(s, \tau) - g(t, \tau)]\} [|h(\tau, x(\tau)) - h(\tau, m(\tau))| \\
 & \quad + h(\tau, m(\tau))]d\tau \\
 & \leq (p(r_0) + H_m) \int_0^s \{|g(s, \tau) - g(t, \tau)| - [g(s, \tau) - g(t, \tau)]\}d\tau.
 \end{aligned} \tag{5.17}$$

Combining the estimates (5.14)–(5.17), using the assumptions (vi) and (vii), and taking into account the definition of the functions  $b$ ,  $c$  and  $d$  in (5.2) we finally get

$$d_\infty(VX) = 0. \tag{5.18}$$

Summarizing, from (5.11), (5.13) and (5.18) we derive the estimate

$$\mu_d(VX) \leq (Ap(r_0) + C)k(r_0)\mu_d(X),$$

where  $\mu_d$  is the measure of noncompactness (3.10). So the second inequality in assumption (viii) guarantees that the operator  $V$  strictly diminishes the measure of noncompactness (3.10), while the first inequality in (viii) shows that  $V$  leaves the bounded closed convex set  $\Omega_{r_0}$  invariant. By Theorem 2.1,  $V$  has at least one fixed point  $x$  in the set  $\Omega_{r_0}$  which of course is a solution of (5.1).

Moreover, keeping in mind Definition 2.2 and the fact that all fixed points of  $V$  in  $\Omega_{r_0}$  belong to  $\ker \mu_d$ , we conclude that all solutions of (5.1) from the set  $\Omega_{r_0}$  are asymptotically stable and ultimately nondecreasing. Finally, the two-sided estimate in the assertion of Theorem 5.1 follows of course from the definition (5.9) of the invariant set  $\Omega_{r_0}$ . This completes the proof.  $\square$

### 6. An example

In this final section we give an example which illustrates the applicability of the abstract Theorem 5.1.

**Example 6.1.** Consider the quadratic functional integral equation of Hammerstein-Volterra type

$$\begin{aligned} x(t) = & 3 + te^{-\frac{t}{2}} + \sin \left[ \pi \frac{3t + 1}{2t + 1} \left( \frac{1}{2} + \frac{x(t)}{9 + x^2(t)} \right) \right] \\ & \times \int_0^t \frac{\tau}{1 + t^2 + \tau^4} \sqrt{\frac{\tau}{1 + \tau^2} - 3 + x(\tau)} d\tau, \end{aligned} \tag{6.1}$$

where  $t \in \mathbb{R}_+$ . Putting

$$\begin{aligned} m(t) = & 3 + te^{-\frac{t}{2}}, & f(t, u) = & \sin \left[ \pi \frac{3t + 1}{2t + 1} \left( \frac{1}{2} + \frac{u}{9 + u^2} \right) \right], \\ g(t, \tau) = & \frac{\tau}{1 + t^2 + \tau^4}, & h(t, u) = & \sqrt{\frac{t}{1 + t^2} - 3 + u}, \end{aligned} \tag{6.2}$$

we see that equation (6.1) is of the form (5.1). We are now going to show that the functions in (6.2) meet all the hypotheses (i)–(viii) of the preceding section.

It is easily seen that the function  $m$  satisfies (i) with  $m_0 = 3$  and  $\|m\| = 3 + 2e^{-1} \approx 3.73576$ ; so we may consider the function  $f$  as in Example 4.2 on

the interval  $J = [3, \infty)$ . As we have seen there,  $f$  satisfies the assumptions (ii) and (iii) with  $k(r) \equiv \frac{1}{72}$ .

In order to check that the function  $g$  satisfies assumption (iv) observe that for any  $T > 0$  and for arbitrary  $t, s$  with  $s > t \geq T$  we have

$$\begin{aligned} & \int_0^s \{|g(s, \tau) - g(t, \tau)| - [g(s, \tau) - g(t, \tau)]\} d\tau \\ &= \int_0^s \left\{ \left| \frac{\tau}{1+s^2+\tau^4} - \frac{\tau}{1+t^2+\tau^4} \right| - \left[ \frac{\tau}{1+s^2+\tau^4} - \frac{\tau}{1+t^2+\tau^4} \right] \right\} d\tau \\ &= 2 \int_0^s \left[ \frac{\tau}{1+t^2+\tau^4} - \frac{\tau}{1+s^2+\tau^4} \right] d\tau \\ &= \frac{1}{\sqrt{1+t^2}} \arctan \frac{s^2}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+s^2}} \arctan \frac{s^2}{\sqrt{1+s^2}} \\ &\leq \frac{1}{\sqrt{1+t^2}} \arctan \frac{s^2}{\sqrt{1+t^2}} \\ &\leq \frac{1}{\sqrt{1+T^2}} \frac{\pi}{2}. \end{aligned}$$

Since the last expression in this estimate tends to 0 as  $T \rightarrow \infty$ , we deduce that the function  $g$  satisfies assumption (iv).

Observing that  $|h(t, u) - h(t, v)| \leq \sqrt{|u - v|}$  for  $t \in \mathbb{R}_+$  and for arbitrary  $u, v \in J$ , we see that the function  $h$  satisfies assumption (v) with  $p(r) = \sqrt{r}$  for  $r \in \mathbb{R}_+$ .

Now we show that the assumptions (vi) and (vii) are satisfied as well. For  $t \in \mathbb{R}_+$  we have

$$f(t, m(t)) = \sin \left[ \pi \frac{3t+1}{2t+1} \left( \frac{1}{2} + \frac{m(t)}{9+m^2(t)} \right) \right] \leq 1$$

and

$$h(t, m(t)) = \sqrt{\frac{t}{1+t^2} - 3 + 3 + te^{-t/2}} \leq \sqrt{\frac{1}{2} + \frac{2}{e}} =: \eta \approx 1.1116,$$

and so (vi) is satisfied. Moreover, the function  $a$  appearing in assumption (vii) has here the form  $a(t) = \int_0^t \frac{\tau}{1+t^2+\tau^4} d\tau = \frac{1}{2\sqrt{1+t^2}} \arctan \frac{t^2}{\sqrt{1+t^2}}$ . Clearly,  $a(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Since

$$c(t) \leq a(t) \sqrt{\frac{1}{2} + \frac{2}{e}} \leq \frac{\eta}{2\sqrt{1+t^2}} \arctan \frac{t^2}{\sqrt{1+t^2}},$$

by (5.2), for the constants defined in (5.4) we get the estimates

$$A \leq \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}, \quad B \leq \frac{\pi}{4}, \quad C \leq \eta \frac{\pi}{4} < 1, \quad D \leq C < 1. \quad (6.3)$$

It remains to show that (viii) is satisfied for some  $r_0 > 0$ . The two inequalities from assumption (viii) have here the form

$$\begin{cases} \frac{Ar\sqrt{r}}{72} + B\sqrt{r} + \frac{Cr}{72} + D \leq r \\ \frac{A\sqrt{r} + C}{72} < 1. \end{cases} \quad (6.4)$$

In view of (6.3), each positive solution  $r$  of the inequalities

$$\begin{cases} \frac{\pi r\sqrt{r}}{288} + \frac{\pi\sqrt{r}}{4} + \frac{r}{72} + 1 \leq r \\ \frac{\pi\sqrt{r}}{288} + \frac{1}{72} < 1 \end{cases} \quad (6.5)$$

is certainly a solution of the inequalities (6.4). But it is easy to check that  $r_0 = 3$ , say, solves (6.5), and so also solves (6.4).

Summarizing, we see Theorem 5.1 applies to equation (6.1), and so (6.1) has a solution  $x \in BC(\mathbb{R}_+)$  which satisfies the two-sided estimate

$$3 + te^{-\frac{t}{2}} \leq x(t) \leq 6 + te^{-\frac{t}{2}}$$

for any  $t \in \mathbb{R}_+$ , since it belongs to the set  $\Omega_4$ , see (5.9). Moreover, this solution is, as any other solution of (6.1) from the set  $\Omega_4$ , both asymptotically stable and ultimately nondecreasing.

**Acknowledgement.** The second author (J. Banaś) is supported by Ministerio de Ciencia y Tecnología (Spain) under the project number MTM 2007-65706.

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Received September 19, 2008