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Trace Operators in Besov and Triebel-Lizorkin Spaces

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Abstract. We determine the trace of Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ – characterized via atomic decompositions – on hyperplanes $\mathbb{R}^m$, $n > m \in \mathbb{N}$, for parameters $0 < p, q \leq \infty$ and $s > \frac{n-m}{p}$. The limiting case $s = \frac{n-m}{p}$ is investigated as well. Our results remain valid considering the classical spaces $\mathbf{B}^s_{p,q}$, $\mathbf{F}^s_{p,q}$ – defined via differences. Finally, we include some density assertions, which imply that the trace does not exist when $s < \frac{n-m}{p}$.

Keywords. Besov spaces, Triebel–Lizorkin spaces, traces, dichotomy

Mathematics Subject Classification (2000). 46E35

1. Introduction

In this article we investigate traces of Besov and Triebel–Lizorkin spaces of positive smoothness – sometimes briefly denoted as B- and F-spaces in the sequel. A clarification of this problem is of crucial interest for boundary value problems of elliptic differential operators.

Besov spaces have been studied for many decades already, resulting, for instance, from the study of partial differential equations, interpolation theory, approximation theory, harmonic analysis. Triebel–Lizorkin spaces were introduced independently by Triebel and Lizorkin in the early 1970s. For a detailed treatment together with historical remarks we refer to Triebel [7, 8]. If

$$0 < p, q < \infty \quad \text{and} \quad s > \tfrac{1}{p} + \max\Big(0, (n-1)\big(\tfrac{1}{p} - 1\big)\Big)$$

the trace of these spaces – on hyperplanes $\mathbb{R}^{n-1}$ – has been known to be a Besov space for a long time, cf. [7, Section 2.7.2]. Since modern subatomic characterizations admit new insights into the nature of these spaces, we are

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now able to extend these results to $s > \frac{1}{p}$. We deal with the most recent definition $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ relying on *atomic decompositions* containing those $f \in L_p(\mathbb{R}^n)$ which can be represented as

$$f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

with coefficients $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ belonging to some appropriate sequence spaces $b^s_{p,q}$ and $f^s_{p,q}$, respectively. In particular, $s > 0$, $0 < p \leq \infty$ ($p < \infty$ for the F-spaces), $0 < q \leq \infty$, and $a_{j,m}(x)$ are normalized atoms. Furthermore,

$$\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| := \inf \|\lambda|b^s_{p,q}\| \quad \text{and} \quad \|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| := \inf \|\lambda|f^s_{p,q}\|,$$

where the infimum is taken over all admissible representations (1.1).

Our results naturally extend the ones previously known, i.e., concerning traces on the hyperplane $\mathbb{R}^{n-1}$ we prove for $s > \frac{1}{p}$,

$$\operatorname{Tr} \mathfrak{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}) \quad \text{and} \quad \operatorname{Tr} \mathfrak{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}).$$

In the limiting case $s = \frac{1}{p}$ we obtain

$$\operatorname{Tr} \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}), \quad 0 < q \leq \min(p, 1)$$

and

$$\operatorname{Tr} \mathfrak{F}^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}), \quad 0 < p < 1.$$

Our results may be extended to more general hyperplanes $\mathbb{R}^m$, $n > m \in \mathbb{N}$.

In particular, all our trace results for Besov spaces $\mathfrak{B}^s_{p,q}$ remain valid for the classical Besov spaces $\mathbf{B}^s_{p,q}$ as well. With some restrictions on the parameters this is also true for Triebel–Lizorkin spaces $\mathbf{F}^s_{p,q}$.

We conclude with the observation that the spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ either have a trace in $L_p(\mathbb{R}^{n-1})$ or the collection of all C^∞ functions in $\mathbb{R}^n$ with compact support in $\mathbb{R}^n \setminus \mathbb{R}^{n-1}$ is dense in them. Related dichotomy numbers are introduced and calculated.

The paper is organized as follows. In Section 2 we present two different approaches to Besov and Triebel–Lizorkin spaces of positive smoothness and briefly discuss their connection. In Section 3 we recall the concept of how to understand traces on hyperplanes $\mathbb{R}^m$ in these function spaces defined on $\mathbb{R}^n$. With the help of the atomic approach we derive our main results for B- and F-spaces, when $s > \frac{n-m}{p}$ as well as for the limiting case $s = \frac{n-m}{p}$. Finally, Section 4 contains some density assertions in form of dichotomy numbers, which imply the non-existence of a trace when $s < \frac{n-m}{p}$.

2. Besov and Triebel-Lizorkin spaces with positive smoothness on $\mathbb{R}^n$

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. The set of multi-indices $\beta = (\beta_1, \dots, \beta_n)$, $\beta_i \in \mathbb{N}_0$, $i = 1, \dots, n$, is denoted by $\mathbb{N}_0^n$, with $|\beta| = \beta_1 + \cdots + \beta_n$, as usual. Moreover, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ we put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.

We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_k \leq b_k \leq c_2\, a_k \quad \text{or} \quad c_1\, \varphi(x) \leq \psi(x) \leq c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $\{a_k\}_k$, $\{b_k\}_k$ are non-negative sequences and φ, ψ are non-negative functions. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of a.

Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both dx and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel.

Let $Q_{j,m}$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-j}m$, and with side length 2^{-j+1}. For a cube Q in $\mathbb{R}^n$ and $r > 0$, we denote by rQ the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q. Furthermore, $\chi_{j,m}$ stands for the characteristic function of $Q_{j,m}$.

2.1. Definitions and basic properties. We give an atomic characterization of Besov and Triebel–Lizorkin spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$. This provides a constructive definition expanding functions f via atoms – excluding any moment conditions – and suitable coefficients, where the latter belong to certain sequence spaces denoted by $b^s_{p,q}$ and $f^s_{p,q}$. According to [10, Prop. 9.14] based on [3], it turns out that these spaces essentially coincide with the well-known classical Besov and Triebel–Lizorkin spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ – defined via differences.

First we introduce the relevant sequence spaces.

Definition 2.1. Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and $\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

(i) Then

$$b^s_{p,q} = \left\{ \lambda : \|\lambda|b^s_{p,q}\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}$$

(with the usual modification if $p=\infty$ and/or $q=\infty$).

(ii) Furthermore

$$f^s_{p,q} = \left\{ \lambda : \|\lambda|f^s_{p,q}\| = \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsq} |\lambda_{j,m}|^q \chi_{j,m}(\cdot) \right)^{\frac{1}{q}} |L_p \right\| < \infty \right\}.$$

Now we define the atoms.

Definition 2.2. Let $K \in \mathbb{N}_0$ and $d > 1$. A K-times differentiable complex-valued function a on $\mathbb{R}^n$ (continuous if $K = 0$) is called a K-atom if for some $j \in \mathbb{N}_0$

$$\operatorname{supp} a \subset dQ_{j,m} \quad \text{for some } m \in \mathbb{Z}^n, \tag{2.1}$$

and

$$|\mathrm{D}^\alpha a(x)| \le 2^{|\alpha| j} \quad \text{for } |\alpha| \le K. \tag{2.2}$$

It is convenient to write $a_{j,m}(x)$ instead of $a(x)$ if this atom is located at $Q_{j,m}$ according to (2.1). Furthermore, K denotes the smoothness of the atom, cf. (2.2). We take the atomic characterization of function spaces of type $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ as a definition.

Definition 2.3. Let $0 < p \le \infty$, $0 < q \le \infty$, and $s > 0$. Let $d > 1$ and $K \in \mathbb{N}_0$ with $K \ge (1 + [s])$ be fixed.

(i) Then $f \in L_p(\mathbb{R}^n)$ belongs to $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \tag{2.3}$$

where the $a_{j,m}$ are K-atoms ($j \in \mathbb{N}_0$) with $\operatorname{supp} a_{j,m} \subset dQ_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and $\lambda \in b^s_{p,q}$, convergence being in $L_p(\mathbb{R}^n)$. Furthermore,

$$\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| = \inf \|\lambda|b^s_{p,q}\|, \tag{2.4}$$

where the infimum is taken over all admissible representations (2.3).

(ii) Then $f \in L_p(\mathbb{R}^n)$ belongs to $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \tag{2.5}$$

where the $a_{j,m}$ are K-atoms ($j \in \mathbb{N}_0$) with $\operatorname{supp} a_{j,m} \subset dQ_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and $\lambda \in f^s_{p,q}$, convergence being in $L_p(\mathbb{R}^n)$. Furthermore,

$$\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| = \inf \|\lambda|f^s_{p,q}\|, \tag{2.6}$$

where the infimum is taken over all admissible representations (2.5).

Remark 2.4. According to [10], based on [3], the spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ are independent of d and K. This may justify our omission of K and d in (2.4) and (2.6).

Moreover, the atomic approaches for B- and F-spaces are strongly linked with the *classical approaches* which introduce $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathbf{F}^s_{p,q}$ as those subspaces of $L_p(\mathbb{R}^n)$ such that

$$\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq}\omega_r(f,t)^q_p \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$

and

$$\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left\|\left(\int_0^1 t^{-sq} d^r_{t,p}f(\cdot)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} |L_p(\mathbb{R}^n)\right\|$$

are finite, respectively, where $0 < p \le \infty$, ($p < \infty$ for F-spaces), $0 < q \le \infty$ (with the usual modification if $q = \infty$), $s > 0$, $r \in \mathbb{N}$ with $r > s$. Here $\omega_r(f,t)_p$ stands for the usual r-th modulus of smoothness of a function $f \in L_p(\mathbb{R}^n)$,

$$\omega_r(f,t)_p = \sup_{|h|\le t} \|\Delta^r_h f \mid L_p(\mathbb{R}^n)\|, \quad t > 0,$$

and $d^r_{t,p}f(\cdot)$ denotes the ball means of $f \in L_p(\mathbb{R}^n)$,

$$d^r_{t,p}f(x) = \left(t^{-n}\int_{|h|\le t} |(\Delta^r_h f)(x)|^p \mathrm{d}h\right)^{\frac{1}{p}}, \quad x \in \mathbb{R}^n,\ t > 0,$$

where

$$(\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta^{r+1}_h f)(x) = \Delta^1_h(\Delta^r_h f)(x), \quad h \in \mathbb{R}^n.$$

Recent results by Hedberg, Netrusov [3] on atomic decompositions and by Triebel [10, Section 9.2] on the reproducing formula prove coincidences

$$\mathbf{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^s_{p,q}(\mathbb{R}^n), \quad s > 0,\ 0 < p,q \le \infty,$$

and

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \quad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{p}\right),\ 0 < p < \infty,\ 0 < q \le \infty,$$

(in terms of equivalent quasi-norms). In particular, this implies that all our results for Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ could as well be stated in terms of the classical spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$. The same is true for the F-spaces with the above restriction on the parameter s.

The following result will be needed later on.

Proposition 2.5. *Let $0 < p, q \le \infty$, $s > 0$, $k \in \mathbb{N}$ with $k > s$.*

(i) (*Diffeomorphisms*)
Let ψ be a k-diffeomorphism. Then $f \longrightarrow f \circ \psi$ is a linear and bounded operator from $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ onto itself.

(ii) (*Pointwise multipliers*)
Let $h \in C^k(\mathbb{R}^n)$. Then $f \longrightarrow hf$ is a linear and bounded operator from $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ into itself.

Proof. We make use of the atomic decomposition according to Definition 2.3 with $K = k$. Concerning (i), if a is a K-atom in the sense of Definition 2.2, then $a \circ \psi$ is also a K-atom based on a new cube, and multiplied with a constant depending on ψ. But this is just what we need and we arrive at the desired assertion.

Similar for (ii). The atomic decomposition (2.3) multiplied with $h \in C^k$ gives an atomic decomposition of hf, which completes the proof. In particular,

$$\|hf|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \le \sum_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |\mathrm{D}^\alpha h(x)| \cdot \|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|. \qquad \square$$

2.2. Embeddings. We recall some embeddings for the spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ that were proven in [4, 6] and which will subsequently be needed. Let $\mathfrak{A} \in \{\mathfrak{B}, \mathfrak{F}\}$.

Proposition 2.6. *Let $s > 0$, $0 < p \le \infty$ ($p < \infty$ for F-spaces), $0 < q \le \infty$.*

(i) *Let $\varepsilon > 0$, $0 < u \le \infty$, and $q \le v \le \infty$. Then*

$$\mathfrak{A}^{s+\varepsilon}_{p,u}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \quad \text{and} \quad \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^s_{p,v}(\mathbb{R}^n). \tag{2.7}$$

(ii) *Let $0 < p_0 < p < p_1 \le \infty$, $s_0, s_1 > 0$ such that*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1},$$

and $0 < q, u, v \le \infty$. If $0 < u \le p \le v \le \infty$, then

$$\mathfrak{B}^{s_0}_{p_0,u}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{s_1}_{p_1,v}(\mathbb{R}^n). \tag{2.8}$$

In terms of boundedness we have the following results which may be also be found in [4, 6].

Proposition 2.7. *Let* $0 < p \leq \infty$ *(with* $p < \infty$ *for F-spaces), and* $0 < q \leq \infty$*. Then*

$$\mathfrak{F}^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \text{if, and only if,} \quad 0 < p \leq 1,\, 0 < q \leq \infty, \tag{2.9}$$

and

$$\mathfrak{B}^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \text{if, and only if,} \quad 0 < p < \infty,\, 0 < q \leq 1, \tag{2.10}$$

where L_∞ *in* (2.9) *and* (2.10) *can be replaced by* C.

Moreover, by (2.7) and (2.8) we obtain

$$\mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n), \quad s > \frac{n}{p},\, 0 < p, q \leq \infty, \tag{2.11}$$

(with $p < \infty$ if $\mathfrak{A} = \mathfrak{F}$), where L_∞ can be replaced by C, too.

3. Traces on hyperplanes in $\mathbb{R}^n$

Let $\mathfrak{A}^s_{p,q}$ denote one of the spaces $\mathfrak{B}^s_{p,q}$ or $\mathfrak{F}^s_{p,q}$. We briefly explain our understanding of the trace operator on hyperplanes, since when dealing with L_p functions the pointwise trace has no obvious meaning. If $x = (x_1, \ldots, x_n)$ put $x' = (x_1, \ldots, x_{n-1})$. We ask for the trace of $f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ on the hyperplane

$$\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x = (x', 0)\}.$$

Obviously, any $\varphi \in S(\mathbb{R}^n)$ has a pointwise trace

$$(\mathrm{Tr}\,\varphi)(x) := (\mathrm{Tr}_{\mathbb{R}^{n-1}}\varphi)(x) = \varphi(x', 0) \quad \text{on } \mathbb{R}^{n-1}.$$

Let $Y(\mathbb{R}^{n-1})$ be either some space $\mathfrak{A}^\sigma_{u,v}(\mathbb{R}^{n-1})$ or $L_u(\mathbb{R}^{n-1})$. Then the trace operator

$$\mathrm{Tr}\, : \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^{n-1})$$

is to be understood in the following sense. One asks, whether there is a constant $c > 0$ such that

$$\|\varphi(\cdot, 0)|Y(\mathbb{R}^{n-1})\| \leq c\|\varphi|\mathfrak{A}^s_{p,q}(\mathbb{R}^n)\|, \quad \text{for all } \varphi \in S(\mathbb{R}^n). \tag{3.1}$$

Since the embedding $S(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ is dense for $0 < p, q < \infty$ one approximates $f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ by $\varphi_j \in S(\mathbb{R}^n)$, where $j \in \mathbb{N}$. If one has (3.1), then $\{\varphi_j(x', 0)\}_{j=1}^\infty$ is a Cauchy sequence in $Y(\mathbb{R}^{n-1})$. Its limit element – which by (3.1) is independent of the approximating sequence $\{\varphi_j\}_{j=1}^\infty \subset S(\mathbb{R}^n)$ – is called the *trace* of $f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ and denoted by $\mathrm{Tr}\, f$. Completion implies

$$\|\mathrm{Tr}\, f|Y(\mathbb{R}^{n-1})\| \leq c\|f|\mathfrak{A}^s_{p,q}(\mathbb{R}^n)\|, \quad f \in \mathfrak{A}^s_{p,q}(\mathbb{R}^n),$$

and $\mathrm{Tr}\, : \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^{n-1})$ is a linear and bounded operator.

Remark 3.1. We can extend (3.1) to spaces $\mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ with $p = \infty$ and/or $q = \infty$ in the following way. If $p = \infty$, then by (2.11), $\mathfrak{B}^s_{\infty,q}(\mathbb{R}^n)$ with $s > 0$ is embedded in the space of continuous functions and Tr makes sense pointwise. If $q = \infty$, then one has by (2.7)

$$\mathfrak{A}^s_{p,\infty}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^{s-\varepsilon}_{p,1}(\mathbb{R}^n) \quad \text{for any } \varepsilon > 0.$$

Let $s > \frac{1}{p}$ and $\varepsilon > 0$ be small enough such that one has $s > s - \varepsilon > \frac{1}{p}$. Since by [11, Remark 13] traces are independent of the source spaces and of the target spaces one can now define Tr for $\mathfrak{A}^s_{p,\infty}(\mathbb{R}^n)$ by restriction of Tr for $\mathfrak{A}^{s-\varepsilon}_{p,1}(\mathbb{R}^n)$ to $\mathfrak{A}^s_{p,\infty}(\mathbb{R}^n)$. Hence (3.1) is always meaningful. We refer also to [12, Section 6.4.2, pp. 218/219] for a detailed discussion.

3.1. The trace problem in $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$.

Now we state our main result concerning traces in Besov spaces on hyperplanes in $\mathbb{R}^n$.

Theorem 3.2. *Let $n \geq 2$, $0 < p, q \leq \infty$, and $s - \frac{1}{p} > 0$. Then* $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{R}^{n-1}}$ *is a linear and bounded operator from $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ onto $\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$,*

$$\mathrm{Tr}\, \mathfrak{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}).$$

Proof. Our constructions follow closely [1, Section 5].

`Step 1`. By Definition 2.3 every $f \in \mathfrak{B}^s_{p,q}$ has an optimal atomic decomposition of the form

$$f(x) = \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \quad x \in \mathbb{R}^n,$$

with $\|f|\mathfrak{B}^s_{p,q}\| \sim \|\lambda|b^s_{p,q}\|$. In this step we wish to prove that

$$\mathrm{Tr}\, \mathfrak{B}^s_{p,q}(\mathbb{R}^n) \subset \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}). \tag{3.2}$$

According to (3.1) and the explanations given thereafter we may restrict ourselves to smooth functions f. For $f \in S(\mathbb{R}^n)$ and the trace operator $\mathrm{Tr}\, f(x) = f(x', 0)$, assumption (3.2) is equivalent to

$$\|f(\cdot, 0)|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| \leq c\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|, \tag{3.3}$$

for some $c > 0$ independent of $f \in \mathfrak{B}^s_{p,q}$. Considering the trace operator we see that for $m = (m', m_n) \in \mathbb{Z}^n$

$$\mathrm{Tr}\, f(x) = f(x', 0) = \sum_j \sum_{m'} \sum_{m_n\in I} \lambda_{j,(m',m_n)} a_{j,(m',m_n)}(x', 0),$$

where for fixed j we only sum over a finite index set $I = I(j, m')$ (depending on $d > 1$) with $\operatorname{supp} a_{j,(m',m_n)} \cap \mathbb{R}^{n-1} \neq \emptyset$ if $m_n \in I$. We define new atoms via

$$b_{j,m'}(x') := \begin{cases} \dfrac{\sum_{m_n \in I} \lambda_{j,(m',m_n)} a_{j,(m',m_n)}(x',0)}{\sum_{m_n \in I} |\lambda_{j,(m',m_n)}|}\,, & \text{if } \displaystyle\sum_{m_n \in I} |\lambda_{j,(m',m_n)}| \neq 0 \\ 0\,, & \text{otherwise.} \end{cases}$$

For our construction only atoms $a_{j,m}$ are of interest with supports in cubes $dQ_{j,m}$, for which $dQ_{j,m}$ has a non-empty intersection with the hyperplane $\mathbb{R}^{n-1}$.

Let $Q = Q_{j,m}$ be one of these cubes and let Q' be the projection of Q on that hyperplane (now being identified with $\mathbb{R}^{n-1}$), i.e., $Q' = Q'_{j,m'}$. Furthermore

$$\eta_{j,m'} := \sum_{m_n \in I} |\lambda_{j,(m',m_n)}|, \quad j \in \mathbb{N}_0,\ m' \in \mathbb{Z}^{n-1}.$$

The restriction (or trace) of f to $\mathbb{R}^{n-1}$ is now

$$\operatorname{Tr} f(x) = f(x',0) = \sum_j \sum_{m'} \eta_{j,m'} b_{j,m'}(x') \tag{3.4}$$

whenever the sum converges. In fact, we have absolute convergence in $L_p(\mathbb{R}^{n-1})$ for all $f \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, cf. Step 2 below.

We show that $b_{j,m'}$ represent suitable atoms according to Definition 2.2. Observe that $b_{j,m'}$ are again $C^K(\mathbb{R}^{n-1})$ functions that additionally satisfy

$$\operatorname{supp} b_{j,m'} \subset \left(\bigcup_{m_n \in I} dQ_{j,(m',m_n)} \right) \cap \mathbb{R}^{n-1} = dQ'_{j,m'}, \tag{3.5}$$

and for $\alpha' \in \mathbb{N}_0^{n-1}$, $|\alpha'| \leq K$,

$$|\mathrm{D}^{\alpha'} b_{j,m'}(x')| \leq \frac{|\sum_{m_n} \lambda_{j,m} \mathrm{D}^{(\alpha',0)} a_{j,m}(x',0)|}{\sum_{m_n} |\lambda_{j,m}|} \leq \frac{2^{j|\alpha'|} \sum_{m_n} |\lambda_{j,m}|}{\sum_{m_n} |\lambda_{j,m}|} = 2^{j|\alpha'|}, \tag{3.6}$$

which establishes that $b_{j,m'}$ is a suitable atom for our representation of $\operatorname{Tr} f$ in $\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$.

Using our new coefficients $\eta = \{\eta_{j,m'}\}_{j,m'}$ we calculate for the norm

$$\begin{aligned} \left\| \operatorname{Tr} f | \mathfrak{B}^{s-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}) \right\| &\leq \left\| \eta | b^{s-\frac{1}{p}}_{p,q} \right\| \\ &= \left(\sum_j 2^{j\left((s-\frac{1}{p}) - \frac{n-1}{p}\right)q} \left(\sum_{m'} |\eta_{j,m'}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &= \left(\sum_j 2^{j(s-\frac{n}{p})q} \left(\sum_{m'} \left| \sum_{m_n \in I} |\lambda_{j,m}| \right|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \end{aligned}$$

and hence

$$\left\| \mathrm{Tr}\, f | \mathfrak{B}_{pq}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}) \right\| \leq c \left(\sum_j 2^{j(s-\frac{n}{p})q} \left(\sum_m |\lambda_{j,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \sim \| f | \mathfrak{B}_{p,q}^s(\mathbb{R}^n) \|$$

(with obvious modifications if $p = \infty$ and/or $q = \infty$), where the sequence spaces $b_{p,q}^{s-\frac{1}{p}}$ are defined according to Definition 2.1 (i) with index set in $\mathbb{N}_0 \times \mathbb{Z}^{n-1}$. We used in the 4th line, that the cardinality of the index set $I = I(j, m')$ is actually independent of j, m'. This proves (3.3).

`Step 2`. The existence, or non-existence, of the trace of $\mathfrak{B}_{p,q}^s(\mathbb{R}^n)$ is equivalent to the question whether we can make sense of the sums in (3.4) whenever (3.5) and (3.6) hold, since any such expression can arise from a suitable $f \in \mathfrak{B}_{p,q}^s(\mathbb{R}^n)$.

If $p \geq 1$ it is known from older results that for $s - \frac{1}{p} > 0$ the sums in (3.4) always converge in L_p (in particular in S') and therefore the trace exists, cf. [7, Section 2.7.2].

Suppose now $0 < p < 1$. Then (3.4) does converge in L_p (but not necessarily in S'), which may be seen calculating

$$\begin{aligned}
\left\| \sum_{j=0}^{\infty} \sum_{m'} \eta_{j,m'} b_{j,m'}(x') | L_p(\mathbb{R}^{n-1}) \right\|^p &\leq \sum_j \sum_{m'} |\eta_{j,m'}|^p \| b_{j,m'} | L_p(\mathbb{R}^{n-1}) \|^p \\
&\leq \sum_j \sum_{m'} |\eta_{j,m'}|^p |dQ'_{j,m'}| \\
&\leq c' \sum_j 2^{-j(n-1)} \sum_m |\lambda_{j,m}|^p \\
&= c' \sum_j 2^{-j(sp-1)} 2^{j(sp-1)} 2^{-j(n-1)} \sum_m |\lambda_{j,m}|^p \\
&\leq c'' \left(\sum_j 2^{j((sp-1)-(n-1))\frac{q}{p}} \left(\sum_m |\lambda_{j,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \\
&\sim \| f | \mathfrak{B}_{p,q}^s(\mathbb{R}^n) \|^p,
\end{aligned}$$

where in the second but last line when $\frac{q}{p} \leq 1$ we can use the embedding $\ell_{\frac{q}{p}} \hookrightarrow \ell_1$, and in the case $\frac{q}{p} > 1$ an application of Hölder's inequality gives the desired result, since $sp - 1 > 0$. This proves the absolute convergence in L_p.

`Step 3`. It is fairly easy to see that the trace map Tr is onto $\mathfrak{B}_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1})$, since any K-atom $b_{j,m'} \in C^K(\mathbb{R}^{n-1})$ satisfying (2.1), (2.2) can be obtained as the restriction of a K-atom $a_{j,m} \in C^K(\mathbb{R}^n)$ (simply construct $a_{j,m}$ by multiplying $b_{j,m'}$ with a suitable K-atom b_{j,m_n} defined on $\mathbb{R}$ with $b_{j,m_n}(0) = 1$).

To establish the extension property we show that for given $g \in \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ there exists a function $f \in \mathfrak{B}^{s}_{p,q}(\mathbb{R}^{n})$ with

$$f(x',0) = g(x') \quad \text{and} \quad \|f|\mathfrak{B}^{s}_{p,q}(\mathbb{R}^{n})\| \le c\|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\|.$$

In order to obtain a bounded extension operator

$$\mathrm{Ex}\colon \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}) \to \mathfrak{B}^{s}_{p,q}(\mathbb{R}^{n}), \quad (\mathrm{Ex}\, g)(x) = f(x),$$

with $\mathrm{Tr} \circ \mathrm{Ex} = \mathrm{id}_{\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})}$, let $g \in \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ with optimal atomic decomposition, i.e.,

$$g(x') = \sum_{j}\sum_{m'} \lambda_{j,m'} b_{j,m'}(x') \quad \text{and} \quad \|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| \sim \|\lambda|b^{s-\frac{1}{p}}_{p,q}\|. \tag{3.7}$$

We set

$$f(x',x_n) = \sum_{j}\sum_{m'}\sum_{m_n} \lambda_{j,m} a_{j,m}(x',x_n),$$

with coefficients

$$\lambda_{j,m} = \begin{cases} \lambda_{j,m'}, & m_n = 0 \\ 0, & m_n \neq 0\,, \end{cases}$$

and $a_{j,m}(x',x_n) = b_{j,m'}(x')b_{j,m_n}(x_n)$, where b_{j,m_n} are K-atoms according to Definition 2.2 satisfying $b_{j,m_n}(0) = 1$ and $\operatorname{supp} b_{j,m_n} \subset [2^{-j}(m_n-1), 2^{-j}(m_n+1)]$. Therefore $a_{j,m}$ are K-atoms and we see that $f(x',0) = g(x')$, $x' \in \mathbb{R}^{n-1}$. Furthermore, we estimate

$$\begin{aligned} \|f|\mathfrak{B}^{s}_{p,q}(\mathbb{R}^{n})\| &\le \left(\sum_{j} 2^{j(s-\frac{n}{p})q} \left(\sum_{m} |\lambda_{j,m}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\ &= \left(\sum_{j} 2^{j\left[(s-\frac{1}{p})-\frac{n-1}{p}\right]q} \left(\sum_{m'} |\lambda_{j,m'}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\ &\sim \|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\|. \end{aligned}$$

Hence we have established the existence of a bounded (but not linear – cf. Remark 3.3) extension operator Ex from the trace space into the original space, which finally completes the proof. □

Remark 3.3. Note that the constructed extension operator in Step 3 is bounded but not linear. In [12, Chapter 6.2] it is shown that

$$\left(\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\right)' = \{0\}, \qquad 0 < s - \frac{1}{p} < (n-1)\left(\frac{1}{p} - 1\right),$$

which implies the impossibility of frame representations in these spaces and therefore the optimal coefficients $\lambda_{j,m'}$ as well as the atoms $a_{j,m}$ in (3.7) are not linear with respect to g in this case.

Alternatively we could use the subatomic approach as described in [10, Section 9.2] instead of atomic decompositions in Step 3 of Theorem 3.2. Then again the constructed extension operator turns out to be bounded but not linear – but in this case linearity fails only in terms of the coefficients but not for the building blocks. We sketch the proof.

For a given $g \in \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ we need to construct a function $f \in \mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ such that

$$f(x',0) = g(x') \quad \text{and} \quad \|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \le c\left\|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\right\|.$$

Let $g \in \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ with optimal subatomic decomposition, i.e.,

$$g(x') = \sum_{\beta'}\sum_{j}\sum_{m'} \lambda^{\beta'}_{j,m'} k^{\beta'}_{j,m'}(x') \quad \text{and} \quad \|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\| \sim \|\lambda|b^{s-\frac{1}{p},\varrho}_{p,q}\|,$$

where $\varrho > 0$ is chosen later on. Put

$$(\operatorname{Ex} g)(x) = f(x) = \sum_{\beta',j,m'} \sum_{m_n=-1}^{-2^J} \lambda^{\beta'}_{j,m'} k^{\beta'}_{j,m'}(x') k^0_{j,m_n}(x_n),$$

where $k^0_{j,m_n}(x_n)$ are 1-dimensional (standardized) building blocks. It is easy to see that $f(x',0) = g(x')$, since $\sum_{m_n=-1}^{-2^J} k^0(0-m_n) = 1$. The following calculation for $\alpha = (\alpha', \alpha_n) \in \mathbb{N}_0^n$

$$\begin{aligned}
&\left|\mathrm{D}^\alpha k^{\beta'}_{j,m'}(x') k^0_{j,m_n}(x_n)\right| \\
&= \left|\sum_{\gamma'+\delta'=\alpha'} \mathrm{D}^{\gamma'}(2^{-J}(2^j x' - m'))^{\beta'} \mathrm{D}^{\delta'} k(2^j x' - m') \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} k(2^j x_n - m_n)\right| \\
&\le \sum_{\gamma'+\delta'=\alpha'} 2^{j|\gamma'|} 2^{(c-\varepsilon)|\beta'|} 2^{j|\delta'|} \sup_{z'\in\mathbb{R}^{n-1}} \left|\mathrm{D}^{\delta'} k(z')\right| 2^{j\alpha_n} \sup_{z\in\mathbb{R}} \left|\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} k(z)\right| \\
&\le c_{k,K} 2^{r|\beta'|} 2^{j|\alpha|}, \qquad \text{since} \quad |\gamma'| + |\delta'| + \alpha_n = |\alpha|,
\end{aligned}$$

together with $\operatorname{supp} k^{\beta'}_{j,m'}(x') k^0_{j,m_n}(x_n) \subset dQ_{j,m}$ shows that $\frac{k^{\beta'}_{j,m'}(x') k^0_{j,m_n}(x_n)}{c_{k,K} 2^{r|\beta'|}}$ represent suitable atoms according to Definition (2.2). Furthermore we estimate for $\eta \leq 1$,

$$\begin{aligned}
\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\|^\eta &\leq \sum_{\beta'} \left\|f^{\beta'}|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\right\|^\eta \\
&\leq \sum_{\beta'} \left(\sum_j 2^{j(s-\frac{n}{p})q} \left(\sum_{m'} \sum_{m_n=-1}^{-2^{-J}} \left|\lambda^{\beta'}_{j,m'} c_{k,K} 2^{r|\beta'|}\right|^p \right)^{\frac{q}{p}} \right)^{\frac{\eta}{q}} \\
&\leq c \sum_{\beta'} 2^{\eta r|\beta'|} \left(\sum_j 2^{j\left[(s-\frac{1}{p})-\frac{n-1}{p}\right]q} \left(\sum_{m'} |\lambda^{\beta'}_{j,m'}|^p \right)^{\frac{q}{p}} \right)^{\frac{\eta}{q}} \\
&\leq c \left(\sum_{\beta'} 2^{-\delta|\beta'|} \right) \sup_{\beta'} 2^{\varrho|\beta'|} \left(\sum_j 2^{j\left[(s-\frac{1}{p})-\frac{n-1}{p}\right]q} \left(\sum_{m'} |\lambda^{\beta'}_{j,m'}|^p \right)^{\frac{q}{p}} \right)^{\frac{\eta}{q}} \\
&\leq c' \left\|g|\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\right\|^\eta,
\end{aligned}$$

where we set $\varrho = \eta r + \delta$ in the second but last line.

Remark 3.4. So far we only considered $\mathrm{Tr}_{\mathbb{R}^{n-1}} f = \mathrm{Tr}\, f$. But it is obvious that traces on hyperplanes of dimension $1, 2, \ldots, n-2$ can be obtained by iteration of Theorem 3.2. Let $n > m \in \mathbb{N}$ and $\mathrm{Tr}_{\mathbb{R}^m} f = \mathrm{Tr}\, f$. Then

$$\mathrm{Tr}\, \mathfrak{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{n-m}{p}}_{p,q}(\mathbb{R}^m) \quad \text{when} \quad s > \frac{n-m}{p}, \quad 0 < p, q \leq \infty.$$

We now discuss what happens in the limiting case $s = \frac{1}{p}$.

Corollary 3.5. *Let* $0 < p < \infty$*,* $0 < q \leq \min(1,p)$*. Then*

$$\mathrm{Tr}\, \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}).$$

Proof. `Step 1`. Using the same construction as in Theorem 3.2, we need to show that $\mathrm{Tr}\, \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n) \subset L_p(\mathbb{R}^{n-1})$, i.e., the sums in (3.4) converge in $L_p(\mathbb{R}^{n-1})$ if $f \in \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$, $0 < q \leq \min(1,p)$. If $0 < p < 1$ and $q \leq p$ this is observed by the following calculation

$$\begin{aligned}
\|\mathrm{Tr}\, f|L_p(\mathbb{R}^{n-1})\|^p &\leq \sum_j \sum_{m'} |\eta_{j,m'}|^p \int_{dQ'_{j,m'}} |b_{j,m'}(x')|^p \mathrm{d}x' \\
&\leq c \sum_j 2^{-j(n-1)} \sum_m |\lambda_{j,m}|^p \\
&\leq c' \left(\sum_j 2^{-j(n-1)\frac{q}{p}} \left(\sum_m |\lambda_{j,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \\
&\sim \left\|f|\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)\right\|, \qquad (3.8)
\end{aligned}$$

where in the second but last line we used the embedding $\ell_{\frac{q}{p}} \hookrightarrow \ell_1$. When $p \geq 1$ and $0 < q \leq 1$ we obtain

$$\begin{aligned}
\|\mathrm{Tr}\, f|L_p(\mathbb{R}^{n-1})\| &\leq \sum_j \left(\int_{dQ'_{j,m'}} \left| \sum_{m'} \eta_{j,m'} b_{j,m'}(x') \right|^p \mathrm{d}x' \right)^{\frac{1}{p}} \\
&\sim \sum_j \left(\sum_{m'} |\eta_{j,m'}|^p \int_{dQ'_{j,m'}} |b_{j,m'}(x')|^p \mathrm{d}x' \right)^{\frac{1}{p}} \\
&\leq c \sum_j 2^{-j\frac{(n-1)}{p}} \left(\sum_m |\lambda_{j,m}|^p \right)^{\frac{1}{p}} \\
&\leq c' \left(\sum_j 2^{-j\frac{(n-1)q}{p}} \left(\sum_m |\lambda_{j,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\sim \left\| f|\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n) \right\|,
\end{aligned} \tag{3.9}$$

using the fact that we only have a controlled overlap of the atoms $b_{j,m'}$ for fixed j, and $\ell_q \hookrightarrow \ell_1$ in the second but last line. Now (3.8) and (3.9) prove that Tr is a bounded (and linear) operator from $\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$ into $L_p(\mathbb{R}^{n-1})$.

`Step 2`. In order to see that Tr is onto $L_p(\mathbb{R}^{n-1})$, it is sufficient to show that each $h \in L_p(\mathbb{R}^{n-1})$ has a decomposition

$$h(x') = \sum_j \sum_{m'} \eta_{j,m'} b_{j,m'}, \tag{3.10}$$

where the $b_{j,m'}$'s satisfy $|\mathrm{D}^{\alpha'} b_{j,m'}(x')| \leq 2^{j|\alpha'|}$, $|\alpha'| \leq K$, $\alpha' \in \mathbb{N}_0^{n-1}$, and $\mathrm{supp}\, b_{j,m'} \subset Q'_{j,m'}$ – since any such representation can be obtained as the restriction of the trace operator applied to an $f \in \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$. Additionally we require that

$$\left(\sum_j 2^{-j\frac{(n-1)}{p}q} \left(\sum_{m'} |\eta_{j,m'}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq c\|h|L_p(\mathbb{R}^{n-1})\|, \tag{3.11}$$

since for $f(x', 0) = h(x')$ this leads to – cf. Step 3 of Theorem 3.2 –

$$\|f|\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)\| \leq c\|h|L_p(\mathbb{R}^{n-1})\|.$$

Our proof follows closely [1, Theorem 5.1]. To establish such a decomposition, start by picking a $\kappa \in C_0^\infty(\mathbb{R}^{n-1})$ satisfying $\mathrm{supp}\, \kappa \subset [0,1]^{n-1} =: \Omega$, $0 \leq \kappa(\cdot) \leq 1$, and $\|1 - \kappa|L_p(\Omega)\| \leq \min\left(\frac{1}{5}, \left(\frac{1}{5}\right)^{\frac{1}{p}}\right)$. If $Q'_{j,m'} = \{x' : m_i 2^{-j} \leq x_i < (m_i + 1)2^{-j}$, $i = 1, \ldots, n-1\}$, put

$$b_{j,m'}(x') := C \cdot \kappa(2^j x' - m'), \tag{3.12}$$

such that $\operatorname{supp} b_{j,m'} \subset Q'_{j,m'}$, where $m' = (m_1, \ldots, m_{n-1})$ and C is chosen small enough for $b_{j,m'}$ to satisfy $|\mathrm{D}^{\alpha'} b_{j,m'}(x')| \leq 2^{j|\alpha'|}$, $|\alpha'| \leq K$, $\alpha' \in \mathbb{N}_0^{n-1}$. Fix a non-negative $h \in L_p(\mathbb{R}^{n-1})$. (It suffices to prove the assumption for such functions, since an arbitrary $h \in L_p(\mathbb{R}^{n-1})$ can be reduced to the sum of two real-valued functions $h = \mathfrak{Re}\, h + i\mathfrak{Im}\, h$ and any real-valued function $h \in L_p(\mathbb{R}^{n-1})$ can be decomposed into two non-negative functions h_+, h_- such that $h(x) = h_+(x) - h_-(x)$, where $h_+ := \max(h, 0)$, $h_- := \max(-h, 0)$. The full generality of (3.10), (3.11) for arbitrary $h \in L_p(\mathbb{R}^{n-1})$ then follows by standard arguments.)

By choosing the side length 2^{-j_1} small enough, it is possible to find a simple function

$$e_1(x') = \sum_{m'} r_{j_1,m'} \chi_{j_1,m'}(x')$$

such that

$$e_1 \geq 0 \quad \text{and} \quad \|h - e_1|L_p(\mathbb{R}^{n-1})\| \leq \min\left(\frac{1}{4}, \left(\frac{1}{4}\right)^{\frac{1}{p}}\right) \|h|L_p(\mathbb{R}^{n-1})\|.$$

We define the smooth version

$$\tilde{e_1}(x') = \sum_{m'} \eta_{j_1,m'} b_{j_1,m'}(x'),$$

where the $b_{j_1,m'}$'s are given by (3.12) and $\eta_{j_1,m'} = \frac{r_{j_1,m'}}{C}$ with the same constant C. Setting

$$D = \frac{C}{\min\left(\frac{5}{4}, \left(\frac{5}{4}\right)^{\frac{1}{p}}\right)}$$

for $p \geq 1$ we see that

$$\begin{aligned}\left(\sum_{m'} 2^{-j_1(n-1)} |\eta_{j_1,m'}|^p\right)^{\frac{1}{p}} &= \frac{\|e_1|L_p(\mathbb{R}^{n-1})\|}{C} \\ &\leq \frac{\|h - e_1|L_p(\mathbb{R}^{n-1})\| + \|h|L_p(\mathbb{R}^{n-1})\|}{C} \\ &\leq \frac{\|h|L_p(\mathbb{R}^{n-1})\|}{D},\end{aligned}$$

and when $0 < p < 1$ we obtain the same estimate via

$$\begin{aligned}\sum_{m'} 2^{-j_1(n-1)} |\eta_{j_1,m'}|^p &= \frac{\|e_1|L_p(\mathbb{R}^{n-1})\|^p}{C^p} \\ &\leq \frac{\|h - e_1|L_p(\mathbb{R}^{n-1})\|^p + \|h|L_p(\mathbb{R}^{n-1})\|^p}{C^p} \\ &\leq \frac{\|h|L_p(\mathbb{R}^{n-1})\|^p}{D^p}.\end{aligned}$$

We picked κ so that $\|e_1-\tilde{e_1}|L_p(\mathbb{R}^{n-1})\| \le \min\left(\frac{1}{5},\left(\frac{1}{5}\right)^{1/p}\right)\|e_1|L_p(\mathbb{R}^{n-1})\|$. Hence for $p\ge 1$,

$$\begin{aligned}\|h-\tilde{e_1}|L_p(\mathbb{R}^{n-1})\| &\le \|h-e_1|L_p(\mathbb{R}^{n-1})\| + \|e_1-\tilde{e_1}|L_p(\mathbb{R}^{n-1})\| \\ &\le \left\{\min\left(\frac{1}{4},\left(\frac{1}{4}\right)^{\frac{1}{p}}\right)+\min\left(\frac{1}{5},\left(\frac{1}{5}\right)^{\frac{1}{p}}\right)\frac{C}{D}\right\}\|h|L_p(\mathbb{R}^{n-1})\| \quad (3.13)\\ &\le \frac{1}{2}\|h|L_p(\mathbb{R}^{n-1})\|.\end{aligned}$$

Similar for $0<p<1$. If this process is repeated with h replaced by $h-\tilde{e_1}$, we obtain $\tilde{e_2}=\sum_{m'}\eta_{j_2,m'}b_{j_2,m'}$ such that

$$\left(\sum_{m'}2^{-j_2(n-1)}|\eta_{j_2,m'}|^p\right)^{\frac{1}{p}} \le \frac{\|h-\tilde{e_1}|L_p(\mathbb{R}^{n-1})\|}{D} \le \frac{\|h|L_p(\mathbb{R}^{n-1})\|}{2D}$$

and

$$\|h-\tilde{e_1}-\tilde{e_2}|L_p(\mathbb{R}^{n-1})\| \le \frac{\|h-\tilde{e_1}|L_p(\mathbb{R}^{n-1})\|}{2} \le \frac{\|h|L_p(\mathbb{R}^{n-1})\|}{4},$$

where we used (3.13). We can also arrange that $j_2>j_1$. Continuing this process inductively we obtain the functions $\tilde{e_i}=\sum_{m'}\eta_{j_i,m'}b_{j_i,m'}$, $i=1,2,\ldots$, satisfying

$$\left(\sum_{m'}2^{-j_i(n-1)}|\eta_{j_i,m'}|^p\right)^{\frac{1}{p}} \le \frac{\|h|L_p(\mathbb{R}^{n-1})\|}{2^{i-1}D}, \tag{3.14}$$

$$\left\|h-\sum_{i=1}^{m}\tilde{e_i}|L_p(\mathbb{R}^{n-1})\right\| \le 2^{-m}\|h|L_p(\mathbb{R}^{n-1})\|, \quad m=1,2,\ldots, \tag{3.15}$$

and $j_{i+1}>j_i$ for every i. The required decomposition of h is $h(x')=\sum_{i=1}^{\infty}\tilde{e_i}(x')$. By (3.15) this sum converges in $L_p(\mathbb{R}^{n-1})$ and from (3.14) we see that

$$\left(\sum_{i=1}^{\infty}\left(\sum_{m'}2^{-j_i(n-1)}|\eta_{j_i,m'}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \le c\|h|L_p(\mathbb{R}^{n-1})\|.$$

This completes the proof. □

Remark 3.6. We actually proved a bit more than stated. Note that Step 3 in the proof of Theorem 3.2 together with Step 2 of Corollary 3.5 establish the existence of a bounded extension operator, i.e. for given $g\in L_p(\mathbb{R}^{n-1})$ there exists a function $f\in\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)$ with

$$f(x',0)=g(x') \quad \text{and} \quad \left\|f|\mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n)\right\| \le c\|g|L_p(\mathbb{R}^{n-1})\|.$$

In particular, we have

$$\mathrm{Ex}: L_p(\mathbb{R}^{n-1}) \to \mathfrak{B}^{1/p}_{p,q}(\mathbb{R}^n), \quad (\mathrm{Ex}\, g)(x)=f(x),$$

with $\mathrm{Tr}\circ\mathrm{Ex}=\mathrm{id}_{L_p(\mathbb{R}^{n-1})}$.

Remark 3.7. As in Remark 3.4 we obtain similar results for the limiting case when dealing with hyperplanes $\mathbb{R}^m$, $n > m \in \mathbb{N}$. Using Theorem 3.2 and Corollary 3.5, by iteration we obtain

$$\operatorname{Tr} \mathfrak{B}_{p,q}^{\frac{n-m}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^m), \quad 0 < p < \infty,\ 0 < q \leq \min(1,p).$$

Remark 3.8. Our results are best possible in the sense that the sums in (3.4) do not necessarily converge in

$$L_p + L_\infty := \{f :\ f = g_p + g_\infty,\ g_i \in L_i(\mathbb{R}^n)\},$$

normed by

$$\|f|L_p + L_\infty\| := \inf_{\substack{f=g_p+g_\infty \\ g_i \in L_i}} (\|g_p|L_p(\mathbb{R}^n)\| + \|g_\infty|L_\infty(\mathbb{R}^n)\|),$$

if $s = \frac{1}{p}$ and $q > p$. Therefore the trace does not exist. (Note that Proposition 2.6 (i) establishes $\mathfrak{B}_{p,q}^{1/p}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}_{p,u}^{s}(\mathbb{R}^n)$, $s < \frac{1}{p}$, $0 < u \leq \infty$, from which then also follows that the trace in general does not exist if $s < \frac{1}{p}$.)

This can be seen in the following way. Let $s = \frac{1}{p}$, $q > p$, and pick a sequence $\{\eta_j\}_{j=2}^\infty \in \ell_q \setminus \ell_p$ (or in $c_0 \setminus \ell_p$ if $q = \infty$). Furthermore, we choose a collection of dyadic cubes $\{E_j\}_{j=2}^\infty$ with $E_j \subset [-1,1]^{n-1}$ and length $l(E_j) = 2^{-j}$, that additionally satisfy $E_j \cap E_k \neq \emptyset$ if $j \neq k$. Put

$$\eta_{j,m'} := \begin{cases} 2^{j\frac{n-1}{p}}\eta_j, & Q'_{j,m'} = E_j \\ 0, & \text{otherwise,} \end{cases}$$

and let $b_{j,m'}$ be K-atoms in $\mathbb{R}^{n-1}$, i.e., $\operatorname{supp} b_{j,m'} \subset dQ'_{j,m'}$, $\left|\mathrm{D}^{\alpha'} b_{j,m'}(x')\right| \leq 2^{j|\alpha'|}$, $|\alpha'| \leq K$, for which additionally $b_{j,m'}(x') \geq c$ if $x' \in Q'_{j,m'}$, $c > 0$. Then

$$\left\|\eta|b_{p,q}^{1/p}\right\| = \left(\sum_{j=0}^\infty 2^{-j\frac{n-1}{p}q}\left(\sum_{m'\in\mathbb{Z}^{n-1}} |\eta_{j,m'}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = \left(\sum_{j=2}^\infty |\eta_j|^q\right)^{\frac{1}{q}} < \infty,$$

and it is clear that $\sum_{j,m'} \eta_{j,m'} b_{j,m'}$ would arise as the trace of a suitable $f \in \mathfrak{B}_{p,q}^{1/p}(\mathbb{R}^n)$ if the trace operator was continuous. But if we let

$$g_N(x') := \sum_{j=2}^N \sum_{m'} \eta_{j,m'} b_{j,m'}(x'), \quad N \text{ large,}$$

then $\operatorname{supp} g_N \subset [-1,1]^{n-1}$. Since $L_\infty([-1,1]^{n-1}) \hookrightarrow L_p([-1,1]^{n-1})$ we estimate

$$\begin{aligned}
\|g_N|L_p + L_\infty\| &\geq c\|g_N|L_p\| \\
&= \left\| \sum_{j=2}^{N} \sum_{m'} \eta_{j,m'} b_{j,m'} | L_p \right\| \\
&\sim \left(\sum_{j=2}^{N} 2^{j(n-1)} |\eta_j|^p \int\limits_{dE_j,\, E_j = Q'_{j,m'}} |b_{j,m'}(x')|^p \mathrm{d}x' \right)^{\frac{1}{p}} \\
&\geq c' \left(\sum_{j=2}^{N} |\eta_j|^p \right)^{\frac{1}{p}} \longrightarrow \infty \quad \text{as } N \to \infty.
\end{aligned}$$

Therefore the sum $\sum_{j,m'} \eta_{j,m'} b_{j,m'}$ cannot converge in $L_p + L_\infty$.

A remark on the trace problem in $B^s_{p,q}(\mathbb{R}^n)$. Our results shed new light upon traces of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ defined via the *Fourier-analytical approach*. The spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined as the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \ | B^s_{p,q}(\mathbb{R}^n) \right\| = \left\| \left\| \left\{ 2^{js} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot) \right\}_{j \in \mathbb{N}_0} | L_p(\mathbb{R}^n) \right\| \, | \, \ell_q \right\|$$

is finite, where $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and $\{\varphi_j\}_j$ is a smooth dyadic resolution of unity. For these spaces there are equivalent characterizations in terms of *atomic decompositions* similar to Definition 2.3, cf. [9, Section 13]. The relevant (K, L)-atoms now are defined as the K-atoms in Definition 2.2, where we additionally require moment conditions up to order L, i.e., $\int_{\mathbb{R}^n} x^\beta a(x) \mathrm{d}x = 0$ if $|\beta| \leq L$. Here $L = -1$ means that there are no moment conditions. The atomic characterization of function spaces of type $B^s_{p,q}(\mathbb{R}^n)$ is given by the following result, cf. [9, Theorem.13.8].

Theorem 3.9. *Let $0 < p \leq \infty$, $0 < q \leq \infty$, and $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with $K \geq (1 + [s])_+$ and $L \geq \max(-1, [\sigma_p - s])$ be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}(x), \quad \textit{convergence being in } S'(\mathbb{R}^n),$$

where the $a_{j,m}$ are K-atoms $(j = 0)$ or (K, L)-atoms $(j \in \mathbb{N})$ with $\operatorname{supp} a_{j,m} \subset dQ_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $d > 1$, and $\lambda \in b^s_{p,q}$. Furthermore,

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| \sim \inf \|\lambda|b^s_{p,q}\|,$$

where the infimum is taken over all admissible representations, is an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$.

With the help of Theorem 3.2 we can now extend the results from [5] as follows.

Theorem 3.10. *Let $n \geq 2$ and $0 < p, q \leq \infty$.*
(i) *Let $s - \frac{1}{p} > 0$. Then*

$$\operatorname{Tr} B^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}).$$

(ii) *In the limiting case $s = \frac{1}{p}$, $p < \infty$ and $0 < q \leq \min(1,p)$ we have*

$$\operatorname{Tr} B^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}).$$

Proof. The proof of (i) is similar to the proof of Theorem 3.2. We indicate the necessary changes. Considering the trace operator in Step 1 we loose moment conditions when defining the new atoms $b_{j,m'}$. Nevertheless, considering the trace makes sense in the setting of the spaces $\mathfrak{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ since we have absolute convergence of the defined trace in $L_p(\mathbb{R}^{n-1})$, cf. Step 2 of the proof. Furthermore, as in Step 3 we obtain a suitable extension operator. Note that it is possible to construct atoms $a_{j,m}$ defined on $\mathbb{R}^n$, extending atoms $b_{j,m'}$ defined on $\mathbb{R}^{n-1}$ that have the desired moment conditions. The convergence in $S'(\mathbb{R}^n)$ of the resulting atomic decomposition

$$f(x) = (\operatorname{Ex} g)(x) = \sum_{j,m} \lambda_{j,m} a_{j,m}(x)$$

follows from [9, Corollary 13.9].

A proof for the limiting case (ii) may be found in [1, Theorem 5.1]. □

3.2. The trace problem in $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$. With the help of our previous results on traces in $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ we are now able to investigate the trace problem for the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$. It turns out that the trace is actually independent of the parameter q. We make use of the following Proposition. A proof may be found in [2, Proposition 2.7].

Proposition 3.11. *Let $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and $E_{j,m} \subset Q_{j,m}$ measurable sets with $|E_{j,m}| \sim |Q_{j,m}|$. Then*

$$\|\lambda|f^s_{p,q}\| \sim \left\| \left(\sum_{j,m} 2^{jsq} |\lambda_{j,m}\chi_{E_{j,m}}(\cdot)|^q \right)^{\frac{1}{q}} | L_p(\mathbb{R}^n) \right\|.$$

The next Theorem states our main result.

Theorem 3.12. *Let $n \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$, and $s - \frac{1}{p} > 0$. Then*

$$\operatorname{Tr} \mathfrak{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}).$$

Proof. It is sufficient to show that the trace of $\mathfrak{F}^s_{p,q}$ is independent of q, i.e.

$$\operatorname{Tr} \mathfrak{F}^s_{p,q}(\mathbb{R}^n) = \operatorname{Tr} \mathfrak{F}^s_{p,p}(\mathbb{R}^n) = \operatorname{Tr} \mathfrak{B}^s_{p,p}(\mathbb{R}^n),$$

since then the rest follows immediately from Theorem 3.2. If $0 < q < r \leq \infty$, we have the embedding $\mathfrak{F}^s_{p,q} \hookrightarrow \mathfrak{F}^s_{p,r}$ yielding $\operatorname{Tr} \mathfrak{F}^s_{p,q} \hookrightarrow \operatorname{Tr} \mathfrak{F}^s_{p,r}$. In order to prove the other direction let $f \in \mathfrak{F}^s_{p,r}$ with optimal atomic decomposition

$$f(x) = \sum_{j,m} \lambda_{j,m} a_{j,m}(x), \qquad x \in \mathbb{R}^n,$$

i.e., $\|f|\mathfrak{F}^s_{p,r}\| \sim \|\lambda|f^s_{p,r}\|$. In particular, by Definition 2.3, we have $\operatorname{supp} a_{j,m} \subset dQ_{j,m}$. We need to show that there exists an $\tilde{f} \in \mathfrak{F}^s_{p,q}$ such that $\operatorname{Tr} f = \operatorname{Tr} \tilde{f}$. Set

$$\tilde{\lambda}_{j,m} := \begin{cases} \lambda_{j,m}, & dQ_{j,m} \cap \mathbb{R}^{n-1} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, put $\tilde{a}_{j,m}(x) = a_{j,m}(x)$ and consider

$$\tilde{f}(x) = \sum_{j,m} \tilde{\lambda}_{j,m} a_{j,m}(x). \tag{3.16}$$

From the construction we immediately see that $\operatorname{Tr} f = \operatorname{Tr} \tilde{f}$. Note that in (3.16) we only sum over finitely many $m_n \in I(j, m')$, where the index set is actually independent of j, m'. This can be seen by observing that

$$m_n \in I \quad \text{if, and only if,} \quad dQ_{j,m} \cap \mathbb{R}^{n-1} \neq \emptyset,$$

which is equivalent to

$$m_n \in I \quad \text{if, and only if,} \quad 0 \in \left(2^{-j} m_n - d2^{-j-1}, 2^{-j} m_n + d2^{-j-1}\right).$$

But this yields

$$m_n \in I \quad \text{if, and only if,} \quad 0 \in \left(m_n - \frac{d}{2}, m_n + \frac{d}{2}\right),$$

establishing the independence of the index set I on j and m'.

We want to apply Proposition 3.11. Therefore we wish to construct suitable sets $E_{j,m}$ such that

$$E_{j,(m',m_n)} \subset Q_{j,(m',m_n)} \quad \text{and} \quad |E_{j,(m',m_n)}| \sim |Q_{j,(m',m_n)}|, \tag{3.17}$$

which do not intersect for fixed $m_n \in I$.

If $|m_n| \geq 2$ we can simply choose $E_{j,m} := Q_{j,m}$, cf. Figure 1.
If $m_n = 0$ put

$$E_{j,(m',0)} := \{x \in Q_{j,(m',0)} : 2^{-j-1} < |x_n| < 2^{-j}\},$$

whereas for $|m_n| = 1$ we set

$$E_{j,(m',1)} := \{x \in Q_{j,(m',1)} : 0 < x_n - 2^{-j} < 2^{-j-1}\}$$

and

$$E_{j,(m',-1)} := \{x \in Q_{j,(m',-1)} : -2^{-j-1} < x_n + 2^{-j} < 0\},$$

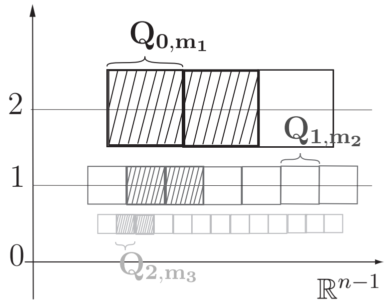


Figure 1: $|m_n| \geq 2$

cf. Figures 2 and 3, respectively.

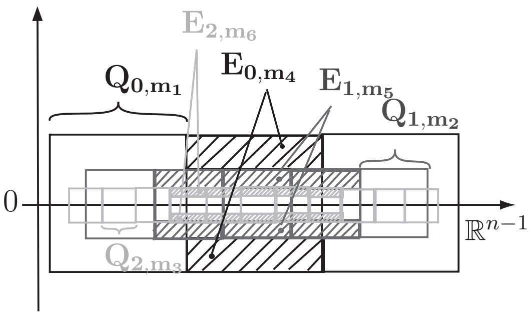


Figure 2: $m_n = 0$

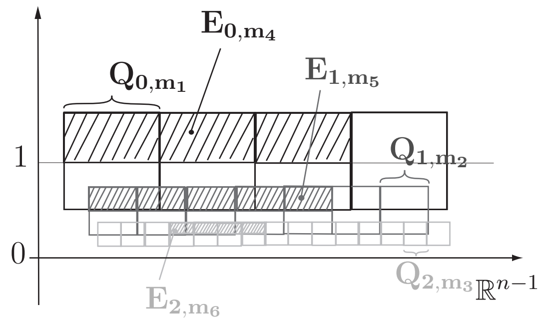


Figure 3: $|m_n| = 1$

Clearly we have (3.17). In particular, for fixed m_n the sets $E_{j,(m',m_n)}$ have pairwise disjoint support for all $j \in \mathbb{N}_0$, $m' \in \mathbb{Z}^{n-1}$. Hence, if $q < \infty$ we calculate

$$\begin{aligned}
\|\tilde{f}|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| \leq \|\tilde{\lambda}|f^s_{p,q}\| &\sim \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsq} |\tilde{\lambda}_{j,m}|^q \chi_{E_{j,m}}(\cdot) \right)^{\frac{1}{q}} |L_p(\mathbb{R}^n) \right\| \\
&\sim \sum_{m_n \in I} \left\| \left(\sum_{j=0}^{\infty} \sum_{m'} 2^{jsq} |\lambda_{j,(m',m_n)}|^q \chi_{E_{j,(m',m_n)}}(\cdot) \right)^{\frac{1}{q}} |L_p(\mathbb{R}^n) \right\| \\
&\sim \sum_{m_n \in I} \left\| \left(\sum_{j=0}^{\infty} \sum_{m'} 2^{jsr} |\lambda_{j,(m',m_n)}|^r \chi_{E_{j,(m',m_n)}}(\cdot) \right)^{\frac{1}{r}} |L_p(\mathbb{R}^n) \right\| \\
&\sim \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsr} |\tilde{\lambda}_{j,m}|^r \chi_{E_{j,m}}(\cdot) \right)^{\frac{1}{r}} |L_p(\mathbb{R}^n) \right\| \\
&\sim \|\tilde{\lambda}|f^s_{p,r}\| \\
&\leq \|f|\mathfrak{F}^s_{p,r}(\mathbb{R}^n)\| < \infty,
\end{aligned}$$

where in the 2nd and 6th step we made use of Proposition 3.11. The q and $\frac{1}{q}$ in line 3 cancel and can be replaced by r and $\frac{1}{r}$, since the sets $E_{j,(m',m_n)}$ have disjoint supports for fixed $m_n \in I$. In particular, $f \in \mathfrak{F}^s_{p,q}$ and therefore $\operatorname{Tr}\mathfrak{F}^s_{p,r} \subset \operatorname{Tr}\mathfrak{F}^s_{p,q}$, which completes the proof. □

We investigate the limiting case when $s = \frac{1}{p}$ as well.

Corollary 3.13. *Let* $0 < p \le 1$ *and* $0 < q \le \infty$. *Then*

$$\operatorname{Tr}\mathfrak{F}^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}).$$

Proof. In Theorem 3.12 we established the independence of the trace of $\mathfrak{F}^s_{p,q}$ on q. Therefore Corollary 3.5 yields

$$\operatorname{Tr}\mathfrak{F}^{1/p}_{p,q} = \operatorname{Tr}\mathfrak{F}^{1/p}_{p,p} = \operatorname{Tr}\mathfrak{B}^{1/p}_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}), \quad 0 < p \le 1.$$ □

Remark 3.14. Again by iteration of Theorem 3.12 and Corollary 3.13 we obtain results for traces on hyperplanes of dimension $1, 2, \ldots, n-2$. Let $n > m \in \mathbb{N}$ and $\operatorname{Tr}_{\mathbb{R}^m} f = \operatorname{Tr} f$. Then for $s > \frac{n-m}{p}$

$$\operatorname{Tr}\mathfrak{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^{s-\frac{n-m}{p}}_{p,p}(\mathbb{R}^m), \quad 0 < p < \infty,\ 0 < q \le \infty$$

and in the limiting case when $s = \frac{n-m}{p}$ we have

$$\operatorname{Tr}\mathfrak{F}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^m), \quad 0 < p \le 1,\ 0 < q \le \infty.$$

4. Dichotomy: traces versus density

4.1. Preliminaries. So far we were concerned with exact traces of spaces $\mathfrak{A}^s_{p,q}(\mathbb{R}^n)$, where $\mathfrak{A} \in \{\mathfrak{B}, \mathfrak{F}\}$, with $n \ge 2$, $s > 0$, and $0 < p, q < \infty$ on hyperplanes $\Gamma = \mathbb{R}^m$, $n > m \in \mathbb{N}$. In the sequel let μ stand for the m-dimensional Lebesgue measure l_m.

We now adopt a slightly more general point of view. Again we understand traces as limits of pointwise traces of smooth functions (recall that $D(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ are dense in all the spaces $\mathfrak{A}^s_{p,q}(\mathbb{R}^n)$, excluding $p = \infty$ and/or $q = \infty$). Therefore, if for some $c > 0$ we have

$$\|\varphi|L_r(\mathbb{R}^m)\| \le c\|\varphi|\mathfrak{A}^s_{p,q}(\mathbb{R}^n)\| \quad \text{for all } \varphi \in S(\mathbb{R}^n), \tag{4.1}$$

the trace operator Tr_Γ,

$$\operatorname{Tr}_\Gamma : \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^m)$$

is the completion of the pointwise trace $(\operatorname{Tr}_\Gamma\varphi)(\gamma) = \varphi(\gamma)$ with $\varphi \in S(\mathbb{R}^n)$ and $\gamma \in \mathbb{R}^m$.

Remark 4.1. In particular, it can be shown that for individual elements f the traces are independent of the source spaces and of the target spaces as long as one has (4.1) and whenever comparison makes sense, cf. [11, Remark 13] and [12, Section 6.4.2, pp. 218/219].

Let $D_\Gamma = D(\mathbb{R}^n \setminus \mathbb{R}^m)$ be as usual the collection of all (complex-valued) C^∞ functions in $\mathbb{R}^n$ with compact support in $\mathbb{R}^n \setminus \mathbb{R}^m$.

One may ask the two mutually exclusive questions (see also Proposition 4.4 below):

(i) In which of the above spaces $\mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ is D_Γ dense?

(ii) For which of the above spaces $\mathfrak{A}^s_{p,q}(\mathbb{R}^n)$ does there exist a linear and bounded trace operator $\mathrm{Tr}_\Gamma : \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^m)$?

It comes out that the above spaces divide sharply in these two contrasting classes (dichotomy).

The well-known inclusion properties of the spaces $\mathfrak{A}^s_{p,q}$ under consideration suggest the following formulation.

Definition 4.2. Let $0 < p < \infty$ and let $\mathfrak{A}_p(\mathbb{R}^n) = \{\mathfrak{A}^s_{p,q}(\mathbb{R}^n) : 0 < q < \infty,\ s > 0\}$. Let $\sigma > 0$. Then $\mathbb{D}(\mathfrak{A}_p(\mathbb{R}^n), L_p(\mathbb{R}^m)) = (\sigma, u)$ with $0 < u < \infty$ is called dichotomy of $\{\mathfrak{A}_p(\mathbb{R}^n), L_p(\mathbb{R}^m)\}$ if

$$\mathrm{Tr}_\Gamma \text{ exists for } \begin{cases} s > \sigma, & 0 < q < \infty \\ s = \sigma, & 0 < q \le u, \end{cases}$$

and

$$D_\Gamma \text{ is dense in } \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \text{ for } \begin{cases} s = \sigma, & u < q < \infty \\ s < \sigma, & 0 < q < \infty. \end{cases}$$

Furthermore, $\mathbb{D}(\mathfrak{A}_p(\mathbb{R}^n), L_p(\mathbb{R}^m)) = (\sigma, 0)$ means that

$$\begin{cases} \mathrm{Tr}_\Gamma \text{ exists for } s > \sigma,\ 0 < q < \infty, \\ D_\Gamma \text{ is dense in } \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \text{ for } s \le \sigma,\ 0 < q < \infty; \end{cases}$$

and $\mathbb{D}(\mathfrak{A}_p(\mathbb{R}^n), L_p(\mathbb{R}^m)) = (\sigma, \infty)$ means that

$$\begin{cases} \mathrm{Tr}_\Gamma \text{ exists for } s \ge \sigma,\ 0 < q < \infty, \\ D_\Gamma \text{ is dense in } \mathfrak{A}^s_{p,q}(\mathbb{R}^n) \text{ for } s < \sigma,\ 0 < q < \infty. \end{cases}$$

Remark 4.3. The above definition makes sense. Let $s \ge \sigma > 0$, $0 < p < \infty$, and $0 < q_1, q_2 < \infty$. From Proposition 2.6 we have the continuous embedding

$$\mathfrak{A}^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow \mathfrak{A}^\sigma_{p,q_2}(\mathbb{R}^n), \tag{4.2}$$

whenever $s \geq \sigma$ and $q_1 \leq q_2$ if $s = \sigma$. If the traces exist in $\mathfrak{A}^{\sigma}_{p,q_2}(\mathbb{R}^n)$, then automatically all spaces on the left-hand side in (4.2) have traces as well.

Furthermore, if D_Γ is dense in $\mathfrak{A}^s_{p,q_1}(\mathbb{R}^n)$, the embedding (4.2) together with the density of $D(\mathbb{R}^n)$ in all spaces in (4.2) imply the density of D_Γ in $\mathfrak{A}^{\sigma}_{p,q_2}(\mathbb{R}^n)$. This can easily be seen. If $\varphi \in D(\mathbb{R}^n)$ and $\psi_j \in D_\Gamma$ is an approximating sequence in $\mathfrak{A}^s_{p,q_1}(\mathbb{R}^n)$, we have

$$\|\varphi - \psi_j|\mathfrak{A}^{\sigma}_{p,q_2}(\mathbb{R}^n)\| \leq c\|\varphi - \psi_j|\mathfrak{A}^s_{p,q_1}(\mathbb{R}^n)\| \longrightarrow 0.$$

Additionally, one has the following almost obvious observation.

Proposition 4.4. *Let $s > 0$, $0 < p, q < \infty$, $0 < r < \infty$ and let D_Γ be dense in $\mathfrak{A}^s_{p,q}(\mathbb{R}^n)$. Then there is no $c > 0$ with*

$$\|\varphi|L_r(\mathbb{R}^m)\| \leq c\|\varphi|\mathfrak{A}^s_{p,q}(\mathbb{R}^n)\|, \quad \varphi \in S(\mathbb{R}^n). \tag{4.3}$$

Proof. We assume that there is a constant $c > 0$ with (4.3). We have $\mathbb{R}^n \subset \bigcup_{l=0}^{\infty} K_l$, where K_l are appropriate compact sets. Put $\Gamma_l := \mathbb{R}^m \cap K_l$, where Γ_l may be interpreted as a subset of $\mathbb{R}^m$ (considered as a space itself and not just a hyperplane of $\mathbb{R}^n$). We approximate a function φ_l which is identically 1 near Γ_l and has support in a neighbourhood of K_l by D_Γ-functions ψ_j, $j \in \mathbb{N}$. Then one has that

$$\mathrm{Tr}_\Gamma \varphi_l = \lim_{j\to\infty} \mathrm{Tr}_\Gamma \psi_j = 0 \quad \mu - a.e.$$

Since $\mu(\mathbb{R}^m) \leq \sum_{l=0}^{\infty} \mu(\Gamma_l)$, this contradicts $\mu(\mathbb{R}^m) > 0$. □

4.2. Dichotomy. Our main result is stated in the theorem below.

Theorem 4.5. *Let $n, m \in \mathbb{N}$, $n > m$, and $0 < p < \infty$. Then*

$$\mathbb{D}\left(\mathfrak{B}_p(\mathbb{R}^n), L_p(\mathbb{R}^m)\right) = \begin{cases} \left(\frac{n-m}{p}, 1\right) & \text{if } p > 1 \\ \left(\frac{n-m}{p}, p\right) & \text{if } p \leq 1 \end{cases} \tag{4.4}$$

and

$$\mathbb{D}\left(\mathfrak{F}_p(\mathbb{R}^n), L_p(\mathbb{R}^m)\right) = \begin{cases} \left(\frac{n-m}{p}, 0\right) & \text{if } p > 1 \\ \left(\frac{n-m}{p}, \infty\right) & \text{if } p \leq 1. \end{cases} \tag{4.5}$$

Proof. The proof is based on ideas from a similar proof in [11] and [12].

`Step 1`. We have to show that the breaking points (σ, u) exist and that they coincide with the right-hand sides of (4.4) and (4.5). By Corollary 3.5, Remark 3.7, and our discussions in Remark 4.3 in case of the B-spaces it remains to prove that

$$D_\Gamma \text{ is dense in } \mathfrak{B}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) \quad \text{if} \quad 0 < p < \infty,\ q > \min(p, 1), \tag{4.6}$$

which will be done in Steps 3 and 5 below. Concerning the F-spaces if $p \le 1$ we have

$$\mathfrak{B}^{\frac{n-m}{p}}_{p,\tilde{q}}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^{\frac{n-m}{p}-\varepsilon}_{p,q}(\mathbb{R}^n), \quad 0 < q < \infty,\ \tilde{q} > p,\ \varepsilon > 0. \tag{4.7}$$

$D(\mathbb{R}^n)$ is dense in both spaces. Using (4.6) (with $\tilde{q}$) now yields that D_Γ is dense in all spaces on the right-hand side of (4.7). This together with Corollary 3.13 and Remark 3.14 already gives the bottom line in (4.5). As for the case $1 < p < \infty$ we have

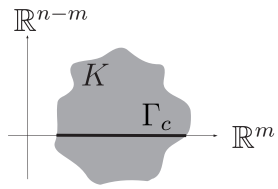
$$\mathfrak{F}^{\frac{n-m}{p}+\varepsilon}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{\frac{n-m}{p}}_{p,1}(\mathbb{R}^n), \quad 0 < q < \infty,\ \varepsilon > 0. \tag{4.8}$$

By Corollary 3.5 all spaces on the right-hand side of (4.8) have traces. It therefore remains to prove in the case of F-spaces that

$$D_\Gamma \text{ is dense in } \mathfrak{F}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) \quad \text{if} \quad 1 < p < \infty,\ 0 < q < \infty, \tag{4.9}$$

which we do in Step 4.

`Step 2`. We begin with a preparation. Let $K \subset \mathbb{R}^n$ be a compact set (which in Step 3 will be chosen to be the support of $f \in D(\mathbb{R}^n)$, the function we wish to approximate) and consider



$$\Gamma_C = \mathbb{R}^m \cap K,$$

cf. the figure aside.

The aim is to construct a sequence $\{\varphi^J\}_{J=1}^\infty \in D(\mathbb{R}^n)$ with

$$\varphi^J(x) = 1 \quad \text{in an open neighbourhood of } \Gamma_C$$

(depending on J) and

$$\varphi^J \longrightarrow 0 \ \text{ in } \mathfrak{B}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n) \quad \text{if } 0 < p < \infty,\ q > 1. \tag{4.10}$$

For given $j \in \mathbb{N}$ we cover a neighbourhood of Γ_C with balls $B_{j,k}$ in $\mathbb{R}^n$ centred at Γ_C and of radius 2^{-j}, where $k = 1, \ldots, M_j$ and $M_j \sim 2^{jm}$ (which is possible since Γ_C is compact) such that there is a resolution of unity,

$$\sum_{k=1}^{M_j} \varphi_{j,k}(x) = 1 \ \text{ near } \Gamma_C, \quad 0 \le \varphi_{j,k} \in D(B_{j,k}), \tag{4.11}$$

with the usual properties,

$$|\mathrm{D}^\gamma \varphi_{j,k}(x)| \le c_\gamma 2^{j|\gamma|}, \quad \gamma \in \mathbb{N}_0^n. \tag{4.12}$$

For $2 \le J \in \mathbb{N}$, let $J' \in \mathbb{N}$ be such that $\sum_{j=J}^{J'+1} r_j = 1$ with $r_j = j^{-1}$ if $J \le j \le J'$ and $0 < r_{J'+1} \le (J'+1)^{-1}$. Then

$$\varphi^J(x) = \sum_{j=J}^{J'+1} r_j \sum_{k=1}^{M_j} \varphi_{j,k}(x), \quad x \in \mathbb{R}^n, \tag{4.13}$$

is an atomic decomposition in $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ according to Definition 2.3 for any $s > 0$, $0 < p < \infty$. Setting $s = \frac{n-m}{p}$ such that $s - \frac{n}{p} = -\frac{m}{p}$ one gets for $q > 1$

$$\begin{aligned} \|\varphi^J|\mathfrak{B}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n)\|^q &\le c \sum_{j=J}^{J'+1} r_j^q 2^{-j\frac{mq}{p}} \left(\sum_{k=1}^{M_j} 1\right)^{\frac{q}{p}} \\ &\le c' \sum_{j=J}^{\infty} j^{-q} \\ &\sim J^{1-q} \longrightarrow 0 \quad \text{as } J \to \infty. \end{aligned} \tag{4.14}$$

This proves (4.10).

`Step 3`. We prove (4.6) for $p > 1$, $q > 1$. It is sufficient to approximate $f \in D(\mathbb{R}^n)$ in $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, $s = \frac{n-m}{p}$, by functions $f^J \in D_\Gamma$. Put

$$K := \operatorname{supp} f \quad \text{and} \quad \Gamma_C := K \cap \mathbb{R}^m.$$

Let φ^J be the functions constructed in Step 2 and

$$f = f_J + f^J \quad \text{with} \quad f_J = \varphi^J f \quad \text{and} \quad f^J = (1-\varphi^J) f \in D_\Gamma.$$

(We choose a different resolution of unity φ_J for every f.) By Proposition 2.5 (ii), using (4.10), one has for some $c > 0$, $f \in D(\mathbb{R}^n)$, and φ^J that

$$\begin{aligned} \|f_J|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| &\le \|f|C^\infty(\mathbb{R}^n)\| \cdot \|\varphi^J|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \\ &\le c\|\varphi^J|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\| \longrightarrow 0 \quad \text{as } J \to \infty. \end{aligned}$$

`Step 4`. We prove (4.9). By Theorem 3.12 and Corollary 3.13 we see that the trace of f in $\mathfrak{F}^s_{p,q}$ is independent of q. Therefore,

$$\left\|\varphi^J|\mathfrak{F}^{\frac{n-m}{p}}_{p,q}(\mathbb{R}^n)\right\| \sim \left\|\varphi^J|\mathfrak{B}^{\frac{n-m}{p}}_{p,p}(\mathbb{R}^n)\right\| \longrightarrow 0 \quad \text{if } J \to \infty,$$

cf. the constructions in Theorem 3.12. Then one gets (4.9) by the same arguments as in Step 3 for all $1 < p < \infty$ and $0 < q < \infty$.

`Step 5`. We prove (4.6) for the remaining case when $p < q$ (in particular $p \le 1$), constructing now a more refined resolution of unity as in Step 2. We cover the compact set Γ_C, say with $\mu(\Gamma_C) = 1$, for given $L \in \mathbb{N}$ by sets Γ_l such that

$\Gamma_C = \bigcup_{l=L}^{L'} \Gamma_l$, $\mu(\Gamma_l) \sim l^{-1}$, $\sum_{l=L}^{L'} \mu(\Gamma_l) \sim \mu(\Gamma_C) = 1$, where $L \in \mathbb{N}$ with $L' > L$ is appropriately chosen. For the details we refer to [12, Theorem 6.68]. In particular, this can be done in such a way that there are functions $\psi_l \in D(\mathbb{R}^n)$, $\psi_l \geq 0$, $\sum_{l=L}^{L'} \psi_l(\gamma) = 1$ if $\gamma \in \Gamma_C$, $\Gamma_l \subset \operatorname{supp} \psi_l \subset \{y \in \mathbb{R}^n : \operatorname{dist}(y, \Gamma_l) < \varepsilon_l\}$ for some $\varepsilon_l > 0$. Let for given $l \in \mathbb{N}$ (between L and L') and appropriately chosen $j(l) \in \mathbb{N}$, $\sum_{k=1}^{M_{j(l)}} \varphi_{j(l),m}(x) = 1$ near Γ_C, $0 \leq \varphi_{j(l),k} \in D(B_{j(l),k})$ as in (4.11) with the counterpart of (4.12) and $M_{j(l)} \sim 2^{j(l)m}$.

With $j(L) < \cdots < j(l) < j(l+1) < \cdots < j(L')$, we put in analogy to (4.13)

$$\varphi^L(x) = \sum_{l=L}^{L'} \psi_l(x) 2^{-\frac{j(l)m}{p}} \sum_{k=1}^{M_{j(l)}} 2^{\frac{j(l)m}{p}} \varphi_{j(l),k}(x), \quad x \in \mathbb{R}^n.$$

If $j(l)$ is chosen large enough this is an atomic decomposition which can be written as

$$\varphi^L(x) = \sum_{l=L}^{L'} 2^{-\frac{j(l)m}{p}} \sum_{k=1}^{M'_{j(l)}} 2^{\frac{j(l)m}{p}} \tilde{\varphi}_{j(l),k}(x), \quad x \in \mathbb{R}^n,$$

with $M'_{j(l)} \sim \mu(\Gamma_l) 2^{j(l)m} \sim l^{-1} 2^{j(l)m}$, counting only non-vanishing terms, where the equivalence constants are independent of l. We have $\varphi^L(x) = 1$ near Γ_C. Then one gets by Definition 2.3 for $q > p$,

$$\left\| \varphi^L | \mathfrak{B}_{p,q}^{\frac{n-m}{p}}(\mathbb{R}^n) \right\|^q \leq c \sum_{l=L}^{L'} 2^{-j(l)\frac{mq}{p}} \left(\sum_{k=1}^{M'_{j(l)}} 1 \right)^{\frac{q}{p}} \leq c' \sum_{l=L}^{L'} l^{-\frac{q}{p}} \sim L^{1-\frac{q}{p}}.$$

This is the counterpart of (4.14). Proceeding as in Step 3 finally proves the Theorem. □

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