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Non-Local Hamilton-Jacobi Equations Arising in Dislocation Dynamics

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Dedicated to Prof. Italo Capuzzo Dolcetta on the occasion of his 60th birthday

Abstract. We investigate a class of non-local Hamilton–Jacobi equations arising in dislocation dynamics. The class of Hamilton–Jacobi equations treated here is a variation of those studied by N. Forcadel, C. Imbert and R. Monneau in [*Discrete Contin. Dyn. Syst.* 23 (2009)(3), 785 – 826], and the new feature lies in the singularity at the origin of the kernel functions which describe non-local effects. For the class of Hamilton–Jacobi equations, we establish some stability properties of (viscosity) solutions, comparison theorems between subsolutions and supersolutions and existence theorems of solutions.
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1. Introduction

Let $p \in \mathbb{R}^N$ and $0 < T \leq \infty$. Set $Q_T = \mathbb{R}^N \times (0,\, T)$. We consider the functional differential equation of the Hamilton–Jacobi type

$$u_t = (c(x,t) + M_p[u(\cdot,\, t)](x))|p + Du(x,t)| \quad \text{ in } Q_T, \tag{1.1}$$

where $u : \mathbb{R}^N \times [0,\, \infty) \to \mathbb{R}$ is the unknown function, $u_t := \partial u/\partial t$, $Du := (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$ and $c \in C(\mathbb{R}^N \times [0,\infty))$ is a given function. Moreover, the operator M_p is formally given by

$$M_p[\phi](x) = \int_{\mathbb{R}^N} J(z)\big(E(\phi(x+z) - \phi(x) + p\cdot z) - p\cdot z\big)\mathrm{d}z,$$

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where J is a measurable function on $\mathbb{R}^N$ and E is the function on $\mathbb{R}$ given by $E(r) = \lfloor r \rfloor + \frac{1}{2}$. Here $\lfloor r \rfloor$ denotes the greatest integer less than or equal to $r \in \mathbb{R}$.

This type of non-local Hamilton–Jacobi equations have been introduced by Forcadel–Imbert–Monneau [8] as model equations in the level-set approach to dislocation dynamics. They have studied not only the well-posedness of the initial value problem for such Hamilton–Jacobi equations but also its homogenization. We refer to [8] for the connections of (1.1) to dislocation dynamics as well as the solvability and homogenization of (1.1). See also [1, 6] and the references therein for related topics.

In this article we investigate the solvability of the initial value problem for (1.1), with the kernel J having a stronger singularity at the origin, in the framework of viscosity solutions and establish some stability properties of solutions of (1.1), comparison theorems between subsolutions and supersolutions of (1.1) and existence theorems of solutions of the initial value problem for (1.1). We refer to [2, 3, 9, 11] for some results on the well-posedness of general functional-differential equations.

The notion of solution here is defined through those of subsolution and supersolution. It is convenient for us to divide (1.1) into two inequalities:

$$u_t(x,t) \le (c(x,t) + M_p^+[u(\cdot,t)](x))|p + Du(x,t)| \quad \text{in } Q_T \tag{1.2}$$
$$u_t(x,t) \ge (c(x,t) + M_p^-[u(\cdot,t)](x))|p + Du(x,t)| \quad \text{in } Q_T, \tag{1.3}$$

where, for bounded measurable functions $\phi : \mathbb{R}^N \to \mathbb{R}$,

$$\begin{aligned}
M_p^+[\phi](x) &:= \limsup_{\delta\to 0+} \int_{|z|>\delta} E_p^+(\phi(x+z) - \phi(x), z) J(z)\mathrm{d}z \\
M_p^-[\phi](x) &:= \liminf_{\delta\to 0+} \int_{|z|>\delta} (E_p^-(\phi(x+z) - \phi(x), z) J(z)\mathrm{d}z \\
E_p^+(r,z) &:= E^*(r + p\cdot z) - p\cdot z \\
E_p^-(r,z) &:= E_*(r + p\cdot z) - p\cdot z.
\end{aligned}$$

Here and later, given a function f, we denote by f^* (resp., f_*) the upper (resp., lower) semicontinuous envelope of f. Note that $E^* = E$ and that $E^*(r) = -E_*(-r)$ for all $r \in \mathbb{R}$. Note also that $|E_p^{\pm}(r,z) - r| \le \frac{1}{2}$ for all $r \in \mathbb{R}$.

To make the meaning of (1.2) and (1.3) precise, we introduce our assumptions on c and J:

(c1) $c \in \mathrm{BUC}(\overline{Q}_\tau)$ for any $0 < \tau < T$;

(c2) for any $\tau \in (0,\, T)$, there is a constant $L_\tau > 0$ such that

$$|c(x,t) - c(y,t)| \le L_\tau |x - y| \quad \text{for all } x, y \in \mathbb{R}^N \text{ and } t \in [0,\, \tau];$$

(J1) J is nonnegative and measurable on $\mathbb{R}^N$;
(J2) $J(-z) = J(z)$ for all $z \in \mathbb{R}^N \setminus \{0\}$;
(J3) $J \in L^1(B(0,1)^c)$, where $B(0,1)^c := \mathbb{R}^N \setminus B(0,1)$;
(J4) there are constants $\beta < N+1$ and $C_0 > 0$ such that $J(z) \le \frac{C_0}{|z|^\beta}$ for all $z \in B(0,1) \setminus \{0\}$.

Note that if $\beta' > \beta$, then $|z|^{-\beta} \le |z|^{-\beta'}$ for all $z \in B(0,1) \setminus \{0\}$. Hence we may and *do assume* throughout the paper that $\beta > N$ in condition (J4).

A new feature of this article is that condition (J4) allows J to have a singularity, stronger than the one studied in [8], at the origin. Indeed, the main issue here is how to deal with singularities of J at the origin in order to establish stability properties of solutions of (1.1) and comparison and existence results for solutions of the initial value problem for (1.1).

We see that, under assumptions (J3) and (J4), if ϕ is bounded measurable, then the values $M_p^{\pm}[\phi](x)$ are well-defined although they may be $\pm\infty$.

The precise meaning of the above inequalities (1.2) and (1.3) are as follows. Henceforth we deal only with solutions of (1.1), (1.2) or (1.3) which are bounded on $\mathbb{R}^N \times (0,\tau)$ for any $0 < \tau < T$. We denote by $\mathcal{B}(Q_T)$ the space of functions on Q_T which are bounded on Q_τ for any $0 < \tau < T$. A function $u \in \mathcal{B}(Q_T)$ is called a (*viscosity*) *solution* or subsolution of (1.2) or (viscosity) subsolution of (1.1) if whenever $(x,t,\phi) \in \mathbb{R}^N \times (0,T) \times C^2(Q_T)$ and $u^* - \phi$ attains a local maximum at (x,t), we have

$$\phi_t(x,t) \le \begin{cases} \big(c(x,t) + M_p^+[u^*(\cdot,\, t)](x)\big)|p + D\phi(x,t)| & \text{if } p + D\phi(x,\, t) \ne 0 \\ 0 & \text{if } p + D\phi(x,\, t) = 0. \end{cases} \tag{1.4}$$

It will be shown (see Lemma 2.1 below) that if $p + D\phi(x,\, t) \ne 0$, then $M_p^+[u^*(\cdot,\, t)](x) < \infty$ in the above inequality.

Similarly, a function u on Q_T is called a (*viscosity*) *solution* or supersolution of (1.3) or (viscosity) supersolution of (1.1) if whenever $(x,t,\phi) \in Q_T \times C^2(Q_T)$ and $u_* - \phi$ attains a local minimum at (x,t), we have

$$\phi_t(x,\, t) \ge \begin{cases} \big(c(x,t) + M_p^-[u_*(\cdot,\, t)](x)\big)|p + D\phi(x,\, t)| & \text{if } p + D\phi(x,\, t) \ne 0 \\ 0 & \text{if } p + D\phi(x,\, t) = 0. \end{cases} \tag{1.5}$$

Here we also remark (see Remark 2.1 below) that, under (J3) and (J4), if $p + D\phi(x,t) \ne 0$, then $M_p^-[u_*(\cdot,t)](x) > -\infty$.

Finally, a function $u \in \mathcal{B}(Q_T)$ is called a (*viscosity*) *solution* of (1.1) if it is both a solution of (1.2) and of (1.3).

We will be also concerned with PDE of the form

$$u_t + f(x,t) = (c(x,t) + M_p[u(\cdot,\, t)](x))\,|p + Du| \quad \text{in } Q_T,$$

where $f \in C(Q_T)$ is a given function. For this, the above notion of solution, subsolution and supersolution can be easily adapted.

We denote by $\mathcal{S}^+ = \mathcal{S}^+(Q_T)$ (resp., $\mathcal{S}^- = \mathcal{S}^-(Q_T)$ or $\mathcal{S} = \mathcal{S}(Q_T)$) the set of all solutions of (1.3) (resp., (1.2) or (1.1)). By definition, we have $\mathcal{S}^\pm(Q_T) \subset \mathcal{B}(Q_T)$ and $\mathcal{S}(Q_T) \subset \mathcal{B}(Q_T)$.

The above definition of viscosity solutions differs slightly from that of [7] where subsolutions (resp., supersolutions) are assumed to be upper (resp., lower) semicontinuous.

Condition (J4) can be considerably relaxed in one dimension. By modifying the notion of solutions, subsolutions and supersolutions by imposing an extra condition on test functions and taking advantage of the simple geometry of the space $\mathbb{R}$, we will show that the Cauchy problem for (1.1) is well-posed in one dimension without the restriction, $\beta < N+1$. See (J4′) for the replacement of (J4) in one dimension.

The paper is organized as follows. Sections 2–5 are concerned with the well-posedness of (1.1) in general dimension. In Section 2 we establish a couple of estimates on the operators $M_p^\pm[u]$ under some semi-convexity or semi-concavity assumptions on u. In Section 3, we establish some stability properties of solutions of (1.1), (1.2) or (1.3) as well as the Perron method. Section 4 is devoted to the proof of comparison theorems for solutions. In Section 5 we apply results obtained in the previous sections to prove an existence and uniqueness theorem for the initial value problem for (1.1). Section 6 is focused on the well-posedness of the initial value problem for (1.1) in one dimension. We modify the notion of solution, subsolution and supersolution and establish stability properties, comparison and existence theorems for solutions of (1.1) in one dimension.

Notation: for $a,\, b \in \mathbb{R}$ we write $a \vee b := \max\{a,\, b\}$ and $a \wedge b := \min\{a,\, b\}$. For any real-valued function f on X, we write $\|f\|_\infty = \|f\|_{\infty,X} := \sup_X |f|$.

2. Basic estimates on operators $M_p^\pm$

In this section, we give some estimates on operators $M_p^\pm$. Let $p \in \mathbb{R}^N$ be a fixed vector.

Lemma 2.1. *Let u be a bounded measurable function on $\mathbb{R}^N$. Let $x,\, q \in \mathbb{R}^N$, $r > 0$, $\Lambda > 0$ and $C_1 > 0$. Assume that $0 < |p+q| \le \Lambda$ and*

$$u(x+z) \le u(x) + q \cdot z + C_1|z|^2 \quad \text{for all } z \in B(0,r).$$

Then there are constants $\rho > 0$, depending only on r, Λ and C_1, and $C > 0$, depending only on C_0, C_1, β and N, such that for any $0 < \delta \le \rho \wedge \left(\frac{|p+q|}{2C_1}\right)$,

$$M_p^+[u](x) \le \frac{C}{|p+q|}\delta^{N+1-\beta} + \int_{|z|>\delta} J(z) E_p^+(u(x+z) - u(x), z)\mathrm{d}z. \tag{2.1}$$

Remark 2.2. An assertion analogous to Lemma 2.1 holds true for M_p^-. It is the proposition same as Lemma 2.1, except that the assumption that $v(x+z) \geq v(x) + q \cdot z - C_1|z|^2$ for all $x \in \mathbb{R}^N$ replace the corresponding assumption in Lemma 2.1 and the inequality

$$M_p^-[v](x) \geq -\frac{C}{|p+q|}\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^-(v(x+z) - v(x), z)\mathrm{d}z.$$

replaces inequality (2.1) of Lemma 2.1. To see this, we just need to apply Lemma 2.1 to $u = -v$, with $-p$ and $-q$ in place of p and q, respectively. Other propositions in this section stated only for M_p^+ have their analogues valid for M_p^-.

Proof. We set $v = p + q$, choose an orthonormal basis $\{f_1, \ldots, f_N\}$ of $\mathbb{R}^N$ so that $f_N = |v|^{-1}v$, and define the orthogonal matrix F by

$$F = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$$

and observe that for any $z \in \mathbb{R}^N$, $zF = z_1 f_1 + \cdots + z_N f_N$. We have

$$\begin{aligned} u(x+zF) - u(x) + p \cdot zF &\leq (p+q) \cdot zF + C_1|zF|^2 \\ &= v \cdot (z_1 f_1 + \cdots + z_N f_N) + C_1|zF|^2 \\ &= |v|z_N + C_1|z|^2 \quad \text{ for all } z \in B(0,r). \end{aligned} \tag{2.2}$$

Observe that if $|z| \leq \frac{|v|}{2C_1}$ and $z_N < -\frac{2C_1}{|v|}|z'|^2$, where $z = (z', z_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, then

$$|v|z_N + C_1|z|^2 < -C_1|z'|^2 + \frac{|v|}{2}z_N + C_1|z|^2 \leq |z_N|\left(C_1|z| - \frac{|v|}{2}\right) \leq 0. \tag{2.3}$$

Set $\rho = \min\left\{r, \frac{1}{1+2(\Lambda+C_1)}\right\}$, and note that $\rho < 1$ and $\Lambda\rho + C_1\rho^2 < 1$.

Next let $0 < \gamma < \delta \leq \rho \wedge \left(\frac{|v|}{2C_1}\right)$. For any $z \in B(0,\delta)$, we have $|v|z_N + C_1|z|^2 < 1$ and moreover, by (2.3)

$$E^*\left(|v|z_N + C_1|z|^2\right) \leq \begin{cases} -\dfrac{1}{2} & \text{if } z_N < -\dfrac{2C_1}{|v|}|z'|^2 \\ \dfrac{1}{2} & \text{otherwise.} \end{cases} \tag{2.4}$$

Using (2.2) and (2.4), we calculate that

$$\begin{aligned}
&\int_{\gamma<|z|\le\delta} J(z)E^*\big(u(x+z)-u(x)+p\cdot z\big)\mathrm{d}z\\
&=\int_{\gamma<|z|\le\delta} J(zF)E^*\big(u(x+zF)-u(x)+p\cdot zF\big)\mathrm{d}z\\
&\le\int_{\gamma<|z|\le\delta} J(zF)E^*\big(|v|z_N+C_1|z|^2\big)\mathrm{d}z\\
&\le\frac{1}{2}\Big(\int_{U^+} J(zF)\mathrm{d}z-\int_{U^-} J(zF)\mathrm{d}z\Big),
\end{aligned}$$

where $U^+ := \big\{z\in\mathbb{R}^N \mid \gamma<|z|\le\delta,\ z_N\ge -\frac{2C_1}{|v|}|z'|^2\big\}$ and $U^- := \big\{z\in\mathbb{R}^N \mid \gamma< |z|\le\delta,\ z_N< -\frac{2C_1}{|v|}|z'|^2\big\}$. Setting $U_0=\big\{z\in\mathbb{R}^N \,\big|\, \gamma<|z|\le\delta,\ |z_N|\le\frac{2C_1}{|v|}|z'|^2\big\}$ and using the symmetry property of J, we observe that

$$\int_{U^+} J(zF)\mathrm{d}z-\int_{U^-} J(zF)\mathrm{d}z=\int_{-U^+} J(zF)\mathrm{d}z-\int_{U^-} J(zF)\mathrm{d}z \quad =\int_{U_0} J(zF)\mathrm{d}z,$$

to find that

$$\int_{\gamma<|z|\le\delta} J(z)E_p^+\big(u(x+z)-u(x),z\big)\mathrm{d}z=\frac{1}{2}\int_{U_0} J(zF)\mathrm{d}z.$$

Now, recalling that $\delta\le 1$, we observe that if $|z_N|\le\frac{2C_1}{|v|}|z'|^2$ and $\gamma<|z|\le\delta$, then $\gamma^2<|z'|^2+z_N^2\le|z'|^2+|z_N|\le\big(1+\frac{2C_1}{|v|}\big)|z'|^2$, and $|z'|\le\delta$. Setting $\nu=\gamma\big(1+\frac{2C_1}{|v|}\big)^{-\frac{1}{2}}$ we note that $U_0\subset\{z\in\mathbb{R}^N \mid \nu<|z'|\le\delta,\ |z_N|\le\frac{2C_1}{|v|}|z'|^2\}$ and compute that

$$\begin{aligned}
\int_{\gamma<|z|\le\delta} J(z)E_p^+\big(u(x+z)-u(x),z\big)\mathrm{d}z &\le\frac{1}{2}\int_{U_0} J(zF)\mathrm{d}z\\
&\le\frac{C_0}{2}\int_{U_0}|z'|^{-\beta}\mathrm{d}z\\
&=C_0\int_{\nu<|z'|\le\delta}\mathrm{d}z'\int_0^{\frac{2C_1}{|v|}|z'|^2}|z'|^{-\beta}\mathrm{d}z_N\\
&\le\frac{2C_0C_1}{|v|}\int_{\nu<|z'|\le\delta}|z'|^{2-\beta}\mathrm{d}z'\\
&=\frac{2C_0C_1\sigma_N}{|v|}\int_\nu^\delta t^{N-\beta}\mathrm{d}t\\
&<\frac{2C_0C_1\sigma_N}{|v|(N+1-\beta)}\delta^{N+1-\beta},
\end{aligned}$$

where σ_N is a positive constant depending only on N. Finally, we note

$$M_p^+[u](x) \le \frac{2C_0C_1\sigma_N}{|v|(N+1-\beta)}\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+\big(u(x+z)-u(x),z\big)\mathrm{d}z,$$

to conclude the proof. □

Lemma 2.3. *Let u be a bounded measurable function on $\mathbb{R}^N$. Let $x, q \in \mathbb{R}^N$, $r>0$ and $C_1>0$. Assume that $p+q\neq 0$ and $u(x+z)\le u(x)+q\cdot z+C_1|z|^2$ for all $z\in B(0,r)$. Then there are constants $\rho>0$, depending only on r and C_1, and $C>0$, depending only on C_0, C_1, β and N, such that if $|p+q|\le\rho$, then*

$$M_p^+[u](x) \le C|p+q|^{N-\beta} + \int_{|z|>\frac{|p+q|}{2C_1}} J(z)E_p^+(u(x+z)-u(x),z)\mathrm{d}z.$$

Proof. By Lemma 2.1 with $\Lambda=1$, there are constants $\rho_1>0$, depending only on r and C_1, and $C_2>0$, depending only on C_0, C_1, β and N, such that if $0<\delta\le\rho_1\wedge\big(\frac{|p+q|}{2C_1}\big)$ and $|p+q|\le 1$, then

$$M_p^+[u](x) \le \frac{C_2}{|p+q|}\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+(u(x+z)-u(x),z)\mathrm{d}z. \tag{2.5}$$

We set $\rho=(2C_1\rho_1)\wedge 1$, so that $\rho\le 1$ and $\frac{\rho}{2C_1}\le\rho_1$. Now, assume that $|p+q|\le\rho$. Then we have $|p+q|\le 1$ and $\delta:=\frac{|p+q|}{2C_1}\le\rho_1$. Hence, by (2.5), we get

$$\begin{aligned} M_p^+[u](x) \le& \frac{C_2}{|p+q|}\Big(\frac{|p+q|}{2C_1}\Big)^{N+1-\beta} + \int_{|z|>\frac{|p+q|}{2C_1}} J(z)E_p^+\big(u(x+z)-u(x),z\big)\mathrm{d}z \\ =& \frac{C_2}{(2C_1)^{N+1-\beta}}|p+q|^{N-\beta} + \int_{|z|>\frac{|p+q|}{2C_1}} J(z)E_p^+\big(u(x+z)-u(x),z\big)\mathrm{d}z, \end{aligned}$$

which was to be shown. □

Lemma 2.4. *Let u be a bounded measurable function on $\mathbb{R}^N$. Let $x, q\in\mathbb{R}^N$, $r>0$, $0<\lambda\le\Lambda<\infty$ and $C_1>0$. Assume that $\lambda<|p+q|\le\Lambda$ and $u(x+z)\le u(x)+q\cdot z+C_1|z|^2$ for all $z\in B(0,r)$. Then there are constants $\rho>0$, depending only on r, λ, Λ, C_0 and C_1, and $C>0$, depending only on C_0, C_1, λ, β and N, such that for any $0<\delta\le\rho$,*

$$M_p^+[u](x) \le C\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+(u(x+z)-u(x)\,z)\mathrm{d}z.$$

Proof. According to Lemma 2.1, there are constants $\rho_1 > 0$, depending only on r, Λ, and C_1, and $C_2 > 0$, depending only on C_0, C_1, β and N, such that if $0 < \delta \le \rho_1 \wedge \left(\frac{|p+q|}{2C_1}\right)$, then

$$M_p^+[u](x) \le \frac{C_2}{|p+q|}\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+(u(x+z)-u(x), z)\mathrm{d}z. \tag{2.6}$$

Setting $\rho = \rho_1 \wedge \left(\frac{\lambda}{2C_1}\right)$ and noting that $\rho \le \rho_1 \wedge \left(\frac{|p+q|}{2C_1}\right)$, we find from (2.6) that for any $0 < \delta \le \rho$,

$$\begin{aligned} M_p^+[u](x) \le& \frac{C_2}{|p+q|}\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+(u(x+z)-u(x), z)\mathrm{d}z \\ \le& \frac{C_2}{\lambda}\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+(u(x+z)-u(x), z)\mathrm{d}z. \end{aligned}$$

The proof is complete. □

Lemma 2.5. *Let u be a bounded measurable function on $\mathbb{R}^N$. Let x, $q \in \mathbb{R}^N$, $r > 0$, $\Lambda > 0$, $C_1 > 0$ and $C_2 > 0$. Assume that $0 < |p+q| \le \Lambda$, $|u(z)| \le C_2$ for all $z \in \mathbb{R}^N$ and*

$$u(x+z) \le u(x) + q\cdot z + C_1|z|^2 \quad \text{for all } z \in B(0,r).$$

Then there is a modulus ω, depending only on r, Λ, β, C_0, C_1, C_2, $\|J\|_{L^1(B(0,1)^c)}$ and N, such that

$$M_p^+[u](x)|p+q| \le \omega(|p+q|).$$

Proof. By Lemma 2.3, there are numbers $\rho > 0$, depending only on r and C_1, and $C_3 > 0$, depending only on C_0, C_1, β and N, such that if $0 < |p+q| \le \rho$, then

$$M_p^+[u](x) \le C_3|p+q|^{N-\beta} + \int_{|z|>\frac{|p+q|}{2C_1}} J(z)E_p^+(u(x+z)-u(x), z)\mathrm{d}z. \tag{2.7}$$

We may assume, by replacing ρ by a smaller positive number if needed, that $\rho < \Lambda$ and $\frac{\rho}{2C_1} < 1$.

Assume that $0 < |p+q| \le \rho$. We compute that

$$\begin{aligned} &\int_{|z|>\frac{|p+q|}{2C_1}} J(z)E_p^+(u(x+z)-u(x), z)\mathrm{d}z \\ &\le (2C_2+1)\int_{|z|>\frac{|p+q|}{2C_1}} J(z)\mathrm{d}z \\ &\le (2C_2+1)\left(C_0\sigma_N'\int_{\frac{|p+q|}{2C_1}}^1 t^{N-1-\beta}\mathrm{d}t + \|J\|_{L^1(B(0,1)^c)}\right) \\ &\le (2C_2+1)\left(\frac{C_0\sigma_N'}{\beta-N}\left(\frac{|p+q|}{2C_1}\right)^{N-\beta} + \|J\|_{L^1(B(0,1)^c)}\right), \end{aligned}$$

where σ'_N is a constant depending only on N and $B(0,1)^c$. Combining this with (2.7), we get

$$\begin{aligned} M_p^+[u](x)|p+q| \le & C_4\left(|p+q|^{N+1-\beta} + |p+q|\|J\|_{L^1(B(0,1)^c)}\right) \\ \le & C_4\left(1+\rho^{\beta-N}\|J\|_{L^1(B(0,1)^c)}\right)|p+q|^{N+1-\beta}, \end{aligned}$$

where $C_4 > 0$ is a constant depending only on C_0, C_1, β, C_2 and N.

By Lemma 2.4, there are constants $0 < \delta < 1$, depending only on r, ρ, Λ, C_0 and C_1, and $C_5 > 0$, depending only on C_0, C_1, ρ, β and N, such that if $\rho < |p+q| \le \Lambda$, then

$$M_p^+[u](x) \le C_5\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+(u(x+z)-u(x),z)\mathrm{d}z. \tag{2.8}$$

Assume that $\rho < |p+q| \le \Lambda$. As before, we compute that

$$\begin{aligned} &\int_{|z|>\delta} J(z)E_p^+(u(x+z)-u(x),z)\mathrm{d}z \\ &\le (2C_2+1)\left(C_0\sigma'_N\int_\delta^1 t^{N-1-\beta}\mathrm{d}t + \|J\|_{L^1(B(0,1)^c)}\right) \\ &\le (2C_2+1)\left(\frac{C_0\sigma'_N}{\beta-N}\delta^{N-\beta} + \|J\|_{L^1(B(0,1)^c)}\right). \end{aligned}$$

Hence, using (2.8), we get $M_p^+[u](x)|p+q| \le C_6|p+q|$ for some constant $C_6 > 0$ which depends only on r, C_i, with $i = 1, 2, 3$, β, Λ and N.

By replacing C_4 and C_6 by larger numbers if necessary, we may assume that $C_4\left(1+\rho^{\beta-N}\|J\|_{L^1(B(0,1)^c)}\right)\rho^{N+1-\beta} = C_6\rho$. Then we define the function $\omega \in C([0,\infty))$ by setting

$$\omega(t) = \begin{cases} C_4\left(1+\rho^{\beta-N}\|J\|_{L^1(B(0,1)^c)}\right)t^{N+1-\beta} & \text{for } t \le \rho \\ C_6 t & \text{for } t > \rho. \end{cases}$$

This function ω is a modulus having all the required properties. □

3. Stability properties and the Perron method

In this section we establish some stability properties of solutions of (1.2) or (1.3) as well as the Perron method. Analogous stability properties are valid for solutions of (1.3), but we do not give here the details and leave it to the reader to supply them.

Lemma 3.1. *Let $\delta > 0$, $\{p_n\} \subset \mathbb{R}^N$, $\{x_n\} \subset \mathbb{R}^N$ and $\{u_n\} \subset \mathrm{USC}(\mathbb{R}^N)$. Let $p,\, x \in \mathbb{R}^N$ and $u \in \mathrm{USC}(\mathbb{R}^N)$. Assume that $\{u_n\}$ is uniformly bounded on $\mathbb{R}^N$ and that $(p_n,\, x_n,\, u_n(x_n)) \to (p,\, x,\, u(x))$ as $n \to \infty$. Moreover assume that*

$$\lim_{k\to\infty} \sup \bigl\{u_n(y) \mid y \in B(z, k^{-1}),\ n \geq k\bigr\} \leq u(z) \quad \text{for all } z \in \mathbb{R}^N. \tag{3.1}$$

Then

$$\begin{aligned}&\limsup_{n\to\infty} \int_{|z|>\delta} J(z)\, E^+_{p_n}(u_n(x_n+z) - u_n(x_n), z)\mathrm{d}z \\ &\leq \int_{|z|>\delta} J(z)\, E^+_p(u(x+z) - u(x), z)\mathrm{d}z.\end{aligned}$$

Proof. Set

$$\begin{aligned} I_n =& \int_{|z|>\delta} J(z)\, E^+_{p_n}(u_n(x_n+z) - u_n(x_n), z)\mathrm{d}z \\ f_n(z) =& E^+_{p_n}(u_n(x_n+z) - u_n(x_n), z) \quad \text{for } z \in \mathbb{R}^N. \end{aligned}$$

Choose a constant $C > 0$ so that $|u_n(z)| \leq C$ for all $(z,n) \in \mathbb{R}^N \times \mathbb{N}$, and note that $J(z)|f_n(z)| \leq (2C+1)J(z)$ for all $(z,n) \in (\mathbb{R}^N \setminus \{0\}) \times \mathbb{N}$. By the Fatou lemma, we find that

$$\limsup_{n\to\infty} I_n \leq \int_{|z|>\delta} J(z) \limsup_{n\to\infty} f_n(z)\mathrm{d}z.$$

Since E^* is upper semicontinuous and non-decreasing in $\mathbb{R}$, we see that for any $z \in \mathbb{R}^N$,

$$\limsup_{n\to\infty} f_n(z) \leq E^*(\limsup_{n\to\infty} u_n(x_n+z) - u(x) + p\cdot z) - p \cdot z.$$

Using (3.1), we see that $\limsup_{n\to\infty} u_n(x_n+z) \leq u(x+z)$ for all $z \in \mathbb{R}^N$. Hence, we get

$$\limsup_{n\to\infty} f_n(z) \leq E^*_p(u(x+z) - u(x), z) \quad \text{for all } z \in \mathbb{R}^N.$$

Thus we obtain

$$\limsup_{n\to\infty} I_n \leq \int_{|z|>\delta} J(z)\, E^*_p(u(x+z) - u(x), z)\mathrm{d}z,$$

which completes the proof. □

Theorem 3.2. *Let $\mathcal{S}_0$ be a non-empty set of solutions of* (1.2). *Assume that the family $\mathcal{S}_0$ is uniformly bounded on Q_τ for any $0 < \tau < T$. Define the function $u \in \mathcal{B}(Q_T)$ by $u(x,\, t) = \sup\{v(x,\, t) \mid v \in \mathcal{S}_0\}$. Then u is a solution of* (1.2).

Proof. Let $(\hat{x},\, \hat{t}\,) \in Q_T$ and $\varphi \in C^2(Q_T)$, and assume that $u^* - \varphi$ attains a strict maximum at $(\hat{x}, \hat{t}\,)$. By the definition of u^*, there are sequences $\{(x_n, t_n)\} \subset B((\hat{x}, \hat{t}\,), 2r)$, where $r > 0$ is chosen so that $B((\hat{x}, \hat{t}\,), 2r) \subset Q_T$, and $\{v_n\} \subset \mathcal{S}_0$ such that $v_n(x_n, t_n) \to u^*(\hat{x}, \hat{t}\,)$ and $(x_n, t_n) \to (\hat{x}, \hat{t}\,)$ as $n \to \infty$. By the definition of u, we have $v_n^* \le u^*$ in Q_T.

For any $n \in \mathbb{N}$ let $(y_n, s_n) \in B((\hat{x}, \hat{t}\,), 2r)$ be a maximum point, over $B((\hat{x}, \hat{t}\,), 2r)$, of the function $v_n^* - \varphi$. Observe that

$$\begin{aligned}(u^* - \varphi)(\hat{x}, \hat{t}\,) &= \liminf_{n\to\infty} (v_n - \varphi)(x_n, t_n)\\ &\le \liminf_{n\to\infty} (v_n^* - \varphi)(y_n, s_n)\\ &\le \limsup_{n\to\infty} (v_n^* - \varphi)(y_n, s_n)\\ &\le \limsup_{n\to\infty} (u^* - \varphi)(y_n, s_n)\\ &\le (u^* - \varphi)(\hat{x}, \hat{t}\,).\end{aligned}$$

This shows that $v_n^*(y_n, s_n) \to u^*(\hat{x}, \hat{t}\,)$ and $(u^* - \varphi)(y_n, s_n) \to (u^* - \varphi)(\hat{x}, \hat{t}\,)$ as $n \to \infty$. It is now easy to deduce that $(y_n, s_n) \to (\hat{x}, \hat{t}\,)$ as $n \to \infty$.

Passing to a subsequence if necessary, we may assume that $(y_n, s_n) \in B((\hat{x}, \hat{t}\,), r)$ for all n. Since $v_n \in \mathcal{S}^-$, we have

$$\varphi_t(y_n, s_n) \le (c(y_n, s_n) + M_p^+[v_n^*(\cdot,\, s_n)](y_n))|p + D\varphi(y_n, s_n)| \tag{3.2}$$

if $p + D\varphi(y_n, s_n) \neq 0$, and

$$\varphi_t(y_n, s_n) \le 0 \quad \text{ if } p + D\varphi(y_n, s_n) = 0. \tag{3.3}$$

We now separate into two cases.

`Case` 1: $p + D\varphi(\hat{x}, \hat{t}\,) = 0$. In view of Lemma 2.5, there is a modulus ω, which depends on $\|D\varphi\|_{\infty, B((\hat{x},\hat{t}\,),r)}$, $\|D^2\varphi\|_{\infty, B((\hat{x},\hat{t}\,),2r)}$ and $\|v\|_{\infty, \mathbb{R}^N\times[\hat{t}-r,\, \hat{t}+r]}$ but not on n, such that

$$M_p^+[v_n^*(\cdot,\, s_n)](y_n)|p + D\varphi(y_n, s_n)| \le \omega(|p + D\varphi(y_n, s_n)|) \quad \text{ for all } n.$$

We combine this with (3.2) and (3.3) and send $n \to \infty$, to see that $\varphi_t(\hat{x}, \hat{t}\,) \le 0$.

`Case` 2: $p + D\varphi(\hat{x}, \hat{t}\,) \neq 0$. By selecting a subsequence if necessary, we may assume that $|p + D\varphi(y_n, s_n)| \ge \lambda$ for all n and for some constant $\lambda > 0$. By the definition of u, we see that for all $x \in \mathbb{R}^N$,

$$\begin{aligned}&\lim_{k\to\infty} \sup\left\{v_n^*(y,\, s_n) \mid n \ge k,\ y \in B(x,\, k^{-1})\right\}\\ &\le \lim_{k\to\infty} \sup\left\{u^*(y,\, s_n) \mid n \ge k,\ y \in B(x,\, k^{-1})\right\}\\ &\le u^*(x, \hat{t}\,).\end{aligned}$$

We now apply Lemma 2.4, to find that there are constants $\rho_0 > 0$ and $C > 0$ such that for any $0 < \delta \le \rho_0$,

$$M_p^+[v_n^*(\cdot,\, s_n)](y_n) \le C\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+(v_n^*(y_n+z) - v_n^*(y_n), z)\mathrm{d}z.$$

We next apply Lemma 3.1, to get for any $\delta \in (0,\, \rho_0]$,

$$\limsup_{n\to\infty} M_p^+[v_n^*(\cdot, s_n)](y_n) \le C\delta^{N+1-\beta} + \int_{|z|>\delta} J(z)E_p^+(u^*(\hat{x}+z, \hat{t}\,) - u^*(\hat{x}, \hat{t}\,), z)\mathrm{d}z.$$

From this, we easily get $\limsup_{n\to\infty} M_p^+[v_n^*(\cdot,\, s_n)](y_n) \le M_p^+[u^*(\cdot,\, \hat{t}\,)](\hat{x})$, and hence conclude from (3.2) that

$$\varphi_t(\hat{x},\, \hat{t}\,) \le \big(c(\hat{x}, \hat{t}\,) + M_p[u^*(\cdot,\, \hat{t}\,)](\hat{x})\big)|p + D\varphi(\hat{x},\, \hat{t}\,)|.$$

Thus, u^* is a solution of (1.2). □

Theorem 3.3. *Let $\{u_n\}$ be a sequence of solutions of* (1.2). *Assume that the collection $\{u_n\}$ is uniformly bounded on Q_τ for any $0 < \tau < T$. Define $u \in \mathcal{B}(Q_T)$ by*

$$u(x,t) = \lim_{k\to\infty} \sup \big\{u_n(y,s) \mid (y,s) \in B((x,t),\, k^{-1}),\, n \ge k\big\}.$$

Then u is a solution of (1.2).

Proof. We begin by noting that $u \in \mathrm{USC}(Q_T)$. Let $(\hat{x},\, \hat{t}\,) \in Q_T$ and $\varphi \in C^2(Q_T)$, and assume that $u - \varphi$ attains a strict maximum at $(\hat{x}, \hat{t}\,)$. By the definition of u, there are sequences $\{n_k\} \subset \mathbb{N}$, diverging to infinity, and $\{(x_k, t_k)\} \subset B((\hat{x}, \hat{t}\,), 2r)$, where $r > 0$ is chosen so that $B((\hat{x}, \hat{t}\,), 2r) \subset Q_T$, such that $u_{n_k}(x_k, t_k) \to u(\hat{x}, \hat{t}\,)$ and $(x_k, t_k) \to (\hat{x}, \hat{t}\,)$ as $k \to \infty$.

Set $v_k = u_{n_k}$ for $k \in \mathbb{N}$. For any $k \in \mathbb{N}$ let $(y_k, s_k) \in B((\hat{x}, \hat{t}\,), 2r)$ be a maximum point, over $B((\hat{x}, \hat{t}\,), 2r)$, of $v_k^* - \varphi$. We observe that

$$(u - \varphi)(\hat{x}, \hat{t}\,) = \lim_{k\to\infty} (v_k - \varphi)(x_k, t_k) \le \liminf_{k\to\infty} (v_k^* - \varphi)(y_k, s_k). \tag{3.4}$$

Let $(x,t) \in B((\hat{x}, \hat{t}\,), r)$ be an accumulation point of the sequence $\{(y_k, s_k)\}$ and let $\{(y_{k_j}, s_{k_j})\}$ be one of its subsequences converging to (x,t). By the definition of u, we see that

$$\limsup_{j\to\infty} (v_{k_j}^* - \varphi)(y_{k_j}, s_{k_j}) = \limsup_{k\to\infty} v_{k_j}^*(y_{k_j}, s_{k_j}) - \varphi(x,t) \le u(x,t) - \varphi(x,t).$$

This together with (3.4) guarantees that $(x,t) = (\hat{x}, \hat{t}\,)$. That is, the sequence $\{(y_k, s_k)\}$ converges to $(\hat{x}, \hat{t}\,)$. Again, by the definition of u, we see that

$$\limsup_{k\to\infty} (v_k^* - \varphi)(y_k, s_k) \le (u - \varphi)(\hat{x}, \hat{t}\,).$$

It is now clear that $v_k^*(y_k, s_k) \to u(\hat{x}, \hat{t}\,)$ as $k \to \infty$.

The rest of the proof parallels the argument in the proof of Theorem 3.2 where it is divided into two cases, and we omit here the details. The proof is complete. $\square$

To formulate the Perron method, we fix $p \in \mathbb{R}^N$ and let $f \in \mathcal{S}_p^-(Q_T) \cap \mathrm{LSC}(Q_T)$ and $g \in \mathcal{S}_p^+(Q_T) \cap \mathrm{USC}(Q_T)$. Assume that $f \le g$ in Q_T. Set

$$u(x,t) = \sup\big\{v(x,t) \mid v \in \mathcal{S}_p^-(Q_T),\ f \le v \le g \text{ in } Q_T\big\}. \tag{3.5}$$

Note that $u \in \mathcal{B}(Q_T)$.

Theorem 3.4. *The function u given by* (3.5) *is a solution of* (1.1).

Proof. First of all, we note by Theorem 3.3 that $u^* \in \mathcal{S}^-$.

We next show that $u_* \in \mathcal{S}^+$. Let $(\hat{x}, \hat{t}\,) \in Q_T$ and $\varphi \in C^2(Q_T)$. Assume that $u_* - \varphi$ attains a strict minimum at $(\hat{x}, \hat{t}\,)$, with minimum value zero. We need to show that the inequality

$$\begin{aligned}&\varphi_t(\hat{x}, \hat{t}\,)\\ &\ge \begin{cases} (c(\hat{x}, \hat{t}\,) + M_p^-[u_*(\cdot, \hat{t}\,)](\hat{x}))|p + D\varphi(\hat{x}, \hat{t}\,)| & \text{if } p + D\varphi(\hat{x}, \hat{t}\,) \ne 0 \\ 0 & \text{if } p + D\varphi(\hat{x}, \hat{t}\,) = 0 \end{cases}\end{aligned} \tag{3.6}$$

holds.

It is clear by the definition of u that $f \le u \le g$ in Q_T. Consequently we have $f \le u_* \le g_*$ in Q_T. Consider the case where $u_*(\hat{x}, \hat{t}\,) = g_*(\hat{x}, \hat{t}\,)$. Then, since $u_* \le g_*$ in Q_T, it follows that $g_* - \varphi$ attains a minimum at $(\hat{x}, \hat{t}\,)$. By the viscosity property of g, we have

$$\begin{aligned}&\varphi_t(\hat{x}, \hat{t}\,)\\ &\ge \begin{cases} \big(c(\hat{x}, \hat{t}\,) + M_p^-[g_*(\cdot, \hat{t}\,)](\hat{x})\big)|p + D\varphi(\hat{x}, \hat{t}\,)| & \text{if } p + D\varphi(\hat{x}, \hat{t}\,) \ne 0 \\ 0 & \text{if } p + D\varphi(\hat{x}, \hat{t}\,) = 0. \end{cases}\end{aligned} \tag{3.7}$$

But, since $g_* \ge u_*$ and $g(\hat{x}, \hat{t}\,) = u_*(\hat{x}, \hat{t}\,)$, we see that if $p + D\varphi(\hat{x}, \hat{t}\,) \ne 0$, then

$$M_p^-[g_*(\cdot, \hat{t}\,)](\hat{x}) \ge M_p^-[u_*(\cdot, \hat{t}\,)](\hat{x}),$$

from which together with (3.7) we conclude that (3.6) holds.

Next we assume that $u_*(\hat{x}, \hat{t}\,) < g_*(\hat{x}, \hat{t}\,)$. We find by the semicontinuity of g_* that $g_*(x,t) > \varphi(x,t) + \varepsilon$ in a compact neighborhood W $(\subset Q_T)$ of $(\hat{x}, \hat{t}\,)$ for some constant $\varepsilon \in (0\,1)$. Furthermore, we may assume by modifying φ except near the point $(\hat{x}, \hat{t}\,)$, if necessary, that $u_*(x,t) > \varphi(x,t) + 1$ for all $(x,t) \in Q_T \setminus W$.

Define $u_n = u \vee \left(\varphi + \frac{1}{n}\right)$ in Q_T. Note that $(u_n)_*(\hat{x}, \hat{t}\,) = \varphi(\hat{x}, \hat{t}\,) + 1/n > u_*(\hat{x}, \hat{t}\,)$ and therefore $u_n \not\leq u$. Since $\varphi + \varepsilon \leq g$ in Q_T, we see that $f \leq u_n \leq g$ for sufficiently large n, say, $n \geq k$, for some $k \in \mathbb{N}$.

In what follows we are concerned only with u_n, with $n \in \mathbb{N}$ satisfying $n \geq k$. Since $u_n \not\leq u$, by the definition (maximality) of u, we find that $(u_n)^* \notin \mathcal{S}^-$. Thus, for each n there are a point $(x_n, t_n) \in Q_T$ and a function $\psi_n \in C^2(Q_T)$ such that (x_n, t_n) is a maximum point of $u_n^* - \psi_n$ and the inequality

$$a_n > \begin{cases} (c(x_n, t_n) + M_p^+[u_n^*(\cdot, t_n)](x_n))|p + q_n| & \text{if } p + q_n \neq 0 \\ 0 & \text{if } p + q_n = 0 \end{cases} \tag{3.8}$$

holds. Here and later we write $a_n = \psi_{n,t}(x_n, t_n)$ and $q_n = D\psi_n(x_n, t_n)$.

Set $\varphi_n(x, t) = \varphi(x, t) + \frac{1}{n}$ for $(x, t) \in Q_T$ and

$$V_n := \{(x, t) \in Q_T \mid \varphi_n(x, t) > u^*(x, t)\}.$$

Note that V_n is an open subset of Q_T and $u_n = \varphi_n$ on V_n.

We claim that $(x_n, t_n) \in V_n$. Indeed, if this were not the case, then we would have $\varphi_n(x_n, t_n) \leq u^*(x_n, t_n)$, and therefore

$$(u_n)^*(x_n, t_n) = u^*(x_n, t_n) \ \vee \ \varphi_n(x_n, t_n) = u^*(x_n, t_n).$$

Now, since $u_n^* \geq u^*$ in Q_T, we see that (x_n, t_n) is a maximum point of $u^* - \psi_n$. Hence we have

$$a_n \leq \begin{cases} (c(x_n, t_n) + M_p^+[u^*(\cdot, t_n)](x_n))|p + q_n| & \text{if } p + q_n \neq 0 \\ 0 & \text{if } p + q_n = 0. \end{cases} \tag{3.9}$$

Since $(u_n)^*(x_n, t_n) = u^*(x_n, t_n)$ and $u_n^* \geq u^*$ in Q_T, we find that

$$M_p^+[u^*(\cdot, t_n)](x_n) \leq M_p^+[u_n^*(\cdot, t_n)](x_n).$$

From this and (3.9) we obtain

$$a_n \leq \begin{cases} (c(x_n, t_n) + M_p^+[u_n^*(\cdot, t_n)](x_n))|p + q_n| & \text{if } p + q_n \neq 0 \\ 0 & \text{if } p + q_n = 0, \end{cases}$$

which contradicts (3.8). Thus we conclude that $(x_n, t_n) \in V_n$.

As noted above, V_n is an open subset of Q_T and $u_n = \varphi_n$ on V_n. Therefore, we have $a_n = \varphi_t(x_n, t_n)$ and $q_n = D\varphi(x_n, t_n)$. Noting that if $p + q_n \neq 0$, then $M_p^+[u_n^*(\cdot, t_n)](x_n) \geq M_p^-[(u_n)_*(\cdot, t_n)](x_n)$, from (3.8) we get

$$a_n > \begin{cases} (c(x_n, t_n) + M_p^-[(u_n)_*(\cdot, t_n)](x_n))|p + q_n| & \text{if } p + q_n \neq 0 \\ 0 & \text{if } p + q_n = 0. \end{cases} \tag{3.10}$$

Since $(x_n, t_n) \in V_n$, we have $(u_n)_*(x_n, t_n) = \varphi_n(x_n, t_n)$, while $(u_n)_* \geq \varphi_n$ on Q_T by the definition of u_n. Therefore, $(u_n)_* - \varphi_n$ attains a minimum at (x_n, t_n) with minimum value 0. Since $(u_n)_* - \varphi_n \geq u_* - \varphi - \frac{1}{n} > 0$ outside the set $W \subset Q_T$, we find that $(x_n, t_n) \in W$. Recall that $u_* \geq \varphi$ and hence $u_* \leq (u_n)_* \leq u_* + \frac{1}{n}$ in Q_T. From this we see that $(u_n)_*(x, t) \to u_*(x, t)$ uniformly for $(x, t) \in Q_T$ as $n \to \infty$. Since $(\hat{x}, \hat{t}\,)$ is a strict minimum of $u_* - \varphi$, we easily deduce that $(x_n, t_n) \to (\hat{x}, \hat{t}\,)$ as $n \to \infty$.

We now divide our consideration into the following two cases.

Case 1: $\#\{n \mid p + q_n = 0\} < \infty$. We may assume by replacing k by a larger integer if necessary that $p + q_n \neq 0$ for all n. Using the facts that $(u_n)_* - \varphi$ attains a minimum at (x_n, t_n), $(x_n, t_n) \to (\hat{x}, \hat{t}\,)$ as $n \to \infty$ and $(u_n)_* \to u_*$ uniformly on Q_T as $n \to \infty$, we apply Lemmas 2.4 and 3.1 if $|p + D\varphi(\hat{x}, \hat{t}\,)| > 0$ or Lemma 2.5 otherwise, to obtain

$$\begin{aligned} &\liminf_{n\to\infty} M_p^-[(u_n)_*(\cdot, t_n)](x_n)\,|p + q_n| \\ &\geq \begin{cases} M_p^-[u_*(\cdot, \hat{t}\,)](\hat{x})\,|p + D\varphi(\hat{x}, \hat{t}\,)| & \text{if } p + D\varphi(\hat{x}, \hat{t}\,) \neq 0 \\ 0 & \text{if } p + D\varphi(\hat{x}, \hat{t}\,) = 0. \end{cases} \end{aligned}$$

Combining this and (3.10), we conclude that (3.6) is valid.

Case 2: $\#\{n \mid p+q_n = 0\} = \infty$. We may choose a sequence $\{n_j\} \subset \mathbb{N}$ diverging to infinity so that $p + q_{n_j} = 0$ for all $j \in \mathbb{N}$. An immediate consequence is that $p + D\varphi(\hat{x}, \hat{t}\,) = 0$. From (3.10), we have $a_{n_j} = \varphi_t(x_{n_j}, t_{n_j}) > 0$ for all j. Sending $j \to \infty$, we obtain $\varphi_t(\hat{x}, \hat{t}\,) \geq 0$, which shows that (3.6) is valid. The proof is complete. □

4. Comparison theorems

Throughout this section we let $p \in \mathbb{R}^N$ be an arbitrary vector.

Theorem 4.1. *Let* $0 < T < \infty$. *Let* u *and* v *be solutions of* (1.2) *and of* (1.3), *respectively. Assume that* u *and* $-v$ *are upper semicontinuous and bounded on* $\mathbb{R}^N \times [0, T)$ *and that*

$$\lim_{r\to 0+} \sup\bigl\{u(x,t) - v(y,s) \mid (x,t), (y,s) \in \mathbb{R}^N \times [0,T),\, |x - y| \vee t \vee s \leq r\bigr\} \leq 0. \tag{4.1}$$

Then $u \leq v$ *on* $\mathbb{R}^N \times [0, T)$.

We show first the following theorem and then apply it to prove the theorem above.

Theorem 4.2. *Assume in addition to the hypotheses of Theorem* 4.1 *that* u *and* v *are defined on* $\overline{Q}_T$*, that* u *and* $-v$ *are bounded and upper semicontinuous on* $\overline{Q}_T$ *and that* $u(x,t)$*,* $-v(x,t)$ *are semi-convex in* x *on* $\mathbb{R}^N$ *uniformly in* $t \in [0, T]$*, that is, there exists a constant* $C_1 > 0$ *such that for any* $t \in [0, T]$ *the functions*

$$u(x,t) + C_1|x|^2 \quad \text{and} \quad -v(x,t) + C_1|x|^2$$

are convex in x *on* $\mathbb{R}^N$*. Then* $u \le v$ *on* $\mathbb{R}^N \times [0, T)$*.*

Proof. We suppose that $\sup_{\mathbb{R}^N\times[0,T)}(u - v) > 0$ and will get a contradiction.

Fix a constant $C_2 > 0$ so that $|u(x,t)| \vee |v(x,t)| \le C_2$ for all $(x,t) \in \overline{Q}_T$. Let $\varepsilon > 0$ and set

$$u_\varepsilon(x,t) = u(x,t) - \frac{\varepsilon}{T + \varepsilon^2 - t} \quad \text{for } (x,t) \in \overline{Q}_T.$$

Observe that u_ε is a subsolution of

$$u_t + \frac{\varepsilon}{(T + \varepsilon^2)^2} = (c + M_p[u(\cdot,t)](x))|p + Du| \quad \text{in } Q_T,$$

and that if $\varepsilon > 0$ is sufficiently small, then

$$\begin{cases} \sup\limits_{\overline{Q}_T}(u_\varepsilon - v) > 0 \\ u_\varepsilon(x,t) - v(y,s) \le 2C_2 - \dfrac{1}{2\varepsilon} < 0 \quad \text{for all } (x,t), (y,s) \in \mathbb{R}^N \times [T - \varepsilon^2, T]. \end{cases}$$

We fix such a small $\varepsilon > 0$ and we write u for u_ε in what follows. We fix a $\gamma > 0$ so that $\gamma \le \frac{\varepsilon}{(T+\varepsilon^2)^2}$ and that for all $(x,t,y,s) \in \overline{Q}_T^{\,2}$, if

$$\text{either} \quad |x - y| \vee t \vee s \le \gamma \quad \text{or} \quad |x - y| \vee (T - t) \vee (T - s) \le \gamma,$$

then

$$u(x,t) - v(y,s) < 0. \tag{4.2}$$

Note that u is a solution of

$$u_t + \gamma \le (c + M_p[u(\cdot,t)](x))|p + Du| \quad \text{in } Q_T, \tag{4.3}$$

We set

$$\tilde{u}(x,t) = u(x,t) + p \cdot x \quad \text{and} \quad \tilde{v}(x,t) = v(x,t) + p \cdot x \quad \text{for } (x,t) \in \overline{Q}_T.$$

In view of (4.2), replacing $\gamma > 0$ by a smaller number if necessary, we may assume that for any $(x,t,y,s) \in \overline{Q}_T^{\,2}$, if

$$\text{either} \quad |x - y| \vee t \vee s \le \gamma \quad \text{or} \quad |x - y| \vee (T - t) \vee (T - s) \le \gamma,$$

then

$$\tilde{u}(x,t) - \tilde{v}(y,s) < 0. \tag{4.4}$$

Let $\alpha > 1$ be a large constant to be specified later on. We define the function $\Phi = \Phi_\alpha$ on $\overline{Q}_T^{\,2}$ by

$$\Phi(x,t,y,s) = \tilde{u}(x,t) - \tilde{v}(y,s) - \alpha|x-y|^2 - \alpha|t-s|^2.$$

We set $\theta = \theta_\alpha := \sup_{\overline{Q}_T^{\,2}} \Phi$ and note that $\theta \geq \sup_{\overline{Q}_T}(\tilde{u}-\tilde{v}) = \sup_{\overline{Q}_T}(u-v) > 0$. Observe that if $\Phi(x,t,y,s) \geq 0$, then

$$\begin{aligned} 2C_2 &\geq -p\cdot(x-y) + \alpha|x-y|^2 + \alpha|t-s|^2 \\ &\geq -\frac{|p|^2}{2\alpha} + \frac{\alpha}{2}\left(|x-y|^2 + |t-s|^2\right) \\ &\geq -\frac{|p|^2}{2} + \frac{\alpha}{2}\left(|x-y|^2 + |t-s|^2\right). \end{aligned}$$

Fix a constant $R_0 > 0$ so that $R_0^2 \geq 4C_2 + |p|^2$, and note that for any $(x,t,y,s) \in \overline{Q}_T^{\,2}$,

$$(\sqrt{\alpha}|x-y|) \vee (\sqrt{\alpha}|t-s|) \leq R_0 \quad \text{ if } \Phi(x,t,y,s) \geq 0. \tag{4.5}$$

In particular, we have $\theta = \sup\{\Phi(x,t,y,s) \mid (x,t,y,s) \in Q_T^2, \sqrt{\alpha}|x-y| \leq R_0\}$. We denote by $\mathcal{R}_\alpha$ the set of those $r \geq 0$ which satisfy

$$\theta = \sup\left\{\Phi(x,t,y,s) \mid (x,t,y,s) \in \overline{Q}_T^{\,2},\ \alpha|x-y| \leq r\right\},$$

and set $\lambda_\alpha = \inf \mathcal{R}_\alpha$. Since $\sqrt{\alpha}R_0 \in \mathcal{R}_\alpha$, we have $0 \leq \lambda_\alpha \leq \sqrt{\alpha}R_0$. Observe that if $\lambda_\alpha > 0$ and $\lambda_\alpha > r \geq 0$, then

$$\theta > \sup\left\{\Phi(x,t,y,s) \mid (x,t,y,s) \in \overline{Q}_T^{\,2},\ \alpha|x-y| \leq r\right\}$$

and that if $r > \lambda_\alpha$, then

$$\theta = \sup\left\{\Phi_\alpha(x,t,y,s) \mid (x,t,y,s) \in \overline{Q}_T^{\,2},\ \alpha|x-y| \leq r\right\}.$$

We divide our consideration into two cases.

`Case` 1: $\liminf_{\alpha\to\infty} \lambda_\alpha = 0$. Let $\eta > 0$ be a constant to be fixed later. We choose an $\alpha > 1$ so that $\lambda_\alpha < \eta$. By the definition of λ_α, there is a sequence $\{(x_n,t_n,y_n,s_n)\} \subset \overline{Q}_T^{\,2}$ such that $\Phi(x_n,t_n,y_n,s_n) > \theta\big(1-\frac{1}{n}\big)$ and $\alpha|x_n-y_n| \leq \eta$. Since $\Phi(x_n,t_n,y_n,s_n) \geq 0$, by (4.5) we have $|x_n-y_n| \vee |t_n-s_n| \leq \frac{R_0}{\sqrt{\alpha}}$. We may assume, by selecting α large enough if needed, that $\frac{R_0}{\sqrt{\alpha}} < \frac{\gamma}{2}$. By (4.4), we see that

$$t_n, s_n \in \left(\frac{\gamma}{2}, T - \frac{\gamma}{2}\right) \quad \text{ for all } n \in \mathbb{N}. \tag{4.6}$$

By taking a subsequence if necessary, we may assume that $(t_n, s_n) \to (\hat{t}, \hat{s})$ for some $\hat{t}, \hat{s} \in [\frac{\gamma}{2}, T - \frac{\gamma}{2}]$ as $n \to \infty$. We choose a maximum point (ξ_n, τ_n) of the function

$$(x,t) \mapsto \tilde{u}(x,t) - 2\alpha|x - y_n|^2 - \alpha|t - s_n|^2 - \alpha|t - \hat{t}\,|^2 \quad \text{on } \overline{Q}_T.$$

Note that such a maximum point exists since the function above goes to $-\infty$ as $|x| \to \infty$ uniformly for $t \in [0, T]$. We have

$$\begin{aligned}
&\Phi(x_n, t_n, y_n, s_n) - \alpha\left(|x_n - y_n|^2 + |t_n - \hat{t}\,|^2\right) \\
&\le \Phi(\xi_n, \tau_n, y_n, s_n) - \alpha\left(|\xi_n - y_n|^2 + |\tau_n - \hat{t}\,|^2\right) \\
&\le \theta - \alpha\left(|\xi_n - y_n|^2 + |\tau_n - \hat{t}\,|^2\right).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\alpha\left(|\xi_n - y_n|^2 + |\tau_n - \hat{t}\,|^2\right) &\le \theta - \Phi(x_n, t_n, y_n, s_n) + \alpha\left(|x_n - y_n|^2 + |t_n - \hat{t}\,|^2\right) \\
\Phi(x_n, t_n, y_n, s_n) &\le \Phi(\xi_n, \tau_n, y_n, s_n) + \alpha\left(|x_n - y_n|^2 + |t_n - \hat{t}\,|^2\right),
\end{aligned}$$

and consequently

$$\limsup_{n\to\infty} \alpha(|\xi_n - y_n|^2 + |\tau_n - \hat{t}\,|^2) \le \frac{\eta^2}{\alpha}, \quad \liminf_{n\to\infty} \Phi(\xi_n, \tau_n, y_n, s_n) \ge \theta - \frac{\eta^2}{\alpha}.$$

Reselecting α large enough if necessary, we may assume that $\frac{\eta^2}{\alpha} < \frac{\theta}{2}$, so that $\liminf_{n\to\infty} \Phi(\xi_n, \tau_n, y_n, s_n) > \frac{\theta}{2}$. We may choose an $n_0 \in \mathbb{N}$ so that if $n \ge n_0$, then

$$\alpha(|\xi_n - y_n|^2 + |\tau_n - \hat{t}\,|^2) < \frac{4\eta^2}{\alpha} \quad \text{and} \quad \Phi(\xi_n, \tau_n, y_n, s_n) > \frac{\theta}{2}.$$

In what follows we are concerned only with those $n \in \mathbb{N}$ which satisfy $n \ge n_0$. Note that $\alpha|\xi_n - y_n| < 2\eta$ and $\alpha|\tau_n - \hat{t}\,| < 2\eta$.

Once again, reselecting α large enough if needed, we may assume that $2\frac{\eta}{\alpha} \le \frac{\gamma}{2}$, and we have $0 < \tau_n < T$ by (4.6). Now, setting

$$\varphi(x,t) = -p \cdot x + 2\alpha|x - y_n|^2 + \alpha|t - s_n|^2 + \alpha|t - \hat{t}\,|^2 \quad \text{for } (x,t) \in \overline{Q}_T$$

and noting that u is a solution of (4.3) in Q_T, we get

$$\varphi_t(\xi_n, \tau_n) + \gamma \le \left(c(\xi_n, \tau_n) + M_p^+[u(\cdot, \tau_n)](\xi_n)\right)|p + D\varphi(\xi_n, \tau_n)| \tag{4.7}$$

if $D\varphi(\xi_n, \tau_n) \ne 0$, and otherwise

$$\varphi_t(\xi_n, \tau_n) + \gamma \le 0. \tag{4.8}$$

Note that for any $z \in \mathbb{R}^N$, $(u-\varphi)(\xi_n+z,\tau_n) \le (u-\varphi)(\xi_n,\tau_n)$ and hence

$$u(\xi_n+z,\tau_n) - u(\xi_n,\tau_n) \le (-p+4\alpha(\xi_n-y_n))\cdot z + 2\alpha|z|^2.$$

By Lemma 2.5, there is a modulus ω, independent of n, such that

$$M_p^+[u(\cdot,\tau_n)](\xi_n)\,|p+D\varphi(\xi_n,\tau_n)| \le \omega(4\alpha|\xi_n-y_n|) \quad \text{if } p+D\varphi(\xi_n,\tau_n)\ne 0.$$

This together with (4.7) and (4.8) yields $\varphi_t(\xi_n,\tau_n)+\gamma \le \omega(4\alpha|\xi_n-y_n|)$. Hence,

$$\gamma \le \omega(8\eta) - 2\alpha(\hat{t}-s_n) - 4\alpha(\tau_n-\hat{t}\,) \le \omega(8\eta)+8\eta-2\alpha(\hat{t}-s_n).$$

Sending $n\to\infty$, we get

$$\gamma \le \omega(8\eta)+8\eta+2\alpha(\hat{s}-\hat{t}\,). \tag{4.9}$$

Choosing a minimum point of the function

$$(y,s)\mapsto \tilde{v}(y,s)+2\alpha|x_n-y|^2+\alpha|t_n-s|^2+\alpha|s-\hat{s}|^2 \quad \text{on } \overline{Q}_T$$

and repeating an argument similar to the above, we get $0 \ge -\omega(8\eta)-8\eta+2\alpha(\hat{s}-\hat{t}\,)$. Subtracting this from (4.9), we obtain $\gamma \le 2\omega(8\eta)+16\eta$, which gives a contradiction by selecting $\eta>0$ small enough.

`Case` 2: $\liminf_{\alpha\to\infty}\lambda_\alpha > 0$. By the semi-convexity and boundedness assumptions on u and $-v$, we find a constant $L>0$ (see Proposition A.1 in Appendix) such that for all $x,y\in\mathbb{R}^N$ and $t\in[0,T]$,

$$|\tilde{u}(x,t)-\tilde{u}(y,t)| \vee |\tilde{v}(x,t)-\tilde{v}(y,t)| \le L|x-y|. \tag{4.10}$$

Also, by the semi-convexity of u in the variable x , we have

$$u(x+z,t)-u(x,t) \ge q\cdot z - C_1|z|^2 \quad \text{for all } (q,z)\in D_1^-u(x,t)\times\mathbb{R}^N$$

and for all $(x,t)\in\overline{Q}_T$, where $D_1^-u(x,t)$ denotes the subdifferential of the function $u(\cdot,t)$ at x. Similarly, we have

$$v(x+z,t)-v(x,t) \le q\cdot z + C_1|z|^2 \quad \text{for all } (q,z)\in D_1^+v(x,t)\times\mathbb{R}^N \tag{4.11}$$

and for all $(x,t)\in\overline{Q}_T$, where $D_1^+v(x,t)$ denotes the superdifferential of the function $v(\cdot,t)$ at x. Here we note also by the semi-convexity assumption on u and $-v$ that $D_1^-u(x,t)\ne\emptyset$ and $D_1^+v(x,t)\ne\emptyset$ for all $(x,t)\in\overline{Q}_T$.

Now, we choose a constant $\lambda_0>0$ so that $\liminf_{\alpha\to\infty}\lambda_\alpha>\lambda_0$ and also a constant $\alpha_0>1$ so that $\lambda_\alpha>\lambda_0$ for all $\alpha>\alpha_0$.

Let $\delta\in(0,1)$. We consider the function $\Psi=\Psi_{\alpha,\delta}$ on $\overline{Q}_T^{\,2}$ by

$$\Psi(x,t,y,s) = \Phi_\alpha(x,t,y,s) - \delta|x|^2.$$

For each $\alpha > 1$ we may choose $\delta_\alpha \in (0,1)$ so that $\sup_{\overline{Q}_T^2} \Psi_{\alpha,\delta} > 0$ for any $0 < \delta < \delta_\alpha$. It is clear that the function Ψ attains a maximum at some point of $\overline{Q}_T^2$. For each $\delta \in (0,\, \delta_\alpha)$ we fix such a maximum point $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$ which, of course, depends on α and δ. Noting that $\Phi \geq \Psi$, we see from (4.6) and (4.5) as before that if $0 < \delta < \delta_\alpha$ and $\frac{R_0}{\sqrt{\alpha}} \leq \frac{\gamma}{2}$, then $\hat{t}, \hat{s} \in (0, T)$. Replacing α_0 by a larger number if necessary, we may assume that $\frac{R_0}{\sqrt{\alpha}} \leq \frac{\gamma}{2}$ for $\alpha > \alpha_0$. Henceforth we deal only with those α and δ satisfying $\alpha > \alpha_0$ and $0 < \delta < \delta_\alpha$, so that $\lambda_\alpha > \lambda_0$ and $\hat{t},\, \hat{s} \in (0,\, T)$.

We may assume by replacing $\delta_\alpha \in (0,1)$ by a smaller number if necessary that $\alpha|\hat{x} - \hat{y}| > \lambda_0$. Indeed, if this were not the case, we could choose a sequence $\{\delta_j\} \subset (0,1)$ converging to zero such that $\alpha|x_j - y_j| \leq \lambda_0$ for all $j \in \mathbb{N}$, where (x_j, t_j, y_j, s_j) denotes the point $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$ corresponding to $\delta = \delta_j$. Observe that $\Phi \geq \Psi$ in $\overline{Q}_T^2$, which implies that $\sup_{\overline{Q}_T^2} \Psi_{\alpha,\delta_j} \leq \theta$. On the other hand, for any fixed $\eta > 0$, if we choose a point $(\bar{x}, \bar{t}, \bar{y}, \bar{s}) \in \overline{Q}_T^2$ so that $\Phi(\bar{x},\, \bar{t},\, \bar{y},\, \bar{s}) > \theta - \eta$, then we get

$$\sup_{\overline{Q}_T^2} \Psi_{\alpha,\delta_j} > \theta - \eta - \delta_j|\bar{x}|^2 \to \theta - \eta \quad \text{ as } j \to \infty.$$

These observations together yield $\lim_{j\to\infty} \sup_{\overline{Q}_T^2} \Psi_{\alpha,\delta_j} = \theta$. Hence, we have

$$\liminf_{j\to\infty} \Phi(x_j, t_j, y_j, s_j) \geq \lim_{j\to\infty} \sup_{\overline{Q}_T^2} \Psi_{\alpha,\delta_j} = \theta,$$

which implies that $\lambda_\alpha \leq \lambda_0$. But, this contradicts our choice of α_0.

A limiting argument parallel to the above shows that

$$\lim_{\delta\to 0+} \delta|\hat{x}|^2 = 0. \tag{4.12}$$

We observe as usual in viscosity solutions theory that

$$\begin{aligned} (-p + 2\alpha(\hat{x} - \hat{y}) + 2\delta\hat{x},\, 2\alpha(\hat{t} - \hat{s})) \in D^+u(\hat{x},\, \hat{t}\,) \\ (-p + 2\alpha(\hat{x} - \hat{y}),\, 2\alpha(\hat{t} - \hat{s})) \in D^-v(\hat{y},\, \hat{s}), \end{aligned} \tag{4.13}$$

and furthermore, we see by the semi-convexity of u, $-v$ in the variable x that u and v are differentiable, as functions of x, at $(\hat{x},\, \hat{t}\,)$ and $(\hat{y},\, \hat{s})$, respectively. Here $D^{\pm}f(x,t)$ denotes the sub- and superdifferential of the function f at (x,t), respectively. The above inclusions together with (4.10) yield

$$|2\alpha(\hat{x} - \hat{y}) + 2\delta\hat{x}| \vee |2\alpha(\hat{x} - \hat{y})| \leq L. \tag{4.14}$$

Next, by the inequality $\Psi(\hat{x} + z,\, \hat{t},\, \hat{y} + z,\, \hat{s}) \leq \Psi(\hat{x},\, \hat{t},\, \hat{y},\, \hat{s})$ for $z \in \mathbb{R}^N$, we find that

$$u(\hat{x} + z,\, \hat{t}\,) - u(\hat{x},\, \hat{t}\,) \leq v(\hat{y} + z,\, \hat{s}) - v(\hat{y},\, \hat{s}) + 2\delta\hat{x} \cdot z + \delta|z|^2 \quad \text{ for all } z \in \mathbb{R}^N.$$

Combining this with (4.11) and (4.13), we get

$$u(\hat{x}+z,\,\hat{t}\,)-u(\hat{x},\,\hat{t}\,)\leq q\cdot z+(\delta+C_1)|z|^2 \quad \text{for all } z\in\mathbb{R}^N, \tag{4.15}$$

where $q:=-p+2\alpha(\hat{x}-\hat{y})+2\delta\hat{x}$. Similarly, we get

$$v(\hat{y}+z,\,\hat{s})-v(\hat{y},\hat{s})\geq q\cdot z-(\delta+C_1)|z|^2 \quad \text{for all } z\in\mathbb{R}^N, \tag{4.16}$$

where $q:=-p+2\alpha(\hat{x}-\hat{y})$.

In view of (4.12), we may assume by replacing δ_α by a smaller number if needed that $2\delta|\hat{x}|\leq\lambda_0$. Since $\alpha|\hat{x}-\hat{y}|>\lambda_0$, we have

$$(\alpha|\hat{x}-\hat{y}|)\wedge|\alpha(\hat{x}-\hat{y})+\delta\hat{x}|\geq\frac{\lambda_0}{2}.$$

This together with (4.14) yields

$$\lambda_0\leq(2\alpha|\hat{x}-\hat{y}|)\wedge|2\alpha(\hat{x}-\hat{y})+2\delta\hat{x}|\leq L. \tag{4.17}$$

Hence, using (4.15) and (4.16), we deduce by Lemma 2.4 that there are constants $0<\rho_0<1$ and $C_3>0$, independent of our choice of α and δ, such that for any $0<\rho\leq\rho_0$,

$$M_p^+[u(\cdot,\hat{t}\,)](\hat{x})\leq C_3\rho^{N+1-\beta}+\int_{|z|>\rho}J(z)E_p^+(u(\hat{x}+z,\hat{t}\,)-u(\hat{x},\hat{t}\,),z)\mathrm{d}z \tag{4.18}$$

$$M_p^-[v(\cdot,\hat{s})](\hat{y})\geq -C_3\rho^{N+1-\beta}+\int_{|z|>\rho}J(z)E_p^-(v(\hat{y}+z,\hat{s})-v(\hat{y},\hat{s}),z)\mathrm{d}z. \tag{4.19}$$

Now, since u and v are solutions of (4.3) and of (1.3), respectively, using (4.17) again, we have

$$2\alpha(\hat{t}-\hat{s})+\gamma\leq 2\big(c(\hat{x},\hat{t}\,)+M_p^+[u(\cdot,\hat{t}\,)](\hat{x})\big)|\alpha(\hat{x}-\hat{y})+\delta\hat{x}| \tag{4.20}$$

and

$$2\alpha(\hat{t}-\hat{s})\geq 2\big(c(\hat{y},\hat{s})+M_p^-[v(\cdot,\hat{s})](\hat{y})\big)|\alpha(\hat{x}-\hat{y})|. \tag{4.21}$$

Next we note that for any $z\in\mathbb{R}^N$,

$$u(\hat{x}+z,\hat{t}\,)-v(\hat{x}+z,\hat{s})\leq u(\hat{x},\hat{t}\,)-v(\hat{y},\hat{s})+p\cdot(\hat{x}-\hat{y})+\delta|\hat{x}+z|^2-\alpha|\hat{x}-\hat{y}|^2.$$

Therefore, for any $z\in\mathbb{R}^N$, if $\delta|\hat{x}+z|^2<\alpha|\hat{x}-\hat{y}|^2$, then we have

$$u(\hat{x}+z,\hat{t}\,)-u(\hat{x},\hat{t}\,)<v(\hat{x}+z,\hat{s})-v(\hat{y},\hat{s})+p\cdot(\hat{x}-\hat{y}),$$

and moreover

$$E_p^+\big(u(\hat{x}+z,\hat{t}\,)-u(\hat{x},\hat{t}\,),z\big)\leq E_p^-\big(v(\hat{x}+z,\hat{s})-v(\hat{y},\hat{s})+p\cdot(\hat{x}-\hat{y}),z\big).$$

Now, let $0 < \rho < \rho_0$, $0 < \nu < \frac{\rho}{2}$ and $R > 1$. By virtue of (4.12), we may assume by replacing δ_α by a smaller number that $8\delta(|\hat{x}|^2 + R^2) < \lambda_0^2/\alpha$ and $\delta|\hat{x}| < \nu$. Accordingly, thanks to (4.17), we have

$$\delta|\hat{x} + z|^2 \le 2\delta(|\hat{x}|^2 + R^2) < \frac{\lambda_0^2}{4\alpha} \le \alpha|\hat{x} - \hat{y}|^2 \quad \text{for any } z \in B(0,\, R).$$

In view of (4.5), we may assume by replacing α_0 by a larger number if necessary that $|\hat{x} - \hat{y}| + |\hat{t} - \hat{s}| < \nu$. Note that δ_α and α_0 indeed depend also on R and ν and on ν, respectively. Thus, we get

$$\begin{aligned}
&\int_{|z|>\rho} J(z) E_p^+(u(\hat{x}+z,\, \hat{t}\,) - u(\hat{x},\, \hat{t}\,), z)\mathrm{d}z \\
&= \left(\int_{\rho<|z|\le R} + \int_{|z|>R}\right) J(z) E_p^+(u(\hat{x}+z,\, \hat{t}\,) - u(\hat{x},\, \hat{t}\,), z)\mathrm{d}z \\
&\le \int_{\rho<|z|\le R} J(z) E_p^-(v(\hat{x}+z, \hat{s}) - v(\hat{y}, \hat{s}) + p\cdot(\hat{x}-\hat{y}), z)\mathrm{d}z \\
&\quad + (2C_2+1)\int_{|z|>R} J(z)\mathrm{d}z \\
&\le \int_{\rho<|z|\le R} J(z) E_p^-(v(\hat{x}+z, \hat{s}) - v(\hat{y}, \hat{s}),\, \hat{x}-\hat{y}+z)\mathrm{d}z \\
&\quad + (2C_2+1)\int_{|z|>R} J(z)\mathrm{d}z + |p||\hat{x}-\hat{y}|\int_{\rho<|z|\le R} J(z)\mathrm{d}z \\
&\le \int_{\rho<|\hat{y}-\hat{x}+y|\le R} J(\hat{y}-\hat{x}+y) E_p^-(v(\hat{y}+y, \hat{s}) - v(\hat{y}, \hat{s}),\, y)\mathrm{d}y \\
&\quad + (2C_2+1)\int_{|z|>R} J(z)\mathrm{d}z + |p|\nu\int_{\rho<|z|\le R} J(z)\mathrm{d}z.
\end{aligned}$$

Setting

$$\begin{aligned}
I(y) = J(\hat{y}-\hat{x}+y), \quad f(y) = E_p^-(v(\hat{y}+y, \hat{s}) - v(\hat{y}, \hat{s}),\, y), \\
A = \{y \in \mathbb{R}^N \mid \rho < |\hat{y}-\hat{x}+y| \le R\}, \quad B = \{y \in \mathbb{R}^N \mid \rho < |y| \le R\}
\end{aligned}$$

for the moment, we observe that

$$\begin{aligned}
&\int_A J(\hat{y}-\hat{x}+y) E_p^-(v(\hat{y}+y, \hat{s}) - v(\hat{y}, \hat{s}),\, y)\mathrm{d}y \\
&- \int_B J(y) E_p^-(v(\hat{y}+y, \hat{s}) - v(\hat{y}, \hat{s}),\, y)\mathrm{d}y \\
&= \int_A I(y)f(y)\mathrm{d}y - \int_B J(y)f(y)\mathrm{d}y \\
&= \int_{A\cap B} \big(I(y) - J(y)\big) f(y)\mathrm{d}y + \int_{A\setminus B} I(y)f(y)\mathrm{d}y - \int_{B\setminus A} J(y)f(y)\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\le (2C_1+1)\int_B |I(y)-J(y)|\mathrm{d}y + (2C_2+1)\int_{(A\setminus B)\cup(B\setminus A)} (|I(y)|+|J(y)|)\mathrm{d}y\\
&\le (2C_2+1)\int_{\rho<|z|\le R} |J(\hat y-\hat x+z)-J(z)|\mathrm{d}z\\
&\quad + (2C_2+1)\biggl(\int_{\rho-\nu\le|z|\le\rho+\nu} + \int_{R-\nu\le|z|\le R+\nu}\biggr)(|J(\hat y-\hat x+z)|+|J(z)|)\mathrm{d}z.
\end{aligned}
$$

Finally, noting by (4.19) that

$$
\begin{aligned}
&\int_{\rho<|z|\le R} J(z)E_p^-(v(\hat y+z,\hat s)-v(\hat y,\hat s),z)\mathrm{d}z\\
&\le \int_{|z|>\rho} J(z)E_p^-(v(\hat y+z,\hat s)-v(\hat y,\hat s),z)\mathrm{d}z + (2C_2+1)\int_{|z|>R} J(z)\mathrm{d}z\\
&\le M_p^-[v(\cdot,\hat s)](\hat y) + C_3\rho^{N+1-\beta} + (2C_2+1)\int_{|z|>R} J(z)\mathrm{d}z
\end{aligned}
$$

and using (4.18), we obtain

$$
M_p^+[u(\cdot,\hat t\,)](\hat x) \le 2C_3\rho^{N+1-\beta} + M_p^-[v(\cdot,\hat s)](\hat y) + e(\rho,\nu,R), \tag{4.22}
$$

where

$$
\begin{aligned}
e(\rho,\nu,R) := (2C_2+1)\biggl\{&2\int_{|z|>R} J(z)\mathrm{d}z + \sup_{h\in B(0,\nu)}\int_{\rho<|z|\le R} |J(h+z)-J(z)|\mathrm{d}z\\
&+ \sup_{h\in B(0,\nu)}\biggl(\int_{\rho-\nu\le|z|\le\rho+\nu} + \int_{R-\nu\le|z|\le R+\nu}\biggr)\bigl(|J(h+z)|+|J(z)|\bigr)\mathrm{d}z\biggr\}\\
&+ |p|\nu\int_{\rho<|z|\le R} J(z)\mathrm{d}z.
\end{aligned}
$$

Note that $\lim_{R\to\infty}\lim_{\nu\to 0+} e(\rho,\nu,R)=0$ for fixed $\rho>0$.

Subtracting (4.21) from (4.20), we get

$$
\gamma \le 2\bigl(c(\hat x,\hat t\,)+M_p^+[u(\cdot,\hat t\,)](\hat x)\bigr)|\alpha(\hat x-\hat y)+\delta\hat x| - 2\bigl(c(\hat y,\hat s)+M_p^-[v(\cdot,\hat s)](\hat y)\bigr)|\alpha(\hat x-\hat y)|.
$$

Hence, using (4.22) and (4.17) and recalling that $(|\hat x-\hat y|+|\hat t-\hat s|)\vee(\delta|\hat x|)<\nu$, we obtain

$$
\begin{aligned}
\frac{\gamma}{2} &\le \bigl\{|c(\hat x,\hat t\,)-c(\hat y,\hat s)| + M_p^+[u(\cdot,\,\hat t\,)](\hat x) - M_p^-[v(\cdot,\hat s)](\hat y)\bigr\}|\alpha(\hat x-\hat y)|\\
&\quad + \bigl(|c(\hat x,\,\hat t\,)| + M_p^+[u(\cdot,\,\hat t\,)](\hat x)\bigr)\delta|\hat x|\\
&\le \bigl(\omega_c(|\hat x-\hat y|+|\hat t-\hat s|) + 2C_3\rho^{N+1-\beta} + e(\rho,\nu,R)\bigr)L\\
&\quad + \biggl(\|c\|_\infty + C_3\rho^{N+1-\beta} + (2C_2+1)\int_{|z|>\rho} J(z)\mathrm{d}z\biggr)\nu\\
&\le \bigl(\omega_c(\nu) + 2C_3\rho^{N+1-\beta} + e(\rho,\nu,R)\bigr)L\\
&\quad + \biggl(\|c\|_\infty + C_3\rho^{N+1-\beta} + (2C_2+1)\int_{|z|>\rho} J(z)\mathrm{d}z\biggr)\nu,
\end{aligned} \tag{4.23}
$$

where ω_c denotes the modulus of continuity of the function c.

We now fix $\rho \in (0, \rho_0]$ so that $2C_3 L\rho^{N+1-\beta} < \frac{\gamma}{4}$, then R large enough so that

$$\lim_{\nu\to 0+} e(\rho,\nu,R)L < \frac{\gamma}{8},$$

and finally ν small enough so that

$$\begin{cases} e(\rho,\nu,R)L < \dfrac{\gamma}{8} \\ \omega_c(\nu)L + \Big(\|c\|_\infty + C_3\rho^{N+1-\beta} + (2C_2+1)\displaystyle\int_{|z|>\rho} J(z)\mathrm{d}z\Big)\nu < \dfrac{\gamma}{8}, \end{cases}$$

to conclude from (4.23) that $\gamma < 0$, which is a contradiction. The proof is complete. $\square$

Remark 4.3. One of main difficulties arises in the proof of Theorem 4.2 due to the discontinuity of the function E. For this, we used an idea from [6, Theorem 5.2] and [8, Theorem 4.4].

Proof of Theorem 4.1. Define the sup- and infconvolutions of u and v as follows:

$$\begin{aligned} u^\varepsilon(x,t) &:= \sup_{y\in\mathbb{R}^N}\left(u(y,t) - p\cdot(x-y) - \frac{|x-y|^2\mathrm{e}^{Kt}}{2\varepsilon}\right) \\ v_\varepsilon(x,t) &:= \inf_{y\in\mathbb{R}^N}\left(v(y,t) - p\cdot(x-y) + \frac{|x-y|^2\mathrm{e}^{Kt}}{2\varepsilon}\right), \end{aligned}$$

where $0<\varepsilon<1$ and $K := 2\|Dc\|_{L^\infty(Q_T)}$. It is a standard exercise to check that u^ε and v_ε are solutions of (1.2) and of (1.3), respectively. Moreover, we have:

$$\begin{gathered} \|u^\varepsilon\|_\infty \le \|u\|_\infty + \frac{|p|^2}{2}, \quad \|v_\varepsilon\|_\infty \le \|v\|_\infty + \frac{|p|^2}{2}, \\ u^\varepsilon(x,t) \searrow u(x,t) \quad \text{and} \quad v_\varepsilon(x,t) \nearrow v(x,t) \quad \text{as } \varepsilon\to 0. \end{gathered}$$

Note that the functions

$$u^\varepsilon(x,t) + \frac{|x|^2\mathrm{e}^{Kt}}{2\varepsilon} \quad \text{and} \quad -v_\varepsilon(x,t) + \frac{|x|^2\mathrm{e}^{Kt}}{2\varepsilon}$$

are convex in x for any $t\in[0,T]$. Furthermore, we fix any $\gamma>0$ and, in view of (4.1), choose a $\delta>0$ so that $u(\xi,t)-v(\eta,s)\le\gamma$ for all $\xi,\eta\in\mathbb{R}^N$, with $|\xi-\eta|\le\delta$, and $t,s\in[0,\delta]$. Then we select a constant $K_\gamma>0$, depending on γ through δ, so that $\|u\|_\infty+\|v\|_\infty\le K_\gamma\delta^2$, and observe that $u(\xi,t)-v(\eta,s)\le \gamma+K_\gamma|\xi-\eta|^2$ for all $\xi,\eta\in\mathbb{R}^N$ and $t,s\in[0,\delta]$. Using this and noting that

$$|\xi-\eta|^2 \le 3(|\xi-x|^2+|x-y|^2+|\eta-y|^2) \quad \text{for all } x,y,\xi,\eta\in\mathbb{R}^N,$$

we see that if $K_\gamma + \frac{1}{2\gamma} \le \frac{1}{6\varepsilon}$, then

$$\begin{aligned} &u^\varepsilon(x,t) - v_\varepsilon(y,s) \\ &\le \sup_{\xi,\eta\in\mathbb{R}^N} \left(\gamma + K_\gamma|\xi-\eta|^2 + |p||\xi-\eta| - \frac{|x-\xi|^2}{2\varepsilon} - \frac{|y-\eta|^2}{2\varepsilon}\right) + |p||x-y| \\ &\le \sup_{\xi,\eta\in\mathbb{R}^N} \left(\gamma + \frac{\gamma|p|^2}{2} + \left(K_\gamma + \frac{1}{2\gamma} - \frac{1}{6\varepsilon}\right)|\xi-\eta|^2\right) + |p||x-y| + \frac{|x-y|^2}{2\varepsilon} \\ &\le \gamma + \frac{\gamma|p|^2}{2} + |p||x-y| + \frac{|x-y|^2}{2\varepsilon} \qquad \text{for all } x,y\in\mathbb{R}^N,\ t,s\in[0,\ \delta]. \end{aligned}$$

Hence, if ε is sufficiently small, then we have

$$\lim_{r\to0+} \sup\left\{u^\varepsilon(x,t) - v_\varepsilon(y,s) \mid (x,t),(y,s)\in\overline{Q}_T,\, |x-y|\vee t\vee s \le r\right\} \le \gamma\left(1+\frac{|p|^2}{2}\right).$$

We apply Theorem 4.2 to the functions $u^\varepsilon(x,t) - \gamma(1+\frac{|p|^2}{2})$ and v_ε, to find that $u^\varepsilon(x,t) \le v_\varepsilon(x,t) + \gamma(1+\frac{|p|^2}{2})$ for all $(x,t)\in Q_T$. Sending $\varepsilon\to0$ and then $\gamma\to0$ guarantees that $u(x,t)\le v(x,t)$ for all $(x,t)\in Q_T$. □

The above proof is easily modified to show the following theorem.

Theorem 4.4. *Under the hypotheses of Theorem* 4.1*, there is a modulus* ω *such that*

$$u(x,t) - v(y,t) \le \omega(|x-y|) \quad \textit{for all } x,y\in\mathbb{R}^N,\ t\in[0,T).$$

Proof. Let u^ε and v_ε be the sup- and infconvolutions as in the proof of Theorem 4.1. According to the proof of Theorem 4.1, for each $\gamma>0$ we can choose an $\varepsilon=\varepsilon(\gamma)>0$ so that

$$u^\varepsilon(x,t) - v_\varepsilon(x,t) \le \gamma \quad \text{for all } (x,t)\in\mathbb{R}^N\times[0,\ T),$$

which yields

$$u(x,t) - v(y,t) \le \gamma + \frac{\mathrm{e}^{KT}}{2\varepsilon}|x-y|^2 \quad \text{for all } x,y\in\mathbb{R}^N,\ t\in[0,\ T). \tag{4.24}$$

Setting

$$\omega_0(r) = \inf_{0<\gamma<1}\left(\gamma + \frac{\mathrm{e}^{KT}}{2\varepsilon(\gamma)}r^2\right) \quad \text{for } r\ge0$$

and observing that $\omega_0(0)=0$ and $\omega_0\in\mathrm{USC}([0,\ \infty))$, we find that there is a modulus ω such that $\omega_0(r)\le\omega(r)$ for all $r\ge0$. We note by (4.24) that

$$u(x,t) - v(y,t) \le \omega(|x-y|) \quad \text{for all } x,y\in\mathbb{R}^N,\ t\in[0,\ T),$$

to complete the proof. □

5. An existence and uniqueness theorem

As usual we fix $p \in \mathbb{R}^N$ arbitrarily throughout this section.

Theorem 5.1. *Let* $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$*. There exists a unique solution* $u \in C(\overline{Q}_\infty)$ *of* (1.1) *for which* $u(\cdot, 0) = u_0$ *on* $\mathbb{R}^N$ *and* $u \in \mathrm{BUC}(\overline{Q}_T)$ *for any* $0 < T < \infty$*.*

Proof. Uniqueness of a solution $u \in \mathrm{BUC}(\overline{Q}_T)$ for every $0 < T < \infty$ of the initial value problem for (1.1) follows from Theorem 4.1.

In view of the uniqueness result, it is enough to show that for each $0 < T < \infty$, there is a solution $u \in \mathrm{BUC}(\mathbb{R}^N \times [0, T))$ of (1.1) satisfying $u(\cdot, 0) = u_0$. We fix any $0 < T < \infty$. Let ω_0 denote the modulus of continuity of u_0. We define the function $\phi \in C^\infty(\mathbb{R})$ by $\phi(r) = \frac{r^2}{1+r^2}$. Note that the function ϕ and all its derivatives are bounded on $\mathbb{R}$. Noting that ω_0 is bounded on $[0, \infty)$, we see that for each $\varepsilon > 0$ there is a constant $A_\varepsilon > 0$ such that $\omega_0(r) \leq \varepsilon + A_\varepsilon \phi(r)$ for all $r \geq 0$. If we set $\psi(x) = \phi(|x|)$ for $x \in \mathbb{R}^N$, then $\psi \in C^\infty(\mathbb{R}^N)$ and ψ and all its derivatives are bounded on $\mathbb{R}^N$. For any fixed $(\varepsilon, y) \in (0, 1) \times \mathbb{R}^N$, we set

$$f^\pm(x) = f^\pm(x; \varepsilon, y) := u_0(y) \pm (\varepsilon + A_\varepsilon \phi(|x - y|)) \quad \text{for } x \in \mathbb{R}^N.$$

Thanks to Lemma 2.5, for each $\varepsilon \in (0, 1)$ there is a constant $B_\varepsilon > 0$ such that for all $(x, t) \in Q_T$,

$$\begin{aligned} &\big(c(x,t) + M_p^+[f^+](x)\big)\ |p + Df^+(x)| \leq B_\varepsilon \qquad \text{if } p + Df^+(x) \neq 0 \\ &\big(c(x,t) + M_p^-[f^-](x)\big)\ |p + Df^-(x)| \geq -B_\varepsilon \quad \text{if } p + Df^-(x) \neq 0. \end{aligned}$$

Now, we define the functions $F^\pm(\cdot; \varepsilon, y)$ on $\overline{Q}_T$, with $(\varepsilon, y) \in (0, 1) \times \mathbb{R}^N$, by

$$F^\pm(x, t) = F^\pm(x, t; \varepsilon, y) := f^\pm(x, t; \varepsilon, y) \pm B_\varepsilon t.$$

It follows from the above observations that functions $F^+(\cdot; \varepsilon, y)$ and $F^-(\cdot; \varepsilon, y)$ are, respectively, solutions of (1.3) and of (1.2) in Q_T. It is obvious that

$$F^-(x, t; \varepsilon, y) \leq u_0(x) \leq F^+(x, t; \varepsilon, y) \quad \text{for all } (x, t) \in \overline{Q}_T,$$

and

$$F^-(x, 0; \varepsilon, x) + \varepsilon = u_0(x) = F^+(x, 0; \varepsilon, x) - \varepsilon \quad \text{for all } x \in \mathbb{R}^N.$$

Next, we define the functions $g^\pm$ on $\overline{Q}_T$ by

$$\begin{aligned} g^+(x, t) &= \inf \big\{ F^+(x, t; \varepsilon, y) \mid (\varepsilon, y) \in (0, 1) \times \mathbb{R}^N \big\} \\ g^-(x, t) &= \sup \big\{ F^-(x, t; \varepsilon, y) \mid (\varepsilon, y) \in (0, 1) \times \mathbb{R}^N \big\}. \end{aligned}$$

It follows that g^+ and g^- are a sub- and supersolution of (1.1), respectively, and that $g^-(x,t) \le u_0(x) \le g^+(x,t)$ for all $(x,t) \in \overline{Q}_T$, $g^-(x,0) = u_0(x) = g^+(x,0)$ for all $x \in \mathbb{R}^N$ and $g^+, -g^- \in \mathrm{USC}(\overline{Q}_T)$. By Theorem 3.4 there is a solution u of (1.1) such that $g^-(x,t) \le u(x,t) \le g^+(x,t)$ for all $(x,t) \in Q_T$. Note that for all $x, y \in \mathbb{R}^N$, $t \in [0,\, T)$ and $\varepsilon) \in (0,1)$,

$$|u(x,t) - u_0(y)| \le \varepsilon + A_\varepsilon \phi(|x-y|) + B_\varepsilon t.$$

In particular, we find that $\lim_{t\to 0+} u(x,t) = u_0(x)$ uniformly for $x \in \mathbb{R}^N$ and that u is bounded on Q_T. By Theorem 4.1, we see that $u^* \le u_*$ in Q_T and hence $u \in C(Q_T)$. Because of the uniform convergence of $u(x,t)$ to $u_0(x)$ as $t \to 0+$, we may extend u to a continuous function on $\mathbb{R}^N \times [0,\, T)$ by setting $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}^N$. We next apply Theorem 4.4 to u, to find a modulus ω such that

$$u(x,t) - u(y,t) \le \omega(|x-y|) \quad \text{for all } x, y \in \mathbb{R}^N,\ t \in [0,T).$$

It remains to show that the family of functions $u(x,\cdot)$, with $x \in \mathbb{R}^N$, is equi-continuous on $[0,\, T)$. This can be done by adapting the above construction of $g^\pm$. Indeed, following the above argument with ω in place of ω_0, we easily see that for each $\varepsilon \in (0,1)$ there is a constant $C_\varepsilon > 0$ such that

$$|u(x,t) - u(x,s)| \le \varepsilon + C_\varepsilon |t-s| \quad \text{for all } x \in \mathbb{R}^N,\ s,t \in [0,\, T),$$

which guarantees the desired equi-continuity. The proof is complete. □

6. One-dimensional case

In this section we always assume that $N = 1$ and show that the requirement, $\beta < N+1$, in (J4) can be removed if $N = 1$. In what follows we replace condition (J4) by the following.

(J4′) There are constants $\beta > 1$ and $C_0 > 0$ such that

$$J(z) \le \frac{C_0}{|z|^\beta} \quad \text{for all } z \in [-1,\, 0) \cup (0,\, 1].$$

We assume throughout this section that (c1)–(c2), (J1)–(J3) and (J4′) hold. We fix $p \in \mathbb{R}$ arbitrarily. In this section we use the notation: $B(x,r) = [x-r,\, x+r]$ and $\phi_x(x,t) = D\phi(x,t)$.

In order to accommodate the higher singularity of the kernel J at the origin, we introduce "admissible test functions" following for instance [10] and modify the definition of sub-, super- and solutions of (1.1).

Let $\beta > 1$ be the constant from (J4′). We denote by $\mathcal{F}_\beta(Q_T)$ the space of functions $\phi \in C^2(Q_T)$ such that for each $(y,s) \in Q_T$, where ϕ_x vanishes, there exist constants $\delta > 0$ and $C > 0$ such that for all $(x,t) \in B((y,s),\delta)$,

$$|\phi(x,t) - \phi(y,s) - \phi_t(y,s)(t-s)| \le C(|x-y|^{\beta+1} + |t-s|^2).$$

It is clear that the function $\phi(x,t) := a|x-y|^{\beta+1} + \psi(t)$, with any $a \in \mathbb{R}$, $y \in \mathbb{R}$ and $\psi \in C^2((0,T))$, belongs to $\mathcal{F}_\beta(Q_T)$.

We next define $\mathcal{F}_{\beta,p}(Q_T)$ as the space of all functions $\phi(x,t) - px$ on Q_T, with $\phi \in \mathcal{F}_\beta(Q_T)$. We note that for any $\phi \in C^2(Q_T)$, we have $\phi \in \mathcal{F}_{\beta,p}(Q_T)$ if and only if for each $(y,s) \in Q_T$ satisfying $\phi_x(y,s) + p = 0$ there are constants $\delta > 0$ and $C > 0$ such that for all $(x,t) \in B((y,s),\delta)$,

$$|\phi(x,t) + p(x-y) - \phi(y,s) - \phi_t(y,s)(t-s)| \le C(|x-y|^{\beta+1} + |t-s|^2). \quad (6.1)$$

We say in this section that $u \in \mathcal{B}(Q_T)$ is a (viscosity) subsolution (resp., supersolution) of (1.1) if whenever $(x,t,\phi) \in Q_T \times \mathcal{F}_{\beta,p}(Q_T)$ and $u^* - \phi$ (resp., $u_* - \phi$) has a local maximum (resp., minimum) at (x,t), inequality (1.4) (resp., (1.5)) holds. As before, we call a subsolution (resp., supersolution) of (1.1) a solution of (1.2) (resp., of (1.3)) as well. A function $u \in \mathcal{B}(Q_T)$ is called a solution of (1.1) if it is both a subsolution and supersolution of (1.1). Remark that if $u \in \mathcal{B}(Q_T)$ is a subsolution (resp., supersolution, solution) of (1.1) in the sense of the previous sections, then it is a subsolution (resp., supersolution, solution) of (1.1) in the current sense.

We set $f(x) = |x|^{\beta+1}$ for $x \in \mathbb{R}$ and observe that if $|y-x| \le |f'(x)|^{\frac{1}{\beta}}$, then we have $|f'(x)| = (\beta+1)|x|^\beta$, $|f''(y)| = \beta(\beta+1)|y|^{\beta-1}$, and

$$\left(\frac{|f''(y)|}{\beta(\beta+1)}\right)^{\frac{1}{\beta-1}} = |y| \le |x| + |y-x| \le \left(\frac{|f'(x)|}{\beta+1}\right)^{\frac{1}{\beta}} + |f'(x)|^{\frac{1}{\beta}},$$

that is,

$$|f''(y)| \le C_\beta |f'(x)|^{1-\frac{1}{\beta}}, \qquad \text{with } \ C_\beta := \beta(\beta+1)\left(1 + \left(\frac{1}{\beta+1}\right)^{\frac{1}{\beta}}\right)^{\beta-1}.$$

By the Taylor theorem, we find that for all $x \in \mathbb{R}$ and $z \in B\big(x, |f'(x)|^{\frac{1}{\beta}}\big)$,

$$|f(x+z) - f(x) - f'(x)z| \le C_\beta |f'(x)|^{1-\frac{1}{\beta}}.$$

Next fix $y \in \mathbb{R}$ and set $g(x) = f(x-y) - px$ for $x \in \mathbb{R}$. It follows from the above inequality that for any $x \in \mathbb{R}$ and $z \in B\big(x, |g'(x)+p|^{\frac{1}{\beta}}\big)$,

$$|g(x+z) - g(x) - g'(x)z| \le C_\beta |g'(x)+p|^{1-\frac{1}{\beta}} z^2. \quad (6.2)$$

The above observation will be useful in our stability arguments.

Lemma 6.1. *Let* u *be a bounded measurable function on* $\mathbb{R}$*. Let* q, $x \in \mathbb{R}$, $r > 0$ *and* $C_1 > 0$*. Assume that* $0 < |p+q| \le 1$ *and*

$$u(x+z) \le u(x) + q \cdot z + C_1 |p+q|^{1-\frac{1}{\beta}} z^2 \qquad \text{for all } z \in B\big(0, r \wedge |p+q|^{\frac{1}{\beta}}\big).$$

Then there is a constant $0 < \rho < 1$*, depending only on* C_1*, such that for any* $0 < \delta \le r \wedge \big(\rho |p+q|^{\frac{1}{\beta}}\big)$*,*

$$M_p^+[u](x) \le \int_{|z|>\delta} J(z) E_p^+(u(x+z) - u(x), z)\mathrm{d}z.$$

Proof. We follow the proof of Lemma 2.1. We set $v = p + q$ and note that

$$u(x+z) - u(x) + pz \le vz + C_1 |v|^{1-\frac{1}{\beta}} z^2 \qquad \text{for all } z \in B\big(0,\, r \wedge |v|^{\frac{1}{\beta}}\big).$$

Hence, if $v > 0$, then

$$u(x+z) - u(x) + pz < 0 \qquad \text{for all } z \in \Big(-r \wedge \tfrac{|v|^{\frac{1}{\beta}}}{C_1},\, 0\Big),$$

and if $v < 0$, then

$$u(x+z) - u(x) + pz < 0 \qquad \text{for all } z \in \Big(0,\, r \wedge \tfrac{|v|^{\frac{1}{\beta}}}{C_1}\Big).$$

We set $\rho = \frac{1}{2C_1+1}$, and note as before that $\rho < 1$ and that if $|z| \le \rho$, then

$$vz + C_1 |v|^{1-\frac{1}{\beta}} z^2 \le \rho + C_1 \rho^2 < 1.$$

Fix any $0 < \delta \le r \wedge \big(\rho |v|^{\frac{1}{\beta}}\big)$. If $v > 0$, then we get

$$E^*\big(u(x+z) - u(x) + pz\big) \le \begin{cases} -\dfrac{1}{2} & \text{for } -\delta < z < 0 \\ \dfrac{1}{2} & \text{for } 0 < z < \delta. \end{cases}$$

If $v < 0$, then

$$E^*\big(u(x+z) - u(x) + pz\big) \le \begin{cases} \dfrac{1}{2} & \text{for } -\delta < z < 0 \\ -\dfrac{1}{2} & \text{for } 0 < z < \delta. \end{cases}$$

Consequently, we obtain

$$\begin{aligned} M_p^+[u](x) &= \limsup_{\varepsilon \to 0+} \left(\int_{\varepsilon < |z| \le \delta} + \int_{|z|>\delta} \right) J(z) E_p^+(u(x+z) - u(z), z)\mathrm{d}z \\ &\le \int_{|z|>\delta} J(z) E_p^+(u(x+z) - u(x), z)\mathrm{d}z, \end{aligned}$$

which completes the proof. $\square$

The following is a one-dimensional version of Lemma 2.1. Its proof parallels that of Lemma 2.1, once the one-dimensionality is taken into account as in the previous proof. We omit here giving the proof.

Lemma 6.2. *Let u be a bounded measurable function on $\mathbb{R}$. Let $q, x \in \mathbb{R}$, $r > 0$, $\Lambda > 0$ and $C_1 > 0$. Assume that $0 < |p+q| \le \Lambda$ and*

$$u(x+z) \le u(x) + q \cdot z + C_1 z^2 \quad \text{for all } z \in B(0, r).$$

Then there is a constant $\rho > 0$, depending only on C_1, r, Λ, such that for any $0 < \delta \le \rho \wedge \frac{|p+q|}{2C_1}$,

$$M_p^+[u](x) \le \int_{|z|>\delta} J(z) E_p^+(u(x+z) - u(x), z) \mathrm{d}z.$$

Lemma 6.3. *Let u be a bounded measurable function on $\mathbb{R}$. Let q, $x \in \mathbb{R}$, $r > 0$, $C_1 > 0$ and $C_2 > 0$. Assume that $0 < |p+q| \le 1$, $|u(z)| \le C_2$ for all $z \in \mathbb{R}$ and*

$$u(x+z) \le u(x) + q \cdot z + C_1 |p+q|^{1-\frac{1}{\beta}} z^2 \quad \text{for all } z \in B\big(0,\, r \wedge |p+q|^{\frac{1}{\beta}}\big).$$

Then there is a constant $C > 0$, depending only on r, $\|J\|_{L^1(1,\infty)}$, β, C_0, C_1 and C_2, such that

$$M_p^+[u](x)\, |p+q| \le C |p+q|^{\frac{1}{\beta}}.$$

Proof. Let $\rho \in (0,1)$ be the constant from Lemma 6.1. Setting $\delta = r \wedge \big(\rho |p+q|^{\frac{1}{\beta}}\big)$, we have

$$\begin{aligned}
M_p^+[u](x) &\le \int_{|z|>\delta} J(z) E_p^+(u(x+z) - u(x), z) \mathrm{d}x \\
&= \left(\int_{\delta<|z|\le 1} + \int_{|z|>1} \right) J(z) E_p^+(u(x+z) - u(x), z) \mathrm{d}z \\
&\le (2C_2 + 1) \Big(2C_0 \int_\delta^1 z^{-\beta} \mathrm{d}z + 2\|J\|_{L^1(1,\infty)} \Big) \\
&< 2(2C_2+1) \left(\frac{C_0 \delta^{1-\beta}}{\beta - 1} + \|J\|_{L^1(1,\infty)} \right).
\end{aligned}$$

Hence,

$$M_p^+[u](x)\, |p+q| \le C_3 (|p+q|^{\frac{1}{\beta}} + |p+q|) \le (C_3 + 1) |p+q|^{\frac{1}{\beta}},$$

where $C_3 > 0$ is a constant depending only on r, $\|J\|_{L^1(1,\infty)}$, β, C_0, C_1 and C_2. This proves our claim. □

We state stability, comparison and existence results in one dimension, which are parallel to the corresponding results in general dimensions.

Theorem 6.4. *Let $\mathcal{S}_0$ be a non-empty set of solutions of* (1.2)*. Assume that the family $\mathcal{S}_0$ is uniformly bounded on Q_τ for any $0 < \tau < T$. Define the function $u \in \mathcal{B}(Q_T)$ by $u(x, t) = \sup\{v(x, t) \mid v \in \mathcal{S}_0\}$. Then the envelope u^* is a solution of* (1.2)*.*

Proof. Let $(\hat{x}, \hat{t}\,) \in Q_T$, $r > 0$ and $\varphi \in \mathcal{F}_{\beta,p}(Q_T)$, and assume that $B((\hat{x}, \hat{t}\,), 2r) \subset Q_T$ and $u^* - \varphi$ attains a strict maximum at $(\hat{x}, \hat{t}\,)$ over $B((\hat{x}, \hat{t}\,), 2r)$. By the definition of u^*, there are sequences $\{(x_n, t_n)\} \subset B((\hat{x}, \hat{t}\,), 2r)$ and $\{v_n\} \subset \mathcal{S}_0$ such that $v_n(x_n, t_n) \to u^*(\hat{x}, \hat{t}\,)$ and $(x_n, t_n) \to (\hat{x}, \hat{t}\,)$ as $n \to \infty$. By the definition of u, we have $v_n^* \le u^*$ in Q_T.

For any $n \in \mathbb{N}$ let $(y_n, s_n) \in B((\hat{x}, \hat{t}\,), 2r)$ be a maximum point, over $B((\hat{x}, \hat{t}\,), 2r)$, of the function $v_n^* - \varphi$. As usual we see that $v_n^*(y_n, s_n) \to u^*(\hat{x}, \hat{t}\,)$ and $(y_n, s_n) \to (\hat{x}, \hat{t}\,)$ as $n \to \infty$. Passing to a subsequence if necessary, we may assume that $(y_n, s_n) \in B((\hat{x}, \hat{t}\,), r)$ for all n. Since v_n is a subsolution of (1.1), we have

$$\varphi_t(y_n, s_n) \le (c(y_n, s_n) + M_p^+[v_n^*(\cdot,\, s_n)](y_n))|p + \varphi_x(y_n, s_n)| \tag{6.3}$$

if $p + \varphi_x(y_n, s_n) \ne 0$, and

$$\varphi_t(y_n, s_n) \le 0 \quad \text{ if } p + \varphi_x(y_n, s_n) = 0. \tag{6.4}$$

We note that for any $z \in B(0,\, r)$ and $n \in \mathbb{N}$,

$$v_n^*(y_n + z, s_n) - v_n^*(y_n, s_n) \le \varphi(y_n + z, s_n) - \varphi(y_n, s_n). \tag{6.5}$$

We treat the following two cases differently.

`Case` 1: $p + \varphi_x(\hat{x}, \hat{t}\,) = 0$. In view of (6.1) and (6.2), by replacing φ by the function

$$(x, t) \mapsto -px + \varphi_t(\hat{x}, \hat{t}\,)t + C\big(|x - \hat{x}|^{\beta+1} + |t - \hat{t}\,|^2\big),$$

with $C > 0$ sufficiently large, we may assume that for all $(x, t) \in Q_T$ and $z \in B\big(0, |p + \varphi_x(x, t)|^{\frac{1}{\beta}}\big)$,

$$\varphi(x + z, t) - \varphi(x, t) \le \varphi_x(x, t)z + C_1|p + \varphi_x(x)|^{1-\frac{1}{\beta}} z^2.$$

Accordingly, we have for any $n \in \mathbb{N}$ and $z \in B\big(0,\, |p + \varphi_x(y_n, s_n)|^{\frac{1}{\beta}}\big)$,

$$\varphi(y_n + z, s_n) - \varphi(y_n, s_n) \le \varphi_x(y_n, s_n)z + C_1|p + \varphi_x(y_n, s_n)|^{1-\frac{1}{\beta}} z^2.$$

In this case we have $p+\varphi_x(y_n,s_n)\to 0$ as $n\to\infty$. In particular, we may assume that $|p+\varphi_x(y_n,s_n)|^{\frac{1}{\beta}}\le \min\{1,\,r\}$ for all $n\in\mathbb{N}$. Hence, using (6.5), we have for any $n\in\mathbb{N}$ and $z\in B\big(0,\,|p+\varphi_x(y_n,s_n)|^{\frac{1}{\beta}}\big)$,

$$v_n^*(y_n+z,s_n)-v_n^*(y_n,s_n)\le \varphi_x(y_n,s_n)z+C_1|p+\varphi_x(y_n,s_n)|^{1-\frac{1}{\beta}}z^2.$$

By replacing C_1 by a larger number if necessary, we may assume moreover that $|v_n(x,s_n)|\le C_1$ for all $x\in\mathbb{R}$ and $n\in\mathbb{N}$.

According to Lemma 6.3, there is a constant $C_2>0$, which does not depend on n, such that if $|p+\varphi_x(y_n,s_n)|\le 1$,

$$M_p^+[v_n^*(\cdot,\,s_n)](y_n)|p+\varphi_x(y_n,s_n)|\le C_2|p+\varphi_x(y_n,s_n)|^{\frac{1}{\beta}}.$$

We combine this with (6.3) and (6.4) and send $n\to\infty$, to see that $\varphi_t(\hat{x},\hat{t}\,)\le 0$.

Case 2: $p+\varphi_x(\hat{x},\hat{t}\,)\neq 0$. By selecting a subsequence if necessary, we may assume that $|p+\varphi_x(y_n,s_n)|\ge\lambda$ for all n and for some constant $\lambda>0$. Note by (6.5) that there is a constant $C_2>0$ such that for all $z\in B(0,r)$ and $n\in\mathbb{N}$,

$$v_n^*(y_n+z,s_n)-v_n^*(y_n,s_n)\le\varphi_x(y_n,s_n)z+C_2z^2.$$

We apply Lemma 6.2, to find that there is a constant $\rho\in(0,1)$ such that for any $0<\delta\le\rho$,

$$M_p^+[v_n^*(\cdot,\,s_n)](y_n)\le\int_{|z|>\delta}J(z)E_p^+(v_n^*(y_n+z)-v_n^*(y_n),z)\mathrm{d}z.$$

By the definition of u, we see that for all $x\in\mathbb{R}$,

$$\begin{aligned}&\lim_{k\to\infty}\sup\big\{v_n^*(y,\,s_n)\mid n\ge k,\,y\in B(x,\,k^{-1})\big\}\\&\le\lim_{k\to\infty}\sup\big\{u^*(y,\,s_n)\mid n\ge k,\,y\in B(x,\,k^{-1})\big\}\\&\le u^*(x,\hat{t}\,).\end{aligned}$$

We now apply Lemma 3.1, to get for any $\delta\in(0,\,\rho]$,

$$\begin{aligned}&\limsup_{n\to\infty}M_p^+[v_n^*(\cdot,\,s_n)](y_n)\\&\le\int_{|z|>\delta}J(z)E_p^+(u^*(\hat{x}+z,\hat{t}\,)-u^*(\hat{x},\hat{t}\,),z)\mathrm{d}z.\end{aligned}$$

Thus we get

$$\limsup_{n\to\infty}M_p^+[v_n^*(\cdot,\,s_n)](y_n)\le M_p^+[u^*(\cdot,\,\hat{t}\,)](\hat{x}),$$

and conclude from (6.3) that

$$\varphi_t(\hat{x},\,\hat{t}\,)\le\big(c(\hat{x},\hat{t}\,)+M_p[u^*(\cdot,\,\hat{t}\,)](\hat{x})\big)|p+\varphi_x(\hat{x},\,\hat{t}\,)|.$$

That is, u^* is a solution of (1.2). □

Theorem 6.5. *Let* $\{u_n\}$ *be a sequence of solutions of* (1.2)*. Assume that the collection* $\{u_n\}$ *is uniformly bounded on* Q_τ *for any* $0 < \tau < T$*. Define* $u \in \mathcal{B}(Q_T)$ *by*

$$u(x,t) = \lim_{k\to\infty} \sup \big\{u_n(y,s) \mid (y,s) \in B((x,t),\, k^{-1}),\, n \geq k\big\}.$$

Then u *is a solution of* (1.2)*.*

Proof. We note that $u \in \mathrm{USC}(Q_T)$. Let $(\hat{x},\hat{t}\,) \in Q_T$ and $\varphi \in \mathcal{F}_{\beta,p}(Q_T)$, and assume that $u - \varphi$ attains a strict maximum at $(\hat{x},\hat{t}\,)$. As in the proof of Theorem 3.3, we can choose sequences $\{n_k\} \subset \mathbb{N}$, diverging to infinity, and $\{(x_k,t_k)\} \subset Q_T$ so that $u^*_{n_k}(x_k,t_k) \to u(\hat{x},\hat{t}\,)$ and $(x_k,t_k) \to (\hat{x},\hat{t}\,)$ as $k \to \infty$, and for any $k \in \mathbb{N}$, the function $u^*_{n_k} - \varphi$ attains a local maximum at (x_k,t_k).

The rest of the proof parallels the last part of the proof of Theorem 6.4. □

Theorem 6.6. *Let* $f \in \mathrm{LSC}(Q_T)$ *and* $g \in \mathrm{USC}(Q_T)$ *be a subsolution and supersolution of* (1.1)*, respectively. Assume that* $f \leq g$ *in* Q_T*. Set*

$$u(x,t) = \sup\big\{v(x,t) \mid v \textit{ is a subsolution of } (1.1),\ f \leq v \leq g \textit{ in } Q_T\big\}.$$

Then u *is a solution of* (1.1)*.*

The proof of Theorem 3.3 is easily adapted to that of the above theorem, and we leave it to the reader to check the details.

Theorem 6.7. *Let* $0 < T < \infty$*. Let* u *and* v *be solutions of* (1.2) *and of* (1.3)*, respectively. Assume that* u *and* $-v$ *are upper semicontinuous and bounded on* $\mathbb{R} \times [0,\, T)$ *and that*

$$\lim_{r\to 0+} \sup\big\{u(x,t) - v(y,s) \mid (x,t),(y,s) \in \mathbb{R} \times [0,\, T),\, |x-y| \vee t \vee s \leq r\big\} \leq 0.$$

Then $u \leq v$ *on* $\mathbb{R} \times [0,\, T)$*. Moreover there is a modulus* ω *such that*

$$u(x,t) - v(y,t) \leq \omega(|x-y|) \quad \textit{for all } x,y \in \mathbb{R},\, t \in [0,\, T).$$

Outline of proof. We follow the proof of Theorems 4.1 and 4.2 with small variations.

As in the proof of Theorem 4.2, we introduce sup- and infconvolutions of u and v as follows:

$$u^\varepsilon(x,t) := \sup_{y\in\mathbb{R}} \left(u(y,t) - p\cdot(x-y) - \frac{|x-y|^{\beta+1}\mathrm{e}^{Kt}}{(\beta+1)\varepsilon} \right)$$

$$v_\varepsilon(x,t) := \inf_{y\in\mathbb{R}} \left(v(y,t) - p\cdot(x-y) + \frac{|x-y|^{\beta+1}\mathrm{e}^{Kt}}{(\beta+1)\varepsilon} \right),$$

where $0 < \varepsilon < 1$ and $K := (\beta + 1)\|Dc\|_{L^\infty(Q_T)}$. It is easy to check that u^ε and v_ε are solutions of (1.2) and of (1.3), respectively. Noting that for any $x, y \in \mathbb{R}$ and $0 \le t < T$,

$$\begin{aligned} |p||x-y| &\le \frac{\mathrm{e}^{Kt}|x-y|^{\beta+1}}{2(\beta+1)\varepsilon} + \frac{\beta}{\beta+1}\left(|p|\left(2\varepsilon \mathrm{e}^{-Kt}\right)^{\frac{1}{\beta+1}}\right)^{\frac{\beta+1}{\beta}} \\ &\le \frac{\mathrm{e}^{Kt}|x-y|^{\beta+1}}{2(\beta+1)\varepsilon} + |p|^{\frac{\beta+1}{\beta}} 2^{\frac{1}{\beta}} \\ &\le \frac{\mathrm{e}^{Kt}|x-y|^{\beta+1}}{2(\beta+1)\varepsilon} + 2(|p|+1)^2, \end{aligned}$$

we find that

$$\begin{aligned} u^\varepsilon(x,t) &\le \|u\|_\infty + 2(|p|+1)^2, \\ u^\varepsilon(x,t) &= \sup_{y \in B(x,R)} \left(u(y,t) - \frac{\mathrm{e}^{Kt}|x-y|^{\beta+1}}{(\beta+1)\varepsilon} \right) \end{aligned}$$

for some constant $R > 0$. Using these observations, we see that u^ε is bounded on $\mathbb{R} \times [0, T)$ and $u^\varepsilon(x,t)$ is semi-convex in x uniformly in $t \in [0, T)$. Similarly, we find that v_ε is bounded on $\mathbb{R} \times [0, T)$ and $v_\varepsilon(x,t)$ is semi-concave in x uniformly in $t \in [0, T)$. Moreover, it is easily seen that for every $(x,t) \in \mathbb{R} \times [0, T)$,

$$\lim_{\varepsilon \to 0+} u^\varepsilon(x,t) = u(x,t) \quad \text{and} \quad \lim_{\varepsilon \to 0+} v_\varepsilon(x,t) = v(x,t)$$

and that

$$\lim_{\varepsilon \to 0+} \lim_{r \to 0+} \sup \left\{ u^\varepsilon(x,t) - v_\varepsilon(y,s) \mid (x,t),(y,s) \in \overline{Q}_T,\ |x-y| \vee t \vee s \le r \right\} \le 0.$$

Fix any $\mu > 0$ and choose an $\varepsilon_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\lim_{r \to 0+} \sup \left\{ u^\varepsilon(x,t) - v_\varepsilon(y,s) \mid (x,t),(y,s) \in \overline{Q}_T,\ |x-y| \vee t \vee s \le r \right\} \le \mu.$$

As before, in order to prove the theorem, we need only to show that $u^\varepsilon - v_\varepsilon \le \mu$ on $\mathbb{R} \times [0, T)$ for any $\varepsilon \in (0, \varepsilon_0)$. Thus we may assume by replacing u and v by $u^\varepsilon - \mu$ and v_ε, respectively, that the functions $u(x,t)$ and $-v(x,t)$ are semi-convex in x uniformly in $t \in [0, T)$.

We argue by contradiction and hence suppose that $\sup_{\mathbb{R}\times[0,T)}(u - v) > 0$. We may assume that u and v are defined on $\overline{Q}_T$. Moreover, arguing as in the first part of the proof of Theorem 4.1, we may assume that there is a small constant $\gamma > 0$ such that for any $(x,t,y,s) \in \overline{Q}_T^2$, if

$$\text{either} \quad |x-y| \vee t \vee s \le \gamma \quad \text{or} \quad |x-y| \vee (T-t) \vee (T-s) \le \gamma,$$

then

$$u(x,t) - v(y,s) < 0, \tag{6.6}$$

and that u is a solution of

$$u_t + \gamma \le (c + M_p[u(\cdot,t)](x))|p + Du| \quad \text{in } Q_T, \tag{6.7}$$

We set

$$\tilde{u}(x,t) = u(x,t) + p\cdot x \quad \text{and} \quad \tilde{v}(x,t) = v(x,t) + p\cdot x \qquad \text{for } (x,t) \in \overline{Q}_T.$$

In view of (6.6), replacing $\gamma > 0$ by a smaller number if necessary, we may assume that for any $(x,t,y,s) \in \overline{Q}_T^{\,2}$, if

$$\text{either} \quad |x-y| \vee t \vee s \le \gamma \quad \text{or} \quad |x-y| \vee (T-t) \vee (T-s) \le \gamma,$$

then

$$\tilde{u}(x,t) - \tilde{v}(y,s) < 0. \tag{6.8}$$

Let $\alpha > 1$ be a large constant to be selected later and define the function $\Phi = \Phi_\alpha$ on $\overline{Q}_T^{\,2}$ by

$$\Phi(x,t,y,s) = \tilde{u}(x,t) - \tilde{v}(y,s) - \alpha|x-y|^{\beta+1} - \alpha|t-s|^2.$$

We set $\theta = \theta_\alpha := \sup_{\overline{Q}_T^{\,2}} \Phi$ and note that $\theta \ge \sup_{\overline{Q}_T}(\tilde{u} - \tilde{v}) = \sup_{\overline{Q}_T}(u-v) > 0$. Choose a constant $C_2 > 0$ so that $|u| \vee |v| \le C_2$ on $\overline{Q}_T$ and observe that if $\Phi(x,t,y,s) \ge 0$, then

$$\begin{aligned} 2C_2 &\ge -p\cdot(x-y) + \alpha|x-y|^{\beta+1} + \alpha|t-s|^2 \\ &\ge -\frac{\beta}{\beta+1}|p|^{1+\frac{1}{\beta}} - \frac{1}{\beta+1}|x-y|^{\beta+1} + \alpha\left(|x-y|^{\beta+1} + |t-s|^2\right) \\ &\ge -|p|^{1+\frac{1}{\beta}} + \frac{\alpha}{2}\left(|x-y|^{\beta+1} + |t-s|^2\right). \end{aligned}$$

Fix a constant $R_0 > 0$ so that $R_0^{\beta+1} \wedge R_0^2 \ge 4C_2 + 2|p|^{1+\frac{1}{\beta}}$, and note that for any $(x,t,y,s) \in \overline{Q}_T^{\,2}$,

$$\left(\alpha^{\frac{1}{\beta+1}}|x-y|\right) \vee \left(\sqrt{\alpha}|t-s|\right) \le R_0 \quad \text{if } \Phi(x,t,y,s) \ge 0. \tag{6.9}$$

We define $\mathcal{R}_\alpha \subset [0,\infty)$ as the set of all $r \ge 0$ which satisfy

$$\theta = \sup\left\{\Phi(x,t,y,s) \mid (x,t,y,s) \in \overline{Q}_T^{\,2},\ \alpha|x-y|^\beta \le r\right\},$$

and set $\lambda_\alpha = \inf \mathcal{R}_\alpha$. We note that $0 \le \lambda_\alpha < \infty$.

We divide our argument into two cases.

`Case` 1: $\liminf_{\alpha\to\infty}\lambda_\alpha = 0$. Let $\eta > 0$ be a small constant. We choose an $\alpha > 1$ so that $\lambda_\alpha < \eta$. There is a sequence $\{(x_n, t_n, y_n, s_n)\} \subset \overline{Q}_T^2$ such that

$$\Phi(x_n, t_n, y_n, s_n) > \theta\Big(1 - \frac{1}{n}\Big) \quad \text{and} \quad \alpha|x_n - y_n|^\beta \le \eta.$$

We may assume, by choosing α large enough if needed, that $R_0/\alpha^{\frac{1}{\beta+1}} < \frac{\gamma}{2}$, so that $|x_n - y_n| \vee |t_n - s_n| \le \frac{\gamma}{2}$ by (6.9) and, by (6.8),

$$t_n, s_n \in \Big(\frac{\gamma}{2}, T - \frac{\gamma}{2}\Big) \quad \text{for all } n \in \mathbb{N}. \tag{6.10}$$

By taking a subsequence if necessary, we may assume that $(t_n, s_n) \to (\hat{t}, \hat{s})$ for some $\hat{t}, \hat{s} \in [\frac{\gamma}{2}, T - \frac{\gamma}{2}]$ as $n \to \infty$. We choose a maximum point (ξ_n, τ_n) of the function

$$(x,t) \mapsto \tilde{u}(x,t) - \Big(\alpha + \alpha^{\frac{1}{\beta}}\Big)\, |x - y_n|^{\beta+1} - \alpha|t - s_n|^2 - \alpha|t - \hat{t}\,|^2 \quad \text{on } \overline{Q}_T.$$

We have

$$\begin{aligned}
&\Phi(x_n, t_n, y_n, s_n) - \alpha^{\frac{1}{\beta}}|x_n - y_n|^{\beta+1} - \alpha|t_n - \hat{t}\,|^2 \\
&\le \Phi(\xi_n, \tau_n, y_n, s_n) - \alpha^{\frac{1}{\beta}}|\xi_n - y_n|^{\beta+1} - \alpha|\tau_n - \hat{t}\,|^2 \\
&\le \theta - \alpha^{\frac{1}{\beta}}|\xi_n - y_n|^{\beta+1} - \alpha|\tau_n - \hat{t}\,|^2.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\alpha^{\frac{1}{\beta}}|\xi_n - y_n|^{\beta+1} + \alpha|\tau_n - \hat{t}\,|^2 &\le \theta - \Phi(x_n, t_n, y_n, s_n) \\
&\qquad + \alpha^{\frac{1}{\beta}}|x_n - y_n|^{\beta+1} + \alpha|t_n - \hat{t}\,|^2 \\
\Phi(x_n, t_n, y_n, s_n) &\le \Phi(\xi_n, \tau_n, y_n, s_n) + \alpha^{\frac{1}{\beta}}|x_n - y_n|^{\beta+1} + \alpha|t_n - \hat{t}\,|^2,
\end{aligned}$$

and consequently

$$\limsup_{n\to\infty}\Big(\alpha^{\frac{1}{\beta}}|\xi_n - y_n|^{\beta+1} + \alpha|\tau_n - \hat{t}\,|^2\Big) \le \limsup_{n\to\infty}\alpha^{\frac{1}{\beta}}|x_n - y_n|^{\beta+1} \le \frac{\eta^{1+\frac{1}{\beta}}}{\alpha}$$

$$\liminf_{n\to\infty}\Phi(\xi_n, \tau_n, y_n, s_n) \ge \theta - \frac{\eta^{1+\frac{1}{\beta}}}{\alpha}.$$

Reselecting α large enough if necessary and choosing $n \in \mathbb{N}$ large enough, we have

$$\Big(\alpha^{\frac{1}{\beta}}|\xi_n - y_n|^{\beta+1} + \alpha|\tau_n - \hat{t}\,|^2\Big) < \frac{(2\eta)^{1+\frac{1}{\beta}}}{\alpha} \quad \text{and} \quad \Phi(\xi_n, \tau_n, y_n, s_n) > \frac{\theta}{2}.$$

Note that $\alpha|\xi_n - y_n|^\beta < 2\eta$ and $\alpha|\tau_n - \hat{t}\,| < (2\eta)^{\frac{1}{2}+\frac{1}{2\beta}}$.

Once again, reselecting α large enough if needed, we may assume that

$$\Big(\frac{2\eta}{\alpha}\Big)^{\frac{1}{\beta}} \vee \frac{(2\eta)^{\frac{1}{2}+\frac{1}{2\beta}}}{\alpha} < \frac{\gamma}{2},$$

and, by (6.10), we have $0 < \tau_n < T$. Setting

$$\varphi(x,t) = -p\cdot x + (\alpha + \alpha^{\frac{1}{\beta}})|x-y_n|^{\beta+1} + \alpha|t-s_n|^2 + \alpha|t-\hat{t}\,|^2 \quad \text{for } (x,t)\in \overline{Q}_T$$

and noting that $\varphi \in \mathcal{F}_{\beta,p}(Q_T)$ and u is a solution of (6.7) in Q_T, we get

$$\varphi_t(\xi_n,\tau_n) + \gamma \le \big(c(\xi_n,\tau_n) + M_p^+[u(\cdot,\tau_n)](\xi_n)\big)\,|p+\varphi_x(\xi_n,\tau_n)| \tag{6.11}$$

if $\varphi_x(\xi_n,\tau_n) \neq 0$, and otherwise

$$\varphi_t(\xi_n,\tau_n) + \gamma \le 0. \tag{6.12}$$

Note that for any $z \in \mathbb{R}$, $(u-\varphi)(\xi_n+z,\tau_n) \le (u-\varphi)(\xi_n,\tau_n)$ and for any $(x,t) \in Q_T$,

$$|p+\varphi_x(x,t)| = (\alpha+\alpha^{\frac{1}{\beta}})(\beta+1)|x-y_n|^\beta \le 2\alpha(\beta+1)|x-y_n|^\beta.$$

Hence, if $|z| \le |p+\varphi_x(\xi_n,\tau_n)|^{\frac{1}{\beta}}$, then

$$\begin{aligned} u(\xi_n+z,\tau_n) - u(\xi_n,\tau_n) &\le \varphi_x(\xi_n,\tau_n)\cdot z \\ &\qquad + \big(\alpha+\alpha^{\frac{1}{\beta}}\big)(\beta+1)\beta(|\xi_n-y_n|+|z|)^{\beta-1}|z|^2 \\ &\le \varphi_x(\xi_n,\tau_n)z + C_3|p+\varphi_x(\xi_n,\tau_n)|^{1-\frac{1}{\beta}}z^2, \end{aligned}$$

where $C_3 > 0$ is a constant depending only on α and β. By Lemma 6.3, there is a modulus ω, independent of n, such that if $0 < |p+\varphi_x(\xi_n,\tau_n)| \le 1$, then

$$M_p^+[u(\cdot,\tau_n)](\xi_n)\,|p+\varphi_x(\xi_n,\tau_n)| \le \omega(|p+\varphi_x(\xi_n,\tau_n)|) \le \omega(4(\beta+1)\eta).$$

This together with (6.11) and (6.12) yields $\varphi_t(\xi_n,\tau_n)+\gamma \le \omega(4(\beta+1)\eta)$ if n is large enough. Hence, for n sufficiently large, we have

$$\gamma \le \omega(4(\beta+1)\eta) - 2\alpha(\hat{t}-s_n) + 4(2\eta)^{\frac{1}{2}+\frac{1}{2\beta}}.$$

Sending $n \to \infty$, we get

$$\gamma \le \omega(4(\beta+1)\eta) + 2\alpha(\hat{s}-\hat{t}\,) + 4(2\eta)^{\frac{1}{2}+\frac{1}{2\beta}}. \tag{6.13}$$

Choosing a minimum point of the function

$$(y,s) \mapsto \tilde{v}(y,s) + (\alpha+\alpha^{\frac{1}{\beta}})|x_n-y|^{\beta+1} + \alpha|t_n-s|^2 + \alpha|s-\hat{s}|^2 \quad \text{on } \overline{Q}_T$$

and repeating an argument similar to the above, we get

$$0 \geq -\omega(4(\beta+1)\eta) - 4(2\eta)^{\frac{1}{\beta}} + 2\alpha(\hat{s} - \hat{t}\,)$$

Subtracting this from (6.13), we obtain $\gamma \leq 2\omega(4(\beta+1)\eta) + 8(2\eta)^{\frac{1}{2}+\frac{1}{2\beta}}$, which gives a contradiction by selecting $\eta > 0$ small enough.

`Case` 2: $\liminf_{\alpha\to\infty} \lambda_\alpha > 0$. The argument for Case 2 of the proof of Theorem 4.1 applies to get a contradiction only with obvious modifications caused by the term $\alpha|x-y|^{\beta+1}$ in the definition of Φ_α. We leave it to the interested reader to check the details. □

The same proposition as Theorem 5.1 holds under our current assumptions.

Theorem 6.8. *Let* $u_0 \in \mathrm{BUC}(\mathbb{R})$*. Then there is a unique solution* $u \in C(\overline{Q}_\infty)$ *of* (1.1) *such that* $u(\cdot,0) = u_0$ *and* $u \in \mathrm{BUC}(Q_T)$ *for any* $0 < T < \infty$.

Proof. The uniqueness assertion is a direct consequence of Theorem 6.7.

To prove the existence of a solution, we will utilize Theorem 6.6. Hence, we have to build appropriate sub- and supersolutions of (1.1).

Fix any $\varepsilon > 0$. Let $A > 0$ and observe that for any $x \in \mathbb{R}$,

$$-px \geq -\frac{\beta|p|^{1+\frac{1}{\beta}}}{(\beta+1)A^{\frac{1}{\beta}}} - \frac{A}{\beta+1}|x|^{\beta+1}$$

and hence

$$-px + A|x|^{\beta+1} \geq -\frac{\beta|p|^{1+\frac{1}{\beta}}}{(\beta+1)A^{\frac{1}{\beta}}} + \frac{\beta A}{\beta+1}|x|^{\beta+1}.$$

We fix $A = A(\varepsilon) > 0$ so large that $\varepsilon > \frac{\beta|p|^{1+\frac{1}{\beta}}}{(\beta+1)A^{\frac{1}{\beta}}}$, and consequently,

$$2\varepsilon - px + A|x|^{\beta+1} \geq \varepsilon + \frac{\beta A}{\beta+1}|x|^{\beta+1} \quad \text{for all } x \in \mathbb{R}.$$

Let ω_0 be the modulus of continuity of the function u_0. By replacing A by a larger number if necessary, we may assume that $\omega_0(r) \leq \varepsilon + \frac{\beta A}{\beta+1} r^{\beta+1}$ for all $r \geq 0$. We have

$$u_0(x) - u_0(y) \leq \varepsilon - p(x-y) + A|x-y|^{\beta+1} \quad \text{for all } x, y \in \mathbb{R}.$$

We choose a constant $C_1 > 0$ so that $|u_0(x)| \leq C_1$ for all $x \in \mathbb{R}$. We set

$$\psi_1(x,y,\varepsilon) = \big(u_0(y) + 2\varepsilon - p(x-y) + A(\varepsilon)|x-y|^{\beta+1}\big) \wedge C_1$$

for $(x,y,\varepsilon) \in \mathbb{R}^2 \times (0,1)$. Observe that for all $(x,y,\varepsilon) \in \mathbb{R}^2 \times (0,1)$,

$$\psi_1(x,x,\varepsilon) = (u_0(x) + 2\varepsilon) \wedge C_1 \quad \text{and} \quad u_0(x) \leq \psi_1(x,y,\varepsilon), \tag{6.14}$$

and that for each $\varepsilon \in (0,1)$, the functions $\psi_1(\cdot, y, \varepsilon)$, with $y \in \mathbb{R}$, are equi-Lipschitz continuous on $\mathbb{R}$.

Let $(x, y, \varepsilon) \in \mathbb{R}^2 \times (0,1)$ be such that $\psi_1(x, y, \varepsilon) < C_1$. As observed before, setting $\psi_2(\xi) = A|\xi - y|^{\beta+1}$, there exists a constant $C_2 > 0$ such that

$$\psi_2(x+z) \le \psi_2(x) + \psi_2'(x)z + C_2|\psi_2'(x)|^{1-\frac{1}{\beta}}z^2 \quad \text{for all } z \in B(0,\, |\psi_2'(x)|^{\frac{1}{\beta}}).$$

Hence, if $|z| \le |p + \psi_{1,x}(x, y, \varepsilon)|^{\frac{1}{\beta}}$, we have

$$\psi_1(x+z, y, \varepsilon) \le \psi_1(x, y, \varepsilon) + \psi_{1,x}(x, y, \varepsilon)z + C_2|p + \psi_{1,x}(x, y, \varepsilon|^{1-\frac{1}{\beta}}z^2.$$

On the other hand, since $\psi(\cdot, y, \varepsilon)$ is semi-concave, there is a constant $C_3 > 0$ such that

$$\psi_1(x+z, y, \varepsilon) \le \psi_1(x, y, \varepsilon) + \psi_{1,x}(x, y, \varepsilon)z + C_3 z^2 \quad \text{for all } z \in \mathbb{R}.$$

Note here that the constants C_2, C_3 can be chosen independently of $y \in \mathbb{R}$. Thus, applying Lemma 6.3 if $0 < |p + \psi_{1,x}(x, y, \varepsilon)| \le 1$ and and Lemma 6.2 if $|p + \psi_{1,x}(x, y, \varepsilon)| \ge 1$, we find a constant $C_4 > 0$, independent of y, such that

$$M_p^+[\psi_1(\cdot, y, \varepsilon)](x)|p + \psi_{1,x}(x, y, \varepsilon)| \le C_4.$$

Next, let $(x, y, \varepsilon) \in \mathbb{R}^2 \times (0,1)$ be such that $\psi_1(x, y, \varepsilon) = C_1$ and $\psi_1(\cdot, y, \varepsilon)$ is subdifferentiable at x. Clearly, we have $\psi_{1,x}(x, y, \varepsilon) = 0$. If $p = 0$, then we have $p + \psi_{1,x}(x, y, \varepsilon) = 0$. Assume for the moment that $p > 0$ and observe that

$$E^+(\psi_1(x+z, y, \varepsilon) - \psi_1(x, y, \varepsilon), z) \le \begin{cases} \dfrac{1}{2} - pz & \text{if } 0 \le z < \dfrac{1}{p} \\ -\dfrac{1}{2} - pz & \text{if } -\dfrac{1}{p} < z < 0 \\ \dfrac{1}{2} & \text{for all } z \in \mathbb{R}. \end{cases}$$

Accordingly, we have

$$\begin{aligned} M_p^+[\psi_1(\cdot, y, \varepsilon)](x)|p + \psi_{1,x}(x, y, \varepsilon)| &\le \frac{1}{2}\int_{|z| > \frac{1}{|p|}} J(z)\mathrm{d}z|p| \\ &\le |p|\|J\|_{L^1(B(0,1/|p|)^c)}. \end{aligned} \tag{6.15}$$

Similarly, we have (6.15) also in the case where $p < 0$. We may assume by replacing C_3 by a larger number if necessary that if $p + \psi_{1,x}(x, y, \varepsilon) \ne 0$, then

$$M_p^+[\psi_1(\cdot, y, \varepsilon)](x)|p + \psi_{1,x}(x, y, \varepsilon)| \le C_3.$$

We now set

$$\varphi(x,t,y,\varepsilon)=\psi_1(x,y,\varepsilon)+C_3(\varepsilon)t \quad \text{for all } (x,t,y,\varepsilon)\in\overline{Q}_\infty\times\mathbb{R}\times(0,1),$$

where the symbol $C_3(\varepsilon)$ is used to emphasize the dependence of C_3 on ε, and observe that for each (y,ε), the function $(x,t)\mapsto\varphi(x,t,y,\varepsilon)$ is a supersolution of (1.1).

Now, we define the function $f^+\in\mathrm{USC}(\overline{Q}_\infty)$ by

$$f^+(x,t)=\inf\{\varphi(x,t,y,\varepsilon)\mid(y,\varepsilon)\in\mathbb{R}\times(0\,1)\}.$$

By a proposition valid for supersolutions analogous to Theorem 6.7, we see that f^+ is a supersolution of (1.1). Moreover we observe by (6.14) that for all $(x,t,\varepsilon)\in\overline{Q}_\infty\times(0,1)$,

$$\begin{aligned}&f^+(x,t)\le\varphi(x,t,x,\varepsilon)=\psi_1(x,x,\varepsilon)+C_3(\varepsilon)t=u_0(x)+2\varepsilon+C_3(\varepsilon)t\\&f^+(x,t)\ge u_0(x).\end{aligned}$$

Similarly to the above, we can find a function $f^-\in\mathrm{USC}(\overline{Q}_\infty)$ having the properties: f^- is a subsolution of (1.1) and

$$u_0(x)\ge f^-(x,t)\ge u_0(x)-2\varepsilon-C_3(\varepsilon)t \quad \text{for all } (x,t,\varepsilon)\in\overline{Q}_\infty\times(0,1).$$

Now, applying Theorem 6.6, with $f=f^-$ and $g=f^+$, we see that there is a solution u, defined on $\overline{Q}_\infty$, of (1.1) such that $f^-(x,t)\le u(x,t)\le f^+(x,t)$ for all $(x,t)\in\overline{Q}_\infty$. It is clear that $u(x,0)=u_0(x)$ for all $x\in\mathbb{R}$ and u is bounded on $\overline{Q}_T$ for any $0<T<\infty$. Since u_0 is uniformly continuous on $\mathbb{R}$ and $|u(x,t)-u_0(x)|\le 2\varepsilon+C_3(\varepsilon)t$ for all $(x,t,\varepsilon)\in\overline{Q}_\infty$, we have

$$\lim_{r\to0+}\sup\big\{u(x,t)-u(y,s)\mid|x-y|\vee t\vee s\le r\big\}=0.$$

Using Theorem 6.7, we find that $u\in C(\overline{Q}_\infty)$. Moreover, we see that for each $0<T<\infty$ there is a modulus ω_T such that $|u(x,t)-u(y,t)|\le\omega_T(|x-y|)$ for all $(x,y,t)\in\mathbb{R}^2\times[0,\,T]$.

Let $0<\tau<T<\infty$. Similarly to the construction of $f^\pm$, we can build functions $f_\tau^\pm$, starting with ω_T in place of ω_0, such that f_τ^+ and f_τ^- are super- and subsolutions of (1.1) in $\mathbb{R}\times(\tau,\,\infty)$, respectively, and that for all $(x,t,\varepsilon)\in\mathbb{R}\times[\tau,\,\infty)\times(0,1)$ and for some constant $C_T(\varepsilon)>0$,

$$u(x,t)\begin{cases}\le f_\tau^+(x,t)\le u(x,\tau)+\varepsilon+C_T(\varepsilon)\\ \ge f_\tau^-(x,t)\ge u(x,\tau)-\varepsilon-C_T(\varepsilon)(t-\tau).\end{cases}$$

It is now obvious that $u\in\mathrm{BUC}(\overline{Q}_T)$ for any $0<T<\infty$. □

Appendix

Proposition A.1. *Let f be a real-valued function on $\mathbb{R}^N$. Let $C > 0$ and assume that $|f(x)| \le C$ for all $x \in \mathbb{R}^N$ and that the function: $x \mapsto f(x) + C|x|^2$ is convex in $\mathbb{R}^N$. Then we have*

$$|f(x) - f(y)| \le 4C|x - y| \quad \text{for all } x, y \in \mathbb{R}^N.$$

Proof. Fix any $y \in \mathbb{R}^n$ and set $g(x) = f(x) + C|x - y|^2$ for $x \in \mathbb{R}^N$. Note that the function $g(x) = f(x) + C|x|^2 - 2Cy \cdot x + C|y|^2$ is also convex in $\mathbb{R}^N$, $g(x) \le 2C$ for all $x \in B(y, 1)$ and $g(y) = f(y)$. Fix any $x \in B(y, 1) \setminus \{y\}$, set $\xi = (x - y)/|x - y| \in \partial B(0, 1)$ and observe by the convexity of g that

$$\begin{aligned} g(x) &= g(y + |x - y|\xi) \\ &\le (1 - |x - y|)g(y) + |x - y|g(y + \xi) \\ &\le (1 - |x - y|)f(y) + 2C|x - y|. \end{aligned}$$

That is, we have

$$\begin{aligned} f(x) + C|x - y|^2 &\le (1 - |x - y|)f(y) + 2C|x - y| \\ &\le f(y) + 3C|x - y| \quad \text{for all } x \in B(y, 1) \setminus \{y\}, \end{aligned}$$

which is obviously valid for $x = y$. Thus, for any $x, y \in \mathbb{R}^N$, we have

$$f(x) - f(y) \le 4C|x - y| \quad \text{if } |x - y| \le 1,$$

which implies that $|f(x) - f(y)| \le 4C|x - y|$ for all $x, y \in \mathbb{R}^N$. □

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