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Smoothness Properties of the Lower Semicontinuous Quasiconvex Envelope

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Abstract. Assume that $\mathrm{K} \subset \mathbb{R}^{nm}$ is a convex body with $o \in \operatorname{int}(\mathrm{K})$ and $\widetilde{f} : \mathbb{R}^{nm} \to \mathbb{R}$ is a Lipschitz resp. C^1-function. Defining the unbounded function $f : \mathbb{R}^{nm} \to \mathbb{R} \cup \{(+\infty)\}$ through

$$f(v) = \begin{cases} \widetilde{f}(v), & v \in \mathrm{K} \\ (+\infty), & v \in \mathbb{R}^{nm} \setminus \mathrm{K}, \end{cases}$$

we provide sufficient conditions in order to guarantee that its lower semicontinuous quasiconvex envelope

$$f^{(qc)}(w) = \sup \left\{ g(w) \;\middle|\; \begin{array}{l} g : \mathbb{R}^{nm} \to \mathbb{R} \cup \{(+\infty)\} \text{ quasiconvex and} \\ \text{lower semicontinuous, } g(v) \leqslant f(v) \; \forall\, v \in \mathbb{R}^{nm} \end{array} \right\}$$

is globally Lipschitz continuous on K or differentiable in $v \in \operatorname{int}(\mathrm{K})$, respectively. An example shows that the partial derivatives of $f^{(qc)}$ do not necessarily admit a representation with a "supporting measure" for f^{qc} in v_0.

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Mathematics Subject Classification (2000). 26B25, 26B35, 26E25, 49J53

1. Introduction

1.1. The lower semicontinuous quasiconvex envelope. The present paper is motivated by the study of multidimensional control problems of Dieudonné-Rashevsky type, which will be obtained from the basic problem of multidimensional calculus of variations

$$F(x) = \int_\Omega r(t, x(t), Jx(t))\, dt \longrightarrow \inf! \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^n), \;\; \Omega \subset \mathbb{R}^m \tag{1.1}$$

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by incorporation of additional restrictions for the partial derivatives of x, e.g.,

$$Jx(t) = \begin{pmatrix} \dfrac{\partial x_1}{\partial t_1}(t) & \dots & \dfrac{\partial x_1}{\partial t_m}(t) \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial t_1}(t) & \dots & \dfrac{\partial x_n}{\partial t_m}(t) \end{pmatrix} \in \mathrm{K} \subset \mathbb{R}^{nm} \quad (\forall)\, t \in \Omega\,. \tag{1.2}$$

Problems of this kind result from the study of underdetermined boundary value problems for nonlinear first-order PDE's (cf. [14, 15] and [16]), as optimization problems for convex bodies under geometrical restrictions (cf. [1] , [2, p. 149 f.]), in elasticity theory (torsion problems) (cf. [23, pp. 531 ff.] , [30, pp. 240 ff.] , [34, p. 531 f.] , [35] and [36, pp. 76 ff.]), in population dynamics (age-structured problems) (cf. [9] and [20]) and, recently, in the framework of image processing (cf. [11] , [22] , [37, pp. 108 ff.] , [44] and [45]). All mentioned applications have in common that the gradient restriction (1.2) is related to a *convex body* K with $o \in \operatorname{int}(\mathrm{K})$. The integrand $r(t, \xi, v)$ in (1.1) is a possibly nonconvex function of v, [1] whose natural range of definition is the subset $\Omega \times \mathbb{R}^n \times \mathrm{K}$ instead of the whole space.

In order to guarantee the existence of global minimizers in Dieudonné-Rashevsky type problems (1.1)–(1.2) with $n \geqslant 2$, $m \geqslant 2$ (and, at the same time, to justify the application of direct methods for their numerical solution), the relaxation of the problems must be based — in analogy to the multidimensional calculus of variations — on a generalized notion of convexity, cf. [39, p. 309, Theorem 1.3] and [40, p. 4, Theorem 1.4] . From the author's previous papers (cf. [39]–[43]), it is known that the case of general integrands $r(t, \xi, v)$ can be reduced to the special case where the integrand depends on v only. [2] Consequently, in the present paper we confine ourselves to the investigation of integrands $f(v) : \mathrm{K} \to \mathbb{R}$, which will be extended by $(+\infty)$ to $\mathbb{R}^{nm} \setminus \mathrm{K}$, and their *lower semicontinuous quasiconvex envelope* as the appropriate semiconvex envelope. More precisely, we study functions within the following class:

Definition 1.1 (*Function class* F_{K})**.** Let $\mathrm{K} \subset \mathbb{R}^{nm}$ be a given convex body with $o \in \operatorname{int}(\mathrm{K})$. We say that a function $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{ (+\infty) \}$ belongs to the class F_{K} iff $f \,\big|\, \mathrm{K}$ belongs to $C^0(\mathrm{K}, \mathbb{R})$ and $f \,\big|\, (\mathbb{R}^{nm} \setminus \mathrm{K}) \equiv (+\infty)$.

The notion of quasiconvexity for functions with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{(+\infty)\}$ will be specified as follows:

[1] In quality of examples, we mention polyconvex regularization terms in the hyperelastic image matching problem (cf. [40, pp. 28 ff.]) and regularization terms of Perona–Malik type in the optical flow problem, cf. [3, pp. 90–93] , [27] and [37, p. 114] .

[2] In further analogy to the multidimensional calculus of variations, cf. [13, pp. 369 ff. and 416 ff.] .

Definition 1.2 (*Quasiconvex function with values in* $\overline{\mathbb{R}}$)**.** [42, p. 73, Definition 2.9][3] A function $f\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{(+\infty)\}$ with the following properties is said to be quasiconvex:

1) $\operatorname{dom}(f) \subseteq \mathbb{R}^{nm}$ is a nonempty Borel set;
2) $f \,\big|\, \operatorname{dom}(f)$ is Borel measurable and bounded from below on every bounded subset of $\operatorname{dom}(f)$;
3) for all $v \in \mathbb{R}^{nm}$, f satisfies Morrey's integral inequality:

$$\frac{1}{|\,\Omega\,|} \int_\Omega f(v + Jx(t))\,dt \quad \forall\, x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \tag{1.3}$$

or equivalently

$$f(v) = \inf\left\{ \frac{1}{|\,\Omega\,|} \int_\Omega f(v + Jx(t))\,dt \,\middle|\, x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n) \right\}. \tag{1.4}$$

Here $\Omega \subset \mathbb{R}^m$ is the closure of a bounded strongly Lipschitz domain.

For the lower semicontinuous quasiconvex envelope of a possibly unbounded function, we adopt the following definition:

Definition 1.3 (*Lower semicontinuous quasiconvex envelope* $f^{(qc)}$)**.** [42, p. 76, Definition 2.14, 2)] For any function $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\, (+\infty)\,\}$ bounded from below, we define its lower semicontinuous quasiconvex envelope by

$$f^{(qc)}(v) = \sup\left\{ g(v) \,\middle|\, \begin{array}{l} g : \mathbb{R}^{nm} \to \overline{\mathbb{R}} \text{ quasiconvex and lower} \\ \text{semicontinuous, } g(v) \leqslant f(v)\ \forall\, v \in \mathbb{R}^{nm} \end{array} \right\}. \tag{1.5}$$

Obviously, Definition 1.3. generalizes the formation of the "usual" quasiconvex envelope for a function f with finite values since, in this case, all quasiconvex functions g below f are continuous from the outset.

1.2. Lipschitz continuity and differentiability of $f^{(qc)}$. In the present paper, we provide some results about Lipschitz continuity and differentiability of the lower semicontinuous quasiconvex envelope $f^{(qc)}$ of a function $f \in F_{\mathrm{K}}$ and compare them with the respective properties of the convex envelope f^c. As separately convex functions, $f^{(qc)}$ as well as f^c are locally Lipschitz continuous on $\operatorname{int}(\mathrm{K})$ (cf. Theorems 2.2. and 2.9) and, consequently, λ^{nm}-a. e. differentiable on $\operatorname{int}(\mathrm{K})$. [4] For the convex envelope, these assertions can be sharpened in the following way:

Theorem 1.4. *Let a convex body* $\mathrm{K} \subset \mathbb{R}^{nm}$ *with* $o \in \operatorname{int}(\mathrm{K})$ *and* $\partial\mathrm{K} = \operatorname{ext}(\mathrm{K})$ *and a function* $f \in F_{\mathrm{K}}$ *be given.*

[3] This is a specification of [5, p. 228, Definition 2.1] in the case $p = (+\infty)$.
[4] As a consequence of Rademacher's theorem, cf. [19, p. 81, Theorem 2].

1) (Global Lipschitz continuity of f^c) *Assume that*
   a) *for every point* $v_0 \in \partial \mathrm{K}$*, there exists an affine function* $\varphi(v, v_0) = \langle a(v_0), v - v_0\rangle + b(v_0)$ *with* $\varphi(v, v_0) \leqslant f(v)\ \forall\, v \in \mathrm{K}$*, and*
   b) $\sup_{v_0 \in \partial \mathrm{K}} |\, a(v_0)\,| < (+\infty)$.

   Then the convex envelope f^c *is globally Lipschitz continuous on* K.
2) (Differentiability of f^c on int (K)) [25, p. 701, Corollary 3.1] [5] *Assume that the function* $f \in F_{\mathrm{K}}$ *is defined through*

$$f(v) = \begin{cases} \widetilde{f}(v)\,, & v \in \mathrm{K} \\ (+\infty)\,, & v \in \mathbb{R}^{nm} \setminus \mathrm{K}\,, \end{cases} \tag{1.6}$$

   where $\widetilde{f} : \mathbb{R}^{nm} \to \mathbb{R}$ *is a continuous function, which is continuously differentiable on* K. *Then the convex envelope* f^c *is continuously differentiable on* int (K).

In the case of the quasiconvex envelope of a finite function $f : \mathbb{R}^{nm} \to \mathbb{R}$ bounded from below, Ball, Kirchheim and Kristensen [4, p. 334, Theorem A] have been proved that the differentiability of f together with some growth conditions implies the differentiability of f^{qc}. Then the partial derivatives of f^{qc} admit a representation

$$\frac{\partial f^{qc}}{\partial v_{ij}}(v_0) = \int_{\mathbb{R}^{nm}} \frac{\partial f}{\partial v_{ij}}(v)\, d\nu(v)\,, \quad 1 \leqslant i \leqslant n\,,\, 1 \leqslant j \leqslant m\,, \tag{1.7}$$

with a "supporting measure" for f^{qc} in v_0, i.e., a positive measure ν resulting as the weak*-limit of a sequence of probability measures $\{\, \nu^N \,\}$ with $f^{qc}(v_0) \leqslant \int_{\mathbb{R}^{nm}} f(v)\, d\nu^N(v) \leqslant f^{qc}(v_0) + \frac{1}{N}$ and $(v_0)_{ij} = \int_{\mathbb{R}^{nm}} v_{ij}\, d\nu^N(v)$.

The proof of analogous assertions for the lower semicontinuous quasiconvex envelope of a function $f \in F_{\mathrm{K}}$ is confronted with serious difficulties. However, we were able to prove the following sufficient conditions for global Lipschitz continuity and differentiability of $f^{(qc)}$:

Theorem 1.5 (Sufficient condition for global Lipschitz continuity of $f^{(qc)}$). [6] *Let a convex body* $\mathrm{K} \subset \mathbb{R}^{nm}$ *with* $o \in \mathrm{int}\,(\mathrm{K})$ *and* $\partial\mathrm{K} = \mathrm{ext}\,(\mathrm{K})$ *and a function* $f \in F_{\mathrm{K}}$ *be given, which is globally Lipschitz continuous on* K. *Assume further that:*

a) *for every point* $v_0 \in \partial \mathrm{K}$*, there exists an affine function* $\varphi(v, v_0) = \langle a(v_0), v - v_0\rangle + b(v_0)$ *with* $\varphi(v, v_0) \leqslant f(v)$ *for all* $v \in \mathrm{K}$*; and*

[5] Loc. cit., it has been assumed instead that $\partial\mathrm{K}$ coincides with a $(nm-1)$-dimensional C^1-manifold. This will be implied by the stronger condition $\partial\mathrm{K} = \mathrm{ext}\,(\mathrm{K})$, cf. [7, p. 26] .

[6] By this theorem, [37, p. 76, Theorem 5.6] will be corrected. In the assertions [37, pp. 97 ff.] , the premisses must be adapted in the same way.

b) $\sup_{v_0 \in \partial \mathrm{K}} |\, a(v_0)| < (+\infty)$.

Then the lower semicontinuous quasiconvex envelope $f^{(qc)}$ *is globally Lipschitz continuous on* K *as well.*

Theorem 1.6 (Sufficient condition for differentiability of $f^{(qc)}$ in $v_0 \in \mathrm{int}\,(\mathrm{K})$)**.** [7] *Assume that a function* $f \in F_{\mathrm{K}}$ *is defined through*

$$f(v) = \begin{cases} \widetilde{f}(v)\,, & v \in \mathrm{K} \\ (+\infty)\,, & v \in \mathbb{R}^{nm} \setminus \mathrm{K}\,, \end{cases} \tag{1.8}$$

where $\widetilde{f} : \mathbb{R}^{nm} \to \mathbb{R}$ *is a continuous function, which is continuously differentiable on some open neighbourhood of* K. *Assume further that, in relation to a point* $v_0 \in \mathrm{int}\,(\mathrm{K})$*, there exist a probability measure* $\nu_0 \in \mathrm{S}^{(qc)}(v_0)$*, a function sequence* $\{\, x^N \,\}\,, W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ *and a number* $0 < \mu < 1$ *with the following properties:*

a) $f^{(qc)}(v_0) \;=\; \int_{\mathrm{K}} f(v)\, d\nu_0(v)\,,$

b) *the constant generalized control* $\boldsymbol{\nu} = \{\, \nu_0 \,\}$ *is generated by the sequence* $\{\, v_0 + Jx^N \,\}\,,$

c) *for almost all* $t \in \Omega$ *and all* $N \in \mathbb{N}$*, it holds that* $v_0 + Jx^N(t) \in \mu\,\mathrm{K}$.

Then $f^{(qc)}$ *is differentiable in* v_0*, and for all indices* $1 \leqslant i \leqslant n$*,* $1 \leqslant j \leqslant m$*, it holds that*

$$\frac{\partial f^{(qc)}}{\partial v_{ij}}(v_0) = \int_{\mathrm{K}} \frac{\partial \widetilde{f}}{\partial v_{ij}}(v)\, d\nu_0(v)\,. \tag{1.9}$$

The set $\mathrm{S}^{(qc)}(v_0) \subseteq \big(C^0(\mathrm{K}, \mathbb{R})\big)^*$ *will be described in Definition* 2.11 *below. In particular, it contains all "supporting measures" for* $f^{(qc)}$ *in* v_0 *(cf. Theorem* 2.12*).*

The differentiability of $f^{(qc)}$ can be ensured further in all points $v \in \mathrm{int}\,(\mathrm{K})$ where f and $f^{(qc)}$ coincide. This is the case, in particular, in those global minimizers of f, which are situated in the interior of K (Theorem 3.2). Finally, we provide an example showing that the partial derivatives of $f^{(qc)}$, even in the case of their existence, do not necessarily admit a representation of the type (1.7):

Theorem 1.7 (Counterexample for the representation of $\nabla f^{(qc)}$ through a "supporting measure")**.** *There exist a convex body* $\mathrm{K} \subset \mathbb{R}^{2\times 2}$ *and a function* $f \in F_{\mathrm{K}}$ *such that one can find a point* $v_0 \in \mathrm{int}\,(\mathrm{K})$ *with the following properties:* $f^{(qc)}$ *is differentiable in* v_0 *but for every probability measure* $\nu_0 \in \mathrm{S}^{(qc)}(v_0)$

[7] This corrects [37, p. 76, Theorem 5.5.]. I am indebted to Prof. Kirchheim (Düsseldorf) who identified a mistake in the proof ibid., p. 78 f.: Lemma 5.9 does not imply (5.37).

with $f^{(qc)}(v_0) = \int_{\mathrm{K}} f(v)\, d\nu_0(v)$*, there exists at least one pair* (i,j) *of indices* $1 \leqslant i \leqslant 2$*,* $1 \leqslant j \leqslant 2$ *with*

$$\frac{\partial f^{(qc)}}{\partial v_{ij}}(v_0) \neq \int_{\mathrm{K}} \frac{\partial f}{\partial v_{ij}}(v)\, d\nu_0(v)\,. \tag{1.10}$$

The same example shows that the assumption $\partial\mathrm{K} = \mathrm{ext}\,(\mathrm{K})$ cannot be removed from Theorem 1.4, 1) and Theorem 1.5 (Lemma 3.5, 3)).

We close this section with a synopsis of notations and abbreviations to be used in the paper. In Section 2, we collect first some tools from generalized convexity and the theory of generalized controls ("Young measures"). Then we summarize the present knowledge about the analytical and structural properties of the lower semicontinuous quasiconvex envelope $f^{(qc)}$. Section 3 contains the announced theorems and proofs.

1.3. Notations and abbreviations. Let $k \in \{0, 1, \ldots, \infty\}$ and $1 \leqslant p \leqslant \infty$. Then $C^k(\Omega, \mathbb{R}^r)$, $L^p(\Omega, \mathbb{R}^r)$ and $W^{k,p}(\Omega, \mathbb{R}^r)$ denote the spaces of r-dimensional vector functions whose components are k-times continuously differentiable, belong to the $L^p(\Omega)$ or to the Sobolev spaces of $L^p(\Omega)$-functions with weak derivatives up to kth order in $L^p(\Omega)$, respectively. In addition, functions within the subspaces $C_0^k(\Omega, \mathbb{R}^r) \subset C^k(\Omega, \mathbb{R}^r)$ and $W_0^{1,p}(\Omega, \mathbb{R}^r) \subset W^{1,p}(\Omega, \mathbb{R}^r)$ are compactly supported; the components of $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^r)$ admit a Lipschitz continuous representative [19, p. 131, Theorem 5] with zero boundary values. By $\frac{\partial x}{\partial t_j}$ we denote the classical partial derivative of x by t_j. In the abbreviation Jx for the Jacobi matrix of x, however, we will not distinguish between classical and weak derivatives.

We denote by $\mathrm{int}\,(\mathrm{A})$, $\partial\mathrm{A}$, $\mathrm{cl}\,(\mathrm{A})$, $\mathrm{co}\,(\mathrm{A})$ and $|\,\mathrm{A}\,|$ the interior, the boundary, the closure, the convex hull and the r-dimensional Lebesgue measure of the set $\mathrm{A} \subseteq \mathbb{R}^r$, respectively. Further, we define $\overline{\mathbb{R}} = \mathbb{R} \cup (+\infty)$ and equip $\overline{\mathbb{R}}$ with the natural topological and order structures where $(+\infty)$ is the greatest element.

Throughout the whole paper, we consider only *proper functions* $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$, assuming that $\mathrm{dom}\,(f) = \{v \in \mathbb{R}^{nm} \,\big|\, f(v) < (+\infty)\}$ is always nonempty. The restriction of the function f to the subset A of its range will be denoted by $f \,\big|\, \mathrm{A}$. If a function $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ belongs to the function class F_{K} defined above then its restriction $f \,\big|\, \mathrm{K}$ is bounded and (even uniformly) continuous. Thus F_{K} and the Banach space $C^0(\mathrm{K}, \mathbb{R})$ are isomorphical and isometrical. Due to the compactness of K, the dual space $\big(C^0(\mathrm{K}, \mathbb{R})\big)^*$ is isomorphical to the space $rca\,(\mathrm{K})$ of the signed regular measures acting on the σ-algebra of the Borel subsets of K. The subset of the probability measures will be denoted by $rca^{\,pr}\,(\mathrm{K})$.

A convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ is understood as a convex, compact set with nonempty interior (we follow [10] and [33]). A point $v \in \mathrm{K}$ is called extremal point of K iff $v = \lambda' v' + \lambda'' v''$, λ', $\lambda'' > 0$, $\lambda' + \lambda'' = 1$, v', $v'' \in$ K always imply $v' = v'' = v$. The set of all extremal points of K is denoted by ext (K). Every convex body possesses at least one extremal point.

We close with the introduction of the following *three nonstandard notions.* "$\{ x^N \}$, A" denotes a sequence $\{ x^N \}$ with members $x^N \in$ A. If A $\subseteq \mathbb{R}^r$, then the abbreviation "$(\forall)\, t \in$ A" has to be read as "for almost all $t \in$ A" resp. "for all $t \in$ A except a r-dimensional Lebesgue null set". The symbol o denotes, depending on the context, the zero element resp. the zero function of the underlying space.

2. Tools for the investigation of $f^{(qc)}$

2.1. Generalized notions of convexity. We start with an overview of the generalized convexity notions to be used in the present paper.

Definition 2.1.

1) (*Polyconvex function*) We consider $v \in \mathbb{R}^{nm}$ as a (n, m)-matrix and collect all subdeterminants of v within a vector $T(v)$ with dimension $\tau(n, m)$. A function $f :\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is said to be polyconvex iff there exists a convex function $g :\ \mathbb{R}^{\tau(n,m)} \to \overline{\mathbb{R}}$ with $f(v) = g(\,T(v))\ \forall\, v \in \mathbb{R}^{nm}$.
2) (*Rank one convex function*) A function $f :\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is said to be rank one convex iff Jensen's inequality is satisfied in any rank one direction: for all v', $v'' \in \mathbb{R}^{nm}$ (considered as (n, m)-matrices) it holds that
$$\begin{aligned} &\operatorname{Rg}(v' - v'') \leqslant 1 \implies \\ &f(\lambda' v' + \lambda'' v'') \leqslant \lambda' f(v') + \lambda'' f(v'') \quad \forall\, \lambda',\, \lambda'' \geqslant 0\,,\ \lambda' + \lambda'' = 1\,. \end{aligned} \tag{2.1}$$
3) (*Separately convex function*) A function $f :\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is said to be separately convex iff it is convex in every variable v_{ij} while the other arguments are fixed.

For functions $f :\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$, the following implications hold: f convex $\implies$ f polyconvex $\implies$ f rank one convex $\implies$ f separately convex, cf. [13, p. 159 f., Theorem 5.3, (i), and Remark 5.4, (iii)]. [8] f^c, f^{pc}, f^{qc} and f^{rc} denote the convex, polyconvex, quasiconvex (in the usual sense; cf. [13, p. 156 f., Definition 5.1 (ii)]) and rank one convex envelope of a given function f, i.e., the largest function below f with the respective convexity property. The following theorem states that local Lipschitz continuity can be guaranteed even for separately convex functions.

[8] The notion of quasiconvexity cannot be classified within this sequence without additional assumptions (note, however, Theorem 2.9 below, as well as [12]).

Theorem 2.2 (Local Lipschitz continuity of separately convex functions)**.** [13, p. 47, Theorem 2.31] *Every separately convex function* $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *is locally Lipschitz continuous on* $\operatorname{int}(\operatorname{dom}(f))$.

From [4], we take the following differentiability theorem:

Theorem 2.3 (Differentiability of separately convex functions)**.** [4, p. 341, Corollary 2.5] *Consider the closed ball* $\mathrm{K}(v_0, \delta) \subset \mathbb{R}^{nm}$. *Let two functions* φ', $\varphi'' : \mathrm{K}(v_0, \delta) \to \mathbb{R}$ *with* $\varphi'(v_0) = \varphi''(v_0)$ *and* $\varphi'(v) \leqslant \varphi''(v)$ *for all* $v \in \mathrm{K}(v_0, \delta)$ *be given. Assume further that* φ' *is separately convex, and that for* φ'' *there exists a vector* $a \in \mathbb{R}^{nm}$ *with*

$$\limsup_{w \to o} \frac{1}{|\, w\,|} \Big(\varphi''(v_0 + w) - \varphi''(v_0) - a^{\mathrm{T}} w \Big) \leqslant 0\,. \tag{2.2}$$

Then φ' *as well as* φ'' *are differentiable in* v_0 *with* $\nabla \varphi'(v_0) = \nabla \varphi''(v_0)$.

2.2. Generalized controls. A measure-valued map $\boldsymbol{\mu} : \Omega \to rca^{pr}(\mathrm{K})$ with $t \longmapsto \mu_t$ is called a *generalized control* ("Young measure") if, for any continuous function $g \in C^0(\mathrm{K}, \mathbb{R})$, the function $h_g(t) = \int_{\mathrm{K}} g(v)\, d\mu_t(v)$ is Borel measurable on Ω, cf. [24, pp. 23 ff.] and [31, pp. 115 ff.]. Two generalized controls $\boldsymbol{\mu}' = \{ \mu'_t \}$ and $\boldsymbol{\mu}'' = \{ \mu''_t \}$ will be identified if $\mu'_t \equiv \mu''_t$ holds for almost all $t \in \Omega$. The set of all equivalence classes of generalized controls will be denoted by $\boldsymbol{\mathcal{Y}}(\mathrm{K})$. The convergence of a sequence $\{ \boldsymbol{\mu}^N \}$, $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ towards the limit $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ is defined through

$$\begin{aligned} \boldsymbol{\mu}^N \to \boldsymbol{\mu} \iff & \int_\Omega \int_{\mathrm{K}} f(t)\, g(v) \big(d\mu_t^N(v) - d\mu_t(v) \big) dt \to 0 \\ & \text{for all } f \in L^1(\Omega, \mathbb{R}),\ g \in C^0(\mathrm{K}, \mathbb{R}). \end{aligned} \tag{2.3}$$

Definition 2.4 (*Generating sequences for generalized controls*)**.** [32, pp. 96 ff.] We say that the sequence $\{ u^N \}$, $L^\infty(\Omega, \mathbb{R}^{nm})$ generates the generalized control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ if $u^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall N \in \mathbb{N}$ and

$$\begin{aligned} \lim_{N \to \infty} \int_\Omega f(t)\, g(\, u^N(t)\,)\, dt &= \lim_{N \to \infty} \int_\Omega \int_{\mathrm{K}} f(t)\, g(v)\, d\delta_{u^N(t)}(v)\, dt \\ &= \int_\Omega \int_{\mathrm{K}} f(t)\, g(v)\, d\mu_t(v)\, dt \end{aligned} \tag{2.4}$$

for all $f \in L^1(\Omega, \mathbb{R})$ and $g \in C^0(\mathrm{K}, \mathbb{R})$.

Definition 2.5 (*Generalized gradient controls, "gradient Young measures"*)**.** [28, p. 333] and [31, p. 126, Definition 4.1] A measure-valued map $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ is called a generalized gradient control if it is generated (in the sense of Definition 2.4.) by a sequence $\{ Jx^N \}$, $L^\infty(\Omega, \mathbb{R}^{nm})$ with $x \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ and $Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall N \in \mathbb{N}$. The set of equivalence classes of generalized gradient controls will be denoted by $\boldsymbol{\mathcal{G}}(\mathrm{K}) \subseteq \boldsymbol{\mathcal{Y}}(\mathrm{K})$.

Theorem 2.6 (Properties of the spaces $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ and $\boldsymbol{\mathcal{G}}(\mathrm{K})$)**.**

1) [31, p. 115 f., Theorem 3.1] *Every sequence* $\{ u^N \}$, $L^\infty(\Omega, \mathbb{R}^{nm})$ *with* $u^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall N \in \mathbb{N}$ *admits a weak*-convergent subsequence, which generates a generalized control* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$*.*
2) [43, p. 450, Theorem 2.8, 1)] *Every sequence* $\{ x^N \}$, $W^{1,\infty}(\Omega, \mathbb{R}^n)$ *with* $\| x^N \|_{L^\infty(\Omega,\mathbf{R}^n)} \leqslant C$, $Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall N \in \mathbb{N}$ *admits a subsequence* $\{ x^{N'} \}$ *with* $x^{N'} \to^{C^0(\Omega,\mathbb{R}^n)} x \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ *and* $Jx^{N'} \xrightarrow{\ *\ }{}^{L^\infty(\Omega,\mathbb{R}^{nm})} Jx \in L^\infty(\Omega, \mathbb{R}^{nm})$*. Consequently,* $\{ Jx^{N'} \}$ *generates a generalized gradient control* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$*.*
3) [6, p. 144, Proposition 1, (i)] [9] *With respect to the topology from* (2.3)*, the set* $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ *is sequentially compact.*
4) [43, p. 450, Theorem 2.8, 2)] *The set* $\boldsymbol{\mathcal{G}}(\mathrm{K})$ *of the generalized gradient controls forms a sequentially compact subset of* $\boldsymbol{\mathcal{Y}}(\mathrm{K})$*.*

The mean value theorem of Kinderlehrer and Pedregal allows the following extension for generalized gradient controls $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$:

Theorem 2.7 (Mean value theorem for generalized gradient controls)**.** [43, p. 450, Theorem 2.9] [10] *Assume that* $\Omega \subset \mathbb{R}^m$ *is the closure of a strongly Lipschitz domain with* $, o \in \mathrm{int}\,(\Omega)$*. We consider sequences* $\{ w^N \}, \mathrm{K}$ *and* $\{ x^N \}, W_0^{1,\infty}(\Omega, \mathbb{R}^n)$*, which satisfy:*

a) $w^N \to w \in \mathrm{K}$ *(*w^N *and* $w \in \mathbb{R}^{nm}$ *have to be understood as* (n, m)*-matrices),*
b) $w^N + Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall N \in \mathbb{N}$*,*
c) $\{ w^N + Jx^N \}$ *generates a generalized gradient control* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$*.*

Then there exists a sequence of Lipschitz functions $\{ \widetilde{x}^N \}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ *with the following properties:*

1) $\lim_{N\to\infty} \| \widetilde{x}^N \|_{C^0(\Omega,\mathbb{R}^n)} = 0$*;*
2) $w^N + J\widetilde{x}^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall N \in \mathbb{N}$*;*
3) *The sequence* $\{ w^N + J\widetilde{x}^N \}$ *generates a constant generalized gradient control* $\boldsymbol{\nu} = \{ \nu \} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$*, which may be understood as the average of* $\boldsymbol{\mu}$ *with respect to* t*:*

$$\begin{aligned} \lim_{N\to\infty} \int_\Omega g(w^N + Jx^N(t))\, dt &= \int_\Omega \int_\mathrm{K} g(v)\, d\mu_t(v)\, dt \\ &= \lim_{N\to\infty} \int_\Omega g(w^N + J\widetilde{x}^N(t))\, dt \\ &= \int_\Omega \int_\mathrm{K} g(v)\, d\nu(v)\, dt \quad \forall\, g \in C^0(\mathrm{K}, \mathbb{R})\,; \end{aligned} \tag{2.5}$$

[9] Independently proved again in [29, p. 391, Theorem 4].
[10] As a generalization of [28, p. 334, Theorem 2.1].

4) *It holds that*

$$w = \begin{pmatrix} \int_{\mathrm{K}} v_{11}\, d\nu(v) & \dots & \int_{\mathrm{K}} v_{1m}\, d\nu(v) \\ \vdots & & \vdots \\ \int_{\mathrm{K}} v_{n1}\, d\nu(v) & \dots & \int_{\mathrm{K}} v_{nm}\, d\nu(v) \end{pmatrix}. \tag{2.6}$$

Theorem 2.7 justifies the definition of an average operator $A : \boldsymbol{\mathcal{G}}(\mathrm{K}) \to rca^{pr}(\mathrm{K})$, which assigns to any generalized gradient control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ a probability measure $A(\boldsymbol{\mu}) = \nu$ as its t-average.

2.3. Properties of the lower semicontinuous quasiconvex envelope $f^{(qc)}$.

The following results have been obtained in [42]:

Theorem 2.8 (Semicontinuity and continuity of $f^{(qc)}$)**.** [42, p. 89, Theorems 3.14 and 3.16, together with p. 95, Theorem 4.1] *Let a function $f \in F_{\mathrm{K}}$ be given.*

1) *The function $f^{(qc)} : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is lower semicontinuous.*
2) *$f^{(qc)}$ is continuous in every point $v \in \mathrm{int}\,(\mathrm{K})$.*
3) *Moreover, the restriction $f^{(qc)} \,\big|\, \mathrm{K}$ is continuous in every point $v \in \mathrm{ext}\,(\mathrm{K})$, and there the equations $f^c(v) = f^{(qc)}(v) = f(v)$ hold.*

Consequently, from $\partial\mathrm{K} = \mathrm{ext}\,(\mathrm{K})$ it follows that $f^{(qc)} \,\big|\, \mathrm{K}$ is continuous on the whole set K. Then together with f, $f^{(qc)}$ belongs to F_{K} as well.

Theorem 2.9 (Quasiconvexity and rank one convexity of $f^{(qc)}$)**.** [42, p. 93, Theorem 3.19] and [42, p. 95, Theorems 4.1 and 4.2][11] *Let a function $f \in F_{\mathrm{K}}$ be given. The the function $f^{(qc)} : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is quasiconvex (in the sense of Definition* 1.2.*) as well as rank one convex. Moreover, for all $v \in \mathbb{R}^{nm}$ it holds that*

$$f^c(v) \leqslant f^{pc}(v) \leqslant f^{(qc)}(v) \leqslant f^{rc}(v) \leqslant f(v)\,. \tag{2.7}$$

For $n = 1$ or $m = 1$, the envelopes f^c, f^{pc}, $f^{(qc)}$ and f^{rc} coincide.

2.4. Two representation theorems for $f^{(qc)}$.

For a function $f \in F_{\mathrm{K}}$, the envelope $f^{(qc)}$ may be represented in the following way in terms of Jacobi matrices:

Theorem 2.10 (First representation theorem for $f^{(qc)}$)**.** [42, p. 95, Theorem 4.1] *Let a function $f \in F_{\mathrm{K}}$ be given. Then its lower semicontinuous quasiconvex envelope $f^{(qc)} : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ admits the representation*

$$f^{(qc)}(v_0) = \lim_{\substack{v \to v_0 \\ v \,\in\, \mathrm{R} \cap \mathrm{int}\,(\mathrm{K})}} \begin{cases} f^*(v_0), & v_0 \in \mathrm{int}\,(\mathrm{K}) \\ f^*(v), & v_0 \in \partial\mathrm{K} \\ (+\infty), & v_0 \in \mathbb{R}^{nm} \setminus \mathrm{K}\,, \end{cases} \tag{2.8}$$

[11] The inequality $f^{pc}(v) \leqslant f^{(qc)}(v)$ follows from [40, p. 25].

where $\mathrm{R} = \overrightarrow{o\,v_0}$ *denotes the ray through* v_0 *starting from the origin, and* $f^*(v_0)$ *is defined by*

$$f^*(v_0) = \inf\left\{ \frac{1}{|\,\Omega\,|} \int_\Omega f(v_0 + Jx(t))\, dt \;\middle|\; \begin{array}{l} x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,, \\ v_0 + Jx(t) \in \mathrm{K} \;\; (\forall)\, t \in \Omega \end{array} \right\} \in \mathbb{R}\,. \quad (2.9)$$

Another possibility is the representation of $f^{(qc)}$ in terms of probability measures (in analogy to the convex envelope, cf. [37, p. 131, Theorem 10.19, 3)]). For this purpose, we define subsets of probability measures as follows:

Definition 2.11 (*Set-valued map* $\mathrm{S}^{(qc)}$)**.** (Synopsis of [43, p. 452, Definition 3.1 and Lemma 3.2, as well as p. 459, Theorem 3.9, 2)]) For any point $v_0 \in \mathrm{K}$, we define the following set of probability measures:

$$\mathrm{S}^{(qc)}(v_0) = \left\{ \nu \in rca^{pr}\,(\mathrm{K}) \;\middle|\; \begin{array}{l} \text{there exist sequences } \{\, v^N \,\}\,, \,\mathrm{int}\,(\mathrm{K}) \\ \text{and } \{\, x^N \,\}\,,\, W_0^{1,\infty}(\Omega, \mathbb{R}^n) \text{ with a)} - \text{d)} \end{array} \right\}, \quad (2.10)$$

where

a) $\lim_{N\to\infty} v^N = v_0$,
b) $\lim_{N\to\infty} \|\, x^N \,\|_{C^0(\Omega,\mathbb{R}^n)} = 0$,
c) $v^N + Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall\, N \in \mathbb{N}$,
d) $\{\, v^N + Jx^N \,\}$ generates the constant generalized gradient control $\boldsymbol{\nu} = \{\, \nu \,\}$.

Theorem 2.12 (Second representation theorem for $f^{(qc)}$)**.** [43, p. 444, Theorem 1.4] *Let a function* $f \in F_{\mathrm{K}}$ *be given. Then with the set-valued map* $\mathrm{S}^{(qc)} : \mathrm{K} \to P\big(\, rca^{pr}\,(\mathrm{K})\,\big)$ *from Definition* 2.11*, for all* $v_0 \in \mathrm{K}$ *it holds that*

$$f^{(qc)}(v_0) \;=\; \mathrm{Min} \left\{ \int_{\mathrm{K}} f(v)\, d\nu(v) \;\middle|\; \nu \in \mathrm{S}^{(qc)}(v_0) \right\}. \quad (2.11)$$

3. Lipschitz continuity and differentiability of $f^{(qc)}$

3.1. Global Lipschitz continuity of $f^{(qc)}$.

Proof of Theorem 1.4, 1)*.* [12] In order to prove the theorem, it suffices to show that the restriction $f^c \,\big|\, \mathrm{K}$ can be extended as a finite, convex function $h\colon \mathbb{R}^{nm} \to \mathbb{R}$ to the whole space. Indeed, the claimed extension h must be locally Lipschitz, in particular, in the neighbourhood of every point $v \in \mathrm{K}$. Consequently, K may be covered by a family $\{\, \mathrm{K}(v, \delta(v)) \,\}_{v\in\mathrm{K}}$ of open balls in such a way that h is Lipschitz continuous on $\mathrm{K}(v, \delta(v))$ with constant $L(v) > 0$, respectively. Since K is compact, the open covering $\{\, \mathrm{K}(v, \delta(v)) \,\}_{v\in\mathrm{K}}$ contains a finite subcovering

[12] The author was unable to find a proof for Theorem 1.4, 1) in the literature.

with $\mathrm{K} \subset \mathrm{K}(v_1, \delta(v_1)) \cup \ldots \cup \mathrm{K}(v_N, \delta(v_N))$. It follows that $h \big| \mathrm{K} = f^c \big| \mathrm{K}$ is globally Lipschitz on K with the constant $\mathrm{Max}\big(L(v_1), \ldots, L(v^N)\big)$.

It remains to prove that the demanded extension of f^c exists. This will be done by use of the following lemma:

Lemma 3.1. [21, p. 28, Theorem 2.1] *Let a convex body* $\mathrm{K} \subset \mathbb{R}^{nm}$ *and a convex function* $g : \mathrm{K} \to \mathbb{R}$ *be given. Assume that we can assign to every point* $v_0 \in \partial\mathrm{K}$ *another point* $w(v_0) \in \mathrm{int}\,(\mathrm{K})$ *with*

$$\lim_{\tau \to 0+0} \frac{g(v_0 + \tau(w(v_0) - v_0)) - g(v_0)}{\tau} > (-\infty)\,. \tag{3.1}$$

Then g *admits a finite, convex extension* $h : \mathbb{R}^{nm} \to \mathbb{R}$ *to the whole space.*

Let us fix a point $v_0 \in \partial\mathrm{K}$ and choose an arbitrary point $w(v_0) \in \mathrm{int}\,(\mathrm{K})$. Since $\partial\mathrm{K} = \mathrm{ext}\,(\mathrm{K})$, we have $f^c(v_0) = f(v_0)$. Choosing a number $0 < \tau \leqslant 1$, we arrive at the following estimates:

$$\begin{aligned} f^c(v_0 + \tau(w(v_0) - v_0)) - f^c(v_0) &\geqslant \sum_s \lambda_s f(v_s) - f(v_0) \\ &\geqslant \sum_s \lambda_s \big(\varphi(v_s, v_0) - \varphi(v_0, v_0)\big) \\ &= \sum_s \lambda_s \langle a(v_0), v_s - v_0\rangle \\ &= \Big\langle a(v_0), \sum_s \lambda_s v_s - v_0 \Big\rangle \\ &= \langle a(v_0), \tau(w(v_0) - v_0)\rangle\,, \end{aligned} \tag{3.2}$$

and hence

$$\begin{aligned} \frac{f^c(v_0 + \tau(w(v_0) - v_0)) - f^c(v_0)}{\tau} &\geqslant \langle a(v_0), w(v_0) - v_0\rangle \\ &\geqslant -\sup_{v_0 \in \partial\mathrm{K}} |\, a(v_0)\,| \cdot \mathrm{Diam}\,(\mathrm{K}) \\ &> (-\infty)\,, \end{aligned} \tag{3.3}$$

where $v_s \in \mathrm{K}$ and $\lambda_s \in [0, 1]$, $1 \leqslant s \leqslant nm+1$, satisfy $\sum_s \lambda_s = 1$ and $\sum_s \lambda_s v_s = v_0 + \tau(w(v_0) - v_0)$. Since the estimate (3.3) holds independently of τ, we may conclude that $f^c \big| \mathrm{K}$ satisfies the condition from Lemma 3.1. Consequently, there exists a finite, convex extension of $f^c \big| \mathrm{K}$ to the whole space, and the proof is complete. □

Proof of Theorem 1.5. We claim that $f^{(qc)} \big| \mathrm{K}$ is locally Lipschitz in the neighbourhood of every point $v \in \partial\mathrm{K}$. By assumption, $f \big| \mathrm{K}$ is globally Lipschitz continuous, and by Theorem 1.4, 1) the same holds for $f^c \,\big|\, \mathrm{K}$. Denote the maximum of the Lipschitz constants of f and f^c by L. Consider now an arbitrary point $v \in \partial\mathrm{K} = \mathrm{ext}\,(\mathrm{K})$ and fix a number $0 < \varepsilon < 1$. Then by Theorems 2.8, 3) and 2.9, for every point $w \in \mathrm{K} \cap \mathrm{K}(v, \varepsilon)$, $w \neq v$, it holds that

$$f^c(w) \leqslant f^{(qc)}(w) \leqslant f(w) \quad \text{and} \quad f^c(v) = f^{(qc)}(v) = f(v) \quad \Longrightarrow$$

$$\begin{aligned} -L \cdot |\, w - v| &\leqslant -|\, f^c(w) - f^c(v)| \\ &\leqslant f^c(w) - f^c(v) \\ &\leqslant f^{(qc)}(w) - f^{(qc)}(v) \\ &\leqslant f(w) - f(v) \\ &\leqslant |\, f(w) - f(v)| \\ &\leqslant L \cdot |\, w - v|\,. \end{aligned} \tag{3.4}$$

Analogously, we find

$$-f(w) \leqslant -f^{(qc)}(w) \leqslant -f^c(w) \quad \text{and} \quad f^c(v) = f^{(qc)}(v) = f(v) \quad \Longrightarrow$$

$$\begin{aligned} -L \cdot |\, v - w| &\leqslant -|\, f(v) - f(w)| \\ &\leqslant f(v) - f(w) \\ &\leqslant f^{(qc)}(v) - f^{(qc)}(w) \\ &\leqslant f^c(v) - f^c(w) \\ &\leqslant |\, f^c(v) - f^c(w)| \\ &\leqslant L \cdot \big|\, v - w|\,, \end{aligned} \tag{3.5}$$

and together

$$\sup_{\substack{w \in \mathrm{K} \cap \mathrm{K}(v,\varepsilon) \\ w \neq v}} \big|\, f^{(qc)}(v) - f^{(qc)}(w)\, \big| \leqslant L \cdot |\, v - w|\,. \tag{3.6}$$

Consequently, $f^{(qc)} \,\big|\, \mathrm{K}$ is locally Lipschitz continuous not only on $\mathrm{int}\,(\mathrm{K})$ but on $\mathrm{int}\,(\mathrm{K}) \cup \partial\mathrm{K} = \mathrm{K}$. Now the arguments from the proof of Theorem 1.4, 1) can be repeated, and the proof is complete. □

3.2. Differentiability points of $\boldsymbol{f^{(qc)}}$. We study a function $f \in F_{\mathrm{K}}$, which is differentiable on $\mathrm{int}\,(\mathrm{K})$. Then, by use of Theorem 2.3, we can describe certain points where the differentiability of f is carried over to $f^{(qc)}$:

Theorem 3.2. *Assume that a function $f \in F_{\mathrm{K}}$ is differentiable on* $\mathrm{int}\,(\mathrm{K})$. *Then the following assertions hold:*

1) (Differentiability in points with $f = f^{(qc)}$) *The function $f^{(qc)}$ is differentiable in every point $v_0 \in \mathrm{int}\,(\mathrm{K})$ with $f(v_0) = f^{(qc)}(v_0)$, and for all indices $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, it holds that*

$$\frac{\partial f^{(qc)}}{\partial v_{ij}}(v_0) = \frac{\partial f}{\partial v_{ij}}(v_0)\,. \tag{3.7}$$

2) (Differentiability in global minimizers of f) *If* $v_0 \in \operatorname{int}(\mathrm{K})$ *is a global minimizer of* f *then* $f^{(qc)}$ *is differentiable in* v_0*, and for all indices* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$*, it holds that*

$$\frac{\partial f^{(qc)}}{\partial v_{ij}}(v_0) = \frac{\partial f}{\partial v_{ij}}(v_0) = 0\,. \tag{3.8}$$

3) (Differentiability in relation to "supporting measures") [13] *Assume that* $\nu_1 \in \mathrm{S}^{(qc)}(v_1)$ *is a probability measure realizing the minimum from Theorem* 2.12 *in a point* $v_1 \in \mathrm{K}$*, i.e.,*

$$f^{(qc)}(v_1) = \int_{\mathrm{K}} f(v)\, d\nu_1(v) = \operatorname{Min}\Big\{ \int_{\mathrm{K}} f(v)\, d\nu(v) \,\Big|\, \nu \in \mathrm{S}^{(qc)}(v_1) \Big\}. \tag{3.9}$$

Then $f^{(qc)}$ *is differentiable in every point* $v_0 \in \operatorname{supp}(\nu_1) \cap \operatorname{int}(\mathrm{K})$*, and for all indices* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$*, it holds that*

$$\frac{\partial f^{(qc)}}{\partial v_{ij}}(v_0) = \frac{\partial f}{\partial v_{ij}}(v_0)\,. \tag{3.10}$$

Proof. 1): Since $\varphi'(v) = f^{(qc)}(v) \leqslant f(v) = \varphi''(v)$ for all $v \in \mathrm{K}(v_0, \varepsilon) \subset \mathrm{K}$ (Theorem 2.9), the assertion is an immediate consequence of Theorem 2.3.

2): On the one hand, at $v_0 \in \operatorname{argmin}(f) \cap \operatorname{int}(\mathrm{K})$ the inequality $f^{(qc)}(v_0) \leqslant f(v_0)$ is satisfied. On the other hand, the second representation theorem for $f^{(qc)}$ (Theorem 2.12) implies together with the theorem about the convexity of the integral (cf. [8, Chapter IV, § 6, p. 204, Corollaire]) that the range of $f^{(qc)}$ is a subset of the closed convex hull of the (compact) range of f. Consequently, the relation $f^{(qc)}(v_0) < f(v_0) = \operatorname{Min}_{v \in \mathrm{K}} f(v)$ cannot hold, and Theorem 2.3 can be applied again.

3): Let $\nu_1 \in \mathrm{S}^{(qc)}(v_1)$ be a probability measure with the claimed properties. Then by [41, p. 615, Theorem 3.9] , the values of $f^{(qc)}$ and $f(v_0)$ coincide for all $v_0 \in \operatorname{supp}(\nu_1) \cap \operatorname{int}(\mathrm{K})$, and Theorem 2.3 may be applied again. □

3.3. A sufficient condition for the differentiability of $f^{(qc)}$.

Proof of Theorem 1.6. `Step 1.` Assume that a point $v_0 \in \operatorname{int}(\mathrm{K})$, a probability measure $\nu_0 \in \mathrm{S}^{(qc)}(v_0)$, a generating sequence $\{ x^N \}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ and a number $0 < \mu < 1$ satisfy the assumptions a) – c) of the theorem. Then, in particular, it holds that

$$f^{(qc)}(v_0) = \int_{\mathrm{K}} f(v)\, d\nu_0(v)\,, \tag{3.11}$$

[13] Compare with [4, p. 346, Proposition 3.6] .

and the generalized controls $\{\, \delta_{v_0 + Jx^N(t)} \,\}$ converge in the sense of (2.3) to the constant generalized control $\{\, \nu_0 \,\}$. We choose now a further point $w \in \operatorname{int}(\mathrm{K})$. Since $v_0 + Jx^N(t) \in \mu\,\mathrm{K}$ $(\forall)\, t \in \Omega$ $\forall\, N \in \mathbb{N}$, we have

$$v_0 + h(w - v_0) + Jx^N(t) \in \mathrm{K} \quad (\forall)\, t \in \Omega\,,\ \forall\, N \in \mathbb{N}\,, \tag{3.12}$$

for all sufficiently small numbers $h > 0$. Then by Theorem 2.6, 2), a subsequence of the function sequence $\{\, v_0 + h(w - v_0) + Jx^N \,\}$ generates a generalized gradient control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$, whose average $A(\boldsymbol{\mu}) = \nu_h \in rca^{pr}\,(\mathrm{K})$ belongs to $\mathrm{S}^{(qc)}(v_0 + h(w - v_0))$ (we keep the index N). Applying again Theorem 2.7 and Theorem 2.12, we obtain

$$\begin{aligned} f^{(qc)}(v_0 + h(w - v_0)) &\leqslant \int_{\mathrm{K}} f(v)\, d\nu_h(v) \\ &= \lim_{N\to\infty} \frac{1}{|\,\Omega\,|} \int_{\Omega} f\bigl(v_0 + h(w - v_0) + Jx^N(t)\bigr)\, dt\,. \end{aligned} \tag{3.13}$$

Together with

$$f^{(qc)}(v_0) = \int_{\mathrm{K}} f(v)\, d\nu_0(v) = \lim_{N\to\infty} \frac{1}{|\,\Omega\,|} \int_{\Omega} f\bigl(v_0 + Jx^N(t)\bigr)\, dt\,, \tag{3.14}$$

we arrive at the following estimate for the difference quotient of $f^{(qc)}$:

$$\begin{aligned} &D(w - v_0, h) \\ &= \frac{1}{h}\Bigl(f^{(qc)}(v_0 + h(w - v_0)) - f^{(qc)}(v_0)\Bigr) \\ &\leqslant \lim_{N\to\infty} \frac{1}{|\,\Omega\,|} \int_{\Omega} \frac{1}{h}\Bigl(f\bigl(v_0 + Jx^N(t) + h(w - v_0)\bigr) - f\bigl(v_0 + Jx^N(t)\bigr)\Bigr) dt\,. \end{aligned} \tag{3.15}$$

`Step 2.` Since $\widetilde{f}$ is, by assumption, differentiable on some open neighbourhood of K, it admits on K the following Taylor expansion [14]

$$\widetilde{f}(v + hz) - \widetilde{f}(v) - \nabla\widetilde{f}(v)^{\mathrm{T}} hz = R(v, hz) \tag{3.16}$$

for all $v \in \mathrm{K}$, $z \in \mathbb{R}^{nm}$ and all sufficiently small $h > 0$. For fixed z and h, $R(v, h\,z)$ is continuous on K as a function of v. Moreover, the continuous differentiability of $\widetilde{f}$ implies its Fréchet differentiability, which may be expressed as follows: $\forall\, \varepsilon > 0$ $\exists\, \delta(\varepsilon) > 0$ such that for all sufficiently small $0 < h \leqslant 1$ and for all $v \in \mathrm{K}$ and $z \in \mathbb{R}^{nm}$ the implication

$$\bigl|\, h\,z \,\bigr| \leqslant \delta(\varepsilon) \implies \bigl|\, R(v, h\,z) \,\bigr| \leqslant \varepsilon \cdot \bigl|\, hz \,\bigr| \tag{3.17}$$

[14] In order to assure the existence of the Taylor expansion on the whole set K, we had to assume that $\widetilde{f}$ is continuously differentiable even on a *neighbourhood* of K.

holds, cf. [26, p. 36]. On the one hand, (3.17) implies that for fixed $v \in \mathrm{K}$ and $z \in \mathbb{R}^{nm}$

$$\lim_{h \to 0} \frac{R(v, hz)}{h} = 0 \tag{3.18}$$

holds; on the other hand, we observe that for fixed $z \in \mathbb{R}^{nm}$, the function sequence

$$\left\{ \frac{R\big(v, \frac{1}{N} z\big)}{\frac{1}{N}} \right\},\ C^0(\mathrm{K}, \mathbb{R}) \tag{3.19}$$

is uniformly convergent with respect to $v \in \mathrm{K}$, and the sequence possesses a continuous majorant. Consequently, from (3.15) we obtain:

$$\begin{aligned} D(w - v_0, h) \leqslant\ & \lim_{N \to \infty} \frac{1}{|\,\Omega\,|} \int_\Omega \nabla \widetilde{f}(v_0 + Jx^N(t))^{\mathrm{T}} \big(w - v_0\big)\, dt \\ & + \lim_{N \to \infty} \frac{1}{|\,\Omega\,|} \int_\Omega \frac{R\big(v_0 + Jx^N(t), h(w - v_0)\big)}{h}\, dt \end{aligned} \tag{3.20}$$

$$= \int_{\mathrm{K}} \nabla \widetilde{f}(v)^{\mathrm{T}} (w - v_0)\, d\nu_0(v) + \int_{\mathrm{K}} \frac{R(v, h(w - v_0))}{h}\, d\nu_0(v)\,. \tag{3.21}$$

From the majorized convergence $\lim_{h \to 0} \frac{1}{h} R(v, h(w - v_0)) = 0$ for all $v \in \mathrm{K}$ it follows that

$$\begin{aligned} D^+(w - v_0) &= \limsup_{h \to 0+0} D(w - v_0, h) \\ &\leqslant \int_{\mathrm{K}} \nabla \widetilde{f}(v)^{\mathrm{T}} (w - v_0)\, d\nu_0(v) \\ &= E(w - v_0)\,. \end{aligned} \tag{3.22}$$

`Step 3.` We invoke the following lemmata about quasiconvex functions, which may take the value $(+\infty)$:

Lemma 3.3. [42, p. 74, Lemma 2.10, (3)] *Let a point $w \in \mathbb{R}^{nm}$ and a number $\mu > 0$ be given. Together with $f(v) : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$, the function $g(v) = f(\,w + \mu\, v\,)$ is quasiconvex as well.*

Lemma 3.4. [42, p. 74, Theorem 2.12] *Let a convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ and a quasiconvex function $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ with $\mathrm{dom}\,(f) = \mathrm{K}$ be given. Assume that $f \,\big|\, \mathrm{K}$ is bounded. Then the restriction $f \,\big|\, \mathrm{int}\,(\mathrm{K})$ is rank one convex.*

By Lemma 3.3., the function $g(v) = f^{(qc)}(v + h(w - v)) = f^{(qc)}(hw + (1-h)v)$ is quasiconvex with respect to v together with $f^{(qc)}$. Since $\mathrm{dom}\,(g) = \big\{ v \in \mathbb{R}^{nm} \,\big|\, v \in \frac{1}{1-h}\mathrm{K} - \{hw\} \big\}$ and $w \in \mathrm{int}\,(\mathrm{K})$, we obtain $\mathrm{K}(v_0, \delta) \subset \mathrm{int}\,\big(\frac{1}{1-h}\mathrm{K} - \{hw\}\big)$ for a sufficiently small $\delta > 0$ and all sufficiently small $h > 0$. Then by Lemma 3.4., the quasiconvexity of $g(v)$ implies its rank one convexity and separate convexity on $\mathrm{K}(v_0 \delta)$. Consequently, for all $w \in \mathrm{int}\,(\mathrm{K})$ and all sufficiently small $h > 0$, $D(w - v_0, h)$ is separately convex as a function of $(w - v_0)$ on the interior of its (convex) effective domain, and particularly on $(w - v_0) \in \mathrm{K}(o, \delta)$. In the pointwise forming of the upper limit, this property is carried over to $D^+(w - v_0)$. Moreover, D^+ is positively homogeneous as a function of $(w - v_0)$ with $D^+(v_0 - v_0) = 0$ while $E(w - v_0)$ is a linear function of $(w - v_0)$. Now we may apply Theorem 2.3 to $\varphi'(w - v_0) = D^+(w - v_0)$ and $\varphi'' = E^+(w - v_0)$: both functions are differentiable in $(v_0 - v_0)$ with

$$\nabla D^+(v_0 - v_0) = \nabla E^+(v_0 - v_0) = \left(\int_{\mathrm{K}} \frac{\partial \widetilde{f}}{\partial v_{ij}}(v)\, d\nu_0(v) \right)_{i,j} . \tag{3.23}$$

We conclude that the functions D^+ and E^+ coincide for all $w \in \mathrm{K}(v_0, \delta)$ and, consequently, for all $w \in \mathbb{R}^{nm}$. Thus we obtain

$$D^+(w - v_0) = \sum_{i,j} \int_{\mathrm{K}} \frac{\partial \widetilde{f}}{\partial v_{ij}}(v)\, d\nu_0(v) \left(w_{ij} - v_{0,ij} \right) . \tag{3.24}$$

`Step 4.` From Theorem 2.3 we may infer in particular that, for a separately convex function g, the inequality (2.2) implies differentiability at v_0 (inserting $\varphi' = \varphi'' = g$). Thus we apply Theorem 2.3 again in order to confirm the differentiability of $f^{(qc)}$ in v_0 (which is a separately convex function on some convex neighbourhood of $v_0 \in \operatorname{int}(\mathrm{K})$). For this purpose, we claim that the relation

$$\limsup_{w \to o} \frac{1}{|\, w\,|} \Big(f^{(qc)}(v_0 + w) - f^{(qc)}(v_0) - \nabla D^+(v_0 - v_0)^{\mathrm{T}} w \Big) \leqslant 0 \tag{3.25}$$

holds true. Assuming on the contrary that there exist a number $\delta > 0$ and a sequence $\{\, w^N \,\}$, $\operatorname{int}(\mathrm{K}) \to o$ with

$$\delta < \frac{1}{|\, w^N\,|} \Big(f^{(qc)}(v_0 + w^N) - f^{(qc)}(v_0) - \nabla D^+(v_0 - v_0)^{\mathrm{T}} w^N \Big) \quad \forall\, N \in \mathbb{N}\, , \tag{3.26}$$

we may select a convergent subsequence of $\{\, w^N / \,|\, w^N\,|\,\}$ with limit w_0 (we keep the index N). Since $f^{(qc)}$ is locally Lipschitz on $\operatorname{int}(\mathrm{K})$ (Theorem 2.2), along this subsequence it holds that

$$\begin{aligned} \delta &< \frac{1}{|\, w^N|} \Big(f^{(qc)}\left(v_0 + w^N\right) \pm f^{(qc)}\left(v_0 + w_0 |\, w^N|\,\right) - f^{(qc)}(v_0) \Big) \\ &\quad - \nabla D^+(v_0 - v_0)^{\mathrm{T}} \frac{w^N}{|\, w^N|} \\ &\leqslant \frac{L}{|\, w^N|} \cdot \left| \left(v_0 + w^N\right) - \left(v_0 + w_0\,|\, w^N|\right) \right| \\ &\quad + \frac{1}{|\, w^N|} \Big(f^{(qc)}\left(v_0 + |\, w^N| w_0\right) - f^{(qc)}(v_0) \Big) - \nabla D^+(v_0 - v_0)^{\mathrm{T}} \frac{w^N}{|\, w^N|}\, , \end{aligned} \tag{3.27}$$

and hence

$$
\begin{aligned}
0 < \delta < \limsup_{N\to\infty} \ldots \\
= D^+\big((w_0+v_0)-v_0\big) - \nabla D^+(v_0-v_0)^{\mathrm{T}}\big((w_0+v_0)-v_0\big) = 0\,,
\end{aligned}
\tag{3.28}
$$

thus we arrive at a contradiction. Consequently, $f^{(qc)}$ is differentiable in $v_0 \in \mathrm{int}\,(\mathrm{K})$, and the proof is complete. □

Remarks. 1) The technique to characterize the derivatives of semiconvex envelopes with the aid of "supporting measures" has been introduced in [4] in the context of finite functions $f : \mathbb{R}^{nm} \to \mathbb{R}$. The proof of Theorem 1.6 as well as the example from Theorem 1.7 show the difficulties to carry over this approach to the case when f is allowed to take the value $(+\infty)$.

2) The conditions given in Theorem 1.6 resemble the fact that, under the assumptions of Theorem 1.4, 2), the gradient $\nabla f^c(v)$ of the convex envelope f^c equals to $\nabla f(\hat{v}_s)$ if the representation $f^c(v) = \sum_s \lambda_s f(v_s)$ with $v = \sum_s \lambda_s v_s$ and $\sum_s \lambda_s = 1$ contains a point $\hat{v}_s \in \mathrm{int}\,(\mathrm{K})$, cf. [25, p. 698, (3.4)] .

3) In all of the situations described in Theorem 3.2, the Dirac measure $\delta_{v_0} \in \mathrm{S}^{(qc)}(v_0)$ satisfies the conditions a) – c) of Theorem 1.6 together with the sequence $\{\, o \,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ and the number $\mu = 0.5$. In this sense, Theorem 1.6 may be considered as a generalization of Theorem 3.2.

3.4. Example: The derivatives of $f^{(qc)}$ cannot be represented through a "supporting measure". In this subsection, we provide an counterexample where the derivative of $f^{(qc)}$ in some differentiability point $v_0 \in \mathrm{int}\,(\mathrm{K})$ cannot be expressed by the formula (1.7). For this purpose, we take a function, which has been already investigated in [38] . In the following, the points $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ will be considered as $(2,2)$-matrices.

Lemma 3.5. (Cf. [38, p. 241, Definition 7]) *Let the points* $v_1 = \left(\begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$, $v_2 = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$ *and the convex set* $\mathrm{C} = \{\, \left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2} \;\big|\; b^2 + c^2 + d^2 \leqslant 1 \,\}$ *be given. We define* $\mathrm{K}_1 = \mathrm{co}\,\big(\{\, v_1 \,\} \cup \mathrm{C}\big)$, $\mathrm{K}_2 = \mathrm{co}\,\big(\{\, \mathrm{P}_2 \,\} \cup \mathrm{C}\big)$ *and* $\mathrm{K} = \mathrm{K}_1 \cup \mathrm{K}_2 \subset \mathbb{R}^{2\times 2}$. *Further, let the function* $f : \mathbb{R}^{2\times 2} \to \overline{\mathbb{R}}$ *be defined through*

$$
f(v) = \begin{cases} (a^2-1)^2\,, & v \in \mathrm{K} \\ (+\infty)\,, & v \in \mathbb{R}^{nm} \setminus \mathrm{K}\,. \end{cases}
\tag{3.29}
$$

Then the following assertions hold:

1) [38, p. 241, Lemma 1] K *is a convex body with* $o_4 \in \mathrm{int}\,(\mathrm{K})$ *and* $\mathrm{ext}\,(\mathrm{K}) = \{v_1, v_2\} \cup \big(\mathrm{ext}\,(\mathrm{C}) \setminus \{\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)\}\big)$.
2) [38, p. 241 f., Theorem 5] f *belongs to* F_{K}, *and* $f\,\big|\,\mathrm{int}\,(\mathrm{K})$ *is infinitely differentiable.*

3) *Although f satisfies the assumptions a) and b) from Theorem* 1.4, 1), *its convex envelope* $f^c \,\big|\, \mathrm{K}$ *is discontinuous: For all points* $\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) \in \mathrm{ext}\,(\mathrm{C})$ *with* $b \neq (-1)$, *we have* $f^c\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) = 1$ *but* $f^c\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) = 0$.
4) *For all points* $\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) \in \mathrm{ext}\,(\mathrm{C})$ *with* $b \neq (-1)$, *we have* $f^{(qc)}\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) = 1$ *and* $f^{(qc)}\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) = 0$ *as well.*

Proof. 3): Since $f(v) \geqslant 0$ for all $v \in \mathrm{K}$, the assumptions a) and b) from Theorem 1.4, 1) can be satisfied with $\varphi(v_0, v) \equiv 0$ for all $v_0 \in \partial\mathrm{K}$. However, by [38, p. 241 f., Theorem 5, ii)] , we have $f^c\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) = 1$ for all points $\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) \in \mathrm{ext}\,(\mathrm{C})$ with $b \neq (-1)$ and $f^c\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) = 0$.

4): By Part 3) and Theorem 2.9, for all points $\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) \in \mathrm{ext}\,(\mathrm{C})$ with $b \neq (-1)$ it holds that

$$1 = f^c\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) \leqslant f^{(qc)}\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) \leqslant f\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) = 1\,, \tag{3.30}$$

what means equality. Since $\overline{v_1\, v_2}$ is a rank-one segment, we conclude further that

$$f^{rc}\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \leqslant \frac{1}{2} f^{rc}(v_1) + \frac{1}{2} f^{rc}(v_2) \leqslant \frac{1}{2} f(v_1) + \frac{1}{2} f(v_2) = 0\,, \tag{3.31}$$

and again with Theorem 2.9

$$0 = f^c\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \leqslant f^{(qc)}\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \leqslant f^{rc}\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \leqslant 0\,. \tag{3.32}$$

The proof is complete. □

Proof of Theorem 1.7. `Step 1.` *Investigation of* $f^{(qc)}$ *in its discontinuity point* $\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)$. We abbreviate $w_0 = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)$ and choose a further point $w_1 = \left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right) \in \mathrm{ext}\,(\mathrm{C}) \cap \mathrm{ext}\,(\mathrm{K})$ with $b \neq (-1)$ and $|\, w_1 - w_0 \,| \leqslant 0.1$. By Lemma 3.5, 4), we have $f^{(qc)}(w_1) = 1$ and $f^{(qc)}(w_0) = 0$. Due to the radial continuity of $f^{(qc)}$ (Theorem 2.10), there exist numbers $\delta_0, \delta_1 \in \big(0, \frac{1}{2}\,\mathrm{Min}\big(|\, w_0\,|, |\, w_1\,|\big)\big]$ with

$$\begin{aligned} \big|w - w_0\big| \leqslant \delta_0 \quad &\Longrightarrow \quad f^{(qc)}(w) \leqslant 0.05 \qquad \forall\, w \in \mathrm{R}_0 = \overrightarrow{o\; w_0} \\ \big|w - w_1\big| \leqslant \delta_1 \quad &\Longrightarrow \quad f^{(qc)}(w) \geqslant 0.95 \qquad \forall\, w \in \mathrm{R}_1 = \overrightarrow{o\; w_1}\,. \end{aligned} \tag{3.33}$$

Here R_0 and R_1 denote the rays starting from o and passing through w_0 resp. w_1. With $\delta_2 = \mathrm{Min}(\delta_0, \delta_1) > 0$, we determine two points $z_0 \in \mathrm{R}_0$ and $z_1 \in \mathrm{R}_1$ with $|\, z_0 - w_0 \,| = \delta_2$ and $|\, z_1 - w_1 \,| = \delta_2$. Since $f^{(qc)}$ is continuous in $z_0, z_1 \in \mathrm{int}\,(\mathrm{K})$ (Theorem 2.8, 2), there exist numbers $\delta_3, \delta_4 > 0$ with

$$\begin{aligned} \big|z - z_0\big| \leqslant \delta_3 \quad &\Longrightarrow \quad \big|f^{(qc)}(z) - f^{(qc)}(z_0)\big| \leqslant 0.05 \qquad \forall\, z \in \mathrm{int}\,(\mathrm{K}) \\ \big|z - z_1\big| \leqslant \delta_4 \quad &\Longrightarrow \quad \big|f^{(qc)}(z) - f^{(qc)}(z_1)\big| \leqslant 0.05 \qquad \forall\, z \in \mathrm{int}\,(\mathrm{K})\,. \end{aligned} \tag{3.34}$$

Consequently, we may choose

$$0 < \delta_5 \leqslant \mathrm{Min}\bigg(\frac{\delta_2}{2}, \delta_3, \delta_4, \frac{1}{2}\,\big|w_1 - w_0\big|\bigg) \tag{3.35}$$

with $\mathrm{K}(z_0, \delta_5) \subset \mathrm{int}\,(\mathrm{K})$, $\mathrm{K}(z_1, \delta_5) \subset \mathrm{int}\,(\mathrm{K})$ as well as

$$z \in \mathrm{K}(z_0, \delta_5) \implies f^{(qc)}(z) \leqslant 0.1\,, \quad z \in \mathrm{K}(z_1, \delta_5) \implies f^{(qc)}(z) \geqslant 0.9\,. \tag{3.36}$$

`Step 2.` *Construction of a segment where $f^{(qc)}$ is differentiable λ^1-a.e.* We denote by Z the convex set $\mathrm{co}\,\big(\mathrm{K}(z_0, \delta_5) \cup \mathrm{K}(z_1, \delta_5)\big) \subset \mathrm{int}\,(\mathrm{K})$ and by N the λ^4-null set of the points $v \in \mathrm{int}\,(\mathrm{K})$ where the differentiability of $f^{(qc)}$ fails. Together with N, $\mathrm{Z} \cap \mathrm{N}$ is a λ^4-null set as well. Consider now the familiy $\{\,\mathrm{G}_p\,\}_{\,p \in \mathbb{R}^3}$ consisting of all straight lines parallel to the segment $\overline{w_0\, w_1}$. By [17, p. 232, Theorem 13.21.5] , for λ^3-almost all $p \in \mathbb{R}^3$, the intersections $\mathrm{Z} \cap \mathrm{N} \cap \mathrm{G}_p$ form one-dimensional null sets. Consequently, we may choose two points $y_0 \in \mathrm{K}(z_0, \delta_5)$ and $y_1 \in \mathrm{K}(z_1, \delta_5)$ in such a way that its connecting line segment $\mathrm{S} = \overline{y_0\, y_1}$ is parallel to $\overline{w_0\, w_1}$, and $f^{(qc)}$ is differentiable in λ^1-almost all points of S.

`Step 3.` *The claim that the partial derivatives of $f^{(qc)}$ admit a representation* (1.7) *in all differentiability points on* S *leads to a contradiction.* By [4, p. 340, Corollary 2.3] , the derivatives of $f^{(qc)}$ are continuous on its range of definition. Further, they are uniformly bounded on $\mathrm{Z} \setminus \mathrm{N}$ since $f^{(qc)}$ is even global Lipschitz continuous on the compact set Z. Thus the restrictions of the partial derivatives of $f^{(qc)}$ to S belong to the space $L^\infty(\mathrm{S}, \lambda^1)$, and we may apply [18, p. 301, Theorem 4.14] , along S:

$$f^{(qc)}(y_1) - f^{(qc)}(y_0) = \int_{\mathrm{S}} \nabla f^{(qc)}(v)^{\mathrm{T}} e\, dv\,, \tag{3.37}$$

where e denotes the unit vector in direction of $(w_1 - w_0)$. Assume now that the partial derivatives of $f^{(qc)}$ admit in all differentiability points $v_0 \in \mathrm{S}$ a representation

$$\frac{\partial f^{(qc)}}{\partial v_{ij}}(v_0) = \int_{\mathrm{K}} \frac{\partial \widetilde{f}}{\partial v_{ij}}(v)\, d\nu_0(v) \quad 1 \leqslant i \leqslant 2\,,\, 1 \leqslant j \leqslant 2\,, \tag{3.38}$$

where $\widetilde{f}\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) = (\,a^2 - 1\,)^2$ and $\nu_0 \in \mathrm{S}^{(qc)}(v_0)$ is a probability measure with $f^{(qc)}(v_0) = \int_{\mathrm{K}} f(v)\, d\nu_0(v)$. Then it follows that

$$\begin{aligned} \big|\nabla f^{(qc)}(v_0)\big| &= \bigg|\bigg(\int_{\mathrm{K}} \frac{\partial \widetilde{f}}{\partial v_{ij}}(\tilde{v})\, d\nu_0(\tilde{v})\bigg)_{i,j}\bigg| \\ &= \bigg|\int_{\mathrm{K}} \frac{\partial \widetilde{f}}{\partial v_{11}}(\tilde{v})\, d\nu_0(\tilde{v})\bigg| \\ &\leqslant \int_{\mathrm{K}} \bigg|\frac{\partial \widetilde{f}}{\partial v_{11}}(\tilde{v})\bigg|\, d\nu_0(\tilde{v})\,, \end{aligned} \tag{3.39}$$

and since

$$\frac{\partial \widetilde{f}}{\partial v_{11}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 4a(a^2 - 1) \tag{3.40}$$

and $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{K}$ implies $-1 \leqslant a \leqslant 1$, we obtain the inequality

$$\left|\nabla f^{(qc)}(v_0)\right| \leqslant \sup_{-1 \leqslant a \leqslant 1} \left|4a(a^2 - 1)\right| = \frac{8}{9}\sqrt{3} = 1.5396 \ldots < 2\,. \tag{3.41}$$

Together with (3.36), we arrive at the following estimates:

$$\begin{aligned} 0.8 \leqslant f^{(qc)}(y_1) - f^{(qc)}(y_0) &\leqslant \int_{y_0}^{y_1} \left|\nabla f^{(qc)}(v)\right| \cdot |e| \cdot |\cos \sphericalangle(\ldots)|\, dv \\ &\leqslant \left|y_1 - y_0\right| \cdot \sup_{v \in \mathrm{S}} \left|\nabla f^{(qc)}(v)\right| \\ &\leqslant \left(\left|z_1 - z_0\right| + 2\delta_5\right) \cdot 2 \\ &\leqslant 4\left|w_1 - w_0\right| \\ &\leqslant 0.4\,. \end{aligned} \tag{3.42}$$

The contradiction shows that the claim about the possible representation (1.7) of the partial derivative $\frac{\partial f^{(qc)}}{\partial v_{11}}$ along S holds wrong. Consequently, the segment S contains some point v_0 where $f^{(qc)}$ is differentiable but there exists no measure $\nu_0 \in \mathrm{S}^{(qc)}(v_0)$ with $f^{(qc)}(v_0) = \int_{\mathrm{K}} f(v)\, d\nu_0(v)$ and

$$\frac{\partial f^{(qc)}}{\partial v_{11}}(v_0) = \int_{\mathrm{K}} \frac{\partial \widetilde{f}}{\partial v_{11}}(v)\, d\nu_0(v)\,. \tag{3.43}$$

The proof is complete. □

Whether the validity of the representation (1.7) in all differentiability points of $f^{(qc)}$ can be ensured under stronger assumptions about $\partial \mathrm{K}$ remains an open question.

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